Nonlinear Quantum Mechanics

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Chapter 1

Introduction

Quantum mechanics is a wildly successful model of the small world. In this domain, the theory is purely linear, states evolve unitarily and all is well. On the other hand, unitary dynamics fail quite spectacularly to explain the world *we* see around us - the macroscopic world. Loosely speaking, this is the *measurement problem* of quantum mechanics - how does the classical world emerge from the quantum? The typical solution to the issue is to introduce the projection postulate:[1] upon measurement of an observable $A = \sum_a c_a \Pi_a$, the state $|\psi\rangle$ of the system collapses onto an eigenspace of A.[2] This particular process is nonlinear and stochastic; the eigenvalue we measure is a random variable, with $\langle\psi|\Pi_a|\psi\rangle$ as probability distribution[3] - the Born rule. The projection postulate takes care of a specific requirement: we'd like (instantaneously) repeated measurements on a system to yield the same value. The cost of preserving this reasonable requirement is that we have two incompatible dynamical laws governing the evolution of our system. Moreover, because of the ambiguity of the term *measurement*, it is not clear when either dynamical law is active. Evidently, this is not a desirable position to be in.

Suppose for the moment that we'd like to retain the notion of wavefunction collapse while also being able to compute the evolution of a quantum state at any given time. In this case one might imagine introducing a tiny bit of nonlinearity in the otherwise unitary dynamics, which can then break superpositions of states by itself and so give us collapse. As may also be imagined, this is a rather formidable task. Given a nonlinear Schrödinger equation, it is not at all clear how one retains norm conservation, or how one can make the nonlinearity affect only macroscopic systems, leaving microscopic ones untouched.

The idea of introducing nonlinearities into quantum mechanics isn't new; as far back as 1927 de Broglie introduced the first version of what is now usually known as Bohmian mechanics - a highly nonlinear model.[3] Fifty years later, Bialynicki-Birula and Mycielski studied the specific case where one adds the nonlinear term $\psi \ln |\psi|^2$ to the Schrödinger equation.[4] Another ten years later Weinberg constructed an entirely general framework for nonlinear quantum mechanics(NLQM)[5] - due to its flexibility it is his work we will use in our investigations. It should perhaps be mentioned that neither Weinberg nor Bialynicki-Birula and Mycielski were interested in NLQM from a foundational point of view, but rather an interest in the puzzling fact

[1] There are other ways of solving the issue, of course. In the Everett interpretation[1], the state vector evolves unitarily at all times, and there is no collapse. Quantum bayesianists[2] believe the state vector to represent our knowledge of the system in question - the collapse is thus simply the observer updating his knowledge of the system. We'll return to the discussion of various solutions to the quantum-classical transition in Chapter 1.

[2] Perhaps stating the obvious, Π_a is the projection operator onto the eigenspace corresponding to the eigenvalue a.

[3] We assume $\langle\psi|\psi\rangle = 1$; if this is not the case then the probability distribution must be divided by $\langle\psi|\psi\rangle$.

that nonlinearities seem to pervade most physical theories(e.g. general relativity, fluid dynamics) except quantum mechanics.

In any event, it was quickly found out that Weinberg's model suffered from severe issues, as N. Gisin pointed out that the theory violated causality and by so doing spelling its death.[6] Fast forward 20 years, an article written by Bennett et al.[7] claims that Gisin made a mistake in his proof, possibly revitalizing Weinberg's theory. One of our goals is then to check whether this holds true, with the overarching aim of understanding whether a physically consistent model for NLQM can at all be constructed.

Foreshadowing results, we will find that Gisin was in fact correct - in its initial form, Weinberg's framework *does* violate causality. However, the model of Weinberg can be modified to avoid the issue, and this was first done by J. Polchinski[8] - though unfortunately at the cost of allowing communication between branches of the wavefunction. This so-called "Everett phone" can in turn be removed, but at the cost of introducing yet other problems. Aside from these issues, Weinberg's model can be seen to admit observables with more eigenvectors than the dimension of the space it acts in, a non-associative product is introduced which ruins the usual probability interpretation, and so on. We will describe these problems in more detail later, as well as possible solutions. Following the chain of problem-solution to the end, we eventually conclude that the Weinberg theory is - even in a heavily modified form - not physically consistent.

At first glance, this seems to be the end of NLQM. But it should be noted that Weinberg's framework, though general, does not encompass all types of nonlinearities. In particular Weinberg treats only deterministic NLQM, whereas one might well imagine adding a *stochastic* nonlinear term to the Schrödinger equation. This is exactly what is done in the so-called dynamical reduction models, initially developed by P. Pearle[9] as well as G.C. Ghirardi, A. Rimini and T. Weber[10]. An important goal for us is then to check if and why this variety of NLQM is consistent.

As it turns out, we will find that reduction models satisfy the physical principles that laid waste to Weinberg's framework. Reduction models do not come without their own set of issues, however. One unavoidable consequence of dynamically induced collapse in position space is a net increase in energy - as such, many reduction models violate the principle of energy conservation. Solving this issue is superficially possible, but we will see that some proposed solutions are at odds with the very reason for introducing nonlinearities in the first place. Another potential issue is that the stochasticity is "integrated out" to yield regular quantum mechanical predictions. In other words, reduction models are hidden variable theories, which we know to be rather problematic. We'd then like to investigate whether they are of the permissible type - loosely speaking, are the hidden variables non-local or local?

Aside from the main goal of understanding whether reduction models either are or can be made physically consistent, we will also want to investigate a number of their features, with a possible falsification in mind. To start with, reduction models should tell us *when* a system has collapsed. Surprisingly this hasn't been much treated in the literature, so we will spend some time quantifying these concepts. We find that the time of collapse is in some sense still subjective, and in any case a random variable. We numerically estimate its distribution, finding it to be sharply localized at the order of magnitude set by its mean. Furthermore reduction models predict that outcomes of experiments may be reversed, by virtue of the underlying stochastic evolution. We quantify this, finding it to be of little importance in the case of experiments making use of macroscopic measurement apparatuses.

To briefly summarize, the underlying motivation for this thesis is the resolution of the measurement problem. Our main goal is to understand whether this can be achieved through the introduction of nonlinearities in the Schrödinger equation. In particular, can this be done deterministically? If it cannot, which physical principle(s) prevents it? Furthermore, can it be

done stochastically, as it is in dynamical reduction models? If it can, how *exactly* does collapse work in these models? Can we use some of its features for falsification?

To answer these questions, the thesis is structured as follows. In Chapter 2 we set the stage by describing the measurement problem, and briefly some alternative solutions besides NLQM. In Chapter 3 we concern ourselves with deterministic NLQM in the context of Weinberg's framework. We present his model and its issues, attempting to resolve them. The chapter ends as we discover the precise reason for the failure of Weinberg's framework and simultaneously the reason stochastic NLQM *can* work. This leads us to Chapter 4, where we discuss stochastic NLQM in the context of dynamical reduction models. A model known as QMUPL will in particular take most of our attention, and we use it to discuss various features of reduction models in some detail. We leave the potentially two most important issues for last, namely energy (non)-conservation and the underlying hidden variables. We will see that the hidden variables are local and so unproblematic in the context of Bell's theorem, whereas energy (non-)conservation has not yet been satisfactorily solved. We end the chapter and our discussion of reduction models by presenting current experimental bounds.

Chapter 2

The Measurement Problem

As we have briefly mentioned, the measurement problem is connected to several important issues in quantum mechanics. If you believe in wavefunction collapse, an explanation must be given for when it kicks in. And if you do not, the quantum-to-classical transition must still be explained. Nonlinear theories offer some hope in resolving the issue from the standpoint of accepting wavefunction collapse, but before discussing that we need to elaborate on the measurement problem itself in some more detail.

2.1 The von Neumann Measurement Scheme

To illustrate the measurement problem, we will need a model for the measurement process itself. To that end, we will use von Neumann's measurement scheme. It is set up as follows. Consider a quantum system S and a measurement apparatus M. The Hilbert space $\mathcal{H}(M)$ must have dimension at least equal to $\mathcal{H}(S)$. We believe the world to be quantum[1] - thus we must be able to assign a wavefunction to both M and S. Furthermore, we assume S and M to be initially non-interacting and spatially separated. Thus we take their wavefunction to be a product one. We will consider an *ideal* measurement - that is, we assume that the measurement can be represented by a Hermitian operator. Suppose we measure $A = \sum_i c_i \Pi_i = \sum_i c_i \left|s_i\right\rangle \left\langle s_i\right|$:

$$\left|\chi\right\rangle = \left|\psi_0\right\rangle \left|r\right\rangle = \sum_i a_i \left|s_i\right\rangle \left|r\right\rangle \tag{2.1}$$

$\left|r\right\rangle$ denotes the inital or "ready" state of the measuring apparatus. We assume the interaction between S and M to evolve the initial state $\left|\chi\right\rangle$ into a final state $\left|\chi'\right\rangle$ like so

$$\left|\chi\right\rangle \longrightarrow \left|\chi'\right\rangle = \sum_i a_i \left|s_i\right\rangle \left|m_i\right\rangle \tag{2.2}$$

This evolution is typically referred to as *premeasurement*. Note that this type of ideal measurement does not change the state of S *if* the initial state is one of the $\left|s_i\right\rangle$. Of course, if it is not, then

[1] The original Copenhagen interpretation postulates the separate existence of a classical and a quantum realm, each of which must be described within their appropriate frameworks. In other words, classicality does not emerge from the quantum. It follows that there exists a divide - the Heisenberg cut - separating these two realms. This is hard to swallow, considering the by now vast amount of experiments confirming quantum behaviour of successively larger systems - from molecules to micrometer-sized mechanical oscillators.[11] The notoriously vague concept of measurement within this interpretation will therefore be of no interest to us - we accept that the interpretation itself is flawed.

S can be assigned no definite state at all. At this point, the various parts of the measurement problem are now more or less apparent.

2.2 The Problem

Following Schlosshauer[12], we will divide the measurement problem into three parts. They are:

(i) The problem of preferred basis

(ii) The problem of outcomes

(iii) The problem of macroscopic interference

Let us discuss the problem of preferred basis first. Consider the premeasurement state, Eqn. (2.2). It can be rewritten in infinitely many ways, e.g.

$$\sum_i a_i \left|s_i\right\rangle \left|m_i\right\rangle = \sum_i a_i' \left|s_i'\right\rangle \left|m_i'\right\rangle \tag{2.3}$$

The same interaction then seems to constitute a measurement of many different, possibly non-commuting observables. For one, this is quite simply not possible - non-commuting observables cannot simultaneously have sharp values. Furthermore, it is worrying because we know that in practice, measurement apparatuses(and the interactions they induce) are designed to measure a single observable only. Consider for example a particle entering a Stern-Gerlach apparatus. It has a magnetic field oriented along a particular axis, and so will *only* measure the corresponding spin component of the particle. Thus it is peculiar that our model seems to predict that a single interaction can measure multiple observables.

A partial resolution can be obtained, however. We will want our apparatus states(or pointer states, as they are usually called) to be orthogonal[2], e.g. $\langle m_i|m_j\rangle = \delta_{ij}$. Otherwise the apparatus is useless for distinguishing measurement outcomes, which defeats its purpose for existing. Note that under the assumption of an ideal measurement, the states $\{s_i\}$ form a basis for $\mathcal{H}(S)$. Then the Schmidt decomposition

$$\left|\chi'\right\rangle = \sum_i d_i \left|s_i\right\rangle \left|m_i\right\rangle \tag{2.4}$$

is unique if all d_i are distinct. This would then give us the preferred basis, and so tell us what the measurement outcomes are. Of course, we cannot guarantee this to be the case, hence the problem.

Then there is the problem of macroscopic interference. The von Neumann scheme clearly amplifies quantum superpositions. And yet we do not observe interference effects between the possible pointer positions on our measuring apparatus.[3] Indeed, interference effects between *any* kind of macroscopic superposition seem absent, and it is not clear why. [4]

[2] As a first approximation, at any rate. Due to the Heisenberg uncertainty principle, exact position eigenstates carry infinite energy, and so are quite unphysical.

[3] Note importantly that we do not speak of the measuring apparatus as being in a superposition - it can be assigned no definite state. Yet if the set $\{s_i\}$ is not orthogonal, interference effects may still be observed.

[4] The usual explanation is that a particle of mass m has a wavelength $\lambda \propto 1/m$ - the interference is then unobservable because we cannot manufacture gratings with the required dimensions. This is unsatisfactory for two reasons. One, not all interference effects are of this type. We can e.g. have interference between superpositions of macroscopic currents. Second, other methods exist for determining whether interference effects are present. In particular, Zeilinger et al. used the Talbot-Lau effect in their demonstration that C_{70}-molecules exhibit interference effects.[13]

Lastly there is the problem of outcomes. Given $|\chi'\rangle$, why do we observe only a single pointer state, say $|m_4\rangle$? And why is *that* outcome selected, i.e. why do we observe $|m_4\rangle$ and not $|m_2\rangle$? This issue is particularly tricky, because supplying an interpretation of the premeasurement state $|\chi'\rangle$ is essentially the same as providing an interpretation for quantum mechanics itself. And as we know, preference for any given interpretation is highly subjective.

Of course, the above presentation of the measurement problem was based on a highly idealized version of measurement. Many of the steps do not correspond to what happens in an actual measurement. Let us list a few objections:

(1) The measuring apparatus is macroscopic, and we cannot control all its degrees of freedom. Thus it cannot be prepared in precisely the ready state $|r\rangle$.

(2) We have treated the system+apparatus as a closed quantum system. But the apparatus is macroscopic, and must be treated as an open quantum system.

(3) The perfect correlation between initial system states and final apparatus states is in general physically unattainable. Consider e.g. a photon detector - it changes the Fock state from $|1,\ldots\rangle \to |0,\ldots\rangle$. Paraphrasing Wallace[14], physicists usually measure something by slamming it very hard into something else.

These do not, however, imply any non-existence of the measurement problem. In fact, assuming only the ability to prepare (micro)systems in eigenstates of certain observables and measurement apparatuses with some level of reliability, one can prove the amplification of superpositions from the microlevel to the macrolevel.[15] Although the position operator poses problems(we can't prepare a system in any of its eigenstates), any observable with a discrete spectrum will do.

With the problem presented, let us briefly cover some approaches to its resolution. This necessarily involves various interpretations of QM, some of which introduce new postulates in addition to what we will call "bare" QM. To be precise:

Bare QM

1. **States** are represented by rays in a complex Hilbert space. Equivalently, they are elements of the corresponding projective Hilbert space.
2. **Observables** are represented by Hermitian operators acting on the above Hilbert space.
3. **Evolution** is unitary and determined by the Schrödinger equation, $i\partial_t |\psi\rangle = H |\psi\rangle$.
4. **Born rule:** if the state of a system is represented by $|\psi\rangle$ and we measure an observable $A = \sum_a c_a |a\rangle \langle a|$, the probability of getting the outcome c_a is given by $P(c_a) = \frac{|\langle a|\psi\rangle|^2}{|\langle\psi|\psi\rangle|}$.

2.3 Resolving the problem

A widely accepted solution[12, 16] to points (i) and (iii) of the measurement problem exists - it is the phenomenon of decoherence, and its description requires nothing but bare QM. It is unable to address point (ii), however, the consideration of which at present belongs to the realm of the infamous *interpretations* of quantum mechanics. The most well-known of these - we'll call it the

standard interpretation - postulates that there is something wrong with the dynamics. Instead of always evolving unitarily, a system sometimes evolves according to the rules of wavefunction collapse. On the other hand, we have various relative state interpretations that stick with bare QM and simply tell us how premeasurement superpositions are to be interpreted. Yet another approach - and it is the one we will mostly be concerned with - is that of the dynamical reduction models. Like the standard interpretation, they postulate a modified dynamics. Unlike the standard interpretation, the evolution of a system is *unified* - it is determined by a single (stochastic) Schrödinger equation. Of course, these approaches are all quite subjective, in the sense that none of them predict (at present) experimentally detectable deviations from standard QM. If we are to make up our mind one way or the other, it will be of interest to study the strengths and weaknesses of each approach. For completeness, we give a very brief overview of the abovementioned approaches below. Considerably more exhaustive expositions of these interpretations(and others) may be found in the literature. We also give a brief presentation of decoherence,[5] not only for its intimate connection to the measurement problem but also for the striking resemblance its equations bears to those of the dynamical reduction models. We return to this point in Chapter 4.

2.4 Decoherence

Let us begin by defining decoherence. Consider a general superposition state

$$|\phi\rangle = \sum_j a_j |j\rangle \tag{2.5}$$

with density matrix

$$\rho_s = |\phi\rangle \langle\phi| = \sum_{ij} a_j^* a_i |i\rangle \langle j| \tag{2.6}$$

The cross-terms $|i\rangle \langle j|, i \neq j$ are often referred to as interference terms. If we were considering a superposition of energy eigenstates, they would cause interference effects that manifest as an oscillation in the expectation of the energy, for example. The state *decoheres* through the vanishing of these terms; as a consequence, a fully decohered state is represented by a diagonal density matrix. In relation to the measurement problem, this is what we want for macroscopic objects, as we do not observe interference effects at this scale. But what causes the decoherence?

In the 1970's, H. D. Zeh highlighted the role of the environment in relation to the decoherence process.[17] Consider the case where the states $|j\rangle$ in Eqn. (2.5) represent position states of the Earth, each time in a different galaxy. So $|1\rangle$ is Earth here in the Milky Way, $|2\rangle$ is Earth in M31, and so on. Clearly the state of the environment depends on which state the Earth is in - orbits of asteroids and planets are affected, numerous photons are either absored or not, etc.[6] Suppose that Earth and environment start off in the initial state $|\phi\rangle |E_0\rangle$. Through their interaction, they evolve into the state $|\psi(t)\rangle$:

$$|\psi(t)\rangle = \sum_j a_j |j\rangle |E_j(t)\rangle \tag{2.7}$$

[5] See ref. [12] for an introduction to the subject.

[6] The usual terminology for this is that the environment records which-path information, in analogy with the double slit experiment. If we record some amount of information about which path the particles take through the slits, the interference pattern on the screen will be correspondingly weaker.

Very quickly, the environmental states will become approximately orthogonal, $\langle E_i(t)|E_j(t)\rangle \approx \delta_{ij}$. If we now compute the reduced density matrix of the Earth, we find[7]

$$\rho_s(t) = \mathrm{Tr}_{\mathrm{Env}} |\psi(t)\rangle \langle\psi(t)| \sum_{ij} a_j^* a_i |i\rangle \langle j| \langle E_j(t)|E_i(t)\rangle \longrightarrow \sum_i |a_i|^2 |i\rangle \langle i| \tag{2.8}$$

In fact, this solves both points (i) and (iii) of the measurement problem. Macroscopic interference vanishes at the same rate at which the environmental states become approximately orthogonal. Furthermore, the preferred basis is selected by the interaction between Earth and environment[8], consisting of those states that are immune to the environment-induced decoherence. In general the preferred basis(or pointer basis, as it is sometimes called) emerges as the eigenstates of the interaction Hamiltonian that pertains to the system alone - these are the states that remain unperturbed under its action.

We end with two important points about the nature of decoherence. Given that the main ingredient is the (produced) entanglement between system and environment, the quantum coherence is not actually lost but only apparently so - it is *delocalized* into the environment. By recording enough knowledge about the environment, we could in principle reconstruct the interference effects. Secondly, due to the entanglement the system cannot in general be assigned any definite state at all. This is in contrast to the situation in dynamical reduction models, as we shall later see.

2.5 The Standard Interpretation

This is the most traditional way of doing quantum mechanics. We add a new postulate to bare QM, namely:

Projection postulate: Upon measurement of an observable A on a system S in state $|\psi\rangle$, the state *collapses* into one of the eigenspaces of A, with probability given by the Born rule. Precisely, measurement induces the following evolution:

$$|\psi\rangle \longrightarrow \frac{\Pi_a |\psi\rangle}{\|\Pi_a |\psi\rangle\|} \tag{2.9}$$

Where Π_a is a projector onto the eigenspace $A_i = \sum_i |a_i\rangle \langle a_i|$ with eigenvalue a.

Note that historically[9] this postulate was motivated by a desire to have (instantaneously) repeated measurements yield the same outcome. Given that we take observables to be Hermitian operators, the only way to guarantee this is if the state, upon measurement, falls into an eigenspace corresponding to the eigenvalue obtained.

From the projection postulate and the Born rule follows the so-called

Eigenstate-Eigenvalue link: if a system S is in an eigenstate $|o\rangle$ of an observable O, we say that o is a property of S.

[7] Remembering that the environmental states $\{E_i\}$ do not form a basis, we trace out using another arbitrary basis.

[8] Typically referred to as environment-induced superselection or einselection, the emergence of a preferred basis in this way was first studied by W. H. Zurek, see ref. [18].

[9] It was originally introduced by Dirac and von Neumann.[19, 20]

Of course, this minimal ontology is immediately appealing. But in view of operators with a continuous spectrum, it is problematic. Concretely, let us consider the concept of a localized particle. Position eigenstates are out, as they carry infinite energy. Representing it by a Gaussian is unacceptable, because there is a non-zero probability to find it *anywhere* in the universe. Localizing the particle by requiring its wavefunction to have compact support doesn't work either, because then its Fourier transform extends across all of momentum space. A particle which is localized in this sense has completely indefinite momentum, and so it will be delocalized over all space in an arbitrarily short amount of time. In short, the framework supplied is insufficient for understanding localization.[14] This is a major problem, but it can be handled by the introduction of a more general concept of measurement, e.g. the POVM formalism.[21]

Conceptually, the most worrying problem with the standard interpretation is the so-called *trigger problem*. Essentially, because of the vagueness of what we refer to as "measurement", it is not at all clear when we should evolve a system under the Schrödinger equation and when we should evolve it through the projection postulate. In other words, although the postulate claims the system to be in a particular eigenstate *after* a measurement, we cannot know when that is without a precise definition of measurement.

2.6 Relative state interpretations

The most well-known relative state interpretation is the many-worlds interpretation, often mentioned just as the Everett interpretation.[10] As we have mentioned, their most important feature is the lack of a collapse mechanism - the wavefunction evolves unitarily at all times. Indeed, Everett's original motivation for constructing his interpretation was the conflict between linear statevector evolution and nonlinear statevector collapse in the standard interpretation, the difficulties of which are often illustrated in terms of an example called "Wigner's friend". As it also elucidates relative state interpretations, we briefly recall it.[22][11]

Wigner's friend sits inside a closed room, together with his measurement apparatus and a quantum system Q. At some point, he decides to measure an observable on his system, and writes down the result in his notebook. Outside the room, Wigner is shuffling about. Being a good physicist, he has already assigned a state $|\psi\rangle$ to his friend and the quantum system, and he has computed its time evolution. But Wigner's friend is a solitary person, and has asked Wigner to remain outside for a year.

After a year, Wigner looks at his wavefunction. He notes that it is a superposition of friend+quantum states, much like the premeasurement state in Eqn. (2.2). Finally barging through the door and into the room, the wavefunction collapses and Wigner informs his friend about the state he and his quantum system is in. In fact, he says, they haven't really been in any state at all for the entire past year!

This all sounds quite peculiar to Wigner's friend, who wrote down the result of his measurement one year past. But they get no further in their discussion, as a car screeches to a halt beside them. Out jumps a remote aquaintance, declaring the current state of Wigner, Wigner's friend and the quantum system. In fact, he says, they haven't really had any state at all for the entire past year and five minutes!

Of course, the paradox here depends crucially on the concepts of measurement and the subsequent wavefunction collapse. In a relative state interpretation, there is no collapse and

[10] They are not quite the same. Everett never made any mention of "worlds" - this (metaphysical) aspect of the interpretation was introduced by B. DeWitt.

[11] Whether it was Wigner or Everett who came up with the example, it first appeared in Everett's thesis. Wigner later used the example to illustrate his belief that consciousness was a necessary ingredient in the measurement process.[23]

no paradox. Indeed, suppose Wigner assigns the following wavefunction to his friend and the quantum system at the end of the year:

$$|\psi\rangle = \sum_i a_i |o_i\rangle |wf_i\rangle \tag{2.10}$$

where $|o_i\rangle$ are eigenstates of the measured observable O and $|wf_i\rangle$ are the states Wigner's friend would be in had he measured the corresponding eigenvalue o_i. According to the Everett interpretation, this wavefunction describes a completed measurement. All outcomes o_i have occured and are equally real, each belonging to a so-called *branch* of the wavefunction. This allows Wigner to assign a definite state to either his friend or the quantum system, conditioned on the state of the other. Thus if he barges into the room and finds out his friend is in the state $|wf_k\rangle$, he is able to assign the state $|o_k\rangle$ to the quantum system. In the words of DeWitt, a history of such branchings constitute a *world*, thus the many-worlds interpretation.

The many-worlds interpretation has its own weaknesses, of course. It suffers from the preferred basis problem: like in the von Neumann measurement, it is not clear in which basis the branching occurs. This is typically resolved by appealing to decoherence, e.g. the preferred basis is determined by the interaction between system and environment. Closely connected to this is the analogue of the trigger problem in the standard interpretation - it is not obvious *when* the wavefunction branches. One might again appeal to decoherence and argue that a measurement is concluded once the interference terms have decayed. But this decay is exponential and we can identify no clear divide.

Another issue is that of explaining the (apparent) probabilistic nature of quantum mechanics from a deterministic theory in which every outcome occurs. In his thesis, Everett appealed to measurements on ensembles of identically prepared systems.[1] Consider an observer in the ready state $|\psi_0\rangle$ who measures an observable O on N identical systems prepared in the state $|\phi_0\rangle = \sum_j a_j |o_j\rangle$. The $|o_j\rangle$ are eigenstates of O. The initial state is given by

$$|\tau\rangle = |\psi_0\rangle \prod_{i=1}^{N} |\phi_0^i\rangle \tag{2.11}$$

it then evolves into the final state

$$|\tau'\rangle = \sum_{i_1,i_2,\ldots,i_N} a_{i_1} a_{i_2} \ldots a_{i_N} \left|o_{i_1}^1\right\rangle \left|o_{i_2}^2\right\rangle \ldots \left|o_{i_N}^N\right\rangle \left|\psi_{[a_{i_1},a_{i_2},\ldots,a_{i_N}]}\right\rangle \tag{2.12}$$

Where the bracket in the final observer state $\left|\psi_{[\ldots]}\right\rangle$ refers to the outcomes observed. If this observer now looks at his collection of outcomes, it *appears* that each system has jumped into one of the eigenstates $|o_j\rangle$ at random. Thus an illusion of probability *does* exist. The more problematic issue is that of assigning a quantitative measure to this probability, e.g. of deriving the Born rule. Everett's initial attempt was unsuccessful[24], subsequent attempts like those of Albert and Loewer[25](the Many-Minds interpretation) and Deutsch[26](an approach based on decision theory, a close relative of game theory) have likewise failed to gain traction within the many-worlds community.

Lastly we mention a criticism often aimed at the proposed ontology of the interpretation. It postulates the existence of a universe consisting of an *extremely* large number of worlds. But it is a principle of physics that things should be kept as simple as possible(Ockham's Razor), and having what is for all practical purposes an infinity of worlds around seems to go against this principle. On the other hand, the number of postulates of quantum mechanics is reduced, and we get a single law governing the dynamics. Whichever of these seem "simplest" is ultimately up to personal preference.

Chapter 3

Nonlinear Quantum Mechanics

3.1 The Weinberg model for NLQM

In order to discuss Gisin's proof that Weinberg's theory breaks Lorentz invariance we'll need at least a basic understanding of it. In the following we therefore present the required theoretical minimum, as well as some of the more apparent issues the theory faces. The full theory is of course to be found in Weinberg's paper.[5]

3.2 Formalism

States are represented by complex-valued functions ψ, depending on either a continuous or discrete variable. For our purposes it will be sufficient to work with spin-1/2 systems, such that $\psi = \psi_k$, where $k = 1, 2$.

In order to accomodate the possibility of nonlinear observables, they are represented by *functions* of the statevector and its conjugate, e.g. $a(\psi_k, \psi_k^*)$. An observable represented by the linear operator A in regular QM would then be represented by the bilinear function $\psi_k^* A_{kl} \psi_l$ in NLQM. A homogeneity condition is imposed on the observables in order to ensure that ψ and $\alpha\psi$ represent the same physical state, where α is an arbitrary complex constant.

Definition
A function $f : X \to Y$ is homogeneous of degree k in x if

$$f(\alpha x) = z^k f(x) \tag{3.1}$$

where z is a constant. Equivalently, f is homogeneous of degree k in x iff

$$x\partial_x f(x) = k f(x) \tag{3.2}$$

which is readily seen by differentiating Eqn. (3.1) with respect to z. A multivariate function may be homogeneous of any order in any or all of its arguments.

Weinberg imposes homogeneity of degree one in both ψ and ψ^*, such that Eqn. (3.2) reads

$$\psi_k \frac{\partial a}{\partial \psi_k} = \psi_k^* \frac{\partial a}{\partial \psi_k^*} = a \tag{3.3}$$

We'll see how the homogeneity condition takes care of $z\psi \sim \psi$ when we come to eigenvalues. Sums of observables and multiplication by scalars work in the usual way. Multiplication of two observables a and b, however, is defined thus

$$a \star b \equiv \frac{\partial a}{\partial \psi_k} \frac{\partial b}{\partial \psi_k^*} \tag{3.4}$$

This $\star$-product is not commutative or associative. The latter property causes problems for the usual probability interpretation, based on the moments of observables. We return to this later.

In regular QM, an observable A is the generator of some symmetry operation, and under this operation it induces an infinitesimal change in the wavefunction. We generalize this change, generated by a possibly non-bilinear observable a as follows:

$$\delta_a \psi_k = -i\epsilon A_{kl}\psi_l = -i\epsilon \frac{\partial a}{\partial \psi_k^*} \tag{3.5}$$

Where ϵ is some real, infinitesimally small parameter. In the linear case, the latter equality is obvious. We postulate it to hold even when a is non-bilinear. Given the change in ψ_k under a, we can also compute the change in another observable b induced by a. We find

$$\delta_a b = i[a, b] \equiv i(a \star b - b \star a) \tag{3.6}$$

This is of course very much analogous to classical mechanics, where $[\cdot, \cdot]$ defines a Poisson bracket. As always, we assume the Hamiltonian to generate time evolution, e.g.

$$\epsilon \delta_h \psi_k = \psi_k(t + \epsilon) - \psi_k(t) \implies \frac{d\psi_k}{dt} = -i\frac{\partial h}{\partial \psi_k^*} \tag{3.7}$$

Using this generalized Schrödinger equation, we may also find the time dependence of observables

$$\frac{d}{dt}a = [h, a] \tag{3.8}$$

Importantly, both the norm $\psi_k^*\psi_k$ and the Hamiltonian function h are conserved.

Given two non-interacting systems with Hamiltonians h_I and h_{II}, the Hamiltonian for the complete system h_{I+II} is formed like so

$$h_{I+II} = \sum_l h_I(\psi^{(l)}, \psi^{(l)*}) + \sum_k h_{II}(\phi^{(k)}, \phi^{(k)*}) \tag{3.9}$$

where

$$\psi_k^{(l)} \equiv \phi_l^{(k)} \equiv \Psi_{kl} \tag{3.10}$$

Where k, l represent degrees of freedom in system I and II respectively. This way of doing it is desirable in the sense that it reduces to the direct sum for bilinear Hamiltonians; unfortunately it does not, in general, avoid introducing interactions into the total system. Again, we illustrate this later.

The eigenvalue condition is generalized in the by now usual way:

$$A_{kl}\psi_l = \alpha\psi_k \quad \longrightarrow \quad \frac{\partial a}{\partial \psi_k^*} = \alpha\psi_k \tag{3.11}$$

Contracting with ψ_k^*, we find that the eigenvalue is given by $\alpha = a/n$, where n is the norm[1]. Note in particular that, because of the one-homogeneity of observables, $z\psi$ and ψ have the same eigenvalue. Furthermore, expectation values are defined as

$$\langle a \rangle = \frac{a}{n} \tag{3.12}$$

Thus if ψ is an eigenstate of a, the expectation value is the eigenvalue, as we would expect. And, again, the one-homogeneity ensures that the expectation value is the same for $z\psi$ and ψ.

3.3 Example of dynamics

In discussing the theory, it will be useful to have a concrete example at hand. We'll make use of a Hamiltonian that was used in experimental tests of Weinberg's theory, and which Gisin also made use of in his proof. It is given as

$$h_\sigma = \frac{\langle\psi|\sigma_z|\psi\rangle^2}{\langle\psi|\psi\rangle} \tag{3.13}$$

It has two eigenvalues: 1 and 0, the latter of which is infinitely[2] degenerate. The Hamiltonian commutes with σ_z in the sense of Eqn. (3.6), and so it is conserved. This makes it easy to compute the dynamics. The generalized Schrödinger equation gives

$$i\partial_t \psi_k = \frac{\partial h_\sigma}{\partial \psi_k^*} = 2\langle\sigma_z\rangle\sigma_{z,kl}\psi_l - \langle\sigma_z\rangle^2\psi_k \tag{3.14}$$

The two equations are decoupled, and since everything on the RHS is a constant of motion, we may immediately integrate them to obtain the solution

$$|\psi(t)\rangle = c_1 e^{-i(2\langle\sigma_z\rangle - \langle\sigma_z^2\rangle)t}|\uparrow_z\rangle + c_2 e^{-i(-2\langle\sigma_z\rangle - \langle\sigma_z^2\rangle)t}|\downarrow_z\rangle \tag{3.15}$$

$$= c_1 e^{-i\omega_1 t}|\uparrow_z\rangle + c_2 e^{-i\omega_2 t}|\downarrow_z\rangle \tag{3.16}$$

The corresponding time evolution operator is simple:

$$U_\sigma(t, t_0 = 0) = \begin{pmatrix} e^{-i\omega_1 t} & 0 \\ 0 & e^{-i\omega_2 t} \end{pmatrix} \tag{3.17}$$

Although it is clear that U_σ depends on the state it acts on, it is the same for all states with the same value of $\langle\sigma_z\rangle$, e.g. it does not change form within an orbit.

To better make use of our example Hamiltonian h_σ, it will be useful to visualize the motion of arbitrary states on the Bloch sphere, see Fig. 3.1. Some peculiarities are worth noting. First, the angular frequency is state-dependent. Secondly, states on the upper and lower hemispheres have an opposite sense of rotation. In fact, a quick computation of for example the x-component of the Bloch vector shows that the angular frequency varies linearly with z from $\Omega = 4 \to \Omega = -4$. At the equator, all states have $\Omega = 0$. Note also that, due to the state dependency, the motion does not conserve the inner product.[3] An explicit example is shown in Section 3.4.2.

[1] Due to the homogeneity requirement, many equations will have the norm inserted, although it is always unity. To avoid clutter, an unqualified n will mean the norm from now on.

[2] Any state with $\langle\sigma_z\rangle = 0$ will do.

[3] Interestingly, this causes the entropy to oscillate in time - a seeming violation of the second law. It is, however, not clear how entropy should even be defined in a nonlinear context.[27]

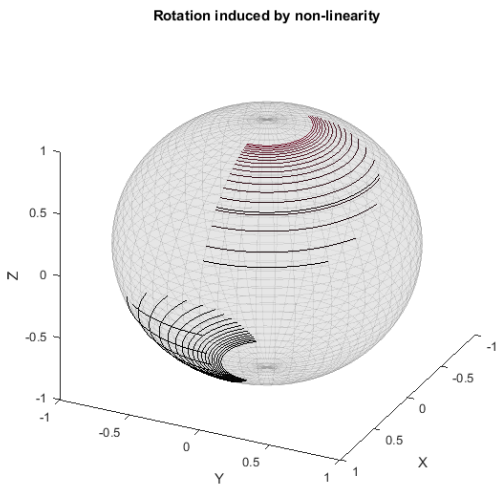

Figure 3.1: Motion of states on the Bloch sphere. All states are started off in the plane $y = 0$, with varying x and z-coordinates. Note that the rotation frequency Ω depends on the state, and that $\Omega \to 0$ as $z \to 0$. We see also that the sense of rotation is opposite on the two sides of the hemisphere.

3.4 Issues

It is interesting in itself to see the issues that a model for nonlinear quantum mechanics faces. In particular, depending on the severity of the issue, it may offer arguments or even proof that QM must be linear. Indeed, this is what the original Gisin paper claimed to do. In this section we collect some problems of smaller severity that the Weinberg model faces; the worst problems we present later.

3.4.1 Basis dependent observables

As first pointed out by Czachor, the value of an observable may depend on the choice of basis for the corresponding Hilbert space.[28] In order to see it, we consider two non-interacting systems S and R, where R has $h_r \propto n$. For simplicity, we set it to zero. Furthermore we take the corresponding Hilbert spaces $\mathcal{H}(\{S, R\})$ to be of dimension two. S is goverened by the Hamiltonian h_σ.

Consider now a general state ${\psi_s}^r$ of the combined system. According to Eqn. (3.9), the total Hamiltonian is

$$h_{\sigma+r} = \sum_r \frac{(|{\psi_1}^r|^2 - |{\psi_2}^r|^2)^2}{|{\psi_1}^r|^2 + |{\psi_2}^r|^2} \tag{3.18}$$

Consider the state

$$|\psi\rangle = {\psi_s}^r \, |s, r\rangle = {\psi_1}^1 \, |1, 1\rangle + {\psi_2}^2 \, |2, 2\rangle \tag{3.19}$$

In this case

$$h_{\sigma+r} = |{\psi_1}^1|^2 + |{\psi_2}^2|^2 \tag{3.20}$$

Now rotate the basis for $\mathcal{H}(R)$ by $\pi/4$, i.e. $\{|i\rangle\} \to \{|i'\rangle\}$, such that our state becomes

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\psi_1{}^1\,|1\rangle + \psi_2{}^2\,|2\rangle\right) \otimes |1'\rangle + \frac{1}{\sqrt{2}}\left(\psi_1{}^1\,|1\rangle - \psi_2{}^2\,|2\rangle\right) \otimes |2'\rangle \tag{3.21}$$

$$= \phi_1{}^1\,|11'\rangle + \phi_2{}^1\,|21'\rangle + \phi_1{}^2\,|12'\rangle + \phi_2{}^2\,|22'\rangle \tag{3.22}$$

And so

$$h_{\sigma+r} = \frac{(|\psi_1{}^1|^2 - |\psi_2{}^2|^2)^2}{|\psi_1{}^1|^2 + |\psi_2{}^2|^2} \tag{3.23}$$

Thus by changing the basis for $\mathcal{H}(R)$ we have changed the total energy in $S + R$. Since h_r is linear(in fact, zero), regular QM applies and we know that the basis change cannot change the energy in R. Thus we have remotely changed the energy in another, non-interacting system through a change of basis. Evidently, this is not a desirable property. The problem can however be remedied by implementing a modification of the Weinberg theory, first proposed by Polchinski. We'll return to this later.

3.4.2 Probability interpretation

In regular QM, all powers of any observable O commute, any may thus be measured simultaneously. Therefore, they may be used to define a probability distribution for the eigenvalues. Furthermore, if a system is in an eigenstate $|o\rangle$ of O, the n'th moment of O is then simply the n'th power of the corresponding eigenvalue - in other words, we are guaranteed that this eigenvalue is what we get when we measure O, and nothing else.

In Weinberg's model, things are different. In particular, the $\star$-product we encountered earlier is not associative - thus the powers of any observable do not commute, and consequently we cannot even define higher moments.[4] It follows that we cannot use the moments to define a probability distribution, though it may of course still exist. The more problematic issue is that even if we are in an eigenstate of some observable A, its expectation value is not uniquely defined. Weinberg was aware of it and proposed a solution, though not without its own issues. Before describing it, let us also mention another interesting argument for why the regular probability interpretation fails.

Consider our usual Hamiltonian h_σ given in Eqn. (3.13). We have seen that it has infinitely many eigenstates. These must then be linearly dependent, and so a given arbitrary state cannot be uniquely expanded in terms of them. The Born rule[5] tells us that $|\langle\psi|i\rangle|^2$ is the probability of getting the outcome i. However, $|\langle\psi|i\rangle|^2$ is now non-unique, and so cannot be interpreted as a probability. The contradictory behaviour is also seen by considering a repeated measurement of h_σ. No matter which eigenvalue we first get, we will have a non-zero probability of subsequently measuring at least one other eigenvalue. By way of example, suppose our first measurement yields eigenvalue zero, so we expect to be in one of the states with $\langle\sigma_z\rangle = 0$. The eigenspace corresponding to eigenvalue one is the entire Hilbert space, thus according to the Born rule we will measure it with unit probability.[6] This is at odds with our view that a system in an eigenstate has a *definite* value of the corresponding eigenvalue. Equivalently, it makes the

[4] Note that $O \star O$ *is* uniquely defined, so we have a notion of variance. $O \star O \star O$ can also be uniquely defined by requiring it to be real. For higher powers however, no unique definition can be obtained.

[5] Weinberg doesn't explicitly define an inner product, but he does define the norm, from which we get the usual inner product.

[6] Note also that we have a non-zero overlap with almost all other states satisfying $\langle\sigma_z\rangle = 0$. It is then clear that the probabilities do not add up to one.

projection postulate inconsistent in the sense that it no longer guarantees a definite outcome for repeated measurements.[7]

Another type of contradictory behaviour that arises if we accept the Born rule is that forward and backward transition probabilities are no longer the same. This was noticed by Helou & Chen in an article claiming to show that the Born rule is ambiguous in NLQM.[29] Their argument was based on the fact that the time-evolution operator U is state-dependent and so $|\langle\psi U|\phi\rangle| \neq \left|\langle\psi\middle|U^\dagger\phi\rangle\right|$. If true, it would not be clear how to interpret an expression like $\langle\phi|U|\psi\rangle$ in NLQM. We believe this conclusion is mistaken, for the following reason.

Suppose we wish to generalize the expression $\langle\phi|U|\psi\rangle$ to NLQM. Helou & Chen claim it can be interpreted either as $\left\langle\phi U_\phi^\dagger\middle|\psi\right\rangle$ or $\langle\phi|U_\psi\psi\rangle$. But it is only the latter expression that makes sense, because the time evolution operator in NLQM is not unitary. This is so because nonlinear mappings are in general not isometries, and so cannot be unitary. Thus $U_\phi^\dagger|\phi\rangle$ does not describe backwards time evolution - it is not clear if it describes any physical operation at all. That said, the conclusion that backwards and forwards transition probabilities do not coincide holds, and we describe this next.

In regular QM, for two states $|i\rangle$ and $|f\rangle$, and a time evolution operator U, we have

$$p_{i\to f} = |\langle f|Ui\rangle|^2 \qquad\qquad p_{i\leftarrow f} = \left|\langle i\middle|U^{-1}f\rangle\right|^2 \tag{3.24}$$

Since U is unitary, $p_{i\to f} = p_{i\leftarrow f}$. But as mentioned, U is not unitary in NLQM, such that $p_{i\to f} \neq p_{i\leftarrow f}$ in general. Let us illustrate by way of our example Hamiltonian h_σ. Choose $|i\rangle = |\uparrow_x\rangle$ and $|f\rangle = \frac{1}{\sqrt{1+\epsilon^2}}\left(|\downarrow_x\rangle + \epsilon\,|\uparrow_z\rangle\right)$. Then

$$p_{i\to f} = |\,\langle f|U_i|i\rangle\,|^2 = |\,\langle f|i\rangle\,|^2 = \frac{1}{1+\epsilon^2}\frac{\epsilon^2}{2} \tag{3.25}$$

but

$$p_{i\leftarrow f} = |\,\langle i|U_f^{-1}|f\rangle\,|^2 = \frac{1}{1+\epsilon^2}\left(1+\frac{\epsilon}{\sqrt{2}}\right)^2 \tag{3.26}$$

where we have chosen the evolution time $t_\pi = \frac{\pi}{4\epsilon}\frac{1+\epsilon^2}{\sqrt{2}+\epsilon}$ to correspond to a rotation by π on the Bloch sphere. Note also that as $\epsilon \to 0$, $p_{i\to f} \to 0$ while $p_{i\leftarrow f} \to 1$. This is rather unpleasant - if we are in the state $|f\rangle$, with probability approaching one we came from state $|i\rangle$ a time t_π ago. And yet if we are in the state $|i\rangle$, we will with probability approaching 0 be in the state $|f\rangle$ after t_π. The example may also be visualized with the help of Fig. 3.1. We see that $|i\rangle$, lying on the equator, does not rotate at all. $|f\rangle$, however, will always rotate due to its non-zero component along the $z-$direction. Evolving $|i\rangle$ will thus have no effect, giving us the inital overlap which we choose close to zero. On the other hand, we can easily evolve $|f\rangle$ to its antipodal point, ensuring an overlap with $|i\rangle$ close to unity.

3.4.3 Measurement of nonlinear observables

Weinberg instead proposes a probability interpretation based on a type of indirect measurement. Consider a beam of particles in some state ψ_k. Let it pass through an external field that couples (transverse) position x to the observable a we wish to measure, such that $h = gxa$, where g is a coupling constant. Weinberg then assumes the system to be integrable[8], such that the solution

[7] Equivalently, h_σ cannot be expressed in terms of a set of commuting projectors.

[8] Roughly, a system of differential equations is integrable when the number of conserved quantites is maximal. In this case, one may in principle identify a canonical transformation to a set of coordinates where the conjugate momenta are all cyclic. These may then be easily integrated to yield the solution.

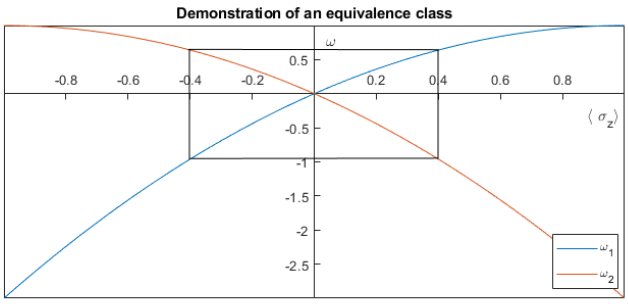

Figure 3.2: The two eigenfrequencies ω_1 and ω_2 are plotted as a function of $z = \langle\sigma_z\rangle$. The black rectangles illustrate a single equivalence class, in the following sense. Consider a state $|\alpha\rangle$ with $\langle\sigma_z\rangle = 0.4$. We wish to find its equivalence class $[\alpha]$. Evidently, all other states $|\beta\rangle$ with the same value of $\langle\sigma_z\rangle$ belong to it. To find the others, we start by drawing a vertical black line through the $\langle\sigma_z\rangle$-axis, interesecting the two eigenfrequency curves. Any state sharing either of these eigenfrequencies belongs to $[\alpha]$. To find them, we draw horizontal lines. Where they intersect an eigenfrequency curve, we draw a vertical line to identify the new state. The figure implies that the states with $\langle\sigma_z\rangle = \pm 1$ have two eigenfrequencies - this is of course not the case, as they only have one non-zero component.

will be quasi-periodic:

$$\psi_k(t) = \sum_n c_n \exp\left(-igxt\omega_{n,\psi}\right) \tag{3.27}$$

Where the $\omega_{n,\psi}$ are in general a linear combination of the eigenvalues of a. The subscript ψ indicates that they are also state dependent. After spending a time τ in the field a particle will have transverse momentum $p = -g\omega_{n,\psi}\tau$.[9] Assuming a distinct set of $\omega_{n,\psi}$, we measure this momentum with probability $P = |c_n|^2$. This measurement is then taken to represent a measurement of the original observable a, whose possible outcomes are $\omega_{n,\psi}$.

This is satisfactory in the sense that we give a prescription for the possible values of outcomes when we measure the observable a. Moreover, somewhat miraculously, the expectation computed by means of $\langle a\rangle = \sum_n P(\omega_{n,\psi})\omega_{n,\psi}$ agrees with our previous definition in Eqn. (3.12).

However we do not think that it is fully satisfactory, for the following reason. Dirac introduced the projection postulate to ensure that instantaneously repeated experiments yield the original result with probability one. This requirement is not met by the measurement scheme above. Let us see why. As always, we work with h_σ as our example. To be concrete, consider the equivalence relation[10]

$$\psi \sim \phi \iff \omega_{n,\psi} = \omega_{m,\phi} \text{ for at least one pair } (n,m) \tag{3.28}$$

The equivalence classes(see Fig 3.2) collect those states that share at least one eigenfrequency. Now, unless the equivalence class $[\Psi]$ contains an eigenstate, *any* $\psi \in [\Psi]$ will have two distinct

[9] The position operator is the generator of momentum, $e^{ixp'}|p\rangle = |p+p'\rangle$ with $|p\rangle$ a momentum eigenstate.

[10] This construction is not useful in QM, since it does not actually define an equivalence relation - it is not transitive. Concretely, consider a spin-1/2 particle. We have $|\uparrow_z\rangle \sim |\phi\rangle$, where $|\phi\rangle$ is any state that is not an eigenstate of σ_z. But $|\phi\rangle \sim |\downarrow_z\rangle$, so if $\sim$ was transitive we would have $|\uparrow\rangle \sim |\downarrow\rangle$. However, $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ belong to different equivalence classes by definition.

eigenfrequencies. Suppose at least one $\psi \in [\Psi]$ has eigenfrequency Ω. Given any other state ϕ, the probability of measuring Ω is zero unless $\phi \sim \psi$, or $\phi \in [\Psi]$.

Suppose now that we measure an eigenfrequency $\omega_{i,\phi}$ on a state ϕ. We assume that it is not an eigenvalue. If we immediately measure it again, the state after the first measurement must belong to $[\phi]$ - the probability of obtaining $\omega_{i,\phi}$ is zero otherwise. But since the states in any equivalence class not containing an eigenstate share at least two distinct frequencies, we cannot get $\omega_{i,\phi}$ with probability one.

This leads us back to the equivalence classes containing eigenstates, since these are the only sets of states that yield a certain eigenfrequency with probability one. And yet this is not feasible either, as we have seen, since the eigenvalues cannot be assigned a unique probability.

Similar difficulties with nonlinear observables have been noted by Jordan[30] and to some extent Czachor[11][28], though following different arguments. Jordan attempted to define a nonlinear observable p that is idempotent, e.g. $p \star p = p$. However, under the assumption that the Weinberg $\star$-product is the correct one and with attention restricted to Hilbert spaces of dimension 2,[12] no such observable exists. In particular, this shows that no nonlinear observable can project onto *any* subspace, which is what we need for repeated measurements to yield the same outcome.

3.4.4 Failure of mixed state description

The theory of Weinberg does not give a prescription for handling mixed states. To do so, the theory must be extended in some way. Polchinski [8] proposed one way of doing this. According to him, observables should be functions only of the corresponding system's density matrix.[13] There is strong argumentation for this, as we shall see later. Even accepting this modification of the theory, all is not well. As Polchinski noted, if we know the observable in terms of its action on pure states, the mapping $A(|\psi\rangle) \mapsto A(\rho)$ is not unique. For example, h_σ may be written equally well as

$$h'_\sigma = \frac{(\operatorname{Tr} \rho\sigma_z)^2}{\operatorname{Tr} \rho} \qquad \text{or} \qquad h''_\sigma = \frac{\operatorname{Tr}(\rho\sigma_z\rho\sigma_z)}{\operatorname{Tr} \rho} \tag{3.29}$$

Both reduce to h_σ for $\rho = |\psi\rangle \langle\psi|$, but that they have different values for *mixed* states. Thus it appears Weinberg's theory cannot be uniquely extended to handle mixed states. This implies that we cannot treat composite systems uniquely either. Equivalently we are unable to treat correlated states uniquely, as they give rise to density matrices on the subsystem level. To solve the problem, Weinberg's theory should be reformulated from the ground up in terms of density matrices.[14]

3.5 Faster-than-light communication in NLQM

As we have mentioned, N. Gisin published an article showing how Weinberg's theory is in conflict with the special theory of relativity. It is not entirely clear how serious the issue is, however, as Weinberg's theory makes no claim to be a relativistic theory. Indeed, faster-than-light phenomena occur in regular QM too. Consider for example a person who measures the momentum of a particle to be $p = p_0$. This spreads the wavefunction evenly across the entire universe, yielding a non-zero probability for the particle to subsequently be absolutely anywhere. In other words,

[11] By way of concrete examples.

[12] Weinberg gives a handy transformation(see appendix A in ref. [5]) in this case, allowing us to express a general observable as $h = n\bar{h}(a)$, where $a = |\psi_2|^2/n$. This considerably simplifies calculations.

[13] Note that this solves the problem of basis-dependent observables, too.

[14] This has been done by Jordan, and we will discuss it in Section 3.13.

QM allows particles to propagate with arbitrarily high speed. In a non-relativistic setting where we have a single global frame, this does not violate causality. But in a relativistic theory, superluminal propagation of any kind of information does imply a violation of causality and is unacceptable. The same problem is in a sense also present in field theory. Particles, or more properly field excitations, have a non-zero amplitude to propagate outside their forward lightcone. The resolution is well-known[31]: taking into account the amplitude for the corresponding anti-particle to travel in the opposite direction, we find that their field operators commute at spacelike separations. And as commuting operators can be simultaneously measured, this ensures that causality is not violated.

More serious are violations of relativity wherein no concept of spacetime intervals are needed. Potential superluminal communication by entanglement is one such example - if this was possible in regular QM, it would carry over to QFT and invalidate the theory.[15] Another example is the one given by Gisin in the context of NLQM. As we shall now see, it makes use of entanglement supplemented by (local) nonlinear evolution. This prohibits a relativistic generalization of Weinberg's theory, and it is in this sense that it is invalidated.

3.6 Experimental protocol

We first give a quick description of Gisin's proof that Weinberg's theory violates Lorentz invariance. To do so, he constructs a telephone that can send faster-than-light(FTL) signals. To prepare it, we dispatch pairs of particles from a source to our sender Alice and receiver Bob. We take the particles to be spin-1/2, prepared in the singlet state and travelling along the y-axis of our coordinate system(see Fig. 3.3). To send the message Alice uses a Stern-Gerlach apparatus, which she rotates in either the z- or u-direction. The u- and z-axes are 45 degrees apart, and both lie in the xz-plane. After this, she lets her particle impinge on a detector. The particles flowing toward Bob are then in a mixture of half-half spin-up/spin-down along either the z- or u-directions. We note that these mixtures are equal in the sense that they are represented by the same density matrix.

To read the message Bob lets his particles enter a region of field with Hamiltonian h_σ, the dynamics of which we computed in Section 3.3. He then measures $\langle\sigma_y\rangle$ on his particles. For the σ_z eigenstates, this is zero at all times. But the σ_u eigenstates have, due to the nonlinearity, opposite directions of rotation, and after a given time yield a non-zero value for $\langle\sigma_y\rangle$. Communication is thus established as Bob can determine whether Alice chose to align her Stern-Gerlach apparatus along the z- or u-axis.

3.7 Important principles

The telegraph as described is primarily based on the fact that a nonlinear evolution may, in general, allow us to distinguish two initially equal mixtures. In notation, we have a nonlinear mapping g_t, acting on pure states such that

$$|\psi_i\rangle\langle\psi_i| \to g_t(|\psi_i\rangle\langle\psi_i|) \tag{3.30}$$

We shall assume that its action on a mixture is the action on each of its components, an assumption we will return to later. Consider, then, two ensembles that are initially equal, in the

[15] Depending on the amount of respect we have for causality, that is.

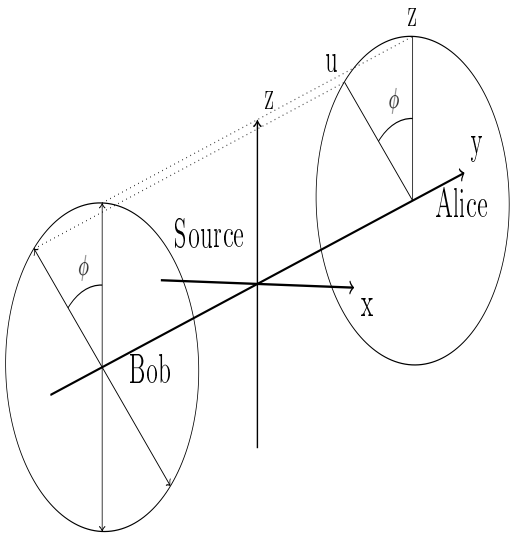

Figure 3.3: The setup of the Gisin telegraph is shown. Alice sends messages to Bob by rotating her apparatus along either the $z-$ or u-axis and measuring the spin of her particle. We *assume* that this collapses Bob's particle to the corresponding spin along that axis. A subsequent non-linear evolution will enable Bob to tell ensembles of spin along the $u-$ and $z-$axes apart. Note that in Gisin's paper $\phi = \pi/4$, but looking back at Fig. 3.1 we see that any angle that isn't an integer multiple of $\pi/2$ will work.

sense that their action on an arbitrary state vector $|\alpha\rangle$ is the same

$$\sum_i x_i \left|\psi_i\right\rangle \left\langle\psi_i|\alpha\right\rangle = \sum_j y_j \left|\phi_j\right\rangle \left\langle\phi_j|\alpha\right\rangle \tag{3.31}$$

Considering for a moment the linear case, it is evident that they remain equal for all time - the above statement is for example time-independent in the Heisenberg picture. In the non-linear case, the argument is not so clear due to the strangeness of the Heisenberg picture we have there.

However, in general, when working out the time evolution of an arbitrary mixture, we expand in energy eigenstates and use the linearity of the Hamiltonian. In the nonlinear case this fails, and hence we expect

$$\sum_i x_i g_t \left(\left|\psi_i\right\rangle \left\langle\psi_i\right|\right) \neq \sum_j y_j g_t \left(\left|\phi_j\right\rangle \left\langle\phi_j\right|\right) \tag{3.32}$$

This is indeed the case in Weinberg's theory, and it is what Gisin exploits. In summary:

Pure state nonlinear evolution does not uniquely determine mixed state evolution

Also of importance is that these identical mixtures can instantly be prepared at arbitrary separation.[16] Gisin proves this[32], but it is fairly evident in our case, as we use the rotationally invariant singlet state. Hence the reduced density matrices of any subsystem will be the same, no matter which axis the other particle is measured along. The more contentious point is the

[16]This stems from the fact that the purification process is not unique. That is, given some mixed state on a Hilbert space $\mathcal{H}_1$, we can extend it to a pure state on a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. That state is in general not unique.

invocation of the collapse postulate,[17] but any discussion of this point goes into the interpretations of quantum mechanics and we avoid it for the time being.

In Weinberg's theory, the mechanism that allows g_t to distinguish between initally equal mixtures can be stated in numerous equivalent ways. It is the non-conservation of the inner product, state-dependent eigenfrequency, and so on - ultimately, all that is required is a nonlinear Hamiltonian.

3.8 The linearity trap

The paper of Gisin proves that a nonlinear evolution of mixtures will in general lead to FTL communication, violating Lorentz invariance. However, it is not entirely clear that his proof is correct. The first objection we will discuss is the claim of Bennett[7] et al. They state that Gisin falls victim to what he calls the "linearity trap": the evolution of a mixture does not in general equal the mixture of evolutions, if the evolution is nonlinear. Denoting the mixture by $\rho = \sum p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$, the linearity trap may be written as

$$g_t(\rho) \neq \sum p_i g_t\Big(\left|\psi_i\right\rangle \left\langle\psi_i\right| \Big) \tag{3.33}$$

In Gisin's telegraph, the receiver gets mixtures proportional to the identity. But then Gisin proceeds to compute the evolution of these mixtures in terms of their component states. In particular, Gisin explicitly writes[32]

$$g_t\Big(\sum p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \Big) = \sum p_i g_t\Big(\left|\psi_i\right\rangle \left\langle\psi_i\right| \Big) \tag{3.34}$$

It seems very much like Gisin has fallen victim to the linearity trap.

He has not, however. The important assumption is that Gisin takes the nonlinearity to be of fundamental origin, i.e. applying individually to each particle. Thus we are computing pure state evolution, *not* the evolution of a general mixture. Nature itself has access to the pure states, and the evolution is given in terms of them. Gisin does not even tell us how we are to evolve an arbitrary density matrix. Thus we are working only on the RHS of Eqn. (3.33), and Eqn. (3.34) is meant to illustrate exactly this fact. Gisin himself writes "... the density operator formalism would not be appropriated for the description of the evolution of mixtures, but it could remain useful for the computation of mean values of observables." Note however that, if we were to deduce an evolution g'_t for mixtures, Eqn. (3.34) would force it to be linear.

We have already noted that Weinberg's theory, as it stands, cannot handle mixed states. Consequently the proof forces us to take the stance that entangled partners collapse to pure states upon measurement. This is, however, an interpretational issue. Since the time of measurement is relative, there are good arguments that the density matrix of a system is all there is to it. It is then interesting to see what happens to the proof if we take this stance. We return to this problem in Section 3.10. First, we need to discuss how to modify Weinberg's theory to work with density matrices.

3.9 The Polchinski formalism

The main result of Polchinski's paper[8] is that observables of a system S should be functions only of that system's density matrix. In the current section we illustrate the arguments leading

[17] That is, we assume implicitly that upon measurement the entangled partner collapses to a pure state, too.

to this conclusion. Polchinski furthermore claims that this interpretation leads to the possibility of constructing an "Everett phone", e.g. communcation between branches of the wavefunction can now be established. This will be the topic of Section 3.11. Lastly, we discuss a possible inconsistency in his reasoning regarding the Everett phone.

Let us begin by considering the time evolution of observables in Weinberg's theory. It is given by a generalized Poisson bracket[18]

$$\partial_t A = -i\{A, H\} \tag{3.35}$$

Consider now the case of a system $S = S_1 + S_2$, where the subsystems are widely separated. Let the total Hamiltonian of this system be the Hamiltonian of S_1, and let A be an observable in S_2. Evidently, it should not be influenced by S_1. This leads to the conclusion that A, *when extended to the composite system*, should be in involution with H,

$$\{A, H\} = 0 \tag{3.36}$$

Furthermore, this should hold for *all* observables A_i in S_2, each generating its own 1-parameter canonical transformation. The involution statement may then be reformulated to say that H is invariant under all canonical transformations pertaining to the Hilbert space of S_2. To make this point clearer, consider a canonical transformation with an associated generating function(or observable, if you will) A_i. The variation in H under A_i is given as

$$\delta H = \frac{\partial H}{\partial \psi_k}\delta\psi_k + \frac{\partial H}{\partial \psi_k^*}\delta\psi_k^* = \frac{\partial H}{\partial \psi_k}\frac{\partial A_i}{\partial \psi_k^*} - \frac{\partial H}{\partial \psi_k^*}\frac{\partial A_i}{\partial \psi_k} \tag{3.37}$$

And clearly vanishes if $\{A_i, H\} = 0$. Let us then see if this invariance puts any restrictions on our observables. In classical mechanics, canonical transformations are defined to be those that leave Hamilton's equations invariant. Equivalently, they leave the Poisson bracket invariant. In taking canonical transformations over to the quantum realm, it is this property that is generalized. Thus we require the transformations to be represented by operators T such that

$$x \mapsto TxT^{-1} \qquad\qquad p \mapsto TpT^{-1} \tag{3.38}$$

We note also that it seems to be a common assumption to take T unitary, though this is not necessary for the invariance of the commutator.[19]

Recalling the form of Weinberg's observables, we have

$$H = H(\psi_i{}^j, \psi_k^{*\ l}) \tag{3.39}$$

Associating the upper index with S_2, invariance under canonical transformations in this system immediately implies that the upper indices should be contracted. And because T mixes up all components of the wavefunction, any occurence of the second index must be summed over. This allows us to finally conclude that observables of subsystems must be functions of the pertaining density matrix *only*. Polchinski concludes that this is a necessary and sufficient condition to disable faster-than-light communication. We come back to whether this is correct or not in the next section.

We note also that using density matrices in our formalism has an unintended side-effect - observables become basis-independent again. To see this, consider an arbitrary density matrix

[18] Where the canonical coordinates are given by ψ_k, ψ_l^*.

[19] We mention this because Polchinski writes that $\{A, \cdot\}$ generates "unitary rotations" - this need not be the case.

ρ for the system $S_1 + S_2$. A change of basis in S_1 may be represented by a unitary operator U, such that

$$\rho \mapsto U \otimes \mathbf{1}\ \rho U^\dagger \otimes \mathbf{1} \tag{3.40}$$

Expectation values of observables are given as $\mathrm{Tr}(\rho A)$, hence cyclicity of the trace ensures that they remain invariant under basis change.

3.10 The telegraph in the Polchinski formalism

Let us now see if the Gisin telegraph is eliminated in the Polchinski formalism. To do so, we need an evolution equation for the density matrix. We find it in the usual way:

$$i\dot{\rho}_{kl} = i\dot{\psi}_k\psi_l^* - (-i\psi_k\dot{\psi}_l^*) = \frac{\partial h}{\partial \psi_k^*}\psi_l^* - \psi_k\frac{\partial h}{\partial \psi_k} \tag{3.41}$$

$$= \frac{\partial^2 h}{\partial\psi_k^*\partial\psi_n}\psi_n\psi_l^* - \psi_k\psi_n^*\frac{\partial^2 h}{\partial\psi_n^*\partial\psi_k} \tag{3.42}$$

$$= [H,\rho]_{kl} \tag{3.43}$$

where we have used the homogeneity condition in the second step. Note that we can in a sense take H to be the "non-linear matrix" representing the Hamiltonian function h. Consider now the Gisin telegraph, evaluated on the nonlinear subsystem. We shall take

$$h_\sigma = \frac{\langle\psi|\sigma_z|\psi\rangle^2}{\langle\psi|\psi\rangle} \to \frac{\mathrm{Tr}(\rho\sigma_z)^2}{\mathrm{Tr}\,\rho} \tag{3.44}$$

although, as we have noted, this is not unique. Computing the commutator most terms cancel, and we are left with

$$i\dot{\rho} = 2\frac{\mathrm{Tr}\,\rho\sigma_z}{\mathrm{Tr}\,\rho}[\rho,\sigma_z] \tag{3.45}$$

We note that this evolution cannot distinguish between different decompositions of a mixture, and so the Gisin telegraph is eliminated. But the situation is not yet clear. While the involution requirement demands that we give the dynamics in terms of density matrices, it does not actually force us to choose an interpretation with or without the projection postulate.

Suppose we choose an interpretation *with* wavefunction collapse. Then we can arrange it so that the mixture of which we are computing the evolution is purely statistical in nature. Then the nonlinear evolution of the density matrix causes statistical interference, which is physically unacceptable.[20] This is a severe problem and seems to suggest that we must have linear dynamics. If we instead choose a no-collapse interpretation, that leads to another strange phenomenon - the Everett phone - which we will discuss in the next section. Taken together, it seems that the Weinberg theory is now entirely dead as we are unable to provide an interpretation of it without running into highly unpleasant situations.

Fortunately for the Weinberg theory, it turns out that there is a third option - due to Jordan - that could save it from both issues with statistical interference and Everett phones. We will return to it in Section 3.13, but we first promised a look at the mysterious Everett phone.

[20] Let us clarify what we mean by this. Suppose we have an ensemble of spin-1/2 particles, half of which are in the spin up state and half of which are in the spin down state. Clearly we do not expect any interference between the spin up and spin down states. Yet this is exactly what we get under nonlinear evolution, and we refer to it as "statistical interference".

3.11 The Everett Phone

Polchinski claims to have eliminated the Gisin telegraph, but at the cost of introducing another; one allowing communication between branches of the wavefunction. Let us see how this comes about. Polchinski constructs it in four steps:

(1) A beam of spin-1/2 particles is sent into an SG apparatus, coupling to the z-component. An observer notes which way the particles go, then merges the two beams back. The wavefunction branches into the two worlds $W_\uparrow$ and $W_\downarrow$.

(2) The observer in $W_\uparrow$ does nothing. The observer in $W_\downarrow$ either does nothing(action A) or rotates the state to spin-up in the x-direction(action B).

(3) The particle now enters the nonlinearity described by the Hamiltonian

$$h = f\frac{\mathrm{Tr}(\rho\sigma_x)^2}{\mathrm{Tr}\,\rho} \tag{3.46}$$

(4) The observer in $W_\uparrow$ measures the z-component of the particle spin.

Let us now follow the evolution of our particle through these steps. If our observer has decided upon action A, particle and observer are in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow O_\uparrow\rangle + |\downarrow O_\downarrow\rangle\Big) \tag{3.47}$$

after step (2). If the observer decided upon action B, they are in the state

$$|\phi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow O_\uparrow\rangle + \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + |\downarrow\rangle\Big)|O_\downarrow\rangle\Big) \tag{3.48}$$

To compute the evolution of the observer+particle state through the field we must extend the Hamiltonian given by Eqn. (3.46) to the composite system. It may be tempting to still let it depend only on the reduced density matrix pertaining to the particle. But we have just seen that observables of a system should depend on the corresponding density matrix of that system only, so consequently we must use either the density matrix $\rho_\psi = |\psi\rangle\langle\psi|$ or $\rho_\phi = |\phi\rangle\langle\phi|$. The evolution is then computed as in Eqn. (3.43), and we get

$$i\dot{\rho} = 2f[\sigma_x, \rho]\frac{\mathrm{Tr}\,\rho\sigma_x}{\mathrm{Tr}\,\rho} \tag{3.49}$$

In the case of action A, we see that $\mathrm{Tr}_O\,\rho_\psi \propto \mathbb{1}$. Then $\mathrm{Tr}\,\rho_\psi\sigma_x = 0$, and the density matrix is constant in time. In $W_\uparrow$, the particle then remains in the spin up state. Now let us see what happens if action B was taken. Note that ρ_ϕ contains non-negative crossterms of the sort $|\uparrow\rangle\langle\downarrow|$, ensuring that $\mathrm{Tr}\,\rho_\phi\sigma_x \neq 0$. Taking the expectation of Eqn. (3.49) wrt. $|O_\uparrow\rangle$ and denoting $\langle O_\uparrow|\,\rho_\phi\,|O_\uparrow\rangle$ by $\rho_\uparrow$, we find

$$\partial_t\rho_\uparrow = 2if\left[\rho_\uparrow, \frac{\sigma_x}{2}\right] \tag{3.50}$$

This may be integrated by series expansion; the only thing we need to remember is the Pauli matrix commutators. Note also that $[\rho_\uparrow, \sigma_x/2] = [\sigma_z/2, \sigma_x/2]$. We get

$$\rho(t) = \rho(0) + \frac{\mathrm{d}\rho}{\mathrm{d}t}t + \frac{1}{2}\frac{\mathrm{d}^2\rho}{\mathrm{d}t^2}t^2 + \dots \tag{3.51}$$

$$= \rho_\uparrow + 2ift\left[\rho_\uparrow, \frac{\sigma_x}{2}\right] + \frac{1}{2}(2ift)^2\left[\left[\rho_\uparrow, \frac{\sigma_x}{2}\right], \frac{\sigma_x}{2}\right] + \dots \tag{3.52}$$

$$= \frac{\cos(2ft)}{2}\sigma_z - \frac{\sin(2ft)}{2}\sigma_y + \frac{1}{2}\mathbb{1} \tag{3.53}$$

Polchinski now takes $\tau f = \pi/2$, where τ denotes the time spent in the nonlinear field. Thus we get

$$\rho(\tau) = |\downarrow\rangle\langle\downarrow| \tag{3.54}$$

and we conclude that our observer now measures spin down with certainty. The disturbing point is that the measurements in step (1) and (4) occur in $W_\uparrow$. The decision between actions A and B, however, was made in $W_\downarrow$. Consequently, the measurement apparatus is of the magic kind that depends on unrealized outcomes. If the arguments hold, we can send messages from $W_\downarrow$ to $W_\uparrow$ - between worlds.

3.12 Transmission problems

As Polchinski himself notes, his argument has a weakness in that it only takes into consideration a quite finite number of wavefunction branching events. This is evident in for example step (2), where the observer in $W_\downarrow$ decides between actions A and B. There is a unitary evolution connecting these observer states[21]

$$U(t_f, t_i)\,|O_{\downarrow,\text{undecided}}\rangle\,|\downarrow\rangle \to \frac{1}{\sqrt{2}}\Big(\,|O_{\downarrow,A}\rangle + |O_{\downarrow,B}\rangle\,\Big)\,|\downarrow\rangle \tag{3.55}$$

Where the subscript $A(B)$ denotes the observer state wherein the observer has decided on action $A(B)$. Since these states are presumably macroscopically distinct, the wavefunction must branch also here.

Seeing as Polchinski disregards this branching, he considers the evolution of the state of observer+particle to be *either* $A \to |\psi\rangle$ *or* $B \to |\phi\rangle$. This is also at odds with the Everett interpretation. Both states are equally real and part of the total wavefunction, but must be taken relative to the observer states in which a certain decision was made. Rather, we believe the evolution should be as follows:

$$|\text{Spin}\rangle\,|\text{Obs}\rangle_{\text{ready}} \to \frac{1}{\sqrt{2}}\big(\,|\uparrow O_\uparrow\rangle + |\downarrow O_\downarrow\rangle\,\big) \tag{3.56}$$

$$\to \frac{1}{\sqrt{2}}\left(\,|\uparrow O_\uparrow\rangle + \frac{1}{\sqrt{2}}\Big[\,|\downarrow O_{\downarrow,A}\rangle + |\downarrow O_{\downarrow,B}\rangle\,\Big]\right) \tag{3.57}$$

$$\to \frac{1}{\sqrt{2}}\left(\,|\uparrow O_\uparrow\rangle + \frac{1}{\sqrt{2}}\left[\,|\downarrow O_{\downarrow,A}\rangle + \frac{1}{\sqrt{2}}\Big(\,|\uparrow O_{\downarrow,B}\rangle + |\downarrow O_{\downarrow,B}\rangle\,\Big)\right]\right) \tag{3.58}$$

[21] We assume for the moment that the decision is made in $W_\downarrow$, i.e. after spin down has been measured. This is in line with what Polchinski writes.

Note that, taking ρ to represent the above state, we have $\mathrm{Tr}\,\rho\sigma_x \neq 0$. This is in contrast to Polchinski's approach, where he computed using the two distinct ensembles $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$, yielding different traces. Now, our observer in $W_\uparrow$ will always see his spin rotate regardless of the action taken in $W_\downarrow$. It *seems* the phone is dead - but the magic measurement apparatus has just been replaced by an equally magical spin-1/2 particle, whose behaviour also depends on unrealized outcomes. Consider e.g. the situation where the actions A and B are the same. Then the spin would no longer rotate. But then we could re-establish the phone by making another decision at an earlier time, namely whether $A = B$. The rather unsettling part is that this argument can be repeated ad-infinitum, back to the initial state of the universe. This means that any conversation taken over the Everett phone is superdeterministic, which isn't great news. In any case, we must conclude that Polchinski's argument *is* correct - even though he has disregarded earlier branchings of the wavefunction, the phone stays connected if we include them.

3.13 The Jordan Reformulation

We have now seen the need to modify the original Weinberg theory, introducing density matrices. We saw also that this eliminated Gisin's telegraph, though at the cost of producing a highly interpretation-dependent theory. In particular, it seems that we cannot apply a collapse-interpretation as this leads to statistical interference. Similarly, applying the Everett interpretation leads to a coupling between worlds. We mentioned a possible saving grace, and it is the topic of this section.

Based on Polchinski's conclusion, T.F. Jordan reformulated Weinberg's theory in terms of density matrices.[33] Aware of the superluminal telegraph and the Everett phone, he claimed that they were simply a consequence of an incorrect application of the non-linear theory. According to Jordan, we *must* distinguish between purely statisical mixtures and those arising from entangled states.[22] At first sight, the distinction might seem strange. In linear QM, they are indistinguishable and discussing it is operationally irrelevant. And yet it is quite natural to think that there is a difference between a particle which is in a definite pure state, and one which is in no definite state at all.

In the case of a proper mixture, the system is in a definite pure state, and so the dynamics apply to these states individually. In the case of an improper mixture, however, Jordan insists that we must use the density matrix to compute the dynamics. Thus Gisin makes the mistake of using a proper mixture where he should have used an improper one, and Polchinski uses an improper mixture where he should have used a proper one.[23] If correct, Jordan's point of view removes some of the most serious issues of the (by now quite modified) Weinberg theory. However, taking this point of view entails at least two rather strange consequences, to which we now turn.

The first consequence we mention concerns the evolution of proper mixtures. According to Jordan, they are to be evolved in terms of their component states. Suppose then that we specify

[22] In terminology introduced by d'Espagnat, statistical mixtures are sometimes called proper mixtures, whereas those arising from non-separable states are termed improper mixtures. They are distinct concepts in the sense that improper mixtures cannot be endowed with a statistical interpretation. The idea goes as follows(see e.g. [34]). Consider a composite system $\mathcal{S} + \mathcal{A}$. Assume it to be in a pure but non-separable state. The reduced density matrices are then mixed. But the knowledge of these cannot reproduce the correlations in state of the composite system.

[23] This is done in step (1) of the Everett phone.

the following proper mixture:

$$\rho = \frac{1}{2} |\uparrow_z\rangle \langle\uparrow_z| + \frac{1}{2} |\downarrow_z\rangle \langle\downarrow_z| \tag{3.59}$$

Our states are assumed to be vectors in a Hilbert space; as such, we may add them in the usual way: $|\uparrow_z\rangle = \frac{1}{\sqrt{2}}(\cos\theta |\uparrow_\theta\rangle - \sin\theta |\downarrow_\theta\rangle)$ and similarly for $|\downarrow_z\rangle$. Then the decomposition above is mathematically equivalent to

$$\rho = \frac{1}{2} |\uparrow_\theta\rangle \langle\uparrow_\theta| + \frac{1}{2} |\downarrow_\theta\rangle \langle\downarrow_\theta| \tag{3.60}$$

But these component states will in general evolve quite differently. Then, if we are to stick with Jordan's view, we must extend the description of a proper mixture to include the basis in which it is to be taken, e.g. $\rho \to (\rho, \{|\theta_i\rangle\})$.

Secondly, it is not so clear when a mixture is proper and when it is improper within the Everett interpretation. Let us illustrate. We'll make use of the usual ideal von Neumann measurement of a spin-1/2 particle:

$$\frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) \otimes |O_r\rangle \longrightarrow \frac{1}{\sqrt{2}} (|\uparrow\rangle |O_\uparrow\rangle + |\downarrow\rangle |O_\downarrow\rangle) \tag{3.61}$$

After the measurement, particle and observer are entangled, and so the state of the particle may very well be thought to be represented by an improper mixture. It should not be, however, because these two particles live in different worlds. Thus there is no indefiniteness of the state with respect to the observer - it is always just $|\uparrow\rangle$ or just $|\downarrow\rangle$. A von Neumann type measurement is then quite sufficient to create a proper mixture within the Everett interpretation. It should be noted, however, that we must introduce epistemic probability[24] somewhere. This could be done by making a nonselective measurement, e.g. not looking at the measurement results. But now it seems that all mixtures are proper, leaving the distinction void. We have no good answer to this point, but leave it as a point of contention.

In fact, it is not so clear when a mixture is to be regarded as proper or improper even within a model with wavefunction collapse - it depends entirely on how we model collapse in a relativistic setting. In non-relativistic quantum mechanics, collapse is simply assumed to occur instantaneously across all space. The relativistic analogue is that if collapse is initiated at some point $x^\alpha = (t, x^i)$, then it occurs instantaneously along the spacelike hypersurface given by $t =$ const.[25] Were this the case, however, Gisin would be right in regarding his mixture as proper and we would be back right where we started. But before we can properly discuss these matters, some more knowledge on relativistic collapse models is needed.

3.14 Relativistic wavefunction collapse

In non-relativistic quantum mechanics, wavefunction collapse is assumed to occur instantaneously at the time of measurement. Attempting to carry this model over to relativistic quantum mechanics meets with immediate difficulty, as was - to the best of our knowledge - first noted by I. Bloch[35]. To illustrate, we present an adapted example from ref. [36]. Consider a particle that is prepared in a superposition of two position eigenstates at $t = -\infty$

$$|\psi\rangle = |x_1\rangle + |x_2\rangle \tag{3.62}$$

[24] That is, probability introduced as a consequence of our own ignorance.

[25] We'll discuss this in more detail shortly, but of course there are infinitely many such hypersurfaces.

Suppose a position measurement is carried out at the spacetime point (t_2, x_2), verifying that the particle is *not* there. This causes a collapse to the state $|x_1\rangle$ along the $t = t_2$ hypersurface. Here the first problem already appears. The statement that the particle is *not* found at $x^\mu = (t_2, x_2)$ is covariant. On the other hand, the $t = t_2$ hypersurface is not a surface of simultaneity in almost any other frame. Considering Fig. 3.4, we see that the collapse process is not yet complete at all points in the $t' = t'_2$ hypersurface. Thus an observer in the marked system at $x^\mu = (t'_1, x'_1)$ would not predict that the particle was located at his position with probability one, even though we asserted at $t'_2 < t'_1$ that the particle was definitely *not* at $x^\mu = (t'_2, x'_2)$.

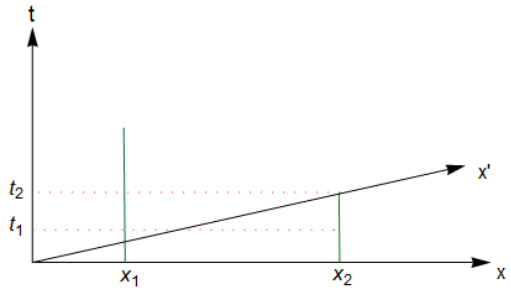

Figure 3.4: A particle is prepared in the superposition state $|\psi\rangle = |x_1\rangle + |x_2\rangle$ at $t = -\infty$. A measurement at $x^\mu = (t_2, x_2)$ verifies that the particle is *not* there. If we postulate collapse to occur along the $t = t_2$ hypersurface, the collapse will be incomplete at $x^\mu = (t'_1, x'_1)$ in the marked system, and the state of the particle assigned at that point will not in general be $|x'_1\rangle$. This contradicts our statement that the particle was not detected at $x^\mu = (t'_2, x'_2)$, since $t'_2 < t'_1$.

Seemingly, then, collapse cannot actually be instantaneous. Based on this another reduction model was proposed by K.E. Hellwig and K. Kraus, in which the collapse process is covariant but not instantaneous.[37] Their idea was that collapse should occur along the backward lightcone of the measurement event, as illustrated in Fig. 3.5. A particle prepared in the state $|\psi\rangle$ at $t = -\infty$, and collapsing to the state $|\phi\rangle$ as a result of a measurement at $x^\mu = (t_2, x_2)$ will thus be ascribed the state $|\phi\rangle$ everywhere outside the past lightcone of $x^\mu = (t_2, x_2)$.

This model has a very particular flaw, of which the authors themselves were aware: it is incompatible with the measurement of non-local observables. Let us illustrate it. Consider two spin-1/2 particles prepared in the state $|\uparrow x_1\rangle_z \otimes |\downarrow x_2\rangle$, traveling along the worldlines $x_1 =$ constant, $x_2 =$ constant. At $x^\mu = (t_2, x_1)$, a measurement of the total spin operator J^2 is performed. According to Hellwig and Kraus, this means that the state assigned by an observer at $x^\nu = (t_1, x_2)$ is either of $|\uparrow x_1\rangle \otimes |\downarrow x_2\rangle \pm |\downarrow x_1\rangle \otimes |\uparrow x_2\rangle$. But then the claim is also that the state assigned by an observer at $x^\sigma = (t_1, x_1)$ is still $|\uparrow x_1\rangle \otimes |\downarrow x_2\rangle$, and this is a contradiction. At the time, this was considered a non-issue, as Landau and Peierls had shown in the 1930's that the measurement of non-local observables was in conflict with causality. [38]

3.14.1 The collapse model of Aharonov and Albert

In the early 1980's it was shown by Aharonov and Albert that certain non-local observables are in fact measurable. Furthermore, they showed that by weakening the concept of measurement slightly the set of "measurable" non-local observables can be extended.[39, 36, 40] Thus the flaw in the Hellwig-Kraus model cannot be dismissed, and so their model fails. Aharonov and Albert proceeded to construct a new model, and we briefly present it before returning to our discussion about proper and improper mixtures.

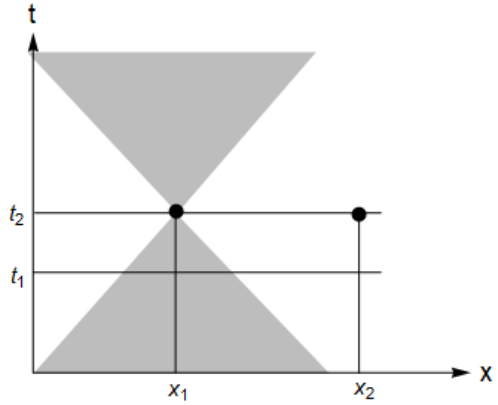

Figure 3.5: The Hellwig-Kraus collapse occurs along the backward lightcone. If the initial state at $t = -\infty$ was $|\psi\rangle$, and it subsequently collapsed to $|\phi\rangle$ as a result of a measurement at $x^\mu = (t_2, x_2)$, the state ascribed to the particle is $|\phi\rangle$ everywhere outside the backward lightcone of the measuring event. Inside the backward lightcone, it is always $|\psi\rangle$.

Earlier we argued that collapse cannot be covariantly instantaneous, as this leads to an ambiguous definition of the state as a function of spacetime, e.g. $\psi(x^\mu)$ is multivalued. However, this flaw can be avoided by extending the concept of state: instead of being a function of spacetime, we will now consider it a functional of spacelike hypersurfaces.[26] That is,

$$\psi(x^\mu) \to \psi[\sigma] \tag{3.63}$$

Suppose now that a measurement occurs at the point y^α. Aharonov and Albert then postulate that statevector reduction occurs instantaneously across all spacelike hypersurfaces σ_i containing y^α.[27] This scheme is clearly covariant - although different observers will in general assign different states to a system, no preferred frame is singled out. What is not so clear is how this scheme allows for the measurement of (certain) non-local observables without contradicting causality - indeed, it seems that in this sense it has the same problems as the usual model in which collapse occurs along a preferred hypersurface. To make this point clearer, we present an example from ref. [39]. Suppose we prepare two spin-1/2 particles in the state $|\uparrow\uparrow\rangle_z \otimes |x_1 x_2\rangle$ at $t = -\infty$ and distribute them to two observers, Alice and Bob(see Fig. 3.6) At $x^\mu = (t_1, x_2)$, Bob may choose to flip the spin of his particle. At x^ν, Bob measures the total spin operator J^2. If he did a spinflip, the state of the particles collapses to either the singlet or triplet state $|\alpha_\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\uparrow\downarrow\rangle)$. Then, at $x^\sigma = (t_3, x_1)$, Alice measures the $z-$component of the spin of her partice. If Bob didn't flip the spin of his particle, she will measure $+1/2$ with probability one. If Bob *did* flip the spin, she will now measure $+1/2$ with probability $1/2$. Thus Bob can send superluminal messages to Alice, and causality is violated.

[26] These are assumed to form a foliation of spacetime, such that each point x^μ belongs to one and only one hypersurface. Any observer moving at constant speed along a worldline $y(\tau) = n^\mu \tau$ defines such a foliation, wherein the points $x^\mu \in \sigma_\tau$ in an equal-time hypersurface is determined by the equation $n^\mu(x_\mu - y_\mu(\tau)) = 0$.

[27] Free evolution of the state functional is governed by the Schwinger-Tomonaga equation: $\frac{\delta|\psi[\sigma]\rangle}{\delta\sigma(x)} = -i\mathcal{H}(x)\,|\psi[\sigma]\rangle$. See ref. [41] for a brief introduction.

3.14.2 Quantum state verification

As it turns out, the only mistake in this example is the assumption that J^2 is measurable. In fact, even in the framework of Aharonov and Albert, J^2 belongs to the class of non-measurable operators. However, it *is* measurable in a weaker sense. It allows what Aharonov and Albert refer to as *quantum state verification*. Namely, although we cannot perform a regular projective measurement of J^2 without violating causality, we *can* verity whether the particles are in the singlet state or not.

To illustrate how state verification differs from regular projective measurements, it is useful to first introduce the notion of a *quantum operation*. Denoting the space of density matrices by $S(\mathcal{H})$, such operations are represented by maps $\phi : S(\mathcal{H}) \to S(\mathcal{H})$. Furthermore, it is a theorem of Kraus[42, 21] that if the map ϕ satisfies the following properties:[28]

1. Linearity[29]
2. Trace preserving (Probability conservation)
2. Complete positivity (Probability, composite systems)

Then there exists operators Ω_k such that[30]

$$\rho \to \phi(\rho) = \sum_k \Omega_k \rho \Omega_k^\dagger \tag{3.64}$$

In the case of a nonselective projective measurement of some observable $A = \sum a_i \Pi_a$, the operation will simply be

$$\phi(\rho) = \sum_a \Pi_a \rho \Pi_a \tag{3.65}$$

where the Kraus operators are the projectors Π_a. On the other hand, suppose we wish to perform a verification of the state $|\Psi_0\rangle$. In this case, we require that both $|\Psi_0\rangle$ and its orthogonal complement remain unchanged, but importantly we *do* allow states to change within the orthogonal complement. Defining the projectors Π_{Ψ_0} and $\Pi_\perp$, we find that the operation is given by

$$\phi(\rho) = \Pi_{\Psi_0} \rho \Pi_{\Psi_0} + \Pi_\perp \rho \Pi_\perp \tag{3.66}$$

To see how such state verification operations avoid causality problems, let us again consider the example illustrated in Fig. 3.6. We generalize slightly by allowing Alice to measure an arbitrary observable W, and letting Bob perform not just a spinflip but any unitary operation U. The measurement of J^2 is replaced by a verification of the state $|\Psi_0\rangle$. Causality then requires Alice's measurement to be independent of whatever Bob does. Assuming that the two spin-1/2 particles start off in the state $|\Psi\rangle$ we formulate the following *causality principle*:

$$\langle\Psi|U^\dagger \Pi_{\Psi_0} W \Pi_{\Psi_0} U|\Psi\rangle = \langle\Psi|\Pi_{\Psi_0} W \Pi_{\Psi_0} U|\Psi\rangle \tag{3.67}$$

[28] Depending on terminology, these conditions are too strict. Some authors only require quantum operations to be non-trace-increasing, defining instead quantum operations that do preserve the trace as *quantum channels*.

[29] As we have seen in Appendix A, this is required by the isolated system assumption.

[30] Often referred to as Kraus operators. In the context of measurement theory, the operators $F_k = \Omega_k \Omega_k^\dagger$ are called *effects* and partition unity, e.g. $I = \sum_k F_k$. They are not required to be mutually orthogonal, and as such they generalize the usual projective measurements.

A broad class of state verification operations respect this principle, as we now show with the help of a theorem due to Vaidman and Popescu.[43]

Information-erasure theorem. Consider two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and two states $|\psi\rangle, |\phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. $|\psi\rangle$ has the Schmidt decomposition $|\psi\rangle = \sum_d \sqrt{a_d} \left|\alpha_d^{(1)}\right\rangle \left|\alpha_d^{(2)}\right\rangle$. The space spanned by the set $\{|\alpha_i\rangle\}$ we denote by $\widetilde{\mathcal{H}}_i$. Suppose a state verification of $|\psi\rangle$ is performed on the state $|\phi\rangle$. Then any subsequent measurement of an observable $W \otimes \mathbb{1}$ will satisfy

$$\operatorname{Tr} W |\psi\rangle \langle\psi| = \operatorname{Tr} W |\phi\rangle \langle\phi| \tag{3.68}$$

Provided that $|\phi\rangle \in \widetilde{\mathcal{H}}_1 \otimes \mathcal{H}_2$.

This shows that under certain state verification operations, *all* information about the initial state is lost and so there can be no conflict with causality. Now, if Alice and Bob attempt to verify that they have a singlet state, then $\widetilde{\mathcal{H}}_1 \otimes \mathcal{H}_2 = \mathcal{H}_1 \otimes \mathcal{H}_2$ and so the condition for the information-erasure theorem to apply is trivially satisfied for any initial state.

To finish up this section we use the information-erasure theorem to derive a condition on the class of measurable non-local operators. The argument presented here is more streamlined than the original argument of Aharonov, Albert and Vaidman[40] and is due to Breuer and Petruccione[41]. Firstly, operators whose eigenstates are all of product form are clearly measurable. Thus we may assume that at least one eigenstate $|\Psi\rangle$ is entangled. As before, we consider the space $\widetilde{\mathcal{H}}_1 \otimes \mathcal{H}$ given by its Schmidt decomposition. Assume it to be D^2-dimensional. Pick an eigenstate basis $\{|\Psi_\nu\rangle\}$ for it. By the information-erasure theorem, we have

$$\operatorname{Tr} W |\Psi_\nu\rangle \langle\Psi_\nu| = \operatorname{Tr} W |\Psi\rangle \langle\Psi| \tag{3.69}$$

Or equivalently

$$\operatorname{Tr}_2 |\Psi_\nu\rangle \langle\Psi_\nu| = \operatorname{Tr}_2 |\Psi\rangle \langle\Psi| \tag{3.70}$$

Summing over the basis vectors $|\Psi_\nu\rangle$, we get

$$\sum_{\nu=1}^{D^2} \operatorname{Tr}_2 |\Psi_\nu\rangle \langle\Psi_\nu| = \operatorname{Tr}_2 \mathbb{1}_{\widetilde{\mathcal{H}}_1 \otimes \mathcal{H}_2} = D \mathbb{1}_{\widetilde{\mathcal{H}}_1} = D^2 \operatorname{Tr}_2 |\Psi\rangle \langle\Psi| \tag{3.71}$$

So we see that $|\Psi\rangle$ is maximally entangled. Then Eqn. (3.70) implies that also all other eigenstates $|\Psi_\nu\rangle \in \widetilde{\mathcal{H}}_1 \otimes \mathcal{H}_2$ are maximally entangled, and this must hold for all measurable non-local observables. Finally, we now see why J^2 is not measurable - considering the singlet state, we have that $\widetilde{\mathcal{H}}_1 \otimes \mathcal{H}_2 = \mathcal{H}_1 \otimes \mathcal{H}_2$. But the measurability criterion we just found would then force all eigenstates of J^2 to be maximally entangled, which they are not.

3.15 On the consistency of Jordan's reformulation

At the end of Section 3.13 we noted the difficulty of deciding when a mixture is proper and when it is improper. We furthermore noted the connection with relativistic wavefunction collapse,

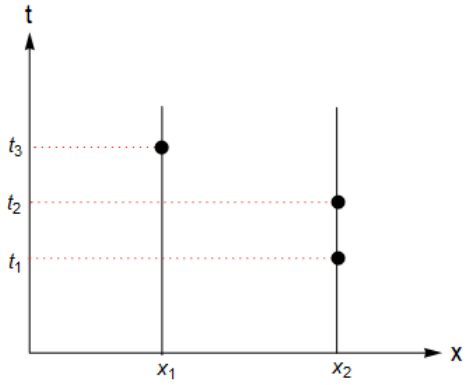

Figure 3.6: Two spin-1/2 particles are prepared in the state $|\uparrow\uparrow\rangle_z \otimes |x_1 x_2\rangle$ at $t = -\infty$. At $x^\mu = (t_1, x_2)$ an observer can choose to perform a spinflip. At $x^\nu = (t_2, x_2)$ the total spin J^2 is measured. If a spinflip has been performed, the particles collapse to either a singlet or triplet state along the $t = t_2$ hypersurface. At t_3 an observer measures the z-component of the spin of his particle. If no spinflip is performed, he gets $+1/2$ with probability one. If a spinflip is done, the probability of measuring $+1/2$ is reduced to $1/2$. Thus causality seems to be violated. [39].

which we can now clarify. To that end, let us consider a composite system $S_1 + S_2$ which is in an entangled state. We have two observers O_1 and O_2, each in the vicinity of either subsystem, each with their own frame F and F' respectively. O_1 performs a measurement on his system at some spacetime point $x^\mu = (t_1, x_1)$. According to what we have learned, the wavefunction collapses both along the hypersurface $t = t_1$ in F and along the hypersurface $t' = t'_1$ in F'. Hence O_2 must compute the evolution of S_2 using an improper density matrix for $t < t'_1$ and a proper density matrix for $t > t'_1$. The situation is analogous for O_1, of course.

As it turns out, this *isn't* compatible with Jordan's point of view. Jordan claims Gisin made a mistake in using a proper mixture when he should have used an improper one, and yet in the case of Gisin's phone both observers are in the same frame, experiencing collapse at the same time. The point is that Jordan implicitly claims another type of wavefunction collapse. Referring back to Gisin's phone as described in Section 3.6, Bob should compute the evolution of his particles using an improper mixture until such a time as when a light signal could have reached him from Alice.[31] This type of collapse isn't occuring along equal-time surfaces, but instead along the future lightcone of measurement events.

This type of collapse is quite similar to that of Hellwig and Kraus, which was taken along the backward lightcone. It is of course Lorentz invariant, for the same reason their collapse model was. But also for the same reasons, it predicts multi-valued wavefunctions $\psi(x^\mu)$ which we have seen to be in conflict with the measurement of non-local observables. By Aharonov and Albert, this conflict was unacceptable because they held the belief that the wavefunction somehow is a part of reality. As such collapse is equally real, and multi-valuedness is a physical aberration.

We're not absolutely *required* to believe that the wavefunction is real, however. Just like the QBists, we can assume it to be simply a representative of an observers' knowledge - that is,

[31] In fact, in [44] Jordan refers to an article by Kent[45] wherein it is suggested that the evolution of a system should always be computed with regards to the so-called local state ρ^{loc}. It is defined as the best possible description of the state, e.g. it is the state an observer would ascribe to it given perfect knowledge of all measurements that have occured in the past light cone of the particle as well as its initial state. Such a state is clearly epistemic in nature.

we assume the wavefunction to be of epistemic nature.[32] And indeed, the preceeding argument shows that if we are to take Jordan's point of view seriously, we are forced to accept that it is.

This can however be quite unpleasant, depending on your point of view. Returning to the notation at the beginning of this section, suppose $S_1 + S_2$ consists of two spin-1/2 particles in the singlet state. Suppose furthermore that they were prepared at a point P inside the lightcones of both O_1 and O_2. Up until collapse, both observers must give the state of their respective system as a diagonal density matrix - just like in linear quantum mechanics(LQM), of course. The difference is that as soon as the particles are separated, a believer in epistemic states can no longer talk of the state of the composite system. Such a state is ill-defined, as it contains non-local information which any single observer cannot obtain by measurement. Unfortunately this means that if Alice and Bob were to compare their data at some point, they would be quite unable to explain the correlations appearing therein. So while this doesn't falsify the notion of epistemic states, it does mean it has less predictive power than the notion of ontic states. As for the non-local measurements, they are no longer an issue for precisely the same reason: there are no non-local states!

Epistemic states are also problematic when it comes to the measurement problem. From a QBists' point of view, the measurement problem isn't one - naturally, collapse occurs as we obtain more knowledge. But this point of view is entirely incompatible with models wherein collapse physically occurs, as in e.g. the dynamical reduction models we shall soon be looking at. So if one is interested in (deterministic[33]) NLQM because it could potentialy be used for modelling wavefunction collapse, what we have discovered is that it is a futile effort.

Now we might be interested in NLQM for other reasons, so suppoe we accept the extended view of mixtures and the epistemic nature of the wavefunction. Is the Weinberg theory then internally consistent? We would say yes, were it not for the last remaining issue - that of the probability interpretation of non-linear observables. We encountered severe difficulties when we attempted to give one in Section 3.4.2. Of course, our difficulties does not mean such a probability interpretation cannnot be constructed - it just means that it has not yet been done. If we wish to salvage the theory in absence of one, we must then discard nonlinear observables. Note that this has a particular consequence for the Hamiltonian as it is both the operator representing energy and the generator of time evolution. So to avoid a complete return to LQM this implies that we must have one for each - one(let us denote it by H^g) generating time translations, and one(H^e) corresponding to energy.[34] More importantly, in restricting ourselves to linear observables and nonlinear dynamics, we have opened ourselves to another attack which we will now discuss.

3.16 The isolated system assumption

To do physics, we assume the ability to describe isolated systems. If we did not, we could not compute the motion of a car without considering inspiralling black holes twenty galaxies distant. Let us formalize it somewhat:

Isolated System Assumption: Consider two isolated, non-interacting but possibly

[32] Of course, Gisin's article about the telegraph is written from the point of view that the wavefunction is ontic(and collapses along equal-time hypersurfaces). But it is still interesting to see if his argument can be dismissed if we take another point of view.

[33] Recall that the problem of distinguishing between proper and improper mixtures arose because the evolution of density matrices was non-linear. If one could somehow arrange for wavefunctions to evolve non-linearly but still retain linear evolution of density matrices, our current discussion would be moot. This is of course exactly what dynamical reduction models manage to do, as we will see in Chapter 4.

[34] To ensure conservation of energy, this means we must choose H^e such that $[H^e, H^g] = 0$. For h_σ, the natural choice would be $H^e = \sigma_z$, for example.

> correlated systems S_1 and S_2. Then we may compute the dynamics of either system regardless of what happens in the other system. In other words, no knowledge of S_2 is required to compute the dynamics of S_1.

Assuming that the above holds true, that all observables are linear and that the Born rule holds, Jordan showed that the dynamics of the density matrix must be linear.[46] We provide a somewhat more streamlined version of his proof in Appendix A. This is already enough to kill the Weinberg theory, as it gives a non-linear evolution law for density matrices. But in the above proof, Jordan further claims that this implies that the underlying statevector dynamics must be linear too. We believe there is a mistake in his proof, such that this statement is not true. This particular mistake turns out to be quite interesting in view of dynamical reduction models, which we will soon discuss. As such, we'll spend a little time analyzing the proof.

Concretely Jordan's statement is that if the density matrix evolves under some function g_t which is linear and injective, then pure states are mapped to pure states and inner products are preserved. These are in turn the assumptions of a proof by Wigner and Bargmann[47, 48] showing that the underlying dynamics must be linear.

The issue is that linearity and injectivity is not enough to guarantee that pure states are mapped to pure states. In his proof, Jordan assumes the existence of an inverse g_t^{-1} to prove that pure states map to pure states. His argumentation goes like so: consider the mixed state

$$\rho = p\rho_1 + (1-p)\rho_2 \tag{3.72}$$

where $p \in [0,1]$ and ρ_1, ρ_2 are pure states. If g_t is injective, then there exist distinct ρ_1 and ρ_2 such that $\rho_1' = g_t(\rho_1)$ and $\rho_2' = g_t(\rho_2)$. Then it is clear that mixed states are mapped to mixed states. Now, *if* the inverse can be defined, the above reasoning shows that if $g_t(\rho)$ is mixed, so is ρ.

However, a linear map $g : V \to W$ only has an inverse $g^{-1} : W \to V$ if it is a bijection. If g is just injective, the inverse must be restricted to the image $g(V)$. Thus the mappings Jordan considers will not in general have inverses on the whole space of density matrices. Consider for example the following map

$$g^d(\rho(t)) = \gamma\left(\frac{1}{d}\mathbb{1} - \rho(t)\right) \tag{3.73}$$

generating the time evolution of $\rho(t)$ through

$$\partial_t \rho(t) = g^d(\rho(t)) \tag{3.74}$$

Here $\gamma \in \mathbb{R}$ is a scale factor and d denotes the dimension of the Hilbert space. This mapping[35] is injective but not surjective. Note that no inverse exists for the maximally mixed state. Furthermore, the inverse mapping defined on mixed states is does not preserve positivity, and so does not map the set of density matrices into itself. The inverse so defined would then map pure states out of the space of density matrices.

Thus the proof, if it is to go through, must require that g_t is not only injective but bijective. This ensures that pure states are mapped to pure states. More importantly, this hole in the proof shows that even if the mapping g_t of density matrices is linear, an underlying nonlinear evolution of state vectors is acceptable as long as g_t is injective but not surjective. Note also that the requirement of bijectivity for g_t is the same as determinism, e.g. the evolution does not introduce new epistemic uncertainty at any point. Geometrically speaking the evolution moves on surfaces of constant $\mathrm{Tr}\,\rho^2$ (see e.g. [49]). Thus the hole above is actually equivalent to requiring nonlinear statevector evolution to be indeterministic or stochastic.

[35] More precisely, the map obtained by integrating the above equation.

3.17 Concluding deterministic NLQM

Now we mentioned that the Weinberg theory is dead as it gives rise to non-linear evolution of density matrices. But there is one last possbility of survival which we should mention. Recall that the equation of motion for the density matrix was derived from the evolution of states, see Eqn. (3.41). This is worrisome because it conflates proper and improper mixtures, the distinction of which we have accepted. To elaborate slightly, we compute the evolution of a proper mixture and assume it applies to improper mixtures - and then we subsequently claim proper mixtures should be evolved componentwise! Thus one might argue that the evolution law we use for improper mixtures is somehow not the right one. We have no good answer to this problem, and in his reconstruction of Weinberg's theory Jordan has made no mention of the issue.

Let us recap our discussion of the Weinberg theory. We got rid of several of its issues by extending the theory to handle density matrices. This introduced an Everett phone, which we in turn got rid of by extending our concept of mixtures, e.g. distinguishing between proper and improper mixtures and demanding that proper mixtures now come with a label denoting the basis in which they are to be taken. We found that this distinction necessitated an epistemic interpretation of the wavefunction. With no solution to the issue of the probability interpretation of the theory, we postulated that all observables are linear and only the dynamics are nonlinear. This opened us to a proof by Jordan, which demands that the evolution law of the density matrix must be linear. If we accept all previous steps, this is the end of the Weinberg theory. We have noted three options for saving NLQM. One is the construction of a probability interpretation for non-linear observables.[36] The second is the construction of an evolution law for improper mixtures which despite deterministic non-linear statevector dynamics turns out to be linear. The third is rejecting deterministic non-linear statevector dynamics and embracing stochasticity - and this is the option which we will pursue in the next chapter.

[36] Note that Jordan's proof implicitly assumes the regular probability interpretation; if another interpretation were provided for non-linear observables then non-linear dynamics might be permissible.

Chapter 4

Dynamical Reduction Models

Having considered deterministic nonlinear theories and found them to suffer from severe problems, we now turn our attention to the dynamical reduction models. They are interesting to study for several reasons, but the main draw is evidently their ability to resolve the measurement problem. We'll be discussing how they achieve this extensively in the coming sections. It should however also be mentioned that the theory involved does bear strong resemblance to that appearing in several other fields, e.g. the study of open quantum systems[41], continuous measurements[50] and decoherence[12].[1]

As briefly mentioned all reduction models make use of stochastic non-linear dynamics in order to break the superposition principle and induce collapse. As we have just seen, this stochasticity is exactly what we needed to avoid Jordan's proof demanding linear statevector dynamics. We will also see that all reduction models give rise to a non-surjective linear evolution of density matrices, as required. While this makes it clear that a Gisin-type phone is no longer viable, it is still not clear whether these models allow faster-than-light signalling in some other way, a point which we shall investigate in some detail. We will find that it is once again the stochasticity that comes to the rescue, and we discuss this point in the last part of this thesis, Section 4.7. To prepare for this we'll spend some time familiarizing ourselves with reduction models in general. As an introduction, we study the original GRW(or QMSL)[2] model in Section 4.2. Therein we will see qualitatively how the measurement problem is resolved and introduce the ontology of the model, which applies to the more sophisticated reduction models as well. We finish the introduction off by presenting some problems of the QMSL model, motivating the introduction of the so-called QMUPL model.[3] We introduce it in Section 4.4, correcting some mistakes we have found in the literature. Subsequently a thorough analysis of the collapse mechanism is given, and we take the opportunity to discuss some phenomena which have received little attention in the literature.[4] To end our discussion of reduction models, we briefly present some open problems and sketch the current experimental situation.

[1] As an example, we shall see that the so-called GRW theory may equally well be interpreted as a sequence of POVM-type measurements occuring at random times. Furthermore all models we encounter give a Lindbladian evolution of the density matrix, which is exactly what we get when treating open quantum systems.

[2] The authors are Ghirardi, Rimini and Weber, hence it is often known by their initials. We will typically refer to the model as QMSL - QM with Spontaneous Localization.

[3] Short for QM with Universal Position Localization.

[4] We will see that these phenomena are of little relevance when considering collapse induced by macroscopic measurement apparatuses. Nevertheless they are a experimental signatures of the model and could potentially be observed in experiments looking at collapse in mesoscopic systems.

4.1 The Origin of Collapse Theories

The idea of unifying the dynamics of QM seems to have originated with Pearle[9], who proposed to add a deterministic nonlinear term to the Schrödinger equation such that macroscopic superpositions are driven into one of their component states. Though for different reasons than those we have given, Pearle abandoned this particular approach and subsequently gave a model where the modified term was of stochastic nature.[51] This approach was also independently investigated by N. Gisin[32]. However, no clear consensus was reached as to the particular form of the stochastic term that was added. Then the original QMSL model[10] came along, in which the wavefunction is postulated to suffer so-called "hits", or localization processes.[5] It was remarkable in that it was the first truly dynamical reduction model, wherein the process responsible for collapse was active at all times. This is no easy feat, as the process must have negligible effect on microscopic systems while still actively collapsing macroscopic systems. Pearle subsequently combined ideas from QMSL with his own into the first collapse model wherein the localization processes are continuous in time.[52] This particular model - now known as QMUPL - has received considerable attention in the literature, in part due to it being substantially easier to analyze than its field-theoretic counterpart(known as CSL[6]) first given by Pearle, Ghirardi and Rimini.[53]. For the same reason, we will also mainly concern ourselves with QMUPL, though we shall also very briefly present the fundamental equation of CSL.

Though we will not(and cannot) summarize everything that has been done, we mention two further milestones in the development of reduction models. Firstly, all reduction models must inherently struggle with the problem of energy non-conservation.[7] Considerable effort has been expended to remedy the issue. Attempts include explicitly adding dissipative mechanisms in the models[54] and changing the nature of the stochasticity used[55]. Secondly, much work has been done to generalize the model to a relativistic setting[56, 57], most recently by Tumulka[58] and Bedingham.[59, 60] The latter model, constructed by Bedingham, is remarkable in that it is both covariant and manages to satisfy the principle of energy conservation. The model does however have many free parameters, and it is a subject of current research.

Before diving into concrete collapse models, we wish to emphasize two important points. One is that dynamical reduction models talk about what *is* without requiring any concept of measurement.[8] This is in stark contrast to the standard interpretation, wherein we may not speak of what is except when a measurement has just been performed.[9] In other words, collapse models provide a much richer ontology.

Secondly the rather vague terms "measurement" and "collapse" are no longer needed to formulate a model that conforms to reality - they emerge from the dynamics. In fact reduction models need less postulates than does "Bare QM", as we presented it in Section 2.2. To be precise, the postulates we need are

[5] In less ponderous language, the wavefunction is multiplied by a Gaussian at random times.

[6] Short for Continuous Spontaneous Localization. Historically this was the term Pearle gave to the model which we referred to as QMUPL. As time progressed and the field-theoretic version of Pearle's model was developed, CSL was interchangably used for both his original model and the field-theoretic variety; hence the introduction of "QMUPL".

[7] We will explain why in due course.

[8] We speak here about "what is" in the sense of a presumed universe existing independently of ourselves and our actions.

[9] As we discussed in Section 2.5, the eigenstate-eigenvalue link permits only a notion of "what is" whenever the system under consideration is in an eigenstate of an observable.

Postulates of Collapse Models

1. **States** are represented by rays in a complex Hilbert space.
2. **Evolution** is stochastic unitary and determined by a modified Schrödinger equation, $i\partial_t |\psi\rangle = (H + \lambda S(\psi))|\psi\rangle$. Here λ is a parameter setting the strength of the collapse term, the specifics of which are contained in S.
3. **Ontology**:[10] the matter density of an N-particle system is given by $\rho_N(x,t) = \sum_{n=1}^{N} m_n \int dx_1 \dots dx_N \delta(x - x_n)|\psi(x_1, \dots, x_N)|^2$ where m_n is the mass of the n'th particle.

From these postulates the effective counterparts of the Born rule and wavefunction collapse appear entirely on their own. Note that we say effective counterparts - collapse does not occur in the sense we are used to, and the probabilities we compute from reduction models are not *quite* what we get from LQM. What is more, the Hermitian operators we typically associate with these terms are not a necessary ingredient in collapse models, though they are still convenient to work with.[62] We emphasize especially that the square modulus of the wavefunction is *not* a probability density. All this will hopefully become clearer as we discuss reduction models; in the current section we hope to elucidate their conceptual aspects, while we have attempted to quantify matters more in our discussion of the QMUPL model. With that said, let us begin with QMSL.

4.2 The QMSL model

Recall that our primary goal is the unification of quantum dynamics - we want a single dynamical law. In other words, we either expect wavefunction collapse to emerge from this law, or that it somehow is a manifest part of it. The latter is the case for QMSL, where collapse is modelled by so-called hits. That is, GRW suggest the following postulate:

Hit postulate: Given a system of n distinguishable particles, each particle undergoes a localization process at random times t_k, drawn from a Poisson distribution with mean frequency λ.[11]

At all other times, the wavefunction evolves unitarily. Of course, we must be precise with what exactly we mean by a "localization process". GRW further postulate that it is represented by an operator L_x^i, such that

$$|\psi\rangle \longrightarrow \frac{L_x^i |\psi\rangle}{\|L_x^i |\psi\rangle\|} \tag{4.1}$$

where

$$L_x^i \equiv \left(\frac{\alpha}{\pi}\right)^{3/4} \exp\left[-\frac{\alpha}{2}(r_i - x)^2\right] \tag{4.2}$$

[10] An alternative ontology exists for the QMSL, known as the flash ontology.[61] We will not concern ourselves with it, as it does not extend to continuous collapse models.

[11] In general the model allows distinct frequencies for distinct particles, but we will lose nothing in the discussion by assuming this frequency is the same for all particles.

The index i labels the particle suffering the hit, x is the center of the hit and r_i is the position operator of particle i. Given Eqn. (4.1), the density matrix evolves like so

$$|\psi\rangle\langle\psi| \to \int d^3x\ P(x)\frac{L_x^i|\psi\rangle\langle\psi|L_x^i}{\||L_x^i|\psi\rangle\|^2} \tag{4.3}$$

$$\stackrel{!}{=} \int d^3x\ L_x^i|\psi\rangle\langle\psi|L_x^i \tag{4.4}$$

$$\equiv T[|\psi\rangle\langle\psi|] \tag{4.5}$$

Given that we know this evolution must be linear, we can deduce that $P(x) = \kappa\||L_x^i|\psi\rangle\|^2$. Conserving the trace of the density matrix further restricts us to $\kappa = 1$, as is evident from Eqn. (4.7) below. Let us emphasize some important points. First, note that this evolution is exactly that of a non-selective, approximate position measurement.[12] Thus it is in a sense just regular QM, except for the very important fact that it is not specified *what* causes the hits, and this will cause us some trouble. Secondly, two new constants of nature are introduced. One is the frequency λ of the Poisson process governing the hits and the other is the localization scale, given by $1/\sqrt{\alpha}$.[13] Finally, let us give the evolution law for the density matrix including unitary evolution. Infinitesimally, there is a probability λdt for a hit to occur during the interval dt. Thus

$$\rho(t+dt) = (1-\lambda dt)\big[\rho(t) - i[H,\rho(t)]dt\big] + \lambda dt T[\rho] \tag{4.6}$$

Or equivalently

$$\partial_t\rho = -i[H,\rho] - \lambda\big[\rho - T[\rho]\big] \tag{4.7}$$

Interestingly, this equation is of Lindblad form. More explicitly, the above is equivalent to

$$\partial_t\rho = -i[H,\rho] + \lambda\int d^3x\left[L_x^\dagger\rho L_x - \frac{1}{2}\{L_x^\dagger L_x,\rho\}\right] \tag{4.8}$$

$$= -i[H,\rho] - \frac{\lambda}{2}\int d^3x\ [L_x,[L_x,\rho]] \tag{4.9}$$

Since the effects $L_x^\dagger L_x$ sum to unity. This is the fundamental equation of QMSL, and gives us the unified dynamics. As mentioned this equation appears in various other subjects, and in particular it is the very same equation that is used to model decoherence.[12] However, the underlying models give drastically different interpretations to the density matrix.[63] In the case of decoherence, it is an improper mixture obtained by tracing out the environment. On the other hand, the density matrix of QMSL is proper - a statistical mixture of states that have suffered hits at every position x. The significance of this is that QMSL *does* allow us to assign definite states to a system, whereas a decoherence model does not. But of course, the importance of this point is debatable. Note also that the evolution equation preserves the trace, such that it maps the space of density matrices into itself. The equation is by construction non-surjective: any initial pure state evolves in infinitesimal time into a mixed state, and as such the evolution will never map onto pure states. A quick computation confirms this:

[12] To describe this, we have to resort to POVMs. In this context, the operators L_x^i are referred to as the *effects*. These must partition identity, e.g. $\int L_x^\dagger L_x = I$, which we see is satisfied.

[13] More precisely, consider $\langle x|L_x^i|\psi\rangle = L(x)\psi(x)$. The FWHM of $L(x)$ is given by $2\sqrt{\frac{2\ln 2}{\alpha}} \propto 1/\sqrt{\alpha}$.

$$\partial_t \operatorname{Tr}(\rho^2) = -2\lambda \sum_{x'x''} \left[1 - e^{-\frac{\alpha}{4}(x'-x'')^2}\right] |\langle x'|\rho|x''\rangle| \leq 0 \tag{4.10}$$

That is, pure states are mapped to mixed states, which asymptotically become fully mixed - in absence of Hamiltonian evolution, of course. With the fundamental equation presented, we next cover the important point of wavefunction collapse, or more precisely how it emerges from the model. These considerations will also allow us to estimate the parameters of the model.

4.2.1 Collapse and parameter estimation

The emergence of collapse is perhaps best illustrated through an example, and to this end we will compute the effect of a hit on a superposition of two Gaussian states:

$$\psi(x) = A\left[e^{-\frac{\gamma}{2}(x+a)^2} + e^{-\frac{\gamma}{2}(x-a)^2}\right] \tag{4.11}$$

We will assume that $a \gg 1/\sqrt{\alpha}$, e.g. that a hit will only affect one of the states in the superposition.[14] We also assume that $1/\sqrt{\gamma} \ll 1/\sqrt{\alpha}$, e.g. that these states are considerably more sharply localized than the hitting Gaussian itself. We could equally well assume $1/\sqrt{\gamma} \gg 1/\sqrt{\alpha}$, of course - the final answer would be the same, but with $\gamma \to \alpha$. Suppose now that a hit occurs with center $x = a$. The state is transformed to

$$L_a\psi(x) = \tilde{A}\left[e^{-\frac{1}{2}(\gamma+\alpha)\left[x+a\frac{\gamma-\alpha}{\gamma+\alpha}\right]^2} e^{-\frac{1}{2}(\gamma+\alpha)a^2\left[1-\left(\frac{\gamma-\alpha}{\gamma+\alpha}\right)^2\right]} + e^{-\frac{1}{2}(\gamma+\alpha)(x-a)^2}\right] \tag{4.12}$$

$$\approx \tilde{A}\left[e^{-\frac{1}{2}(\gamma+\alpha)\left[x+a\left(1-2\frac{\alpha}{\gamma}\right)\right]^2} e^{-2\alpha a^2} + e^{-\frac{1}{2}(\gamma+\alpha)(x-a)^2}\right] \tag{4.13}$$

$$\approx \tilde{A}e^{-\frac{\gamma}{2}(x-a)^2} \tag{4.14}$$

Where in the second line we have expanded to first order in α/γ. We see that the state at $x = -a$ is heavily suppressed, whereas the state at $x = a$ is barely affected. Note also that the state localized around $x = -a$ is shifted towards $x = a$ by the hit.

Let us also consider the effect of a hit at $x = 0$. In this case, $|\psi(0)|^2 \propto \exp(-\gamma a^2)$, indicating that such a hit is highly improbable to occur. And even if it does, the assumption $a \gg 1/\sqrt{\alpha}$ ensures that it will have practically no effect. The same reasoning holds for hits where the center b satisfies $b \gg a$, of course. In short, the wavefunction must have a non-negligible amplitude at a hit location for it to be effective.

Having now illustrated the collapse mechanism of QMSL, we must explain how it conforms to reality. Concretely we must require the collapse mechanism to be ineffectual at microscopic scales to agree with LQM, and to explain the fact that experiments have outcomes we must require that it is highly effective at the macroscopic scale. This is typically referred to as the *amplification mechanism*, and it must necessarily be present in any type of reduction model. This mechanism usually consists of choosing the parameters of the model such that the reductive processes are *extremely* weak for single-particle systems; in the case of QMSL they are such that a particle having been around since the big bang will at this point in time have suffered just a single hit. More concretely, the frequency λ of the underlying Poisson process is taken to be

$$\lambda \sim 10^{-16} \text{ sec}^{-1} \tag{4.15}$$

[14] If on the other hand we assume that $a \ll 1/\sqrt{\alpha}$, the wavefunction will be more or less unaltered by the hit. This is an important point too, but it obviously does not illustrate collapse.

Which practically ensures agreement with quantum mechanics at the microscopic scale. However, in the event that a hit *does* occur, one additionally chooses the length scale of the process to be

$$1/\sqrt{\alpha} \sim 10^{-5} \text{ cm} \tag{4.16}$$

Since microscopic systems are typically much smaller than this, an eventual hit, however improbable, will also be quite ineffectual.

As for macroscopic systems, their rapid collapse is ensured by their large number of constituent particles and the helpful hand of entanglement; if one particle localizes then so does the whole system. Effectively the centre of mass of an N-particle system is hit by a Poisson process with frequency $\lambda_N = N\lambda$. This isn't terribly difficult to show[10], but it is even simpler in the context of QMUPL - so we postpone the calculation until the end of Section 4.4.2. As an example of the numbers involved, suppose a macroscopic system is made up of $\sim 10^{24}$ constituent particles. If we then prepare it in a superposition of position states, it will localize in approximately 1 μs.

4.2.2 Measurement in QMSL

We have now presented the unified dynamics of the QMSL model, and given a short calculation illustrating how it affects position state superpositions. We have also shown that the induced collapse of the model only emerges at the macroscopic scale. But we have not yet discussed measurements and the emergence of the Born rule, and this is the topic of the current section. For the moment, we will still assume that by a measurement we mean some sort of interaction between a microscopic and a macroscopic system.[15] Thus the usual von Neumann model will be sufficient for our discussion, and we begin by considering the premeasurement state of a system initally in the state $|\psi(t_0)\rangle$:[16]

$$|\psi(t_0)\rangle\,|r\rangle \to \sum_i c_i(t_0)\,|s_i\rangle\,|m_i\rangle \tag{4.17}$$

Within approximately[17] 1 μs of the formation of the macroscopic superposition, one of its constituents will suffer a hit, immediately localizing the measurement apparatus onto one of the states $|m_s\rangle$. We emphasize that the postmeasurement state is *not* $|s\rangle\,|m_s\rangle$, as we would conclude from the projection postulate. Instead it is highly localized around it, e.g. $|c_s(t_1)|^2 \gg |c_i(t_1)|^2$ for $i \neq s$ and $t_1 > t_0$. Thus it is clear that QMSL does not give a clear-cut solution to the measurement problem in the sense of entirely reducing macroscopic superpositions; we are still required to interpret the state in Eqn. (4.17). To interpret it, we need to invoke the last postulate of the model - namely, that the squared modulus of the wavefunction is related to the matter density. Concretely, given a system consisting of N particles with possibly different masses $m_i,\ i = 1, \ldots, N$, the mass density of particle j is given by

$$\rho(x_j) = m_j \left(\prod_{i \neq j}^{N} \int d^3x_i \right) |\psi(x_1, \ldots, x_N)|^2 \tag{4.18}$$

This means that the postmeasurement states created by the hits describes systems where the *mass density* is highly localized - not the probability density, as we would conclude from the old

[15] One may very well argue that other kinds of measurements may be done wherein no macroscopic system is involved. We will return to this point in the next section.

[16] Note that this assumes only the Schrödinger equation is in play; in other words, we assume the characteristic time required to form the premeasurement state is considerably less than the inverse frequency of the hits for the combined system(effectively only the measurement apparatus).

[17] Again assuming the measurement apparatus to consist of $\sim 10^{24}$ particles.

Born rule. Concretely, suppose we send a spin-1/2 particle of mass m through a Stern-Gerlach apparatus to measure its spin, yielding the state

$$|\psi(t_0)\rangle = \eta(t_0)\,|\uparrow \psi_+\rangle + \beta(t_0)\,|\downarrow \psi_-\rangle \tag{4.19}$$

where $|\psi_\pm\rangle$ are highly localized around the positions $x_\pm$.[18] If we then let the particle interact with a measurement apparatus, we get the premeasurement state

$$|\psi\rangle = \eta(t_1)\,|\uparrow \psi_+ m_\uparrow\rangle + \beta(t_1)\,|\downarrow \psi_- m_\downarrow\rangle \tag{4.20}$$

Once this state is formed, the hits on the measurement apparatus will rapidly force one coefficient to dominate the other - let us assume that $|\eta(t_2)|^2 \gg |\beta(t_2)|^2$. We wish to associate this with a measurement of $|\uparrow\rangle$. If we then compute the mass density of our particle, we find

$$\rho(x) = m\left[|\eta(t_2)\psi_+(x)|^2 + |\beta(t_2)\psi_-(x)|^2\right] \tag{4.21}$$

In the standard interpretation, this would be untenable as it would imply a non-zero probability to subsequently measure the particle to have spin-down. But within the QMSL, this type of state is perfectly acceptable. Consider an experiment designed to verify that the particle indeed is localized around $|x_+\rangle$. Since the mass density is almost completely localized at that point, any type of gravitational(or electromagnetic, since the charge of the particle needs mass to support it) interaction will also conclude with a particle localized around $|x_+\rangle$. Thus we see that in QMSL, the concept of measurement is well-defined, yielding outcomes in a precise way and with a collapse mechanism that does indeed yield states that are compatible with what we expect of repeated measurements.

Though not directly, the above discussion also reveals the emergence of the Born rule. Consider again the state in Eqn. (4.20). As we illustrated, a hit on the apparatus[19] will yield an outcome by localizing the apparatus state to be either $|m_\uparrow\rangle$ or $|m_\downarrow\rangle$. Thus we see that the probability of getting a particular outcome is directly proportional to the probability of a hit localizing the apparatus in the corresponding state for the measurement state. Let us compute the probability of measuring spin up in the above case. We shall assume that the apparatus states $|m_i\rangle$ are highly localized in regions Σ_i with respect to the hits, e.g. the length scale $|m|$ of these states is much less than $1/\sqrt{\alpha}$. This ensures that the individual apparatus states $|m_i\rangle$ are practically unaffected by the hits. We also assume the states $|m_i\rangle$ to be separated by a distance $d \gg 1/\sqrt{\alpha}$, such that a hit will localize only a single state $|m_i\rangle$. Then the probability for the measurement apparatus to localize around $\Sigma_\uparrow$ is

$$P(\Sigma_\uparrow) \approx \int_{\Sigma_\uparrow} \|L_x\,|\psi\rangle\|^2 \approx |\eta(t_1)|^2 \int_{\Sigma_\uparrow} \|L_x\,|m_\uparrow\rangle\|^2 \approx |\eta(t_1)|^2 \tag{4.22}$$

Where the last (approximate) equality holds because $L_x\,|m_\uparrow\rangle \approx 0$ if $x \notin \Sigma_\uparrow$, allowing us to extend the limits to infinity.[20] A similar result holds for localization around $\Sigma_\downarrow$. This is just what we would expect from the Born rule. And since either $|\eta(t_2)|^2 \gg |\beta(t_2)|^2$ or $|\beta(t_2)|^2 \gg |\eta(t_2)|^2$, the post-measurement state will effectively be either of the product states $|\uparrow\rangle\,|m_\uparrow\rangle$ or $|\downarrow\rangle\,|m_\downarrow\rangle$. Thus we see that the Born rule and in a sense also the projection postulate[21] are emergent from the unified dynamics of QMSL.

[18] These evolve in time, of course, but by including it our discussion will only gain in notational clutter, so we disregard it.

[19] We will not consider the case where the spin-1/2 particle suffers a hit; this case is of negligible probability compared to that of the apparatus suffering a hit.

[20] Physically, this just says that the probability of hitting $|m_\uparrow\rangle$ is essentially zero outside $\Sigma_\uparrow$.

[21] In the form: after a measurement of an operator O on a system S, the state of S will *effectively* be one of the eigenstates of O.

4.2.3 The concept of measurement

In the previous section we explicitly assumed that any measurement can be considered as an interaction between a microscopic and a macroscopic system. This need not be the case, a fact that was brought to attention by Albert and Vaidman.[64, 65] By way of example, the authors consider a spin-1/2 particle passing through a Stern-Gerlach apparatus, which subsequently impinges on a fluorescent screen. As the particle hits the screen, it will inevitably excite a number(possibly macroscopic) of atomic electrons, e.g. the state of the incident particle and the electrons in the screen is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big[\,|\uparrow, x_\uparrow\rangle\,(\otimes_i\,|e_{i,\uparrow}, x_i\rangle) + |\downarrow, x_\downarrow\rangle\,(\otimes_i\,|e_{i,\downarrow}, x_i\rangle)\Big] \tag{4.23}$$

where $|e_{i,\uparrow}\rangle$ denotes the energy state of the i'th electron relative to the incident particle being in the $|\uparrow, x_\uparrow\rangle$ state. The point to notice is that this superposition is not of the type affected by the hits as it is a superposition of energy states - thus there is no reduction at this stage. Inevitably, however, the excited electrons will de-excite and emit photons. But QMSL does not permit us to consider an eventual reduction of these photons - they are as relativistic as you can get, and QMSL is explicitly non-relativistic.[22] This means that an eventual reduction in this experiment can only occur when the emitted photons cause a macroscopic position-space superposition, e.g. in the nervous system of a human observer watching the screen. This is worrying at two levels. For one, the measurement described above is typically considered as concluded when the particle impinges on the screen. And secondly, it seems to require the existence of some sentient being with a retina or two watching the screen.

Although it can be shown that a superposition of photons entering the retina will cause reduction and subsequently conclude the measurement[15], it is important to note that Albert and Vaidman's conclusion that a sentient being of some sort is required for the model to make sense is too hasty. For one, sentience is entirely beside the point - all that is required is a macroscopic superposition, whether it occurs in the retina a sophisticated monkey or a brick of concrete is irrelevant. Secondly it is reasonable to think that such a macroscopic superposition will arise long before the photons reach the retina of an observer. In any given experimental situation they will most likely first traverse *some* type of medium(glass, air and so on), creating a superposition therein which may then suffer a collapsing hit.

To conclude, if we wish to accept the QMSL model we are forced to somewhat reconsider the concept of measurement. But this is not too unreasonable of a request - *measurement* as typically considered is a vague concept at the best of times. Let us also emhpasize again that sentient beings have no privileged position within the model - it may simply be that the first collapse-inducing interaction occurs within such a being.

4.3 Issues with QMSL

We have now presented the main mechanisms at work in QMSL, and it is clearly an impressive theory. It is however not without problems, and we present a few known issues below. Some are only apparent and easily rebutted once the proper mindset is adopted, others are severe and indicate the necessity of a more advanced theory. We shall see some of the latter problems remedied later, but others persist even in the most sophisticated reduction models.

[22] Conceivably, a relativistic extension may allow us to consider the reduction of photons. But even this will not help us in this case - the photons emitted from the screen will have overlapping wavepackets long before they suffer reduction - assuming a similar hit frequency.

4.3.1 Indistinguishable particles

QMSL cannot handle indistinguishable particles by construction. The localization process (4.1)) in particular is assumed to act only on one given particle. If our particles are indistinguishable, this is ill-defined as we cannot label any single particle in a multiparticle system. A perhaps naive generalization also fails; suppose we have a system of two indistinguishable particles in the state

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \left(h(x_1)t(x_2) + t(x_1)h(x_2) \right) \tag{4.24}$$

Where $h(x)$ and $t(x)$ are highly localized wavefunctions. If the localization is supposed to affect only a single particle, it seems reasonable that the hit operator should act only in one Hilbert space, e.g. $L_t \to L_t \otimes \mathbb{1}$. If $\psi(x_1, x_2)$ suffers a hit at $t(x_1)$, it then changes to

$$L_t \psi(x_1, x_2) \approxeq t(x_1)h(x_2) \tag{4.25}$$

which is no longer invariant under permutations. There is not much to discuss about this point - it is a clear weakness of the model that must be remedied. However, let us also mention a related oddity. QMSL insists that hits affect particles. But what constitutes a particle? Suppose for example a proton is hit - is it then really one of its constituent valence quarks that is hit? This is not so clear to us.

4.3.2 Energy conservation

In QMSL, the equation of motion for the density matrix of a free particle is analytically solvable. What one finds[10] is that the expectation values of position and momentum are the same as in the case of evolution under the Schrödinger equation, but the expectation of the *square* of position/momentum increases with time. Concretely, the standard deviations are

$$\sigma_x = \sigma_x^s + \frac{\alpha \lambda \hbar^2}{6m^2} t^3 \tag{4.26}$$

$$\sigma_p = \sigma_p^s + \frac{\alpha \lambda \hbar^2}{2} t \tag{4.27}$$

Where the superscript s denotes the standard deviation computed from pure Schrödinger evolution. The energy gain isn't really a surprise, of course. Any type of localization in position space necessarily implies a delocalization in momentum space. In other words, energy gain is inevitable in any type of collapse model. Importantly the energy gain is independent of hit location;[23] as such all states in the ensemble gain energy upon a hit.

What perhaps is more surprising is that the spread in position is strictly increasing, which isn't quite what we'd expect from a model of wavefunction collapse. For this to make sense, there are a couple of details to pay attention to. One, this is the spread as computed for the total *ensemble*, i.e. averaged over all possible hit locations. Suppose for example that we start off with a system in the pure state

$$\rho = \frac{1}{2} \left(|h\rangle \langle h| + |h\rangle \langle t| + |t\rangle \langle h| + |t\rangle \langle t| \right) \tag{4.28}$$

[23] Except perhaps arguably where the wavefunction is of negligible norm - but in this case it will not contribute to the ensemble anyway.

Such that the spread in position is

$$\sigma_x = \frac{1}{4}(h-t)^2 \tag{4.29}$$

Now let a hit occur, and assume for simplicity that it is completely reducing. The hit has a 50/50 chance to reduce the state to either $|h\rangle\langle h|$ or $|t\rangle\langle t|$. As we do not know the hit location, they must be averaged over. Consequently the system is now in the fully mixed state

$$\rho = \frac{1}{2}(|h\rangle\langle h| + |t\rangle\langle t|) \tag{4.30}$$

Which yields the exact same spread as we initially had. The same type of reasoning can be applied as long as Hamiltonian motion is sufficiently weak so as not to interfere with the collapse process - the collapse dynamics themselves preserve the individual diagonal elements of the density matrix for short times and so the variance in position averaged over the ensemble stays the same.

Secondly, we must explain what actually causes the increasing position spread in the ensemble. The answer lies in the energy gain; as energy is transferred to a system upon a hit, its momentum-space wavefunction delocalizes, in turn causing a stronger diffusion than does regular Schrödinger evolution. In any event, this type of feedback effect takes an extremely long time to build up - at least in the macroscopic case. As an example order of magnitude estimate, consider t to be on the order of the collapse time for an object consisting of 10^{23} carbon atoms, e.g. $t \sim \mu s$ and $m \sim 10g$. Then we have $\delta\sigma \sim 10^{-77}$ mm^2 - hardly relevant for collapse dynamics.

Nonetheless, these two manifestations of energy non-conservation are still important. Energy conservation is an extremely successful principle of physics, and cannot easily be put aside. Despite this, we know of no attempts to resolve the issue in the context of QMSL. Presumably this is because the obvious solution - introducing dissipative effects - should ideally be done at the statevector level. However, note that we have not presented an explicit equation of motion for individual states - in fact, none has so far been constructed. As such we'll leave this point for now and return to it when we discuss the QMUPL model.

4.3.3 The tails problem

The tails problem is somewhat vaguely defined, but we attempt to cover a few of the arguments which are typically referred to as "the tails problem". Common to all is that they take issue with the wavefunction not collapsing onto position eigenstates.

First, there is the objection that the collapse induced in QMSL does *not* actually result in localized states.[66] The argument is that as the localization process results in a wavefunction with non-compact support, a system cannot be ascribed a definite position post-collapse as any subsequent measurement might find it anywhere. The argumentation is strange for two reasons. Firstly, as we mentioned in our discussion of the standard interpretation of QM, the concept of localization is difficult to work with even there. In particular, the eigenstate-eigenvalue link does not permit us to speak of particles being anywhere at all. Insisting on reduction onto non-normalizable position eigenstates then seems like an odd request. So if this is a problem, then it is not one unique to QMSL. Secondly the argument is based on the assumption that the standard interpretation of post-measurement states is the correct one, e.g. that the square modulus of the wavefunction is a probability. This is a plain misunderstanding of QMSL - the correct understanding of post-measurement state is the one we gave in Section 4.2.2. And while we admit that the mass densiy of any system will be smeared out across all space after collapse, any type of experiment designed to reveal the position of the system will give a well-defined answer.

A similar argument is the so-called counting problem.[67, 68, 69] Consider a marble which is in a superposition of being inside a box and outside it: $|\psi\rangle = a(t_0)\,|\text{in}\rangle + b(t_0)\,|\text{out}\rangle$. If a constituent atom suffers a hit, the marble will be in a state where $|\psi(x)|^2$ is heavily concentrated either inside the box or outside it. But as it does not have compact support, some of $|\psi(x)|$ will necessarily also be either outside or inside, respectively. Consider then the case of N marbles and N boxes, where the total state of the marbles is

$$|\psi\rangle = \bigotimes_{1\leq i\leq N} \left(a_i(t_0)\,|\text{in}_i\rangle + b_i(t_0)\,|\text{out}_i\rangle\right) \tag{4.31}$$

Within less than a second, all marbles have localized. Suppose that they all localize within their respective box, e.g. each marble is in the state

$$|\psi_i\rangle = a_i(t_1)\,|\text{in}_i\rangle + b_i(t_1)\,|\text{out}_i\rangle \tag{4.32}$$

where $|a_i(t_1)|^2 \gg |b_i(t_1)|^2$. According to the ontology of QMSL, this means that the particles really are inside their boxes, since that is where their mass density is localized. And yet, since $|a_i(t_1)|^2 < 1$, the *probability* of finding all particles inside their box is $|a_i(t_1)|^{2N}$, which we can make arbitrarily small. We would reach the same conclusion in regular QM, of course, but without the claim that the marbles necessarily were in their boxes.

The argument is faulty, of course. Again the conclusion reached by the authors cited above is based on the assumption that the coefficients $|a_i(t)|^2$ denote probabilities for finding the particles somewhere. But this is not how the Born rule works in QMSL - the fact that $|a_i(t_1)|^{2N} \to 0$ simply tells us that it is vanishingly unlikely that all particles subsequently suffer a hit inside their box. At the time t_1, the path of any test particle would reveal that all particles were inside their box, assuming that no further hits occur whilst the test particle is in motion. As such QMSL does not contradict itself.

On the other hand, it is true that for *extremely* large N we cannot guarantee that all marbles remain inside their respective box, as by taking N large enough we can ensure a non-negligible probability for at least one hit occuring outside a box. Thus an experimenter faced with counting some $N = 10^{20}$ marbles - may God have mercy on his soul - will inevitably find that not all of his N marbles were inside their box. But this occurs only because the counting process has a timescale longer than that of the hitting process; we can still state without contradiction that all marbles were inside their box at $t = t_1$. Note also that while it may seem odd that marbles suddenly jump out of their box, it is no more puzzling than is regular tunneling. In LQM there is also a non-zero but rather small probability for marbles to simply tunnel out of their box.

Though the previous two tails problems were easily dismissed, the next one is not. It concerns the effect that the localization process has on the tails themselves, and is referred to as the problem of *structured tails*.[70] The issue here is that if a wavefunction suffers a hit at $x = a$, then all other parts of the wavefunction are shifted, or "kicked" towards $x = a$. In Eqn. (4.13) we computed a first order shift of

$$|\delta x| = 2a\frac{\alpha}{\gamma} \tag{4.33}$$

for a microscopic system(e.g. any system satisfying $\gamma \gg \alpha$). Recall that in this particular example we assumed a hit occuring at $x = a$, such that $2a$ essentially represents the distance between the centre of our microscopic system and the hit centre. Furthermore, the characteristic size of the system is given by $1/\sqrt{\gamma}$. If the displacement length δx exceeds the size of the system:

$$2a\left(\frac{1/\sqrt{\gamma}}{1/\sqrt{\alpha}}\right)^2 > \frac{1}{\sqrt{\gamma}} \tag{4.34}$$

this is evidently bad news.[24] Concretely, consider a macroscopic object consisting of $\sim 10^{23}$ nucleons in a superposition of two Gaussian states separated by a distance of 10 m. Within approximately a microsecond of the formation of this superposition, a constituent nucleon will suffer a hit centred in either of the two Gaussian states. This nucleon will then suffer a shift

$$\delta x = 10 \text{ m} \left(\frac{1 \text{ pm}}{100 \text{ nm}} \right)^2 = 1 \text{ pm} \tag{4.35}$$

That is, it is displaced by an amount of the order of its own length scale - certainly enough to excite it! And if we manage to extend the separation distance by another two orders of magnitude, the constituent nucleons suffering hits will be kicked right out their parent atoms. Again, this is worrying due to energy conservation - the displacement of atoms is equivalent to a net energy gain in the system. Another worrying point is that the energy has to *go* somewhere. As the decay of excited states will typically involve the emission of radiation, and considering that collapse processes occur everywhere and all the time, it is quite the surprise that we aren't all suffering from severe radiation sickness.

4.3.4 The nature of stochasticity

QMSL and reduction models in general are phenomenological by construction. They modify the Schrödinger equation and justify it on the grounds of explanatory power - within experimental bounds, and despite issues like energy non-conservation, it works! But ultimately, a believer in reduction models would like to see his/her model of choice ascend to the status of a fundamental theory. To do so, it is necessary to explain the nature of whatever stochastic term is added to the Schrödinger equation - be it a Poisson process, Brownian motion or something else. Pearle has called this the *legitimization problem*.[71] In the case of QMSL and its underlying Poisson process, we know of no attempts to resolve the issue. This is natural, of course, as QMSL cannot possibly be a fundamental theory - it cannot even handle indistinguishable particles. The situation is different when it comes to QMUPL and CSL. In the case of QMUPL, an early attempt by L. Diosí was made to explain the origin of the noise on gravitational grounds.[72] This is an especially attractive option because gravitational effects are mass-proportional and so provide a natural amplification mechanism.[25] Concretely, he considered a measurement of the gravitional acceleration $g = -\nabla\phi$ by a quantum probe. Performing the measurement over a time T and averaging over a volume V, he found that the uncertainty in g satisfies the following bound:

$$(\delta g)^2 \geq \frac{\hbar G}{VT} \tag{4.36}$$

Where G is the gravitational constant. This then motivates introducing quantum fluctuations around the mean of the gravitational potential, e.g. the appearance of stochasticity in the Schrödinger equation. Note also that this means the resulting reduction model becomes parameter free, as the stochastic element is expressed in terms of known constants like $\hbar$ and G. Unfortunately Diosí's model was quickly rejected, as it was shown by Ghirardi et al. that it would cause matter to emit an amount of radiation measurable with current technology.[74]

It should further be mentioned that the noise appearing in models of the QMUPL type is not easily connected to physical fields. As we will see, this type of model couples a Brownian motion to each particle individually, and it is not easy to see how these processes connect to a field filling

[24] We note that Wallace also reached this inequality, but by somewhat different means.[70]

[25] Interestingly Diosí was inspired by Feynman, who suggested that if the collapse postulate was connected with gravity, then a breakdown of unitary quantum mechanics should occur at the mass scale set by $GM^2/\hbar c = 1$.[73]

spacetime. This particular issue is remedied in some relativistic reduction models, as in the one constructed by Bedingham[60]. The novelty of the model is that a new, so-called pointer field is introduced with an associated non-Hermitian stochastic coupling. The stochastic part of the coupling consists of a field of Brownian motions $W_t(x^\mu)$, and the possibility of explaining the physical origins of this field is more promising.[26] As it stands this remains an open problem to be studied.

4.4 QM with Universal Position Localization

We next turn our attention to the so-called QMUPL model. It is interesting to us mainly because of its similarity to the most sophisticated collapse model(namely, the CSL model) and because it is analytically and numerically tractable(unlike CSL).[27] Of course, it also represents an improvement on the QMSL model, being able to handle indistinguishable particles.[28] We will begin our discussion by considering a general Ito process in Hilbert space. Unsurprisingly, not all such processes are physical - they may not conserve the norm of states, for example. Our first task is then to enforce various physical constraints, and this will eventually lead us to the evolution equation of the QMUPL model. With this in hand, we will discuss the collapse mechanism in some detail. It is fuzzy by construction, but we will see that collapse occurs over a well-defined timescale. The model seemingly allows measurement apparatuses to "redecide" on an outcome; we investigate the phenomenon and find that it has little effect if the apparatus is macroscopic. We will furthermore see that under the evolution equation of QMUPL, the wavefunction of free particles and those in the presence of linear or quadratic potentials will asymptotically converge to a Gaussian.

We begin, as in ref. [15], by considering the general Ito process

$$d\left|\psi_{B_i}(t)\right\rangle = \left(Cdt + A_i dB_i\right)\left|\psi_{B_i}(t)\right\rangle \tag{4.37}$$

Where C and A_i are arbitrary operators, summation convention is implied and B_i are independent Brownian processes whose increments satisfy

$$\mathbb{E}^{\mathbb{R}}(dB_i) = 0 \qquad\qquad \mathbb{E}^{\mathbb{R}}(dB_i dB_j) = \gamma\delta_{ij}dt \tag{4.38}$$

We will furthermore write only $|\psi\rangle$ instead of $|\psi_B(t)\rangle$ to reduce clutter, although this of course opens up the possibility of confusion as to which process $|\psi\rangle$ depends on - it will usually be clear from the context, but some care must be shown. Now, some points about Eqn. (4.37) strike us immediately as peculiar. For one, the equation above is linear, which is unexpected if it is to cause collapse. We also see that the evolution is not manifestly unitary(however one would define this for a stochastic process), which is a troubling prospect for the conservation of probability. It is also somewhat strange that quantum states are now stochastic processes, and it is not clear at this point how this is to be understood. As it turns out, all these issues find their resolution in the consideration of a probability interpretation, and so we will discuss this important point before we begin formulating constraints on Eqn. (4.37).

[26] The pointer field permits energy exchange between itself and collapsing matter, solving the issue of energy conservation in reduction models. As for its nature, one might conjecture that it is connected to dark matter somehow. This possibility has not yet been explored, however - in fact, the pointer field so introduced by Bedingham is quite unconstrained, though it is postulated to be bosonic.

[27] CSL can be explored numerically, of course. But it is computationally expensive compared to QMUPL, which in itself is also rather demanding. For context, some of our simulations of QMUPL required in the range of $10-20$ hours to complete on the authors relatively strong desktop PC.

[28] Strangely, A. Bassi writes that it cannot.[75] On the other hand, N. Gisin and P. Pearle write that it can.[32, 52] It is our point of view that QMUPL *can* handle indistinguishable particles; we will later see that the generator of time evolution commutes with the particle-exchange operator P.

4.4.1 Probability in QMUPL

We mentioned that the statevector is now a stochastic process. This means that given some initial state $|\psi(0)\rangle$, each realization w of the Wiener processes B_i will give rise to a distinct statevector $|\psi_w(t)\rangle$. Thus any state immediately evolves into an ensemble, where the probability of each state $|\psi_w(t)\rangle$ is that of its realization. The measure on the space of realizations - which is really just the space of all continuous functions - we denote by $\mathbb{R}[w]$.

Now, of course, we want our model to cause collapse. Under the assumption that some wavefunctions in our ensemble have in a certain sense collapsed, it is necessary to supply a rule to pick these out of the ensemble and discard the others. To motivate this rule, we first heuristically illustrate how collapse occurs.[29] We start with Eqn. (4.37) and make the replacements[30]

$$C = -\left(iH + \frac{1}{2}\gamma\right) \tag{4.39}$$

$$A_i = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp\left(-\frac{\alpha}{2}(\hat{x} - z_i)^2\right) dz_i \tag{4.40}$$

Here H is the Hamiltonian of the system, $\hat{x}$ is the position operator, γ and α are real constants, the set $\{dz_i\}$ is a discretization of position space and z_i is a point in the i'th interval. For notational convenience, we stick to one dimension. Taking furthermore the coordinate representation and passing to the continuum limit, we obtain

$$d\psi(x) = -idt \int dx' \, \langle x|H|x'\rangle \, \psi(x') + \left[\left(\frac{\alpha}{\pi}\right)^{1/4} \int dz \exp\left(-\frac{\alpha}{2}(x-z)^2\right) dB_z - \frac{\gamma}{2}dt\right] \psi(x) \tag{4.41}$$

For simplicity, we disregard the Hamiltonian evolution.[31] Thus we are left with the simple equation

$$d\psi(x) = \left[\left(\frac{\alpha}{\pi}\right)^{1/4} \int dz \exp\left(-\frac{\alpha}{2}(x-z)^2\right) dB_z - \frac{\gamma}{2}dt\right] \psi(x) \tag{4.42}$$

Now, the Wiener increments dB_z have vanishing expectation and so the vast majority of states in our ensemble will be affected by the second term only, e.g. their norm will decay exponentially. However, there will be some small subset of the ensemble for which, in some region Ω, dB_z will be positive on average and so will dominate the second term. In these cases, the wavefunction grows exponentially. If the wavefunction is also comparatively large in Ω, its norm will quickly exceed that of the other states in the ensemble undergoing exponential decay. Consider then the case of a state consisting of two Gaussian wavepackets. The vast majority of states in the resulting ensemble will quickly have negligible norm. A select few will have experienced growth in the region of either wavepacket, thus collapsing the state. Considerably fewer will also experience growth in the region of *both* wavepackets, thus not collapsing.

[29] We will discuss this in considerably more detail in Section 4.4.4.

[30] The parameter choices here are made so as to turn the equation into one originally proposed by Pearle.[52] In fact, this was the first *continuous* collapse model, and Pearle naturally used the term CSL about it. In more recent literature, his type of model falls under the term QMUPL, and CSL instead refers to field-theoretic collapse models. This means the terminology is unavoidably a little confusing, but we stick with recent conventions here.

[31] This is equivalent to assuming the stochastic evolution to occur on much shorter timescales than the Hamiltonian evolution, an assumption that may not necessarily be realistic. We give an example wherein they occur on the same timescale in Section 4.6.

We now state our probability rule: to pick out those states that have collapsed, we postulate that the probability of getting the statevector $|\psi_w(t)\rangle / \||\psi_w(t)\rangle\|$ is[32]

$$\mathbb{P}[w(t)] = \mathbb{R}[w(t)] \||\psi_w(t)\rangle\|^2 \tag{4.43}$$

Of course, the ensemble of states which occur with physical probability is different from the ensemble of states which occur with the raw probability. To make this explicit, we will refer to these ensembles as being physical and raw, respectively. Concretely,

$$\rho = \int d\mathbb{R}[w] \, |\psi\rangle \langle\psi| = \int \mathcal{D}[w]\mathbb{R}[w] \, |\psi\rangle \langle\psi| \tag{4.44}$$

$$= \int d\mathbb{P}[w] \frac{|\psi\rangle \langle\psi|}{\||\psi\rangle\|^2} = \int \mathcal{D}[w]\mathbb{P}[w] \frac{|\psi\rangle \langle\psi|}{\||\psi\rangle\|^2} \tag{4.45}$$

Where $\mathcal{D}[w]$ denotes a functional measure over sample paths w. We can write this more compactly as

$$\rho = \mathbb{E}^{\mathbb{R}}[|\psi\rangle \langle\psi|] = \mathbb{E}^{\mathbb{P}}\left[\frac{|\psi\rangle \langle\psi|}{\||\psi\rangle\|^2}\right] \tag{4.46}$$

Which illustrates why we call one ensemble physical - its constituent states are normalized. Note also how our probability rule picks out the correct states in our example of the two Gaussian wavepackets above: states whose norm is undergoing exponential decay are heavily suppressed as the norm appears in the probability rule. The states undergoing growth in both regions are also heavily suppressed due to the presence of the raw probability. Thus it is only the states where growth occurs in a single region which have both a comparatively large raw probability and norm. This concludes our discussion of the probability interpretation of the continuous collapse models, and in the next section we shall use it to constrain the form of Eqn. (4.37).

4.4.2 Developing the fundamental equation

Although we have now formulated the relation between the physical and raw probability densities, it is not clear that the physical probability is normalized, e.g. that

$$\int d\mathbb{P}[w(t)] = \int d\mathbb{R}[w(t)] \||\psi\rangle\|_t^2 = \mathbb{E}^{\mathbb{R}}[\||\psi\rangle\|_t^2] = 1 \tag{4.47}$$

There are several ways of using this requirement to constrain the form of Eqn. (4.37). The first we discovered was due to Ghirardi and Bassi[15], unfortunately we could make little sense of its "intuitive" steps. In trying to gain a better understanding, we saw that one of these steps was referred to as a "Girsanov transformation" by Gisin.[76]. Further reading revealed that this in turn imposed the requirement that $\||\psi\rangle\|_t^2$ be a so-called martingale.[77] Piecing things together, we believe we (1) have found Bassi and Ghirardi's (initial) approach to be wrong and (2) have a more coherent and semi-rigorous way of constraining the fundamental equation of the model. As such, the following section contains some calculations that are similar to those found in ref. [15], but we have otherwise not seen a derivation like this in the literature. We mentioned that there are other ways of reaching the same conclusions, so for completeness we include one argument

[32] We will refer to this probability as physical, although it is often referred to as the cooked probability in the literature, e.g. by Bassi and Ghirardi.[15] We prefer physical because "cooking" implies a rather pragmatic attitude. Although we do not pretend to give a rigorous treatment of the model, the "cooking" approach turns out to be so pragmatic that it is flat-out wrong - as we will see.

which does not involve any mention of martingales. A further method[78] we leave off for now - it is needlessly complicated for the case of uncorrelated Brownian motion, but we will return to it in Section 4.6 when it is needed.

Without further ado, we begin by noting that Eqn. (4.47) is naturally satisfied if $\||\psi\rangle\|_t^2$ is martingale. To see this, let (g_t) be the natural filtration[33] of $|\psi\rangle$ and assume that $\||\psi\rangle\|_{t_0}^2 = 1$. Then

$$\mathbb{E}^{\mathbb{R}}[\||\psi\rangle\|_{t+dt}^2] = \mathbb{E}^{\mathbb{R}}[\mathbb{E}^{\mathbb{R}}[\||\psi\rangle\|_{t+dt}^2|g_t]] = \mathbb{E}^{\mathbb{R}}[\||\psi\rangle\|_t^2] \tag{4.48}$$

Where we have used the definition of the conditional expectation(Eqn. (C.18)) to reach the first equality and the martingale condition to reach the last. Using this recursively we find that indeed $\mathbb{E}^{\mathbb{R}}[\||\psi\rangle\|_t^2] = \||\psi\rangle\|_{t_0}^2$. Our objective is then to show that $\||\psi\rangle\|_t^2$ is martingale. Now by linearity of the conditional expectation, we have

$$\mathbb{E}^{\mathbb{R}}\left[\||\psi(t+dt)\rangle\|^2\middle|g_t\right] = \mathbb{E}^{\mathbb{R}}\left[\||\psi(t)\rangle\|^2 + d\||\psi\rangle\|^2\middle|g_t\right] = \||\psi(t)\rangle\|^2 + \mathbb{E}^{\mathbb{R}}\left[d\||\psi(t)\rangle\|^2\middle|g_t\right] \tag{4.49}$$

So the conditional expectation of the variation in the norm squared must vanish. Computing it, we find

$$d\langle\psi|\psi\rangle = (d\langle\psi|)|\psi\rangle + \langle\psi|(d|\psi\rangle) + [\langle d\psi|, |d\psi\rangle]_t \tag{4.50}$$

$$= \langle\psi|\left[(C+C^\dagger)dt + (A_i + A_i^\dagger)dB_i + A_i^\dagger A_i\gamma dt\right]|\psi\rangle \tag{4.51}$$

The processes $B_i(t)$ are Wiener with respect to $\mathbb{R}$, consequently the corresponding increments dB_i are independent of any past history and so not g_t-measurable. The states $|\psi\rangle$ on the other hand *are*, and so can be freely taken out of the expectation. Taking the conditional expectation then kills the middle term and leaves us with

$$\mathbb{E}^{\mathbb{R}}[d\||\psi\rangle\|^2|g_t] = \langle\psi|(C + C^\dagger + A_i^\dagger A_i\gamma)dt|\psi\rangle \tag{4.52}$$

Thus we must require

$$C + C^\dagger = -\gamma A_i^\dagger A_i \tag{4.53}$$

Clearly, this constitutes (twice) the Hermitian part of C. Denoting the anti-Hermitian part by $-iH$(implying but not assuming H to be the Hamiltonian), the basic QMUPL equation turns into

$$d|\psi\rangle = \left[-iHdt + A_i dB_i - \frac{\gamma}{2}A_i^\dagger A_i dt\right]|\psi\rangle \tag{4.54}$$

We remark that this equation in fact describes geometric Brownian motion. If we further impose the requirement that the A_i be Hermitian and time independent, the evolution operator is then nothing else than the stochastic exponential of the process, e.g.

$$|\psi(t)\rangle = T\exp\left[-i\int H\ dt - \gamma t A_i^2 + A_i B_i\right]|\psi(t_0)\rangle \tag{4.55}$$

Where T is the time-ordering operator. If we do *not* notice that the requirement of norm conservation essentially is the martingale condition, things become a little more muddled. We still insist that the norm is unity at $t = t_0$ and that the expected variation in the norm vanishes.

[33] We discuss filtrations and filtered probability spaces in some detail in Appendix C.2.

But without using the conditional expectation, we cannot simply pull the state $|\psi\rangle$ out of the expectation as we did in Eqn. (4.52). So our starting point is

$$\mathbb{E}^{\mathbb{R}}[d\||\psi\rangle\|_t^2] = \left(C + C^\dagger + A_i^\dagger A_i\right) dt \mathbb{E}^{\mathbb{R}}[\||\psi\rangle\|_t^2] + \mathbb{E}^{\mathbb{R}}[\,\langle\psi|A_i + A_i^\dagger|\psi\rangle\, dB_i] \tag{4.56}$$

It's still natural to kill the deterministic part separately, e.g. insist on Eqn. (4.53). This again leaves us with Eqn. (4.54), though we have not yet shown the variation in the norm to vanish. Assuming a time-independent Hamiltonian and that $[H, A_i] = [A_i, A_j] = 0$, this is trivially true given our newly constrained SDE. Using Eqn. (4.55) we have

$$\mathbb{E}^{\mathbb{R}}\left[\,\langle\psi(t)|\psi(t)\rangle\,\right] = \langle\psi(t_0)|\psi(t_0)\rangle\, \mathbb{E}^{\mathbb{R}}\left[\,\exp\left(-2\gamma A_i^2 t + 2A_i dB_i\right)\,\right] = \langle\psi(t_0)|\psi(t_0)\rangle \tag{4.57}$$

Consequently the variation in the norm must vanish and we are again done.

Having now both ensured that our probability distribution is well-defined and constrained the fundamental equation, we turn to the next issue - the states $|\psi\rangle$ do not remain normalized under Eqn. (4.54). Indeed, we have seen that the variation in the norm is proportional to the differential of the Wiener processes B_i. It will however be convenient to have an evolution equation for the normalized states, and finding it is our next task.[34] Remembering our Ito calculus and defining $|\phi\rangle = |\psi\rangle / \||\psi\rangle\|$, we get

$$d\,|\phi\rangle = \frac{d\,|\psi\rangle}{\||\psi\rangle\|} - \left[\frac{1}{2}\frac{1}{\||\psi\rangle\|^3} d\||\psi\rangle\|^2 - \frac{3}{8}\frac{1}{\||\psi\rangle\|^5}\left[d\||\psi\rangle\|^2\right]_t\right]|\psi\rangle + \left[d\,|\psi\rangle\, d\left(\||\psi\rangle\|^{-1}\right)\right]_t \tag{4.58}$$

$$= \left[-iHdt + A_i dB_i - \frac{\gamma}{2}A_i^\dagger A_i dt\right]|\phi\rangle \tag{4.59}$$

$$- \left[\frac{1}{2}\,\langle\phi|A_i + A_i^\dagger|\phi\rangle\, dB_i - \frac{3}{8}\gamma dt(\,\langle\phi|A_i + A_i^\dagger|\phi\rangle)^2\right]|\phi\rangle \tag{4.60}$$

$$- \left[\frac{1}{2}\gamma dt A_i\,\langle\phi|A_i + A_i^\dagger|\phi\rangle\right] \tag{4.61}$$

$$= \left[-iHdt + (A_i - R_i)dB_i + \frac{\gamma}{2}dt\left(3R^2 - A_i^\dagger A_i - 2A_i R_i\right)\right]|\phi\rangle \tag{4.62}$$

Where we have defined

$$R_i \equiv \frac{1}{2}\,\langle\phi|A_i + A_i^\dagger|\phi\rangle \tag{4.63}$$

for notational convenience. This equation *does* conserve the norm, as we may check by a quick computation similar to Eqn. (4.51). We assume that $\langle\psi(t_0)|\psi(t_0)\rangle = 1$, and compute the

[34] It is particularly well suited for numerical simulations, as we will see in Section 4.4.3.

variation of the norm squared to be

$$d\langle\phi|\phi\rangle = \langle d\phi|\phi\rangle + \langle\phi|d\phi\rangle + [\langle d\phi|\,, |d\phi\rangle]_t \tag{4.64}$$

$$= idt\langle H\rangle + dB_i(\langle A_i^\dagger\rangle - R_i) + \frac{\gamma}{2}dt\Big(3R^2 - \langle A_i^\dagger A_i\rangle - 2\langle A_i^\dagger\rangle R_i\Big) \tag{4.65}$$

$$- idt\langle H\rangle + dB_i(\langle A_i\rangle - R_i) + \frac{\gamma}{2}\Big(3R^2 - \langle A_i^\dagger A_i\rangle - 2\langle A_i\rangle R_i\Big) \tag{4.66}$$

$$+ \langle(A_i^\dagger - R_i)(A_i - R_i)\rangle[dB^2]_t \tag{4.67}$$

$$= dB_i(\langle A_i\rangle + \langle A_i^\dagger\rangle - 2R_i) + \gamma dt\Big(3R^2 - \langle A_i^\dagger A_i\rangle - \langle A_i\rangle R_i - \langle A_i^\dagger\rangle R_i\Big) \tag{4.68}$$

$$+ \gamma dt\Big(\langle A_i^\dagger A_i\rangle - R_i(\langle A_i\rangle + \langle A_i^\dagger\rangle) + R^2\Big) \tag{4.69}$$

$$= 0 \tag{4.70}$$

We are almost done, but although Eqn. (4.62) does conserve the norm it is still conceptually displeasing because it is defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{R})$ whereas our physical states are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The dependence on the probability space comes in through the Brownian motion $B_i(t)$, so what we would like is some way of relating this to processes defined with respect to the measure $\mathbb{P}$. This can be done by appealing to the Radon-Nikodym theorem as well as the (Cameron-Martin-)Girsanov theorem. To see how, we need to take a little detour into the theory behind these theorems.[79, 80]

Definition. Given a measurable space $(\Omega, \mathcal{F})$ and two measures $\mathbb{V}$ and $\mathbb{W}$, we say that $\mathbb{W}$ is *absolutely continuous* with respect to $\mathbb{V}$ if $V(f) = 0 \implies W(f) = 0$ for $f \in \mathcal{F}$. We denote this as $\mathbb{W} \ll \mathbb{V}$. If also $\mathbb{V} \ll \mathbb{Q}$, we say that $\mathbb{V}$ and $\mathbb{W}$ are *equivalent*.

The Radon-Nikodym theorem states that if $\mathbb{W} \ll \mathbb{V}$, then there exists a measurable function Z such that

$$\mathbb{W}(f) = \int_f d\mathbb{W} = \int_f Z d\mathbb{V} \tag{4.71}$$

Z is called the *Radon-Nikodym derivative* and is denoted by

$$Z = \frac{\mathbb{d}\mathbb{W}}{\mathbb{d}\mathbb{V}} \tag{4.72}$$

Let now $\{g_t\}_{t\in T}$ be a filtration on the probability space $(\Omega, \mathcal{F}, \mathbb{W})$. This may be used to define a (density) process which we can guarantee to be martingale:

$$Z_t \equiv \mathbb{E}[Z|g_t] \tag{4.73}$$

That it is a martingale is easily shown by using the tower rule of the conditional expectation(see Appendix C.2):

$$\mathbb{E}[Z_{t+dt}|g_t] = \mathbb{E}[\mathbb{E}[Z|g_{t+dt}]|g_t] = \mathbb{E}[Z|g_t] = Z_t \tag{4.74}$$

We can now state the Girsanov theorem. Given the density process Z_t, it guarantees that there exists a g_t-adaptable process Y_t such that $Z_t = \mathcal{E}(Y_t)$. Furthermore, given a process X_t which is $\mathbb{V}$-martingale, the process

$$\tilde{X}_t = X_t - [X_t, Y_t] \tag{4.75}$$

is $\mathbb{W}$-martingale. This is the result we need to write the evolution equation for our states in terms of the physical probability, so let us apply it. From Eqn. (4.43), we see that the physical measure is absolutely continuous with respect to the raw measure, and that the Radon-Nikodym density process can be identified as $Z_t = \||\psi\rangle\|_t^2$. Then by Girsanov, this may be written as the stochastic exponential of a process Y_t. To find it, recall that(see Eqn. (4.50) and Eqn. (4.53)) Z_t satisfies the SDE

$$dZ_t = \langle\psi|A_i + A_i^\dagger|\psi\rangle\, dB_i \tag{4.76}$$

This SDE can be rewritten in exponential form by pulling out the norm:

$$dZ_t = 2R_i Z_t dB_i \tag{4.77}$$

Such that

$$Z_t = \exp\left[2\int_0^t R_i dB_i - 2\gamma\int_0^t R^2\, ds\right] \tag{4.78}$$

From which we can immediately identify $dY_t = 2R_i dB_i$. According to Eqn. (4.75) we can then relate the differentials of the $\mathbb{R}$-Brownian processes B_i to $\mathbb{P}$-Brownian processes $\tilde{B}_i$ through the differential of the quadratic variation $[B_i, Y]_t$, which we compute to be $2\gamma R_i dt$. Thus we finally find

$$d\tilde{B}_i = dB_i - 2\gamma R_i dt \tag{4.79}$$

Inserting this into Eqn. (4.62) and defining $D_i = A_i - R_i$, we get the final QMUPL equation

$$d\,|\phi\rangle = \left[-iHdt + \frac{\gamma}{2}\left(D_i R_i - D_i^\dagger A_i\right)dt + D_i d\tilde{B}_i\right]|\phi\rangle \tag{4.80}$$

If the operators A_i are Hermitian, as they will be in our examples, this simplifies to

$$d\,|\phi\rangle = \left[-iHdt - \frac{\gamma}{2}\left(A_i - R_i\right)^2 dt + (A_i - R_i)d\tilde{B}_i\right]|\phi\rangle \tag{4.81}$$

$$= \left[-iHdt - \frac{\gamma}{2}\left(A_i - \langle A_i\rangle\right)^2 dt + (A_i - \langle A_i\rangle)d\tilde{B}_i\right]|\phi\rangle \tag{4.82}$$

We remark again that our derivation does *not* follow ref [15]. In place of Eqn. (4.79) they instead define $d\tilde{B}_i = dB_i + 2\gamma R_i dt$ and it is not clear to us how they get this sign - it seems to us a mistake. If we were to get this sign, then the Radon-Nikodym derivative would have to be $\|\psi\|^{-2}$ which is not martingale. Furthermore, they make the direct substitution $dB \to d\tilde{B}$ in Eqn. (4.62) and claim that this is how one defines the process $|\phi\rangle$ with respect to the measure $\mathbb{P}$. This then leads to Eqn. (4.82) but with $d\tilde{B}_i \to dB_i$. Again we have to disagree - by direct substitution they change the process $|\phi\rangle$ even with respect to $\mathbb{R}$. Note that both choices do conserve the norm of $|\phi\rangle$, as the variation of the norm is independent of whether the processes B_i are driftless or not(under the measure $\mathbb{P}$). On the other hand, the time t_c^e taken until a superposition collapses *does* depend on whether the process we are using has drift. We can compute this time directly using the linear equation we started with, but we can also get it by numerically simulating the non-linear equation. We will do this at the end of Section 4.5.1, showing that only underlying $\mathbb{P}$-Wiener processes yield consistent results. We also note that the change of measure is done with our choice of sign in refs. [81, 82].

It is important to note that even though the normalized state $|\phi\rangle$ evolves according to a non-linear equation, the density matrix evolves linearly, as required by Jordan's proof. Indeed, given Eqn. (4.46) we can simply compute its evolution by using the (linear) equation for the un-normalized states $|\psi\rangle$. As the measure $\mathbb{R}$ is time-translation invariant, there is no term corresponding to variation in it, hence $d\mathbb{E}^{\mathbb{R}}[|\psi\rangle\langle\psi|] = \mathbb{E}^{\mathbb{R}}[d(|\psi\rangle\langle\psi|)]$. We further have

$$d\rho = \mathbb{E}^{\mathbb{R}}[|d\psi\rangle\langle\psi| + |\psi\rangle\langle d\psi| + [|d\psi\rangle\langle d\psi|]_t] \tag{4.83}$$

$$= i[\rho, H]dt + \mathbb{E}^{\mathbb{R}}\big[dB_i A_i\rho + dB_j\rho A_j^\dagger - \frac{\gamma\ dt}{2}\{A_i^\dagger A_i, \rho\} + \gamma dt A_i\rho A_i^\dagger\big] \tag{4.84}$$

$$= i[\rho, H]dt - \frac{\gamma dt}{2}\big[A_i, [A_i, \rho]\big] \tag{4.85}$$

where the last line follows if we assume the A_i are Hermitian. Incidentally, the density matrix evolves once again according to a Lindblad equation, just like we found in the QMSL model. We haven't seen this mentioned in the literature, but if we can neglect the Hamiltonian then $(\langle x|\psi\rangle\langle\psi|y\rangle)(t)$ is nothing but two-dimensional geometric Brownian motion. This then allows us to find a closed form for $\rho(x, y, t)$. Taking $A_i = x$ and denoting $\rho_r = |\psi\rangle\langle\psi|$, we have

$$d\langle x|\rho_r|y\rangle = \left[2xdB - \frac{\gamma dt}{2}(x-y)^2\right]\langle x|\rho_r|y\rangle \tag{4.86}$$

$$\implies \rho_r(x, y, t) = \rho_r(x, y, 0)\exp\big[-\gamma(x^2+y^2)t + (x+y)B\big] \tag{4.87}$$

$$\implies \rho(x, y, t) = \mathbb{E}^{\mathbb{R}}[\rho_r] = \rho(x, y, 0)\exp\big[-\frac{\gamma}{2}(x-y)^2 t\big] \tag{4.88}$$

Under the assumption of a multi-particle system in a product state, each particle's reduced density matrix will be of this form. In the case of an entangled state, we can give no closed form for the evolution. One might also wan a closed form for the "physical" density matrix $\rho_p(t) = |\phi_t\rangle\langle\phi_t|$. Unfortunately, we have no answer here - it is clear that $|\phi_t\rangle$ is *not* regular geometric Brownian motion due to the presence of the non-linear terms in $d|\phi\rangle$, hence there is no easy way of exponentiating it.

Before moving on we will briefly show how the amplification mechanism works in QMUPL, as we promised in Section 4.2.1. We consider a system of N particles and assume that the Hamiltonian does not give rise to interactions between the centre of mass and the internal degrees of freedom. That is, we assume $H = H_{\text{cm}} + H_{\text{int}}$. It is then natural to also take the system state to be of product form as Hamiltonian evolution will preserve it, e.g. we take $|\phi\rangle = |\phi_{\text{cm}}\rangle \otimes |\phi_{\text{int}}\rangle$. As always we take the collapse operators $A_i = x_i$. It's convenient to work with our non-linear equation(Eqn. (4.82)). We introduce the centre-of-mass operator R and the relative position operators r_i, such that $x_i = R + r_i$. Then the evolution takes the form

$$d|\phi\rangle = \Big\{ -iH_{\text{cm}}dt - \sum_{i=1}^{N}\Big[\frac{\gamma}{2}(R - \langle R\rangle)^2 dt - (R - \langle R\rangle)d\tilde{B}_i\Big] \tag{4.89}$$

$$- iH_{\text{int}}dt - \sum_{i=1}^{N}\Big[\frac{\gamma}{2}(r_i - \langle r_i\rangle)^2 dt - (r_i - \langle r_i\rangle)d\tilde{B}_i\Big] \tag{4.90}$$

$$- \gamma dt\sum_{i=1}^{N}(R - \langle R\rangle)(r_i - \langle r_i\rangle)\Big\}|\phi\rangle \tag{4.91}$$

Now it is apparent that both ϕ_{cm} and ϕ_{int} obey equations similar in form to the one we have for the total system. Tracing out the internal degrees of freedom is particularly interesting; it kills the latter two lines and leaves us with

$$d\left|\phi_{\text{cm}}\right\rangle = \left\{ -iH_{\text{cm}}dt - \sum_{i=1}^{N}\left[\frac{\gamma}{2}\left(R-\langle R\rangle\right)^2 dt - \left(R-\langle R\rangle\right)d\tilde{B}_i\right]\right\}\left|\phi_{\text{cm}}\right\rangle \tag{4.92}$$

$$= \left\{ -iH_{\text{cm}}dt - \left[\frac{\gamma_{\text{cm}}}{2}\left(R-\langle R\rangle\right)^2 dt - \left(R-\langle R\rangle\right)d\tilde{B}\right]\right\}\left|\phi_{\text{cm}}\right\rangle \tag{4.93}$$

Where we have defined $\gamma_{\text{cm}} = N\gamma$ and $dB = \sum_i dB_i$.[35] Hence the centre of mass undergoes collapse with a strength set by the number of constituent particles, and the amplification mechanism is manifest.

Having now presented the fundamental equations of the model, we will want to discuss their implications and predictions. But we are not quite ready yet - an important tool for our analysis will be numerical simulations, the issue of which we now turn to.

4.4.3 Numerical simulation

Before we move on to discuss the contents of our collapse equations, we will first describe how to numerically simulate them. This is particularly interesting in the case of Eqn. (4.82). As it is a non-linear SDE, making analytic progress using it is an unlikely endeavour at the best of times. In fact it also turns out to be our only viable option for numerical simulations. The corresponding linear equation(Eqn. (4.54)) does not conserve the norm, so consequently almost every sample path we generate will have a norm decaying to zero. Getting an actual collapsing sample path where the norm tends to infinity is so exceedingly rare that we cannot reasonably gamble on it.

Picking a numerical scheme that converges isn't entirely straightforward either. Firstly, there are different notions of convergence for a sequence of random variables. To illustrate, let us consider an arbitrary stochastic process $X(t)$. We define it on the interval $[0,T]$ which we discretize into chunks of size Δt. Let X_n be an approximation to $X(t_n)$ generated by some numerical scheme, and let furthemore $\psi(X)$ be an arbitrary smooth function. Then we can define the two errors

$$e_{\text{weak}} \equiv \sup_{0\leq t_n\leq T} \left\|\mathbb{E}[\psi(X_n)] - \mathbb{E}[\psi(X(t_n))]\right\| \qquad e_{\text{strong}} \equiv \sup_{0\leq t_n\leq T} \mathbb{E}[\left\|X_n - X(t_n)\right\|] \tag{4.94}$$

A scheme converges weakly(or strongly) if the corresponding error remains less than or equal to $k\Delta t^p$, with k an arbitrary constant. Intuitively, weak convergence means that the distributions of $\psi(X_n)$ and $\psi(X(t_n))$ become arbitrarily close[36] whereas strong convergence means that the generated random variables X_n become arbitrarily close to $X(t_n)$ as $\Delta t \to 0$. The distinction is relevant to us because one might assume[37] that schemes used to solve ODE's converge with the same order when applied to SDE's. As a concrete example, we are used to forward Euler being a first order scheme when applied to ODE's. When applied to SDE's, it is however *weakly*

[35] Note that the sum of N driftless independent Brownian motions is again a Brownian motion(though typically with different variance; in our case $\text{Var}[B_t] = Nt$) so the centre of mass really does obey the same type of equation we started with.

[36] It's the type of convergence which we encounter in the central limit theorem, for example. The distribution of N iid. random variables will approach the normal distribution as $N \to \infty$.

[37] The author is guilty of this mistake.

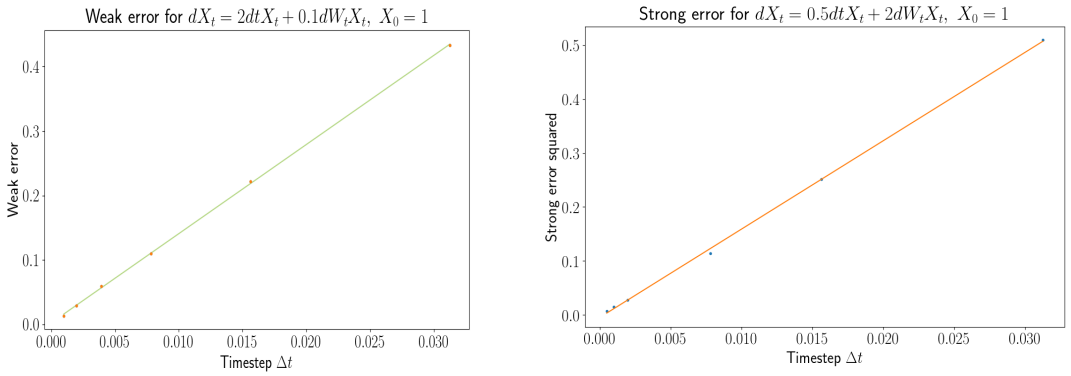

Figure 4.1: Weak and strong errors as defined by Eqn. (4.94) are computed for the process $dX_t = \mu X_t dt + \sigma X_t dW_t$. Parameters chosen for the weak error are $\mu = 2, \sigma = 0.1$ and $X_0 = 1$, for the strong error we chose $\mu = 0.5, \sigma = 2$ and $X_0 = 1$. W_t is a standard Wiener process. The average is taken over 10^5 sample paths. Each path was run over $t \in [0, 1]$ with variable timesteps as shown in the figure. We observe a good linear fit for the weak error, indicating that $e_{\text{weak}} \propto \Delta t$. On the other hand, it is the *square* of the strong error that yields a good linear fit, indicating that $e_{\text{strong}} \propto \sqrt{\Delta t}$. In other words, the Euler scheme is weakly convergent of order 1 and strongly convergent of order $1/2$.

convergent of order one but only strongly convergent of order $1/2$. An example simulation checking the order of convergence for the forward Euler scheme is shown in Fig. 4.1.[38]

Selecting a scheme

Several schemes are familiar to us from solving ODE's numerically: the various Euler schemes, Heun's method, Runge-Kutta methods of differing order and so on. We are also accustomed to selecting e.g. a higher order Runge-Kutta method if the Euler method does not converge, or switching to a more numerically stable scheme. Unfortunately, this intuition does not carry over when we attempt to use similar methods for solving SDEs either. As an example, the fourth order Runge-Kutta method is weakly convergent of order *one*, which is the same as the forward Euler method.[41]

For the specific case of simulating Eqn. (4.82), a first order weak scheme turns out to be insufficient. Consequently we will use a scheme due to Kloeden and Platen which is weakly convergent of order two.[83] To fix notation, we consider a general SDE of the form

$$dX_t = D_1(X_t, t)dt + D_2(X_t, t)dW_t \tag{4.95}$$

with W_t a standard Wiener process. In this notation, the familiar Euler scheme reads

$$X_{n+1} = X_n + D_1(X_n)\Delta t + D_2(X_n)\Delta W_n \tag{4.96}$$

for example. We now define the three "midpoint" values

$$\tilde{X}_n = X_n + D_1(X_n)\Delta t + D_2(X_n)\Delta W_n \tag{4.97}$$

$$X_n^{\pm} = X_n + D_1(X_n)\Delta t \pm D_2(X_n)\sqrt{\Delta t} \tag{4.98}$$

[38] When applied to SDE's, it is sometimes referred to as the Euler-Maruyama method.

According to the scheme of Kloeden and Platen, we then generate X_{n+1} according to the following algorithm:

$$X_{n+1} = X_n + \frac{1}{2}\left(D_1(\tilde{X}_n) + D_1(X_n)\right)\Delta t \tag{4.99}$$

$$+ \frac{1}{4}\left(D_2(X_n^+ + D_2(X_n^-) + 2D_2(X_n)\right)\Delta W_n \tag{4.100}$$

$$+ \frac{1}{4}\left(D_2(X_n^+) - D_2(X_n^-)\right)\left[(\Delta W_n)^2 - \Delta t\right]\Delta t^{-1/2} \tag{4.101}$$

Unfortunately, this algorithm doesn't quite do it for us either. We see that the drift part of the equation is essentially solved using a midpoint Euler method, which we know to be numerically unstable for solving the regular Schrödinger equation. As such we will integrate an eventual Hamiltonian term separately using the Crank-Nicolson scheme, while the stochastic terms are integrated using the above algorithm.

Choice of units

Given our non-linear equation and a sufficiently stable and convergent scheme,[39] we are almost ready to run the simulations. Our last task is to connect the simulation to some physical situation. Typically we will define various characteristic dimensions for the phenomenon we wish to study, where quantities such as energy, time and so on are neither too large nor too small. In atomic phenomena for example, the energy is usually of the order of electronvolts. To determine the timescale, we note that Planck's constant is of order one if time is measured in femtoseconds, hence that is the timescale of interest. In our case, the characteristic length and time scales are linked by the collapse constant γ. Given some system of mass m, the value is typically chosen to be[82]

$$\gamma = \frac{m}{m_0}\gamma_0,\ \gamma_0 \equiv 10^{-2}\ \mathrm{m}^{-2}\ s^{-1} \tag{4.102}$$

where the reference mass m_0 is taken to be 1 amu. The value of γ_0 is chosen to be the product of the two parameters λ and α of the QMSL model(see Eqn. (4.15) and Eqn. (4.16)). Though we will not describe it here, the choice is motivated by the fact that QMUPL can be considered a "continuum" limit of QMSL, wherein the parameters $\lambda \to \infty$ and $\alpha \to 0$. The details may be found in ref. [84]. Though perhaps obvious, neither the mass dependency nor the particular value of γ_0 is by any means definite - it is simply a convenient choice that is in agreement with current experimental data.

The odd thing[40] about γ is that it depends on the system under consideration through its mass. Thus each system has a characteristic collapse scale, which turns out to be quite convenient for simulation. In particular, suppose we wish to study the collapse of C_{70}-molecules. They are of nanometer size and have a mass of 840 amu, giving us the collapse constant

$$\gamma = 8.4\ \mathrm{nm}^{-2}(10^{18}\ s)^{-1} \tag{4.103}$$

which implies that collapse of C_{70}-molecules occurs on timescales of the order of the age of the universe. Hence a given characteristic length and the mass of the system, the characteristic time

[39] Stable schemes do not magnify accumulated numerical errors, which is necessary for the scheme to converge. If a scheme converges, it will generate an approximation which tends to the true solution as the mesh size is decreased.

[40] In the sense of not defining a system-independent scale. It is of course exactly what we want; the mass dependency ensures that collapse is ineffective at microscopic scales and effective at macroscopic scales, which we saw at the end of the previous section.

is determined by the collapse constant. Consequently we will simply pick a fixed value for γ in our simulations, and then the resulting data can be used to describe any mass/lengthscale/timescale combination we want. For the plots generated throughout the thesis, we have chosen the numerical value $\gamma = 10^{-6}$. The characteristic length of the system to be studied we denote by L, and each iteration of our numerical scheme we refer to as a *frame*. A frame may or may not correspond to a unit increment of the characteristic time, if it does not we will explicitly point it out.

4.4.4 The collapse mechanism

In Section 4.4.1 we qualitatively sketched how collapse occurs within this type of model, and we subsequently developed a nonlinear evolution equation - namely Eqn. (4.82). To actually prove that collapse will occur, two strategies are possible. The easiest one is to show that off-diagonal elements of the density matrix vanish, but this is not by itself satisfactory. Consider for example a density matrix ρ_0 representing an ensemble of systems in the Hilbert space spanned by the two states $|\text{here}\rangle$ and $|\text{there}\rangle$, and assume that it evolves like so:

$$\rho_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow \rho_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.104}$$

Does this mean the states in the ensemble have collapsed to a mixture of either $|\text{here}\rangle$ or $|\text{there}\rangle$? No, because ρ_t can equally well represent a mixture of the states $|\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|\text{here}\rangle \pm |\text{there}\rangle)$. So we need a stronger condition. The second strategy most often found in the literature[15, 53, 85] consists of showing that the variance of the operator to whose eigenstates we wish to collapse asymptotically vanishes. Concretely, assume that we have proven that $\langle \Delta O\rangle = \langle O^2\rangle - \langle O\rangle^2 \to 0$ as $t \to \infty$ for some arbitrary operator O, given any state $|\psi(t)\rangle = \sum_n c_n(t)\,|o_n\rangle$.[41] Then we find for the variance that

$$\lim_{t\to\infty} \langle \Delta O\rangle = \lim_{t\to\infty} \sum_n |c_n(t)|^2 (o_n - \langle O\rangle)^2 = 0 \tag{4.105}$$

Now $\langle O\rangle$ can match at most one distinct eigenvalue, which may of course correspond to several eigenstates in the case of degeneracy. Denote the corresponding coefficients $c_{\{k\}}(t)$. Then if the variance is to vanish, all $c_n(t) \neq c_k(t)$ must vanish, and so the statevector has collapsed to the eigenspace corresponding to $o_n = \langle O\rangle$. There are however some caveats here. Typically we can only guarantee that the variance vanishes as $t \to \infty$, e.g. collapse takes infinite time. This is at first glance undesirable, but in the case of position-space collapse, it is perhaps not - to reach a state with infinite energy, it is not so unreasonable that it takes infinite time. Secondly, the above proof only holds in the case where the operator O has a finite spectrum.[86] By way of example, suppose we have the state

$$|\psi(t)\rangle = \alpha \left| x_0 + \frac{1}{t} \right\rangle + \beta \left| x_0 + \frac{1}{t+\delta t} \right\rangle \tag{4.106}$$

[41] Though not as elegant as the proof of Adler[85], this can be shown by brute force. Take Eqn. (4.54), assume the A_i are Hermitian and commute with H(alternatively, set $H = 0$). Note that $d\langle A_i^2\rangle \propto dB_i$, whereas $d\langle A_i\rangle^2$ picks up a time-dependent term $\gamma dt \langle A_i^2\rangle^2$. Thus the variance is always decreasing, and tends to its lower bound - which is always positive - of $\langle \Delta x\rangle$ as $t \to \infty$. If the Hamiltonian has no diffusive terms this lower bound will be zero, as it is in Eqn. (4.107).

Where the states on the right-hand side are position eigenstates, to which we want to collapse. The variance of the position operator in this state is

$$\langle \Delta x \rangle = |\alpha|^2 |\beta|^2 \left(\frac{\delta t}{t(t+\delta t)} \right)^2 \tag{4.107}$$

which clearly vanishes as $t \to \infty$, regardless of the values of α and β. Thus decreasing variance does not necessarily imply collapse; indeed the above state does not even converge to $|x_0\rangle$ in the Hilbert-space norm, as $\| |\psi(t)\rangle - |x_0\rangle \| = \sqrt{2}$ for all time. Since we are mainly interested in position space collapse, this is of interest to us. But does this result imply severe problems for QMUPL? We don't think so. The state $|\psi(t)\rangle$ is unphysical and can never be prepared in practice. If we instead prepare some state $|\phi(t_0)\rangle$ with a smooth coordinate representation, it *will* converge towards a given position eigenstate, albeit in the infinite time limit. And physically, this is just what we want - it should take infinite time for an infinite energy process to take place.

4.4.5 Collapse with non-zero Hamiltonian

The variance proofs found in the literature either set the Hamiltonian to zero or assume that it commutes with the collapse operators A_i. We have also seen that in order to ensure collapse in space, the A_i are typically chosen to be position operators, which do not commute with the kinetic part of any Hamiltonian. The natural question is then how including a kinetic term affects collapse - in particular, does it prevent collapse to an infinitely sharp wavepacket? Based on intuition alone, we would expect that it does - the kinetic term is diffusive, whereas the stochastic terms are localizing, and at some point an equilibrium must be reached where they are equally strong. The question above has been analyzed in great detail by Bassi et al.[86], the results of which we will now summarize. For completeness, we will also give some calculations that have been omitted in the article.

We wish to analyze the asymptotic behaviour of a particle subject to a potential at most quadratic in position. As we know, the corresponding physical wavefunction obeys the non-linear equation, but it is sufficient to consider the asymptotic behaviour of the un-normalized wavefunction as its normalized version will be a solution of the non-linear equation. The strategy will be as follows: first we will link the time evolution operator of Eqn. (4.54) with that of the non-self-adjoint harmonic oscillator(henceforth NSA).[42] We will denote the former evolution operator by U_α and that of the NSA by U_β.[43] To pre-empt the result, we will find that any state asymptotically converges to a Gaussian of fixed width.

To link U_α and U_β, we transform a solution ϕ_0 of Eqn. (4.54) into a solution of an NSA equation. Let $P(p')$ be the operator that translates momentum, and $T(x')$ be the operator that translates position, e.g. $P(p')\psi(p) = \psi(p-p')$ and $T(x')\psi(x) = \psi(x-x')$. We allow p' and x' to be complex. The transformation process goes as follows:

$$\phi_0 \to \phi_1 = P(-iB_t)\phi_0 \tag{4.108}$$

$$\phi_1 \to \phi_2 = \exp\left[-\frac{i}{2m} \int_0^t dB_s^2 \, ds \right] \phi_1 \tag{4.109}$$

$$\phi_2 \to \phi_3 = \exp[-ia_t] T(-b_t) P(c_t) \phi_2 \equiv V_t \phi_2 \tag{4.110}$$

[42] The only difference between the NSA and a regular harmonic oscillator is that the potential term is imaginary.

[43] Incidentally, both Green functions for these equations are known in closed form; we will however not be needing them for our purposes.

where in the last line, a_t, b_t and c_t are as of yet undetermined stochastic processes. For notational convenience, we gather these transformations into a single operator S_t, such that

$$\phi_3 = S_t \phi_0 \tag{4.111}$$

We then claim that, under certain assumptions which we shall clarify, ϕ_3 is a solution of the NSA equation

$$i\partial_t \psi = \frac{p^2}{2m}\psi - i\gamma x^2 \psi \tag{4.112}$$

In other words, $\phi_3(t) = U_\beta(t;t_0)\phi_3(t_0) = S_t\phi_0(t) = S_t U_\alpha(t;t_0)\phi_0(t_0)$. Taking $t_0 = 0$, we have that $S_0 = 1$ and so $\phi_3(t_0) = \phi_0(t_0)$, which implies that the evolution operators are related as

$$U_\alpha = S_t^{-1} U_\beta \tag{4.113}$$

Before we use this relation, let us work through the transformations in some more detail.

The first transformation is designed to remove the stochastic term from Eqn. (4.54). Computing the differential of ϕ_1, we find

$$d\left(e^{-xB_t}\phi_0\right) = \left\{\left(-xdB_t + \frac{\gamma dt}{2}x^2\right) + \left(e^{-xB_t}\left[-i\frac{p^2}{2m}\right]e^{xB_t}dt + xdB_t - \frac{\gamma dt}{2}x^2\right) - \gamma dt x^2\right\}\phi_1 \tag{4.114}$$

where the last term is the usual Ito term. To compute the transformed Hamiltonian we make use of the Hadamard lemma, which states that for two operators A, B we have

$$e^A B e^{-A} \equiv B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \ldots \tag{4.115}$$

Using it we obtain

$$e^{-xB_t}p^2 e^{xB_t} = p^2 + [p^2, xB_t] + \frac{1}{2}\left[[p^2, xB_t], xB_t\right] \tag{4.116}$$

$$= p^2 - 2iB_t p - B_t^2 \tag{4.117}$$

Inserting this, we obtain the differential equation satisfied by ϕ_1:

$$\partial_t \phi_1 = \left[-i\frac{p^2}{2m} - \gamma x^2 - \frac{B_t}{m}p + \frac{iB_t^2}{2m}\right]\phi_1 \tag{4.118}$$

The second transformation - Eqn. (4.109) - ensures that ϕ_2 satisfies the slightly simpler equation

$$i\partial_t \phi_2 = \left[\frac{p^2}{2m} - i\gamma x^2 - \frac{iB_t}{m}p\right]\phi_2 \tag{4.119}$$

where we have multiplied in a factor i for future convenience. The third transformation - Eqn. (4.110) - must now bring us to NSA territory. This is yet again an exercise in using the Hadamard lemma:

$$i\partial_t \phi_3 = i\partial_t\left(V_t\right)\phi_2 + V_t\left[\frac{p^2}{2m} - i\gamma x^2 - \frac{iB_t}{m}p\right]\phi_2 \tag{4.120}$$

The first term is easily computed:

$$i\partial_t(V_t)\phi_2 = \left[\dot{a}_t - \dot{b}_t p + \dot{c}_t(x+b_t)\right]\phi_3 \tag{4.121}$$

Note that we get the factor $(x+b_t)$ and not just x as we must commute $T(-b_t)$ over it to recover ϕ_3. The second term requires some more Hadamard, where we again have

$$V_t p^2 V_t = p^2 + 2c_t p + c_t^2 \qquad\qquad V_t x^2 V_t = x^2 + 2b_t x + b_t^2 \tag{4.122}$$

Gathering terms, we are left with

$$i\partial_t\phi_3 = \left[\dot{a}_t - \dot{b}_t p + \dot{c}_t(x+b_t) + \frac{1}{2m}\left(p^2 + 2c_t p + c_t^2\right) - i\gamma\left(x^2 + 2b_t x + b_t^2\right) - \frac{iB_t}{m}(p+c_t)\right]\phi_3 \tag{4.123}$$

$$= \left[\frac{p^2}{2m} - i\gamma x^2 - \left(\dot{b}_t - \frac{c_t}{m} + \frac{iB_t}{m}\right)p + \left(\dot{c}_t - 2i\gamma b_t\right)x + \left(\dot{a}_t + \dot{c}_t b_t + \frac{c_t^2}{2m} - i\gamma b_t^2 - \frac{iB_t c_t}{m}\right)\right]\phi_3 \tag{4.124}$$

We now impose the condition that the processes a_t, b_t and c_t must satisfy the equations

$$\dot{b}_t - \frac{c_t}{m} + \frac{iB_t}{m} = 0 \tag{4.125}$$

$$\dot{c}_t - 2i\hbar\gamma b_t = 0 \tag{4.126}$$

$$\dot{a}_t + \dot{c}_t b_t + \frac{c_t^2}{2m} - i\gamma b_t^2 - \frac{i\hbar B_t c_t}{m} = 0 \tag{4.127}$$

in which case ϕ_3 will be a solution of the claimed NSA equation.

To further express the evolution of an arbitrary state, we will need some properties of the NSA harmonic oscillator.[87] We assume it is given in the form of Eqn. (4.112). Its eigenvalues are given by

$$\lambda_n = \frac{1-i}{2}\omega_n, \qquad\qquad \omega_n \equiv \left(n + \frac{1}{2}\right)2\sqrt{\frac{\hbar\gamma}{m}} \tag{4.128}$$

and the eigenvectors are

$$\phi_n(x) = \sqrt{z}e^{-z^2x^2/2}\bar{H}_n(zx), \qquad\qquad z^2 \equiv (1-i)\sqrt{\frac{\gamma m}{\hbar}} \tag{4.129}$$

where $\bar{H}_n$ are normalized Hermite polynomials. Due to the complex argument, these form a complete but not orthonormal set such that the evolution operator of the NSA harmonic oscillator is more peculiar than we are used to; it is given by

$$U_\beta = \sum_{n=0}^{\infty} e^{-(1+i)\omega_n t/2}\left|\phi_n\right\rangle\left\langle\phi_n^*\right| \tag{4.130}$$

where we note the unusual projection operator. Furthermore, the sum above converges only for $t > t_c$, where the critical time is given by $t_c = 4c/\omega$ and where c is a constant of order one. As we are only interested in studying asymptotic behaviour, this isn't an issue for us.[44] Using

[44] It could also be that only a finite number of terms is needed, in which case it isn't an issue either.

this, we may now write down the time evolution of an arbitrary wavefunction $\langle x|\theta\rangle$ under Eqn. (4.54):

$$\theta(x,t) = \langle x|U_\alpha|\theta(t_0)\rangle = \langle x|S_t^{-1}U_\beta|\theta(t_0)\rangle = \int dx' \langle x|S_t^{-1}|x'\rangle \sum_{n=0}^{\infty} \langle \phi_n^*|\theta(t_0)\rangle e^{-(1+i)\omega_n t/2}\phi_n(x') \tag{4.131}$$

The exact form of the coordinate representation of S_t^{-1} isn't terribly interesting for us, as we only wish to show asymptotic convergence to a Gaussian. But considering Eqns. (4.108-4.110) it is immediate that it contains a factor $\delta(x'-(x-b_t))$, so we can tidy up a bit by extracting it and defining $S_t^{-1} = \tilde{S}_t^{-1}\delta(x-(x-b_t))$. We also notice that $\ln \tilde{S}_t^{-1}(x,x-b)$ only contains terms that are either independent of or linear in x, such that it may readily be absorbed in a possibly appearing Gaussian. Defining also $\theta_n = \langle \phi_n^*|\theta(t_0)\rangle$, we have

$$\theta(x,t) = \tilde{S}_t^{-1}(x,x-b_t)\sum_{n=0}^{\infty}\theta_n e^{-(1+i)\omega_n t/2}\phi_n(x-b_t) \tag{4.132}$$

$$= \sqrt{z}\theta_0\tilde{S}_t^{-1}(x,x-b_t)e^{-(1+i)\omega_0 t/2}e^{-z^2x^2/2} + \tilde{S}_t^{-1}(x,x-b_t)\sum_{n=1}^{\infty}\theta_n e^{-(1+i)\omega_n t/2}\phi_n(x-b_t) \tag{4.133}$$

Now $\theta(x,t)$ must be normalized to get at the physical state. Notice that the damping factor $e^{-\omega_n t/2}$ is stronger as n increases; consequently only the damping factor of the ground state will be fully cancelled when we divide by the norm. This means that the second term in the equation above will vanish asymptotically, leaving us with exactly what we expected: a Gaussian.

Then there is the issue of how the width of this Gaussian evolves in time. Although it can be computed using the NSA link, it is considerably easier to take a general Gaussian as an ansatz. That is, we assume the following is a solution of Eqn. (4.54):

$$\phi_t(x) = \sqrt{\frac{2\,\mathrm{Re}(a_t)}{\pi}}\exp\left[-a_t(x-x_t)^2 + ik_t x\right] \tag{4.134}$$

where a_t, x_t and k_t are unspecified stochastic processes. We will allow a_t to be complex, while x_t and k_t are real. Inserting this into the aforementioned equation and gathering powers of x, we find the equation obeyed by a_t:

$$da_t = \left[\gamma - \frac{2i}{m}a_t^2\right]dt \tag{4.135}$$

Some more work also reveals that the mean position x_t and momentum k_t obey

$$dx_t = \frac{\hbar}{m}k_t dt + \frac{\sqrt{\gamma}}{2\,\mathrm{Re}(a_t)}\left(dB_t - 2\sqrt{\gamma}x_t dt\right) \tag{4.136}$$

$$dk_t = -\sqrt{\gamma}\frac{\mathrm{Im}(a_t)}{\mathrm{Re}(a_t)}\left(dB_t - 2\sqrt{\gamma}x_t dt\right) \tag{4.137}$$

The latter equations are reassuring in the sense that they give us what we would expect from Ehrenfest's theorem, plus some noise that in any case averages to zero under the physical density. The equation for a_t is deterministic, and its solution is

$$a_t = \frac{1-i}{2}\sqrt{\frac{m\gamma}{\hbar}}\tanh\left[(1+i)\sqrt{\frac{\gamma}{m}}t + c\right] \tag{4.138}$$

where c is a constant that can be determined from the initial conditions. Thus we reach the interesting conclusion that *any* wavefunction asymptotically turns into a Gaussian with spread

$$\sigma_\infty = \sqrt{\frac{1}{2\,\mathrm{Re}(a_\infty)}} = \left(\frac{\hbar}{m\gamma}\right)^{1/4} \tag{4.139}$$

Where we have restored units. We have a few remarks on this result. One, this is exactly the width of the NSA ground state. If we were just interested in the asymptotic width, Eqn. (4.129) gives it immediately. Second, the asymptotic width is in fact only reached in infinite time. As such this cannot save us from energy non-conservation. Lastly the width scales with the square root of inverse mass, such that the larger the system, the more it gets localized. This *is* good news, and it reflects what we would expect from e.g. the de Broglie wavelength of a system(although that scales inversely with mass). Before moving on, it will also be useful to us to have the asymptotic width for a state in a harmonic potential of frequency ω. The derivation is essentially the same as the one given above with the replacement

$$\gamma \to \tilde{\gamma} \equiv \gamma + \frac{i}{2}m\omega^2 \tag{4.140}$$

Apart from this the only change is that the processes a_t, b_t and c_t must satisfy slightly different equations, but it is of no importance for the moment. Inserting this replacement in Eqn. (4.129), picking the real part and again restoring units, we find the asymptotic width

$$\sigma_\infty^\omega = \left(\frac{\hbar}{m\gamma}\right)^{1/4}\left[1+\frac{m^2\omega^4}{4\hbar^2\gamma^2}\right]^{-1/8} \tag{4.141}$$

4.5 Three overlooked aspects of QMUPL

At this point we have seen that QMUPL is a relatively satisfactory collapse model. Asymptotically we get sharply localized states, and we see that the Born rule remains in effect as collapse simply removes off-diagonal elements in the density matrix. However, we have not yet mentioned *when* one may consider a wavefunction to have collapsed - in fact, this question has not been quantitatively addressed in the literature. Motivated by this, we will study it in some detail. Furthermore it is known that stochastic dynamics sometimes reverse collapse. This is potentially worrying because it also means that reduction models predict the reversal of outcomes - for example, we might measure an electron to have spin up, only to find it with spin down some time later. In the literature this phenomenon is usually brushed off as being exceedingly unlikely. Although this is true, we would nonetheless like to quantify it, and we do so. Lastly both QMSL and QMUPL cause all parts of the wavefunction to flow toward the expected position. We'd like to investigate this motion, in particular because the time-varying velocity during collapse should lead to radiative emission in the case of charged particles. We'll study this numerically, again finding it to be a rather negligible effect.

4.5.1 Collapse time

In the previous section we saw that the reduction process is active for all time - if $H = 0$ then reduction to position eigenstates occurs only at $t = \infty$. Similarly if $H \neq 0$, reduction to the smallest-width Gaussian possible occurs only at $t = \infty$. However, as is clear from Eqns. (4.136-4.139), collapse *effectively* finishes in finite time, whereafter the excited states of the NSA harmonic oscillator are of negligible norm and the decrease in width of the ground state is

comparatively small. So it is this effective collapse time which we want to quantify. We remark that this type of collapse is still in a weak sense observer-dependent; different observers may disagree on when exactly the NSA excited states are of negligble norm, or in other words when interference effects are no longer detectable. But of course the collapse process proceeds by itself with no observer required present, as opposed to the standard interpretation of LQM.

We mentioned two main ways of quantifying collapse - either by the strength of off-diagonal elements of the density matrix or the variance in position of the state. Either approach can be made to work, though using the density matrix gives us some additional difficulties we must take care of. On the other hand, using the density matrix gives us an easy check for whether our numerics are correct, and it is typically used in the literature when estimating bounds on the parameters in a given reduction model. As such we will stick with that approach for now.[45]

Again, we are not interested in when the off-diagonal elements vanish entirely but only when they pass a certain (arbitrary) threshold. The collapse time seen in experiment will correspond to the decay of off-diagonal elements of the density matrix given by Eqn. (4.88), as the specific realizations are unknown to us. Suppose we claim collapse has occured whenever

$$\left| \frac{\rho(x,y,t)}{\rho(x,y,t_0)} \right| \leq \exp(-\alpha) \tag{4.142}$$

where α is an arbitrary real number and we exponentiate it for convenience. Solving for the corresponding time, we find that the collapse time t_c^e in this case is simply given by[46]

$$t_c^e = \frac{2\alpha}{\gamma(x-y)^2} \tag{4.143}$$

If we want the actual collapse times of individual realizations, we will have to work a little harder.[47] We will primarily be interested in the collapse times of states figuring in the physical ensemble; unfortunately this means we must use the density matrix computed using Eqn. (4.82), to which we have no analytic solution. If on the other hand we use the density matrix computed using the linear equation, then the collapse times can be determined. In the hope that this knowledge can be transferred to the collapse times of the physical states, we'll study it first. The closed form of the raw density matrix is given by Eqn. (4.87), and using Eqn. (4.142) we find that collapse occurs whenever

$$T_t = \gamma(x^2+y^2)t - (x+y)B_t = \alpha \tag{4.144}$$

Given that B_t is Brownian motion with positive-valued drift, this will happen with probability one. Importantly for the stability of the collapse process, there is also a non-zero probability for it to occur *more* than once. We will discuss this point in more detail in the following section. In any event, we see that we must redefine the collapse time slightly, as it depends on the process B_t. We choose the following definition:

$$t_\alpha^{(f)} = \inf\{t \geq 0 \ \mid T_t = \alpha\} \tag{4.145}$$

which is now a random variable, whose distribution we will want to determine. As it turns out, the above definition of the collapse time is a quantity that is well-known in stochastic analysis

[45] Again, there are potentially problems with the fact that density matrices are one-to-many. But if we know the initial state to be some superposition, this isn't an issue. The density matrix should also be taken relative to the eigenbasis of the collapse operators A_i, which typically means we want it in coordinate basis.

[46] By dimensional analysis, it was already evident that $t_c^e \propto 1/\gamma \propto 1/m$, as the model contains just one constant.

[47] We remark that the determination of their distribution relative to the physical ensemble is raised as an open question at the end of ref. [86]. It should be mentioned that they quantify the collapse time by considering when the variance in position has passed some threshold value, whilst we use the density matrix.

- it is the so-called hitting time of Brownian motion, and it obeys the Wald distribution.[48] Concretely, given Brownian motion with drift and diffusion coefficients μ and σ respectively, the distribution of the hitting time $t_\alpha^{(f)}$ is[88]

$$j_\alpha(t,\mu,\sigma) = \left[\frac{\alpha^2}{2\pi\sigma^2 t^3}\right]^{1/2} \exp\left[-\frac{(t\mu-\alpha)^2}{2t\sigma^2}\right] \tag{4.146}$$

A reader familiar with the distribution will notice that it is not in standard form. In particular, μ and σ as given above do *not* describe the mean and standard deviation respectively. Usually the Wald distribution is expressed in terms of the mean m and a so-called shape parameter λ. In terms of these variables, it takes the form

$$j(t,m,\lambda) = \left[\frac{\lambda}{2\pi t^3}\right]^{1/2} \exp\left[-\frac{\lambda(t-m)^2}{2m^2 t}\right] \tag{4.147}$$

with first and second moments[49]

$$\mathbb{E}[T] = m \qquad\qquad \mathbb{E}[T^2] = \frac{m^2(m+\lambda)}{\lambda} \tag{4.148}$$

The conversion between standard form and our "Brownian motion" form is then

$$m = \frac{\alpha}{\mu} \qquad\qquad \lambda = \frac{\alpha^2}{\sigma^2} \tag{4.149}$$

Reassuringly, we see that the mean of the distribution coincides with the mean collapse time(given by Eqn. (4.143)) we computed directly using the linear equation. Let us also write down the corresponding cdf, which in standard form is given by

$$J(t,m,\lambda) = \Phi\left(\sqrt{\frac{\lambda}{t}}\left(\frac{t}{m}-1\right)\right) + \exp\left(\frac{2\lambda}{m}\right)\Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{m}+1\right)\right) \tag{4.150}$$

where $\Phi(t)$ is the cdf of the standard normal distribution, e.g.

$$\Phi(t) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{t}{\sqrt{2}}\right)\right] \tag{4.151}$$

That is it for the states figuring in the raw ensemble, but we were originally interested in the physical states. Given that we have no analytic solution, we have studied the problem using numerics. Concretely, we have simulated the collapse of a superposition of two Gaussians:

$$\psi(x) = \sqrt{\frac{1}{8\pi\sigma^2}}\left[\exp\left(-\frac{(x-a)^2}{2\sigma^2}\right) + \exp\left(-\frac{(x+a)^2}{2\sigma^2}\right)\right] \tag{4.152}$$

We have neglected the Hamiltonian evolution, but it could easily be included. A plot of the collapse times is shown in Fig. 4.2, and somewhat surprisingly we see that it also follows a Wald distribution. If nothing else, it leaves the option open for the possibility that the physical density matrix is the exponential of an Ito process:

$$\rho_p(x,y,t) = \rho_p(x,y,0)\exp\left[\mu(x,y)t + \sigma(x,y)\tilde{B}\right] \tag{4.153}$$

[48] Also known as the logistic or inverse Gaussian distribution.
[49] A host of the properties of the Wald distribution may be found in ref. [89].

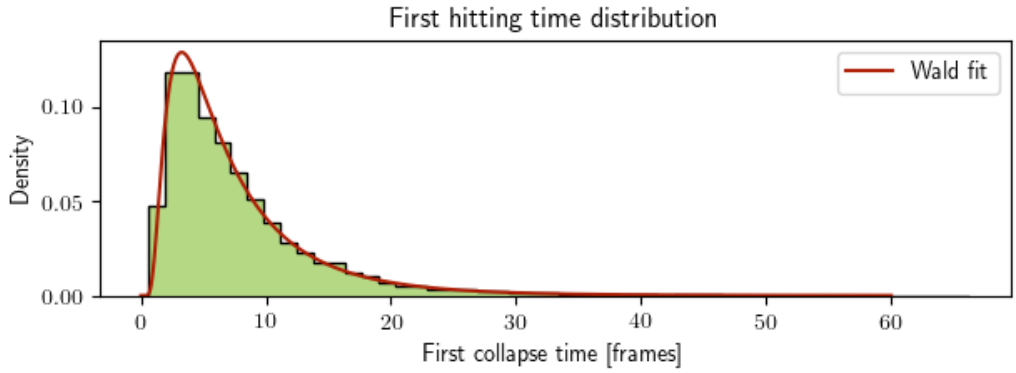

Figure 4.2: Distribution of collapse time $t_\alpha^{(f)}$ as defined by Eqn. (4.145) for the initial state given by Eqn. (4.152). It has been evolved using the non-linear equation. The mapping onto a physical situation is flexible, as described in Section 4.4.3. We have chosen the following numerical values: $\sigma = 1,\ a = 25, \gamma = 10^{-6}$ and $\alpha = 1$. The simulation ran with $\Delta t = 1$ for a total time of $T = N\Delta t$ with $N = 7000$. We have scaled the plot such that one frame corresponds to 100 timesteps. The number of runs was taken to be $M = 10^4$, taking approximately 20 hours to complete on my i7-7700k. The generated collapse times are fitted to a Wald distribution as described by Eqn. (4.147) with parameters $m = 7.59$ and $\lambda = 12.00$.

Where $\tilde{B}$ is $\mathbb{P}$-Wiener as always. It would be desirable to know the expressions for $\mu(x, y)$ and $\sigma(x, y)$. But this would amount to constructing a closed-form solution to the non-linear equation, and although I have certainly tried it is beyond my ability. Unfortunately there are strong indications that the above hypothesis is in any event false, as we shall soon see.

It would however be remiss of us to not give some concrete numbers for collapse times, so let us give an example. Consider the collapse of a macroscopic object of mass 10^{23} amu, prepared in a superposition of two Gaussian states with centers separated by 50 nm. From the definition of γ and the fact that we have chosen its numerical value to be 10^{-6} in simulations, we find the correspondence between frames and realtime:

$$\gamma = 10^{23} \cdot 10^{-20}\ \text{nm}^{-2}\ \text{s}^{-1} = 10^{-6}\ \text{nm}^{-2}\ \text{ns}^{-1} \tag{4.154}$$

The simulation described in Fig. 4.2 yielded $\mathbb{E}^{\mathbb{P}}[t_\alpha^{(f)}] = 7.59$ frames $= 0.76\ \mu s$ for $\alpha = 1$. This should coincide with the expected collapse time computed using the linear equation, which in this case is

$$t_c^e = \frac{2}{\gamma(x-y)^2} = 2\left(10^3 \cdot 50^2\ \text{s}^{-1}\right)^{-1} = 0.80\ \mu s \tag{4.155}$$

Which is in good agreement. The fact that our simulations yield a slightly smaller value will be explained in the next section. Incidentally, this is also the same order of magnitude as we predicted for a similar object using QMSL. As a quick aside, recall that at the end of Section 4.4.2 we claimed that the underlying process in Eqn. (4.82) must be $\mathbb{P}$-Wiener and not $\mathbb{R}$-Wiener. Fig. 4.3 shows the same simulation we did in Fig. 4.2, but for both cases. For an $\mathbb{R}$-Wiener process we find $t_c^e = 27.32$ frames $= 2.73\ \mu s$, clearly not in agreement with Eqn. (4.155).

In this respect, our simulations have yielded nothing new - we could already compute the mean collapse time rather easily. A more interesting question is the probability for collapse to

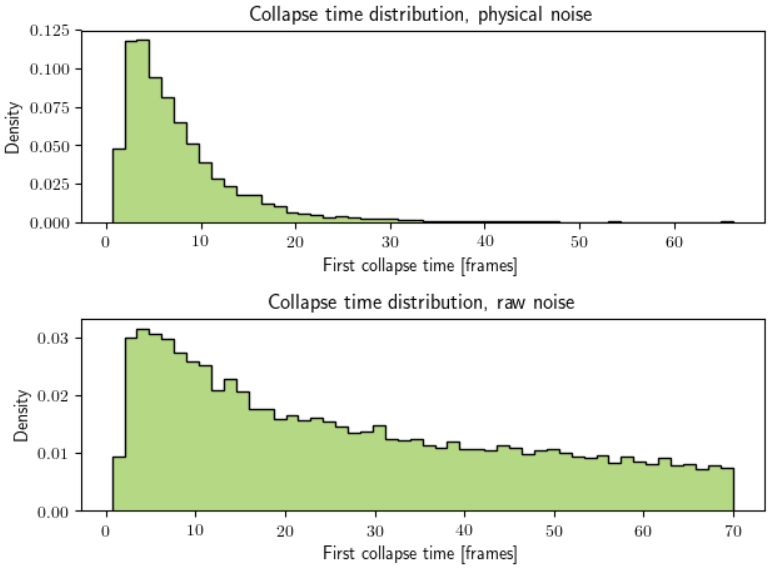

Figure 4.3: The first hitting times $t_\alpha^{(f)}$ of a double-Gaussian state is numerically simulated. All parameters are identical to the ones chosen in Fig. 4.2. We see that the time to collapse is *highly* dependent on whether the underlying processes chosen in Eqn. (4.82) are $\mathbb{P}-$ or $\mathbb{R}$-Wiener. In the plot these are referred to as physical and raw noise. The mean collapse times are $\mathbb{E}^{\mathbb{P}}[t_\alpha^{(f)}] = 7.59$ frames and $\mathbb{E}^{\mathbb{R}}[t_\alpha^{(f)}] = 27.32$ frames respectively.

occur after a certain time. Indeed, we might know from experiment that a measurement done using an apparatus of mass m at threshold β will be complete at most after a time t_k. Then it would be nice to compute the probability for collapse occuring at $t > t_k$, which could falsify the model. This is now easily computed since we have the cdf of the collapse time. We have not studied this in detail as we would need a concrete experimental setup to compare to, but it could be interesting to analyze in the future. Let us in any event give some sample numbers. Suppose we know that in a given situation we have $t_k = 20$ frames, and let us choose the threshold $\beta = 1$. Then $\Pr(t \geq t_k) = 1 - \Pr(t \leq t_k) = 1.4 \cdot 10^{-10}$. Even in the absence of a physical example, this is a good result because it shows that anything higher than roughly twice the expected time is highly, highly unlikely. In other words, the collapse time is sharply confined to a single order of magnitude.

Our discussion of the collapse time so far has been in terms of the *first* hitting time of the exponent of the raw and physical density matrices. This isn't satisfactory for several reasons, to which we now turn.

4.5.2 State revival

As we briefly mentioned, there may be multiple times t_{α_i} such that $T_t = \alpha$. This means that the reduction process is not reliable, as off-diagonal matrix elements may regrow after they have fallen below the threshold α. And if our model predicts a reduction process that is not reliable, then it also predicts measurements that are not reliable. Suppose a willing physicist records the outcome of some experiment, which may be spin down for an electron passing through an SG apparatus for example. This means the apparatus state has collapsed onto the pointer state

corresponding to spin down. Now revival occurs, and the subsequent collapse may be onto the pointer state corresponding to spin up. If the experimenter looks back up from his notebook, he will see that his apparatus now indicates that the electron has spin up - evidently not a great situation.

In the literature, these revivals are neglected on the grounds of being exceedingly unlikely. This is not entirely accurate, but the intuition is of course correct. The argument typically goes as follows[90]: consider a state $|\gamma\rangle = a(t_0)\,|\phi\rangle + b(t_0)\,|\psi\rangle$ which collapses, i.e. $|a(t_1)|^2 \gg |b(t_1)|^2$ for $t_1 > t_0$. If the state is to revive, then the underlying Wiener process must be large and positive for a very long time in order to bring $|b(t)|$ back up to the same size as $|a(t)|$. And as we know, that is unlikely. We can also give a perhaps more compelling argument in view of our discussion of hitting times. Consider the process given by Eqn. (4.144). Since it has positive drift, it will almost certainly pass any threshold we settle on. And given that the Wiener process has zero mean, it will on average only oscillate around the threshold for a short time after it has been passed.

These aren't very strong arguments, however. While it *does* turn out that these revivals are of no importance to the typical measurement situations where the apparatus is macroscopic, one might imagine smaller measurement devices where this effect will become important. On these grounds, we will attempt to give a more quantitative description of revivals.

We will first redefine the collapse time to include revivals. Our starting point will be the assumption of Eqn. (4.153) - namely, that the density matrix may be written as an exponential with the exponent as a Brownian motion process $X_t = \mu(x,y)t + \sigma(x,y)W_t$. We take W_t to be standard $\mathbb{P}$-Wiener, as the autocorrelation strength $\sqrt{\gamma}$ can be absorbed into σ. From our numerics, we have seen that the first hitting time of X_t on a threshold α follows a Wald distribution. To consider revivals, it would instead be interesting to compute the *last* hitting time of X_t - which would take revivals into account. And this we can actually do analytically, given the values of μ and σ, which we can extract numerically from the first hitting time. Let us see how. The last hitting time of a threshold α is defined as

$$t_c^{(l)}(\alpha) \equiv \sup\left\{t \geq 0 \;\; |X_t = \alpha\right\} \tag{4.156}$$

Now we know that first hitting times are Wald-distributed. One might then imagine running the X_t backwards in time, such that the last hitting time becomes a first hitting time. To that end we define the inverse time $u = 1/t$ and rewrite $X_t = \alpha$ in terms of it. We have

$$X_{1/u} = \frac{\mu}{u} + \sigma W_{1/u} = \alpha \implies u\alpha - u\sigma W_{1/u} = \mu \tag{4.157}$$

Defining the new process $B_u \equiv u\alpha - u\sigma W_{1/u}$, we see that whenever $B_u = \mu$, $X_t = \alpha$. And since B_u runs backwards in time from $t = \infty$ to $t = 0$, the first time this occurs will be precisely the last hitting time of X_t. In other words,

$$u_\mu^{(f)} \equiv \inf\left\{u \geq 0 \;\; |B_u = \mu\right\} = \frac{1}{t_c^{(l)}(\alpha)} \tag{4.158}$$

Now, if B_u was Brownian motion we could immediately conclude that $u_\mu^{(f)}$ is Wald-distributed. The term $u\sigma W_{1/u}$ doesn't look very comforting in that respect, however. But it turns out that Wiener processes have a remarkable property known as time-inversion. In fact, if W_t is a Wiener process, then $V_t = tW_{1/t}$ is also a Wiener process(see ref. [91] for a proof). Consequently $u_\mu^{(f)}$ really *is* Wald-distributed, with density $j_\mu(u, \frac{\mu}{\alpha}, \frac{\mu^2}{\sigma^2})$. What we want however, is the distribution of $t_\alpha^{(l)}$. To relate the two, we denote the cdf of $u_\mu^{(f)}$ by $J(u)$ and that of $t_\alpha^{(l)}$ by $K(t)$. The density

of $t_\alpha^{(l)}$ we denote by $k_\alpha(t)$. Then

$$J(u) = \Pr\left(u_\mu^{(f)} \leq u\right) = \Pr\left(t_\alpha^{(l)} \geq \frac{1}{u}\right) = 1 - K\left(\frac{1}{u}\right) \tag{4.159}$$

Differentiating then gives us the densities:

$$j_\mu(u) = \frac{1}{u^2} k_\alpha\left(\frac{1}{u}\right) \implies k_\alpha(t) = \frac{1}{t^2} j\left(\frac{1}{t}\right) \tag{4.160}$$

Inserting the parameters we determined above in Eqn. (4.146), we find the distribution of the last hitting time:

$$k_\alpha(t, \mu, \sigma) = \left[\frac{\mu^2}{2\pi\sigma^2 t}\right]^{1/2} \exp\left[-\frac{(\alpha - \mu t)^2}{2t\sigma^2}\right] \tag{4.161}$$

Considering Fig. (4.4), we see that the last hitting time $t_\alpha^{(l)}$ fits the k_α distribution quite well. Unfortunately, it fits a regular Wald distribution even better. There are at least two possible reasons for this. One is that the exponent of the physical density matrix quite simply isn't Brownian motion. Had it been, the last hitting times *should* have been better fitted to the k_α than the Wald, as we have illustrated in Fig. 4.5 for a known Brownian motion. Second is the possibility that what we have recorded as last hitting times aren't truly last hitting times. By necessity we must simulate for a finite time and so we cannot guarantee that the Brownian process wouldn't have returned to the threshold at some post-simulation time. We can however avoid this issue by ensuring we run our simulations for a sufficiently long time. In this way, the positive drift will make the realizations wherein a return to threshold occurs exceedingly unlikely, e.g. they will belong to the tail of the distribution, and so will not contribute to its shape around the mean. Consequently, we believe that the above is evidence for the fact that the physical density matrix cannot in fact be written in the form of Eqn. (4.153).

To make predictions about revivals we'll make use of the Wald distribution, which as we have seen is an excellent approximation to the true distribution - whatever it may be. Their effects aren't entirely negligible, as we see from Fig. 4.2 and Fig. 4.4. In particular, the inclusion of revivals increases the mean collapse time of the considered system by a factor of 1.45.[50] More important is the fact that the distribution of collapse times including revivals still have the same strongly dampened tail, such that the last hitting times are also highly concentrated at one order of magnitude. In other words, an experimentalist need not fear that his recorded outcomes are suddenly changed *provided* he "waits out" the timescale of collapse. Considering e.g. a macroscopic apparatus of mass 10^{23} amu with a mean collapse time $\mathbb{E}^{\mathbb{P}}[t_\alpha^{(l)}] = 1.1\ \mu s$, we can use the cdf of the Wald distribution to compute the probability of collapse occuring at $t > 10\ \mu s$. To numerical precision we find that $P(t > 10\ \mu s) = 0$, such that by waiting at least this time we can be almost entirely confident in the given experimental outcome.

As a finishing remark to our discussion about revivals, we note that they are a unique experimental signature of the model. In particular, if one could directly observe collapse,[51] then the absence of revivals would falsify the model. Whether this is possible in practice is not so clear to us, even if one could see collapse occur. Suppose one observes the first collapse of a system, presumably with the help of some macroscopic apparatus. Then the act of observation

[50] To be explicit: Fig. 4.2 yields a mean collapse time of 7.6 frames, whereas Fig. 4.4 yields 11.0 frames. Hence the factor 1.45.

[51] To be precise we mean collapse in the sense of reduction models, e.g. collapse must be observed in a system wherein all environmental interactions are sufficiently weak that decoherence plays no role.

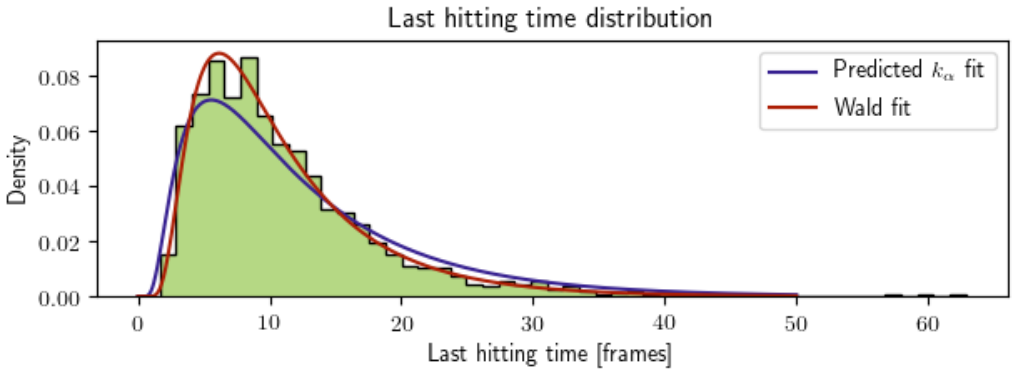

Figure 4.4: Distribution of collapse time $t_\alpha^{(l)}$ as defined by Eqn. (4.156). Initial state and parameter choices are identical to those chosen in Fig 4.2. Using m and λ as determined therein together with Eqn. (4.149) we can determine the drift and diffusion coefficients of the hypothesized Ito process in the exponent of the physical density matrix. They are in turn used to fit $t_\alpha^{(l)}$ to the distribution given by Eqn. (4.161). It is also fitted to a standard Wald distribution with $m = 11.03$ and $\lambda = 26.89$.

will almost instantly and astronomically depress the probability for revival - even if we had a realization where it would be otherwise guaranteed. Now it might be imagined that the observation could be done weakly, e.g. that by tuning the interaction strength between system and apparatus low enough one could extract sufficient information to conclude that collapse occured but without disturbing the system so much as to prevent a subsequent revival. We haven't studied this, but we think it could be an interesting point to investigate in the future.

4.5.3 Noise-induced drift

So far we have quantified collapse using off-diagonal elements of the density matrix. This is satisfactory as long as the off-diagonals correspond to the "same point" of the collapsing wavefunction. If the wavefunction undergoes additional motion - as it does under almost any type of unitary evolution - then the decay of a certain off-diagonal element is not necessarily indicative of collapse. So this is clearly a problem to be considered when we include a Hamiltonian, but it turns out to be a potential issue even without. The reason is that the non-linear Schrödinger equation we use causes wavefunctions to attract towards the expected position $\langle x \rangle$. Note however that the attraction toward $\langle x \rangle$ is only really relevant during actual wavefunction collapse, as this is when a significant portion of the particle mass density experiences acceleration. Concretely, suppose we have a state $|\beta(t)\rangle = \sum_i |g_i(t)\rangle$ where $|g_i(t)\rangle$ are Gaussians centered at some coordinate $x_i(t)$. As the superposition is formed, *all* $|g_i(t_0)\rangle$ have an initial velocity $v_i(t_0)$ toward the expected position $\langle x \rangle_\beta(t_0)$. The particular state $|g_j\rangle$ which survives collapse will be accelerated to a halt as the process finishes. It contains the overwhelming majority of the mass density, and so it is what we would interpret as corresponding to the "real" system we are working with. The other "ghost" states $|g_i\rangle$, $i \neq j$ will also accelerate during the collapse process, but their norm simultaneously undergoes rapid decay. As such they contribute very little to the overall mass density, and are of comparatively small physical significance. For the

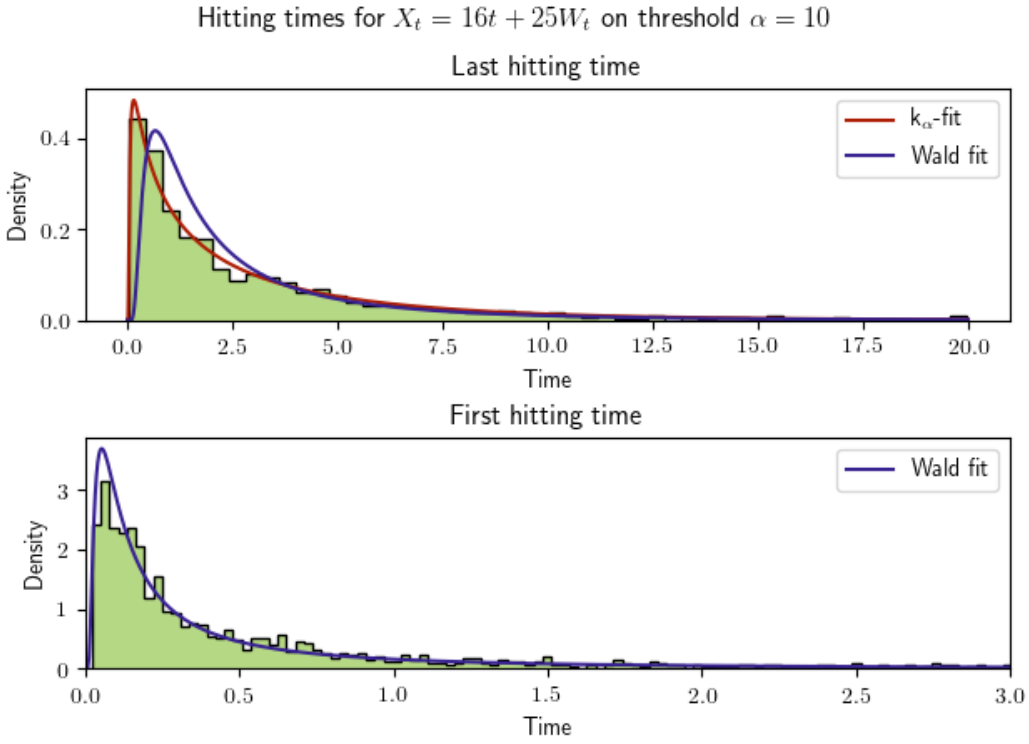

Figure 4.5: First and last hitting times for regular Brownian motion $X_t = \mu t + \sigma W_t$, where $\mu = 16$ and $\sigma = 25$. Threshold taken to be $\alpha = 10$. We ran simulations using a regular forward Euler scheme with a (dimensionless) timestep of $\Delta t = 0.004$ for a total time of $T = N\Delta t = 20$. The histograms contain hitting times for 1400 sample paths. Fits are done to the Wald- and k_α-distribution, given by Eqn. (4.146) and Eqn. (4.161) respectively. Scales may be chosen arbitrarily, consequently I have left them dimensionless.

same reason the drift of the ghosts occuring post-collapse isn't very interesting either.[52]

During collapse it *is* interesting, however. For one, it causes the state $|\beta\rangle$ to exhibit time-varying interference. If the velocity is of sufficient magnitude, then this could potentially be detected and it would be another experimental signature of the model. Secondly, if we are dealing with charged particles, then the acceleration of the mass density will cause the particles to spontaneously emit radiation, which may furthermore lead to a temperature increase in a macroscopic object.[53] Again, the strength of emission and/or temperature increase depends very much on the magnitude of this velocity. Incidentally this is exactly the same type of phenomenon we saw in our discussion of QMSL(cf. the problem of structured tails, Section 4.3.3). Lastly it has consequences for our computations of the collapse time, which we considered only in the case of fixed density matrix elements. To illustrate, suppose we wanted to compute the collapse time of the state $|\beta(t)\rangle$ above. Then we should *not* monitor the fixed element $\rho(x_i(0), x_j(0), t)$ but rather $\rho(x_i(t), x_j(t), t)$, where $x_i(t)$ and $x_j(t)$ denote the center of the i'th and j'th component state respectively.[54]

We're not sure how to estimate the drift velocity analytically, so once again we turn to our numerics. As before, we simulate the collapse of a double-Gaussian state as given by Eqn. (4.152). As we are interested in the drift of the individual Gaussian states, the evolution of the centre of mass is of little interest to us.[55] Instead we record the position of the centres of the Gaussians at each timestep, generating the two sample paths $x_l(t)$ and $x_r(t)$. Having underlying Wiener processes these paths are nowhere differentiable, so in principle they cannot give us a velocity. Consequently we must take the average before differentiating, such that $v_l(t) = \partial_t \mathbb{E}^{\mathbb{P}}[x_l(t)]$ and similarly for the other path. Under the assumption that one could average over all possible sample paths $x_l(t)$, this would ensure a smooth average $\mathbb{E}^{\mathbb{P}}[x_l(t)]$ which we could differentiate. This is evidently not possible in practice, but reasonably good averages can of course still be gotten. Now, as mentioned we are primarily interested in the drift during collapse. As such we will run simulations for a duration only approximately twice the mean collapse time t_c^e, starting at $t = t_0$. For the case of double Gaussians separated by a distance $2a = 50$ L and of width $\sigma = 1$ L, we find the approximate drift velocity

$$v_i(t_0) \approx 10^{-4} \text{ L/frames} \tag{4.162}$$

The details of the fit are shown in Fig. 4.7. Note however that this is the drift velocity of the ghost states - the surviving state, quite surprisingly, does not drift at all! The explanation for this is that the collapse process very rapidly suppresses one state with respect to the other, i.e. it takes very few frames before a sort of preliminary collapse occurs. Consequently the expected position shifts heavily toward one Gaussian and in most realizations[56] remains there until the defined

[52] Asymptotically the ghosts will merge with the surviving state $|g_j\rangle$. This means they will experience acceleration during the merging phase(s). Their norm will however have decayed so significantly at this point in time that they effectively do not contribute to the mass density, and so we expect no physical consequences from the merging process.

[53] This point is true even if we neglect drift. The noise causes the particle position to constantly fluctuate, leading to radiation and temperature increase.

[54] It might be interesting to see if a sort of interaction picture could be introduced wherein all non-collapsing motion was removed. We suspect it cannot, as the terms of type $x - \langle x \rangle$ are needed for collapse but also induce drift. We haven't checked this in detail however.

[55] Note that for the double Gaussian case we have $\partial_t \mathbb{E}^{\mathbb{R}}[\langle x \rangle] = 0$, although in every realization we have $\langle x \rangle(t_0) = 0 \to \langle x \rangle(t_1) \approx \pm x_0$ for some $t_1 > t_0$, where the Gaussians are assumed to be initially located at $\pm x_0$. The reason of course is that both collapses at $\pm x_0$ contribute with equal weight to the ensemble average.

[56] To be a little more precise, consider the case where the collapse threshold is set to $\alpha = 1.2$, and a corresponding "revival" threshold is set to $\beta = 1 - \ln 2$. Through simulating the double-Gaussian state used in this section

threshold is reached, and so the corresponding Gaussian will not move. However, it should be mentioned that there do exist extremely rare realizations where the norm of both Gaussians remain approximately equal for long periods of time. In these cases they do both drift, but of course their rarity drowns out their contribution to the average. Nonetheless, one might suspect they are of relevance as they will cause spontaneous emission of radiation. Unfortunately even in these cases the acceleration due to the noise is far stronger and will mainly be responsible for any radiative emission, as we see from Fig. 4.6.

Even so, we would like to give some estimates regarding both drift distance d_d and acceleration a_d during collapse. For the state we chose, $t_c^e \approx 10^3$ frames, so using Eqn. (4.162) and assuming constant drift velocity we find that $d_d \approx 0.1$ L and $a_d \approx 10^{-7}$ L/frames^2. Now we know that the drift velocity will drop to zero as $x_i(t) \to \langle x \rangle$, so we should really check the approximation of constant velocity. To do so we need to know the dependency of the velocity on the separation $\Delta x_i \equiv |x_i(t) - \langle x \rangle|$. A fit is shown in Fig. 4.6. We find that at least while the Gaussians are fairly well separated, $v_i(\Delta x_i) = 3.9\Delta x_i + 2.3$ $\tilde{\text{L}}$/frames, where $\tilde{\text{L}} = 10^{-6}$ L. Returning to our above example, we see that Δx_i increases from its initial value of 25 L to approximately 50 L at the time of collapse.[57] Then

$$v(t_0) = 9.98 \cdot 10^{-5} \text{ L/frames} \longrightarrow v(t_c^e) = 1.97 \cdot 10^{-4} \text{ L/frames} \tag{4.163}$$

So our approximation of constant velocity during collapse is fairly good. To give a couple of example numbers, assume we have the rare occurence of both Gaussians remaining of equal norm until the time of collapse. Then the velocity should be experimentally measurable - at least in principle - as most of the mass density is in motion. Then $v(t) \approx v(t_0) \approx 10^{-4}$ L/frames for both states. A C_{70}-molecule with L = 1 nm will drift little less than 0.1 nm, a grain of sand might drift 0.1 mm and so on. For the ghost states we'd need to multiply these numbers by a factor $k \in (1, 2)$, but again we cannot expect to experimentally measure their velocity.

Our next point concerns collapse times, as we computed them for fixed density matrix elements. More correctly - if we wish to stick with the density matrix approach - we should track the constituent states in the superposition such that we compute the decay of fixed points on the wavefunction even in the presence of drift. As we mentioned in the beginning of this section, one might for example track the centers of each Gaussian state in a superposition. This is quite doable numerically - we simply need to identify the support of each Gaussian and find the maximal value of the wavefunction therein. A comparison of collapse times with and without tracking the centers is shown in Fig. 4.8. It is a minor correction, but depending on the system it may obviously still be visible.

Let us also stress again that implementing some type of tracking may be essential if Hamiltonian evolution is included. If the unitary dynamics occur on a timescale roughly equal to that of the collapse dynamics, the decay of off-diagonal density matrix elements will almost certainly not be exclusively due to collapse. If furthermore the unitary evolution occurs on a timescale smaller than the collapse dynamics, the density matrix approach fails completely as interference between different parts of the wavefunction ruins any type of tracking. In this case we must use another method to measure collapse - for example by monitoring the variance in position.

---

for a total time of $T = 10^4$ frames, we found only 96/9000 runs containing a revival event - roughly 1.1 %. Unsurprisingly the number of revivals decreases rapidly with increasing threshold. Keeping β fixed and setting $\alpha = 1.5$, only 5/9000 runs contained a revival event.

[57] Strictly speaking there is also a "back-reaction" to take into account. As the states drift toward each other their separation decreases, in turn decreasing their velocity. But for the initial separation we chose, it is a very small correction compared to the change in velocity induced by the collapse itself.

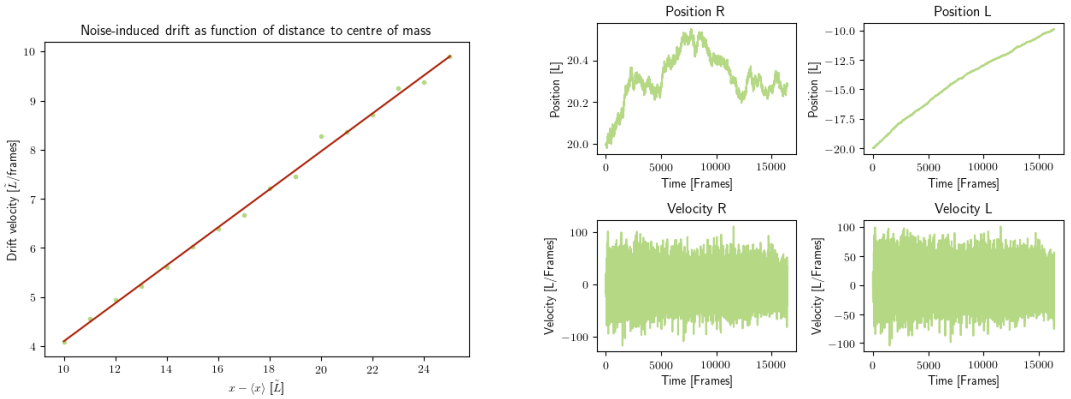

Figure 4.6: *Left plot:* The dependency of the drift velocity on the distance to the centre of mass is shown. We find a good linear fit under the assumption that the two Gaussian states are well-separated. The linear fit yields $v_i(\Delta x_i) = 3.9\Delta x_i + 2.3$ L̃/frames, where $\tilde{\mathrm{L}} = 10^{-6}$ L. Standard deviations are of order 10^{-9} and beyond precision. The intercept is non-zero because our assumption about well-separated states breaks down as $\Delta x_i \to 0$. Note that the current plot implies the sample paths $x_i(t)$ must be approximately quadratic in Δx_i. *Right plot:* Long-term behaviour of double-Gaussian collapse is illustrated. The quadratic behaviour of $x_l(t)$ is vaguely apparent, and that - excepting fluctuations - one state remains at rest. Comparison with Eqn. (4.163) shows that the fluctuations in the velocity dominates the drift velocity by about 6 orders of magnitude. As such any radiative emission will be overwhelmingly due to fluctuations.

4.6 Energy conservation in QMUPL

By now, we have mentioned the issues reduction models have with energy conservation multiple times. In Section 4.3.2 we very briefly referred to a calculation of the mean position/momentum in QMSL, but we haven't given a quantitative result in the context of QMUPL. We had not seen this energy increase computed in the literature when we did it, but it has been done.[92] As they use a different method,[58] we give our calculation below. We begin by computing the differential of the expectation of a general Hamiltonian $H = H_{\text{free}} + V(x,p)$, allowing for electromagnetic interactions by including a part linear in momentum

$$V(x,p) = V_1(x) + V_2(x)p \tag{4.164}$$

Using Ito calculus and Eqn. (4.54) with $A_i = x$, we have

$$\begin{aligned} d\langle\psi|H|\psi\rangle &= \langle d\psi|H|\psi\rangle + \langle\psi|H|d\psi\rangle + [\langle d\psi|H|d\psi\rangle]_t && (4.165)\\ &= i\langle H^2\rangle dt - \frac{\gamma}{2}dt\,\langle\psi|x^2H|\psi\rangle + d\tilde{B}\,\langle\psi|xH|\psi\rangle && (4.166)\\ &\quad - i\langle H^2\rangle dt - \frac{\gamma}{2}dt\,\langle\psi|Hx^2|\psi\rangle + d\tilde{B}\,\langle\psi|Hx|\psi\rangle && (4.167)\\ &\quad + [dB]_t\,\langle\psi|xHx|\psi\rangle && (4.168) \end{aligned}$$

[58] They use a so-called imaginary noise trick.[93] It consists of noticing that expectation values are independent under phase changes in the noise, e.g. taking $\alpha \in \mathbb{R}$ the equation $d|\phi\rangle = \big[-iHdt + \big(e^{i\alpha}A_i - \mathrm{Re}\{e^{i\alpha}\}\langle A_i\rangle\big)d\tilde{B}_i - \frac{\gamma}{2}dt\big(A_i^2 - 2e^{i\alpha}\,\mathrm{Re}\{e^{i\alpha}\}A_i\langle A_i\rangle + \mathrm{Re}\{e^{i\alpha}\}^2\langle A_i\rangle^2\big)\big]\,|\phi\rangle$ will give rise to exactly the same expectation values as will Eqn. (4.82). Taking $\alpha = \pi/2$ not only kills several terms but turns the remaining collapse terms Hermitian, rendering the dynamics effectively unitary. I see no advantage to applying the trick in the present case, however.

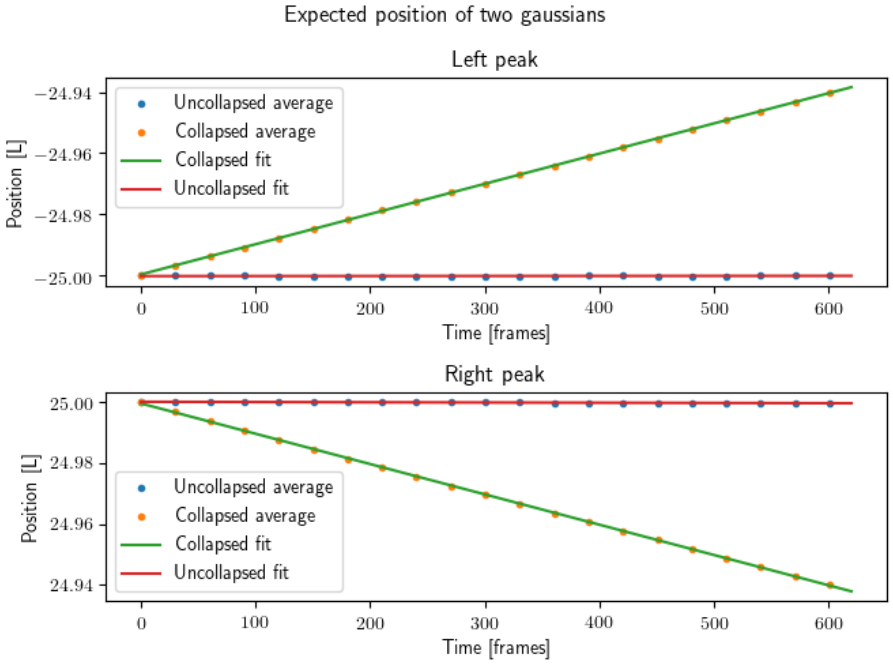

Figure 4.7: Expected positions of the Gaussians described by Eqn. (4.152). Chosen parameters are $\sigma = 1$ and $a = 25$. Simulated at threshold $\alpha = 1$ with timestep $\Delta t = 1$. Denoting the realized first hitting time by $t_\alpha^{(f)} = t_\alpha^1$, each simulation ran for a time $t_m = \max(7000,\ t_\alpha^1 + 2000)$ frames to accelerate data acquisition. The data points shown below are averaged over $N = 10^4$ runs. Only every 30'th data point is shown to avoid clutter. In the plot we denote "collapsed" by the Gaussian which vanished, e.g. it is not the one we collapse *to*. All states are linearly fitted, where the gradient of the collapsed states are $v_l(t) = 9.96 \cdot 10^{-5}$ L/frames and $v_r(t) = 9.89 \cdot 10^{-5}$ L/frames with standard deviation beyond precision. By symmetry we expect $v_l(t) = v_r(t)$; the small disagreement in our results we attribute to the relatively small amount of sample paths simulated. The timescale is too short to capture the quadratic nature of the paths $x_i(t)$, which is shown in Fig. 4.6.

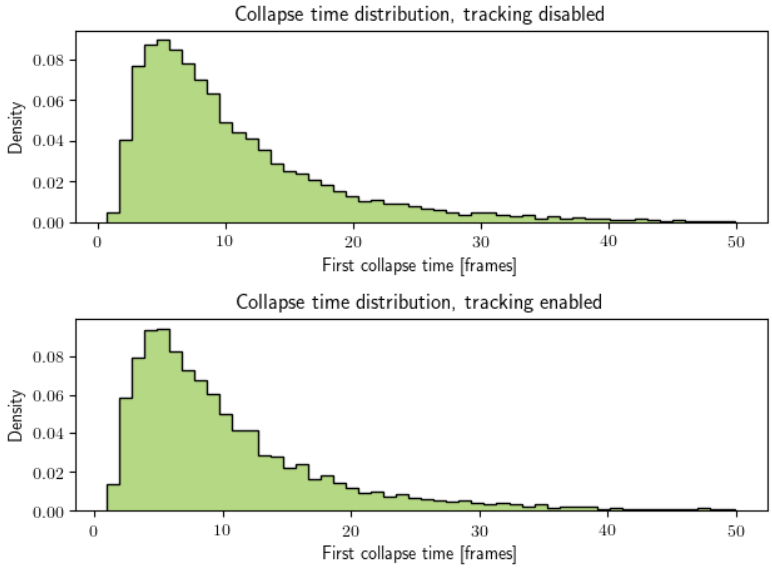

Figure 4.8: Collapse times of a double-Gaussian state with width $\sigma = 1$ and centers $a = \pm 20$. The upper plot is generated without dynamically tracking the centres of the Gaussian states, whereas the lower plot *does* track them. The mean collapse time without tracking is $m = 10.50$ frames, with tracking it is $m = 10.36$ frames.

Note that if we take the expectation this quite close to being a Lindblad type equation, as we have

$$\mathbb{E}^{\mathbb{R}}[d\langle H\rangle] = -\frac{\gamma}{2}dt\langle\big[x,[x,H]\big]\rangle \tag{4.169}$$

The missing term that prevents us from recovering the full Lindblad equation died precisely because we are computing the variation in the expectation of the Hamiltonian; if we instead redo the above computation with an arbitrary operator O we get

$$\mathbb{E}^{\mathbb{R}}[d\langle O\rangle] = i\mathbb{E}^{\mathbb{R}}[\langle[H,O]\rangle]dt - \frac{\gamma}{2}dt\langle\big[x,[x,O]\big]\rangle \tag{4.170}$$

Which is of Lindblad form.[59] Returning to the above calculation, we want to bring the middle terms to the form $\langle\psi|xHx|\psi\rangle$, so we must evaluate the commutator

$$[x,H] = [x,\frac{p^2}{2m} + V_1(x) + V_2(x)p] = \frac{ip}{m} + iV_2(x) \tag{4.171}$$

Using this, the second and fifth terms add up to cancel the last term. Then we are left with

$$d\,\langle\psi|H|\psi\rangle = dB\,\langle\psi|\{x,H\}|\psi\rangle + \frac{i\gamma}{2}dt\,\langle\psi|\left[\frac{p}{m} + iV_2(x), x\right]|\psi\rangle \tag{4.172}$$

$$= dB\,\langle\psi|\{x,H\}|\psi\rangle + \frac{\gamma}{2m}dt \tag{4.173}$$

[59] As an aside, take $O = x$ and $O = p$ and assume $V = V(x)$. Then we find $\partial_t\mathbb{E}^{\mathbb{R}}[\langle x\rangle] = \mathbb{E}^{\mathbb{R}}[\langle p\rangle]/m$ and $\partial_t\mathbb{E}^{\mathbb{R}}[\langle p\rangle] = -\langle\nabla V\rangle$ - which is again Ehrenfest's theorem.

Not unexpectedly, we see that the energy is constantly fluctuating due to the dB term. And as we suspected we see that there is a net increase in energy, given by

$$\delta E(t) = \frac{\gamma\hbar^2}{2m}t \tag{4.174}$$

Where we have restored units. We can use this result a bit further to give an estimate of the radiated power of a free particle with charge q. Assuming the particle to be initially at rest, the kinetic energy at time t is then

$$3\delta E(t) = \frac{1}{2}mv^2 \implies v(t) = \sqrt{\gamma t}\frac{3\hbar}{m} \tag{4.175}$$

Where we have switched to 3-dimensional space as we will want to compare to similar results in the literature. Solving for the acceleration we find

$$a(t) = \frac{3\hbar}{2m}\sqrt{\frac{\gamma}{t}} \tag{4.176}$$

Or in frequency space[60]

$$a(k) = \frac{3\hbar}{2m}\sqrt{\frac{\gamma}{2\pi}}\int dt\; t^{-1/2}e^{-ikt} = e^{-i\pi/4}\frac{3\hbar}{m}\sqrt{\frac{\gamma}{2k}} \tag{4.177}$$

Inserting the square of the acceleration in the Larmor formula, we find that the radiated energy per unit time is

$$\frac{dE}{dt} = \frac{2}{3}\frac{q^2a^2}{4\pi\epsilon_0c^3} = \frac{q^2}{2\pi\epsilon_0c^3}\frac{\hbar^2}{4m^2}\frac{\gamma}{t} \tag{4.178}$$

Where c is the speed of light and ϵ_0 is the vacuum permittivity. Using the Fourier transform of the acceleration and Plancherel's theorem, we can also give the radiated energy per unit frequency

$$\frac{dE}{dk} = 3\pi\frac{q^2\hbar^2}{4\pi^2\epsilon_0c^3m^2}\frac{\gamma}{k} \tag{4.179}$$

A similar result has been derived in the literature in the context of the CSL model by Q. Fu, using field-theoretical methods.[94] Although we won't discuss the model, we simply state that it contains two parameters λ_0 and α_0 in similar fashion to QMSL, taken by Fu to be

$$\gamma_0^{\text{csl}} = 10^{-16}s^{-1} \qquad\qquad \alpha_0^{\text{csl}} = 10^{-7}\text{ m} \tag{4.180}$$

Such that $\gamma_0 = \gamma_0^{\text{csl}}/(\alpha_0^{\text{csl}})^2 = 10^{-2}\text{ m}^{-2}\ s^{-1}$. Fu furthermore does not use a mass-proportional coupling constant, so to compare our result to his we must make the replacement $\gamma \to \gamma_0$ in Eqn. (4.179). If we then rewrite γ_0 in terms of the CSL parameters, we find that our result is a factor 3π different from that of Fu, though pleasingly the dependency on all dimensionful variables is the same. As the models are different(we used QMUPL, Fu used CSL) we did in any case not expect perfect agreement, so we will leave the result as it stands for now.

Another peculiarity related to energy non-conservation that we would like to mention occurs in bound systems. At the end of Section 4.4.4 we showed that the states bound by potentials at most quadratic in postion asymptotically converge to Gaussians. This is particulary strange

[60] The Fourier transform we use is not convergent, but we can regularize it by adding a small real part to the exponential in the integrand.

for bound systems, wherein excited states will have a net transfer of kinetic to potential energy as the collapse mechanism gradually removes the nodes of a given initial state.[61] Our numerics allow us to easily observe this effect. A timelapse is shown in Fig. 4.9 wherein the time evolution of the second excited state in a harmonic oscillator is simulated. Also shown is the evolution of the kinetic, potential and total energy of the state. Fig. 4.10 furthermore shows the evolution of the asymptotic Gaussian, and we see that it - for a time - closely approximates a coherent state.

Having now computed the rate of energy increase and shown it occuring in concrete systems, we'd like to discuss some potential solutions. To do so, we first note that there are two specific issues underlying energy non-conservation:

(I1) The noise we use in our stochastic Schrödinger equations is *white*, i.e. it has a flat spectral density.[62] As this implies infinite power we cannot hope to connect it to a physical process, and we should instead use non-white noise.

(I2) Our model lacks any kind of dissipative mechanism. As such it is not enough to use non-white noise, we must find a way to include dissipation as well.

Beginning with (I1), we'd like to write down a new Schrödinger equation for a noise with a general autocorrelation. This isn't terribly difficult in itself, but the resulting dynamics *do* turn out to be extremely difficult to analyze. Let's quickly show how to derive it, loosely following ref. [78]. We start by considering the SDE

$$d\left|\psi\right\rangle = \Big[-iHdt + A_i \circ dB_i + Idt \Big] \left|\psi\right\rangle \tag{4.181}$$

Where we have written it in Stratonovich form. In the literature the circle symbol is often omitted but the time-derivative is written explicitly, e.g. we can equivalently write it as

$$\partial_t \left|\psi\right\rangle = \Big[-iH + A_i w_i + I \Big] \left|\psi\right\rangle \tag{4.182}$$

Where $w_i = dB_i/dt$ are Gaussian processes with mean zero and correlation $\mathbb{E}^{\mathbb{R}}[w_i(t)w_j(t')] = C_{ij}(t,t')$. This is similar to Eqn. (4.37), but with C split into a Hermitian and an anti-Hermitian part. The operator I is related but not equal to the Ito term; we previously determined its form in the case where $C_{ij}(t,t') = \delta_{ij}\delta(t-t')$ but its general form is unknown to us. Our goal is to determine what it must be for the norm to be conserved. The formal solution to Eqn. (4.182) is

$$\left|\psi(t)\right\rangle = \exp\left[\int_0^t ds \ \left(-iH + I + A_i w_i\right) \right] \left|\psi(t_0)\right\rangle \tag{4.183}$$

To determine I, we again use the fact that the expected variation in the norm squared must vanish. Using Eqn. (4.181) and time-translation invariance of $\mathbb{R}$ we get

$$\partial_t \mathbb{E}^{\mathbb{R}}[\||\psi\rangle\|_t^2] = 2\mathbb{E}^{\mathbb{R}}[\, \langle\psi|A_i|\psi\rangle \, w_i] + \mathbb{E}^{\mathbb{R}}[\, \langle\psi|I|\psi\rangle \, dt] \tag{4.184}$$

The first term on the right hand side is not easy to evaluate, as the state $|\psi\rangle$ also depends upon the processes w_i. But it can be done by employing the Furutsu-Novikov theorem.[63] In general

[61] Asymptotically this kinetic energy will of course be restored in excession of its initial value.

[62] Easily shown by use of the Wiener-Khinchin theorem which relates the spectral density to the Fourier transform of the autocorrelation. White noise is delta-correlated in time by definition, so the result follows.

[63] See ref. [95] for the original derivation, or Appendix B in ref. [96] for some intuition.

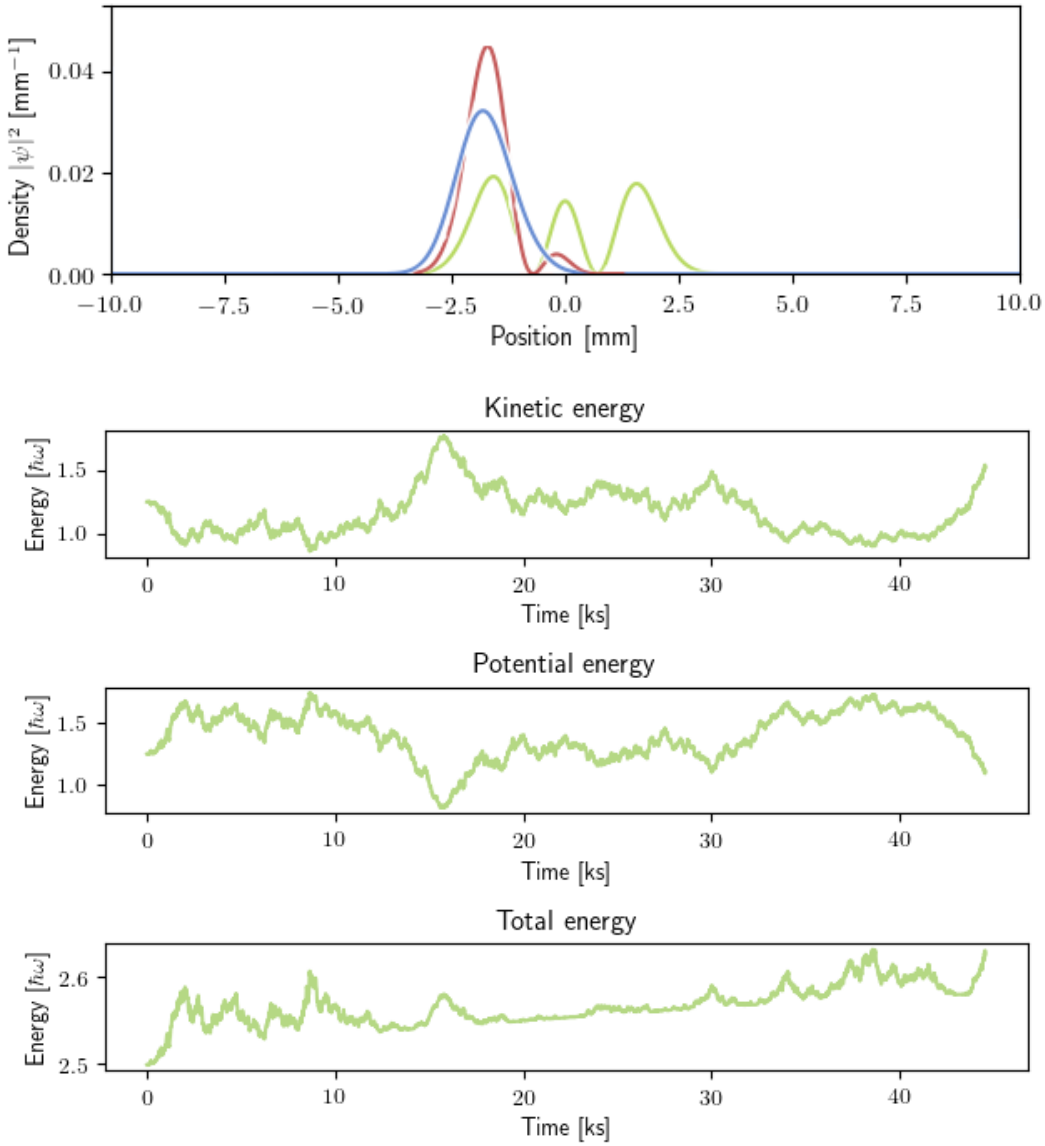

Figure 4.9: The time evolution of the second excited state in a harmonic oscillator is shown. We set $\hbar c = 1$ and consider a particle of mass 10^6 GeV/c$^2 \approx 10^6$ amu. At a mass scale of GeV, $\hbar c$ is of order one when the length scale is chosen to be mm. We further set $\gamma = 10^{-6}$ numerically, which for the given mass and length scales sets the timescale to be 0.1 s - approximately the value of a single frame. We have neglected some constants of order one as we are not mapping onto an exact physical situation. We choose the angular frequency of the oscillator to be $\omega = 10^{-6}$ frames^{-1} = 10^{-5} s^{-1}. Time is measured in kiloseconds(ks) in the plots. *Upper plot:* Three snapshots of the collapse process are shown. Initial state is green, the intermediate state at $t \approx 17$ ks is red and the final state at $t \approx 80$ ks is blue. *Lower plots:* The energy of the particle during the collapse process is shown. As expected the initial energy is $E(t_0) = \hbar\omega(2 + 1/2)$, equally split between kinetic and potential as it should be. A net increase in the total energy is also apparent. At $t \approx 17$ ks we see a dip in kinetic energy and a corresponding increase in potential energy - this is due to a node in the wavefunction being smoothed out. At $t \approx 30$ ks the second node is almost smoothed out, but reappears at $t \approx 40$ ks. Due to size constraints, the remaining plot is shown in Fig. 4.10.

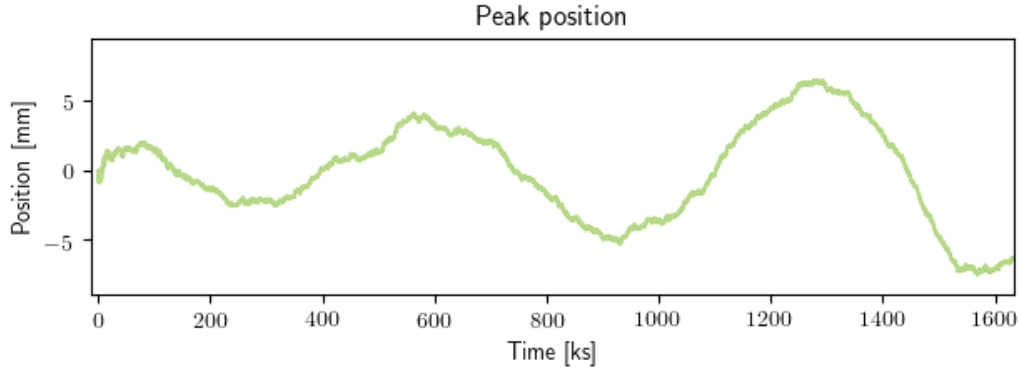

Figure 4.10: Continuation of Fig. 4.9. The position of the asymptotic Gaussian is tracked. Note that it takes shape at about $t \approx 80$ ks, at prior times there is no single peak. The asymptotic width is given by Eqn. (4.141). For our choice of units, this width is considerably smaller than that of a coherent state, as we expect since $\tilde{\gamma} \approx \gamma$(see Eqn. (4.140)). We see that the state asymptotically exhibits the expected oscillatory behaviour. A rough estimate of the period T from the two peaks at $t_1 \approx 580$ ks and $t_2 \approx 1280$ ks yields $T \approx 700$ ks, or equivalently an angular frequency of $\omega \approx 10^{-5}\ s^{-1}$. This tells us that the state is close to being coherent in this time interval, or in other words that it has not yet reached its asymptotic width.

it states that

$$\mathbb{E}^{\mathbb{R}}[F[w]w_i] = \int_{-\infty}^{\infty} dt'\ C_{ij}(t,t')\mathbb{E}^{\mathbb{R}}\left[\frac{\delta F[w(t)]}{\delta w_j(t')}\right] \tag{4.185}$$

Where $F[w]$ is an arbitrary functional of the noises w_i. In our case we identify $F[w(t)] = 2\langle\psi(t)|A_i|\psi(t)\rangle$. We note that it does not depend on processes $w(s)$ for $s > t$, consequently we may reduce the limit on the integral above to $[0,t]$. Applying this we find

$$\partial_t \mathbb{E}^{\mathbb{R}}[\||\psi\rangle\|_t^2] = 4\int_0^t dt'\ C_{ij}(t,t')\mathbb{E}^{\mathbb{R}}\left[\left\langle\psi\middle|A_i\middle|\frac{\delta\psi}{\delta w_j(t')}\right\rangle\right] + 2\mathbb{E}^{\mathbb{R}}[\langle\psi|I|\psi\rangle] \tag{4.186}$$

Where we have assumed the A_i Hermitian. In order that the expression above vanish, we find that I must be given by

$$I = -2\int_0^t dt'\ C_{ij}(t,t')A_i\frac{\delta}{\delta w_j(t')} \tag{4.187}$$

Note that in the case of uncorrelated white noise, we have $I|\psi\rangle = \gamma A_i^2|\psi\rangle$, exactly what we found in Section 4.4.2.[64] Inserting the above we find our final norm-conserving SDE for a process with a general Gaussian noise:

$$\partial_t|\psi\rangle = \left[-iH + A_i w_i - 2\int_0^t dt'\ C_{ij}(t,t')A_i\frac{\delta}{\delta w_j(t')}\right]|\psi\rangle \tag{4.188}$$

[64]The factor 1/2 deserves a comment. We have $C_{ij}(t,t') = \delta_{ij}\delta(t-t')$. Now we typically consider such a delta function to be the limit of smooth functions symmetric about $t' = t$. As it appears in an integral where the upper limit *is* $t' = t$, only half the area of the underlying smooth function contributes. Hence the factor 1/2.

Which we also mention is manifestly non-Markovian.[65] Unsurprisingly the functional derivative makes this equation very hard to analyze, both numerically and analytically. Initial attempts simply considered various ansatze of the form[76]

$$\frac{\delta |\psi(t)\rangle}{\delta w_i(t')} = O(t, t', w_i) |\psi(t)\rangle \tag{4.189}$$

Where O is an arbitrary operator. Interestingly it is shown in the reference just cited that in the case of a spin-1/2 system, Eqn. (4.188) with the choices $H \propto \sigma_z$, $O(t, t', w_i) = f(t, t')\sigma_-$,[66] $A \propto \sigma_-$ and an exponential correlation function causes any initial state to collapse to the ground state in finite time - quite to the contrary of the reduction models we have considered so far. Note however that the collapse seen in this example is purely induced by dissipation(as the reduction operator is taken to be σ_-), which is not what we want in a reduction model.

Eqn. (4.188) has also been analyzed in the context of reduction models. In particular the corresponding Green function has been derived for a free particle, and the resulting collapse dynamics has been studied for the case where the particle is initially in a Gaussian state and where the correlation function is again exponential.[92] Their results confirm what one would expect: the collapse process becomes slower the wider the correlation function is, e.g. the fewer frequency components of the noise participate. Furthermore the asymptotic width of the state remains the same, owing to the lack of a dissipative mechanism.

So let us consider (I2) - introducing dissipation into our collapse process. In this respect, it turns out to be highly useful to take a detour to the study of open quantum systems. As always, one is interested in a system S coupled to an environment E. As the dynamics of the environment is usually extremely complicated, we can't always "just" trace it out to obtain the dynamics of S; consequently an evolution equation for S depending at most on coarse-grained features of E is desirable. Now it can be shown that the (reduced) density matrix of the system obeys the so-called Born-Markov master equation[67]

$$\partial_t \rho_s = -i[H_s, \rho_s] - \sum_\alpha \{[S_\alpha, B_\alpha \rho_s] + [\rho_s C_\alpha, S_\alpha]\} \tag{4.190}$$

Where

$$B_\alpha \equiv \int_0^\infty d\tau \; C_{\alpha\beta}(\tau) S_\beta^{(i)}(-\tau) \tag{4.191}$$

$$C_\alpha \equiv \int_0^\infty d\tau \; C_{\beta\alpha}(-\tau) S_\beta^{(i)}(-\tau) \tag{4.192}$$

Here a superscript (i) denotes that the operator should be taken in the interaction picture. The subscripts s and e refer to system and environment respectively. $C_{\alpha\beta}(t)$ is the environmental autocorrelation, defined as[68]

$$C_{\alpha\beta}(t) = \mathrm{Tr}_e \left(E_\alpha(0) E_\alpha(t) \right) \tag{4.193}$$

The operators S_α and E_α arise in the interaction Hamiltonian, e.g.

$$H_{\mathrm{int}} = S_\alpha \otimes E_\alpha \tag{4.194}$$

[65] We mention that this equation has also been derived in the context of diffusion processes in open quantum systems. It can be seen to arise when one considers the dynamics of a system S coupled to a bath of harmonic oscillators, and where the Markov approximation is not made.[97]

[66] Here $f(t, t')$ is an arbitrary function, but it can be determined.

[67] Derivations may be found in ref. [41] and ref. [12], the latter reference being considerably more extensive.

[68] One assumes the environment to be in thermal equilibrium.

Two important approximations are made in arriving at this master equation. One is the *Born approximation*, which entails the assumption that the system-environment coupling is sufficiently weak that the environment is practically unaltered during evolution. That is, one assumes

$$\rho(t) = \rho_s(t) \otimes \rho_e \tag{4.195}$$

Second is the Markov approximation, which assumes that environmental excitations decay on a timescale considerably smaller than the system evolves on. This is interesting in relation to what we just discussed about non-Markovian reduction models; in fact, as we shall see, one need not reverse the Markov approximation to retain a non-white environment.

Now the Born-Markov master equation is a tad too general. To make proper use of it, we apply it to so-called quantum Brownian motion. It consists of taking the system to be a single particle coupled to an environment of harmonic oscillators; the environment can however be generalized and we will do so. The interaction Hamiltonian is taken to be

$$H_{\text{int}} = x \otimes c_i q_i \tag{4.196}$$

Where x is the position operator of our particle and the q_i are oscillator position operators coupling to the particle with strength c_i. As the environment is fully specified, its autocorrelation may be computed. In generalized form, it is given by[69]

$$C(t) = \nu(t) - i\eta(t) = \int_0^\infty d\omega\ J(\omega)\sin(\omega t) - i\int_0^\infty d\omega\ J(\omega)\coth\left(\frac{\omega}{2k_BT}\right)\cos(\omega t) \tag{4.197}$$

Where $J(\omega)$ is the spectral density of the environment and T is its temperature. The functions $\nu(t)$ and $\eta(t)$ are usually referred to as the noise and dissipation kernels respectively, for reasons that will soon become clear. Using Eqn. (4.196) and Eqn. (4.197) in the Born-Markov master equation, one obtains

$$\partial_t\rho_s = -i[H_s,\rho_s] - \int_0^\infty ds\left\{\nu(s)[x,[x^{(i)}(-s),\rho_s]] - i\eta(s)[x,\{x^{(i)}(-s),\rho_s\}]\right\} \tag{4.198}$$

Which is starting to look a lot like the equations we have used in our reduction models. To get it into a more suitable form, we'll take our system to be a single free particle. Applying the Hadamard lemma, we find that the interaction-picture position operator is

$$x^{(i)}(s) = x + \frac{s}{2m}p \tag{4.199}$$

Inserting it into the master equation and defining the quantities

$$D \equiv \int_0^\infty ds\ \nu(s) \tag{4.200}$$

$$f \equiv -\frac{1}{2m}\int_0^\infty ds\ s\ \nu(s) \tag{4.201}$$

$$\gamma \equiv \frac{1}{2m}\int_0^\infty ds\ s\ \eta(s) \tag{4.202}$$

$$\Omega \equiv -\int_0^\infty ds\ \eta(s) \tag{4.203}$$

[69] In deriving the autocorrelation from an oscillator environment, one finds $J(\omega) = \sum_i \frac{c_i^2}{2m_i\omega_i}\delta(\omega-\omega_i)$, where m_i and ω_i denote mass and frequency of the i'th oscillator respectively.

We get

$$\partial_t \rho_s = -i[H_s + \Omega x^2, \rho_s] - i\gamma[x, \{p, \rho_s\}] - D[x, [x, \rho_s]] - f[x, [p, \rho_s]] \tag{4.204}$$

The third term is immediately recognizable; it is what we have been working with for the last chapter. The other terms are new, and especially the second term is interesting. To see why, let us consider its effect on the expected momentum. Disregarding the other terms for the moment, we have

$$\begin{aligned} \operatorname{Tr} p\partial_t \rho_s &= -i\gamma \operatorname{Tr} p[x, \{p, \rho_s\}] && (4.205)\\ &= -i\gamma \operatorname{Tr} p\left(xp\rho_s + x\rho_s p - p\rho_s x - \rho_s p x\right) && (4.206)\\ &= -i\gamma \operatorname{Tr}(-2ip\rho_s) = -2\gamma \operatorname{Tr} p\rho_s && (4.207) \end{aligned}$$

As the term causes a net decrease in expected momentum, it is exactly the dissipative term we have been looking for. However, given that we are interested in reduction models, we must include the dissipative term in the dynamics of the statevector itself - otherwise there would be absolutely no advantage over regular decoherence. Based on drawing similar parallells to open quantum systems as we did above, Bassi et al. postulated that one should take the reduction operators (appearing in Eqn. (4.82)) and similar) $A_i = x_i + i\alpha p_i$ and phenomenologically modify the original Hamiltonian to include an extra term, such that $H = H_0 + \frac{\gamma\alpha}{2}\{x_i, p_i\}$.[98] Here α constitutes another parameter of our collapse models, and is defined as

$$\alpha = \frac{m_0}{m}\alpha_0 \qquad\qquad \alpha_0 = 10^{-18}\ \mathrm{m}^2 \tag{4.208}$$

This then leads to the following evolution equation for the density matrix:

$$\partial_t \rho = -i[H_0, \rho] - \frac{\gamma}{2}[x_i, [x_i, \rho]] - \frac{\gamma\alpha^2}{2}[p_i, [p_i, \rho]] - i\gamma\alpha[x_i, \{p_i, \rho\}] \tag{4.209}$$

Which is similar to the Born-Markov master equation and contains the desired localizing and dissipative terms. The term $[p_i, [p_i, \rho]]$ has no apparent desirable purpose in our eyes, as it localizes states in momentum space. Though this is the opposite of what we want, its coefficient is fortunately considerably smaller than those of the other terms. As for the dissipative term, it will eventually cause the collapsing system to reach equilbrium with some sort of stochastic medium, in analogy to our discussion above. And indeed, the reference just cited finds such results. The asymptotic spread of Gaussian states is found to be

$$\sigma_\infty = \sqrt{\frac{\hbar}{\sqrt{2}m\omega\sin\theta}} \tag{4.210}$$

Where

$$\theta = \frac{1}{2}\tan^{-1}\left[\frac{\hbar}{2\gamma_0\alpha_0^2 m_0}\right] \tag{4.211}$$

Which is to be compared with the width obtained from our original collapse equations, Eqn. (4.139). Although the dependency on the dissipative strength α_0 is complicated, it is evident that as $\alpha_0 \to \infty$, $\sigma_\infty \to \infty$. This is exactly what we would expect, as a highly dissipative(or cold) medium will equilibrate the wavefunction at a large spread. Furthermore we see that for a weakly dissipative medium $\alpha_0 \ll 1$, $\theta \approx \pi/4$ and we recover the original spread. As for the

energy increase, Bassi et al. finds[70]

$$\partial_t \mathbb{E}[\langle H_0 \rangle] = \frac{\gamma \hbar^2}{2m} - 4\gamma\alpha \mathbb{E}[\langle H_0 \rangle] \tag{4.212}$$

From which it is evident that the system reaches a finite maximal energy at equilibrium. Thus we have found a solution to (I2), and the infinite energy increase is gone. One may furthermore suspect that complete conservation of energy is upheld if one also treats the "environment" - whatever it may be. To be complete, both solutions to (I1) and (I2) should be combined, e.g. our final reduction equation must be non-Markovian and dissipative. This has been done, and the corresponding Green function has also been determined.[54] The combined model offers no new insight however, so we do not present it here.

Before we end our discussion of energy (non-)conservation, we point out that we find the phenomenological introduction of dissipation as done by Bassi et al. a little disturbing. This way of resolving the problem of energy non-conservation is based upon the concession that there must be some sort of interaction between a system and the stochastic medium. Usually interaction Hamiltonians will produce some level of entanglement between the systems involved - disallowing the assignment of a pure quantum state to either subsystem. Nonetheless, this is exactly what the authors above do. We cannot definitely say that it is wrong - it is conceivable that an energy transfer between systems is possible whilst preserving them in a product state. A cursory investigation hints that it may not be the case[99], but this would be interesting to study in more detail in the future.

4.7 Bell's theorem and Reduction Models

We have now seen two examples of dynamical reduction models - QMSL and QMUPL. Common to both is that they introduce underlying stochastic processes, the realizations of which we have no knowledge. In other words, the realization of the field is a hidden variable, responsible for determining the outcomes of experiments. This is somewhat worrying, as hidden variable theories have a troubled history. Their possible existence was first noted by EPR[100], who illustrated through their famous paradox that 1) either quantum mechanics is fundamentally nonlocal or 2) it is not a complete description of reality.[71] Taking the latter point of view is difficult, however, in view of two no-go theorems against hidden variables - namely, the Bell-Kochen-Specker(BKS) theorem and Bell's theorem. And yet this is the point of view we are forced to take, if we wish to seriously consider dynamical reduction models. Now, both of these no-go theorems need a set of assumptions to go through - it is not the case that they categorically dismiss *all* hidden variable theories. For example, they are irrelevant with regards to so-called nonlocal hidden variables[72], they do not apply to *contextual* hidden variables and so on. Of course, our primary interest is to see whether our reduction models survive contact with the no-go theorems. But to do that, we first need to present them and the assumptions they use.

[70] Unsurpsiringly it is the same as Eqn. (4.174), but with exponential damping.

[71] Typically descriptions of the EPR paradox are given in Bohm's version, e.g. by considering two spin-1/2 particles in a singlet state passing through Stern-Gerlach type apparatuses at spacelike separation. If one measures the spin of one particle in the z-direction to be up, QM predicts that a measurement of the other particles' spin along the z-direction must yield down. If one believes in the standard interpretation, these values did not exist prior to measurement. Thus the result of the first measurement somehow affects the second measurement, indicating that QM is fundamentally non-local. However, one isn't forced to take this point of view. If - like Einstein - one insists on locality, then it seems that the only possible explanation is that the outcomes, or some set of variables determining them fully, *did* exist prior to measurement. In other words, insisting on locality implies that QM is incomplete - it is the result of averaging over a more fundamental set of variables, much in the same way as thermodynamics arises from classical mechanics.

[72] We will shortly make precise by what we mean with nonlocal hidden variables.

The Bell-Kochen-Specker theorem

Neither the BKS theorem nor Bell's theorem were the first no-go theorems for hidden variable theories. In fact, as early as 1932 von Neumann purportedly showed that hidden variables are incompatible with quantum mechanics.[20] His reasoning went as follows. Consider three observables A, B, C such that $C = A + B$. For a given state, each of these have definite values given by $v(A), v(B)$ and $v(C)$. Since $C = A + B$, we must also have $v(C) = v(A) + v(B)$. This is the so-called additivity requirement, and von Neumann claimed it to be *incompatible* with quantum mechanics. By way of example, suppose that $(A, B) = (\sigma_x, \sigma_y)$, such that they take the values ± 1. Then by additivity, C - representing the spin along the axis between the $x-$ and $y-$axes - does *not* take the values ± 1, as required by QM.

As may be suspected, the argument is faulty. While the additivity requirement is reasonable to impose in the case where A and B commute(and in which case there is no conflict with QM), it is not so reasonable to impose in the case where they do not - they cannot even be measured simultaneously, so there is no reason to insist on additivity in this case.[73] To drive the point home, Bell constructed an explicit hidden-variable model for a spin-1/2 system.[102] Evidently, von Neumann's proof was wrong.

A considerably stronger case would be made against hidden-variable theories if, somehow, one could show that no such theory can satisfy the additivity requirement even for commuting observables. And indeed, this is the content of the BKS theorem: hidden-variable theories for systems where the corresponding Hilbert space has dimension greater than two cannot satisfy additivity of expectation values of commuting observables.

We will quickly sketch the proof as it was given by Bell[103], but first of all let us mention some terminology regarding states. To distinguish the hidden variable states from the regular quantum mechanical ones, we will refer to them as dispersion free states. That is to say, if we specify a dispersion free state, then the expectation value of *any* projector is either 0 or 1 with respect to this state. If we do not specifically mention that the state be dispersion free, we shall allow both to be the case.

Consider then an orthonormal basis $\{\psi_i\}$ for our Hilbert space $\mathcal{H}$. The projectors $P(\psi_i) = |\psi_i\rangle \langle\psi_i|$ partition unity

$$\sum_i P(\psi_i) = 1 \tag{4.213}$$

We will assume that additivity of the expectation of commuting operators holds, i.e. that

$$\sum_i \langle P(\psi_i)\rangle_\alpha = 1 \tag{4.214}$$

with respect to some state $|\alpha\rangle$. This is evidently true if $|\alpha\rangle$ is a regular quantum mechanical state; the hypothesis is that it is true also if $|\alpha\rangle$ is dispersion free. We shall make two further assumptions:

(A) if $\langle P(\psi)\rangle_\alpha = 1$, then $\langle P(\phi)\rangle_\alpha = 0$ for any state $|\phi\rangle$ orthogonal to $|\psi\rangle$.

(B) If $\langle P(\psi)\rangle_\alpha = \langle P(\phi)\rangle_\alpha = 0$ and $\langle\psi|\phi\rangle = 0$, then also $\langle aP(\psi) + bP(\phi)\rangle_\alpha = 0$, for any complex numbers a, b.

[73] That Neumann's argument was faulty was in fact first noted by Grete Hermann(see ref. [101]) in 1935, but it seems to have passed under the radar of most people. The argument was brought to attention again by Bell[102] in 1964.

Suppose now that

$$\langle P(\psi)\rangle_\alpha = 1 \qquad\qquad \langle P(\phi)\rangle_\alpha = 0 \tag{4.215}$$

for some set of states $|\phi\rangle, |\psi\rangle$ and $|\alpha\rangle$. Our first goal is to show that this implies that $|\phi\rangle$ and $|\psi\rangle$ cannot be arbitrarily close; in fact they must satisfy

$$\||\psi\rangle - |\phi\rangle\| > \frac{1}{2}\||\psi\rangle\| \tag{4.216}$$

In order to show this, we write

$$|\phi\rangle = |\psi\rangle + \epsilon\,|\psi'\rangle \tag{4.217}$$

where $\langle\psi'|\psi'\rangle = 1$ and $\langle\psi|\psi'\rangle = 0$. Pick an additional $|\psi''\rangle$ orthogonal to both $|\psi\rangle$ and $|\psi'\rangle$. Since $\langle P(\psi)\rangle_\alpha = 1$, assumption (A) implies that

$$\langle P(\psi')\rangle_\alpha = \langle P_{\psi''}\rangle_\alpha = 0 \tag{4.218}$$

Since $\langle P(\psi'')\rangle_\alpha = \langle P(\phi)\rangle_\alpha = 0$, assumption (B) implies that also

$$\langle P(\phi + \gamma^{-1}\epsilon\psi'')\rangle_\alpha = 0 \tag{4.219}$$

and

$$\langle P(-\epsilon\psi' + \gamma\epsilon\psi'')\rangle_\alpha \tag{4.220}$$

The last two equations and (B) gives

$$\langle P(\psi + \epsilon(\gamma + \gamma^{-1})\psi'')\rangle_\alpha = 0 \tag{4.221}$$

If $\epsilon < 1/2$, then there exist real γ such that

$$\epsilon(\gamma + \gamma^{-1}) = \pm 1 \tag{4.222}$$

which gives

$$\langle P(\psi + \psi'')\rangle_\alpha = \langle P(\psi - \psi'')\rangle_\alpha = 0 \tag{4.223}$$

Using (B) for the last time, this implies that $\langle P(\psi)\rangle_\alpha = 0$, contrary to what we assumed. Then we must have $\epsilon > 1/2$, and so Eqn. (4.216) is proved.

Suppose now that $|\alpha\rangle$ is dispersion free; the expectation of any projector is then either 0 or 1. Assume again our Hilbert space to be (at least) 3-dimensional. Pick any set of three projectors P_i.[74] Additivity implies that the expectation values of these projectors must be 0, 0 and 1 - in any order. Consider the states corresponding to the projectors. If the expected value of the projection onto that state is 0, we assign it the color blue.[75] If the expected value of the projection onto the state is 1, we assign it the color red. Thus the assignment of the values 0, 0 and 1 to the expected values of our projectors is transformed into coloring the sphere with just two colors - red and blue. Now, if we have just two colors, then there must necessarily be vectors of different color that are arbitrarily close. Given that we just proved that this cannot be the case, the conclusion seems to be that no hidden-variable theory (in more than 2 dimensions) cannot

[74] We assume that they project onto subspaces of dimension one.

[75] Following here an argument given by Mermin.[101]

satisfy additivity in expectation - even for commuting observables - and so cannot reproduce quantum mechanical predictions.

Although the theorem is strong, it does not rule out *all* hidden variable theories. In particular, there are three assumptions in the proof that bear explicit mention. Firstly, we assumed that all observables have a definite value in a non-dispersive state. This is the hallmark of hidden variable theories; given that this is what we are interested in, we won't question it. Second is the assumption that all observables of a system can be represented by (a set of) projectors.[76] The last assumption is in fact implicit in assumption (B), which need not be true for non-dispersive states. Consider the projector $P(\alpha\psi+\beta\phi)$. It does not commute with either $P(\psi)$ or $P(\phi)$ unless $\alpha = 0$ or $\beta = 0$. Thus they cannot be simultaneously measured, and there is no reason to insist on additivity in this case - indeed, measuring them would require distinct experimental setups. In other words, assumption (B) *implicitly* assumes that the values assigned are independent of the experimental setup, or measurement context - a property referred to as *non-contextuality*. More concretely, suppose we wish to measure some observable A. A commutes with two sets of operators $\{B_i\}$ and $\{C_i\}$, where $[B_i, C_j] \neq 0$ for at least one pair (i, j). If a hidden variable theory assigns different values to A depending on whether we simultaneously measure either the set $\{B_i\}$ or $\{C_i\}$, the theory is said to be contextual. If the value of A is independent of all other measurements, the theory is non-contextual.

Given these assumptions, there are at least two possible ways for a hidden variable theory to evade the BKS theorem. Either observables cannot all be represented by projectors, or the theory must be contextual. Now, Bell noticed that the assumption of non-contextuality can in certain cases be replaced by a simpler assumption, and that is the content of Bell's theorem.

Bell's theorem

While investigating Bohm's theory,[77] Bell noticed that the equations of motion were also manifestly non-local. As it turns out, this non-locality is intimately related to non-contextuality.[101] Consider the case where we have a system in an entangled state. We wish to measure either the set of operators $\{A, B_i\}$ or $\{A, C_i\}$, where again $[A, B_i] = [A, C_i] = 0$ and $[B_i, C_j] \neq 0$ for at least one pair (i, j). We perform the measurements at spacelike separation, measuring A on one subsystem and either the set $\{B_i\}$ or $\{C_i\}$ on the other. [78]

Now we may invoke the assumption of locality - turning a knob on the second measurement apparatus to decide on whether we measure $\{B_i\}$ or $\{C_i\}$ should not affect the result of measuring A. In this case, locality implies non-contextuality. This hints at the idea that hidden-variable theories must - in some sense - be non-local in order not to conflict with quantum mechanics. The proof that this is the case is Bell's theorem. In short, assuming a theory to be local, Bell shows that this implies an inequality on expectation values[102], which has now been found to be experimentally violated.[79]

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[76] This is an interesting discussion in the context of reduction models, which do not a priori assign any particular meaning to projectors or Hermitian operators. As we have seen, the act of measurement in reduction models is nothing but interaction between system and apparatus. Thus we must in principle calculate the possible outcomes of an experiment and their probabilities directly from this interaction, a process which can be quite laborious. However, it has been shown by Bassi et al.[62] that Hermitian operators arise naturally as a convenient way to encode this process, e.g. that we may indeed use sets of projectors to represent observables.

[77] Also known as Bohmian mechanics or the pilot-wave theory, see refs. [104, 105].

[78] To be explicit: we are not talking about non-local observables here; $A = A \otimes \mathbb{1}$ and $B_i(C_i) = \mathbb{1} \otimes B_i(C_i)$.

[79] Bell's inequality was first tested experimentally by A. Aspect et al. in 1982, and found to be violated.[106] Though a 5 σ result, it was not conclusive due to problems with the experimental setup. For example, a signal could in principle have propagated between the two measurement events at subluminal speed, a problem known as the communication loophole. Experiments eliminating this loophole(and several others) have recently been performed, see e.g. ref [107].

We will not show how Bell's inequality follows from the assumption of locality; it is not relevant for us. On the other hand, the assumption of locality *is*, and so we will make it more precise. Consider a composite system $S_t = S_1 + S_2$. The state of the system is specified by an element λ of the state space Λ given by our theory of choice. It may or may not be dispersive. On either subsystem we may measure observables whose outcomes are elements of the sets O_1 and O_2. respectively. To carry out the measurements, we need apparatuses. These may be configured in different ways; thus each apparatus has a configuration $c_1 \in C_1$ and $c_2 \in C_2$ respectively.

Consider then the probability for getting the outcomes $o_1 \in O_1$, $o_2 \in O_2$ given that the state of the system is λ, and that the measurement apparatuses have configurations c_1 and c_2:

$$p(o_1, o_2|c_1, c_2, \lambda) = p(o_1|o_2, c_1, c_2, \lambda)p(o_2|c_1, c_2, \lambda) \tag{4.224}$$

Locality amounts to the assumption that the probability of getting the outcome o_1 is independent of both the outcome o_2 *and* the apparatus configuration c_2; events occuring at spacelike separation should have no effect. In Jarrett's terminology, these assumptions are[108]:

$$\text{Outcome independence: } p(o_1|o_2, c_1, c_2, \lambda) = p(o_1|c_1, c_2, \lambda) \tag{4.225}$$

$$\text{Parameter independence: } p(o_1|o_2, c_1, c_2, \lambda) = p(o_1|o_2, c_1, \lambda) \tag{4.226}$$

Applying these assumptions to Eqn. (4.224), we find that

$$p(o_1, o_2|c_1, c_2, \lambda) = p(o_1|c_1, \lambda)p(o_2|c_2, \lambda) \tag{4.227}$$

Which is typically the starting point for deriving the Bell inequality.[80] We note that Bell started directly from this Eqn. (4.227) without mentioning either parameter independence (PI) or outcome independence (OI). The distinction is potentially useful because PI *may* allow superluminal signalling; we will shortly discuss this in some more detail. If a theory is to agree with experiment, it must violate either parameter independence, outcome independence, or both. Regular quantum mechanics is typically held to violate outcome independence, for example. The question is open as to whether our dynamical reduction models violate either of these assumptions, or equivalently if we can show that they are contextual; we turn to this next.

Returning to reduction models

Let us first mention that the possible incompatibility between reduction models and Bell's theorem has been considered by Butterfield et al.[90] They conclude that reduction models violate parameter independence, albeit in a way that turns out to satisfy causality. We agree with this, but with some reservations. In the following we will consider the reduction models where the statevector is a continuous stochastic process, like CSL/QMUPL. The original QMSL model is not of too much interest - we already know that it is incompatible with experiment.

Let us first present Butterfield et al.'s argument for violation of parameter independence. As before, we consider measurements performed at spacelike separation on a composite system $S = S_1 + S_2$. For simplicity, the system in question is taken to be two spin-1/2 particles in the singlet state. Assume that we are in a frame such that the measurement on S_2 is performed prior to any measurement of S_1, and that the measurement apparatuses M_1 and M_2 only have

[80] There are additional assumptions, as for example the assumption that the experimenters responsible are free to change the apparatus configurations up until the last moment - known as the free-will assumption. It leads to the so-called superdeterminism loophole, i.e. that everything in the universe is predetermined from its initial state. This is quite interesting in view of the recent free-will theorem of Conway and Kochen.[109] Incidentally they also claimed that their results prevent any consistent relativistic generalization of dynamical reduction models. This was subseqently shown not to be true by Goldstein et al.[110].

two configurations - ON or OFF. If a result does not depend on the apparatus configuration, we will denote it by $\star$. The outcomes of the measurements are goverened by two independent processes B_1 and B_2 - one for each apparatus.[81] As is often the case, the measured observable is the $z-$component of the spin.

Consider the space of all realizations of the underlying stochastic process B_i. Some subset $b_i^{\uparrow}$ gives rise to the outcome spin-up, a corresponding subset $b_i^{\downarrow}$ gives rise to spin-down. Consider then a measurement performed on S_2. Obviously the probability of either outcome is conditioned on $M_2 = \text{ON}$, but M_1 can be anything as the measurement on S_1 is performed after that on S_2. Then we have[82]

$$p(\uparrow_2 |b_2^{\uparrow},\ \star_1\text{ON}_2) \approx 1/2 \qquad\qquad p(\downarrow_2 |b_2^{\downarrow},\ \star_1\text{ON}_2) \approx 1/2 \tag{4.228}$$

Consider then the subsequent measurement on S_1. Conditioned on the measurement of S_2, we have

$$p(\uparrow_1 |b_1^{\uparrow},\ b_2^{\uparrow},\ \text{ON}_{12}) \approx p(\uparrow_1 |b_2^{\uparrow},\ \star_1\text{ON}_2) \approx 0 \tag{4.229}$$

$$p(\downarrow_1 |b_1^{\downarrow},\ b_2^{\uparrow},\ \text{ON}_{12}) \approx p(\downarrow_1 |b_2^{\uparrow},\ \star_1\text{ON}_2) \approx 1 \tag{4.230}$$

The equalities above follow because the measurement by M_2 precedes that of M_1; if for example $b_2^{\uparrow}$ is the relevant realization, the term $|\uparrow_1\downarrow_2\rangle$ will have vanishingly small norm - consequently $b_1^{\uparrow}$ will be practically ineffectual even if $M_1 = \text{ON}$. Similar equalities hold for the other realization $b_2^{\downarrow}$. Suppose we now set M_2 to OFF. Then

$$p(\uparrow_1 |b_1^{\uparrow},\ b_2^{\uparrow},\ \text{ON}_1\text{OFF}_2) \approx p(\uparrow_1 |b_1^{\uparrow},\ \text{ON}_1\text{OFF}_2) \approx 1/2 \tag{4.231}$$

$$p(\downarrow_1 |b_1^{\downarrow},\ b_2^{\uparrow},\ \text{ON}_1\text{OFF}_2) \approx p(\downarrow_1 |b_1^{\downarrow},\ \text{ON}_1\text{OFF}_2) \approx 1/2 \tag{4.232}$$

From which Butterfield et al. conclude that reduction models exhibit parameter dependence. And yet it seems to us that this argument applies equally well to regular quantum mechanics, from which we would conclude that it is *also* parameter dependent. We typically do not think of QM as a parameter dependent theory however, because the evolution of a system Q does not depend on parameters in any other isolated system Q' - which is the case because the Hamiltonian of Q' commutes with the density matrix of Q. Furthermore, if the above is to be considered a demonstration of parameter dependence, it is nonetheless not of the usable sort. An observer of S_1 does not know whether M_2 has been set to ON or OFF, consequently the state ρ he assigns to his system will the same in either case.[83] Essentially, if the parameter dependence is to be usable for some sort of (superluminal) communication, the dependence must carry over into the probabilities encoded in the (reduced) density matrix. We will show later that this is not the case for the continuous reduction models.

Let us also mention that the authors claim that the above argument primarily applies to the linear QMUPL model, e.g. where the modified Schrödinger equation is given by Eqn. (4.54).[84]

[81] Butterfield et al. perform their analysis in view of the CSL model, so they operate only with a single stochastic field in spacetime. We use the QMUPL model, and in this case there will be a stochastic process for every operator onto whose eigenstates we wish to collapse. That is, we have $d|\phi\rangle \supset \left[(x_1 - \langle x_1\rangle)dB_1 + (x_2 - \langle x_2\rangle)dB_2\right]|\phi\rangle$ - assuming that the apparatuses can be described in terms of their center of mass coordinate x_i. Although these two models are not entirely the same in terms of what they predict, for our calculations the equations will turn out the same and it makes no difference which model we use for the analysis.

[82] We write approximate equalities as state revival prevents the probabilities from being precisely those of LQM.

[83] Consider also the situation in light of the BKS theorem. If M_2 is set to ON, it measures some set of observables $\{O_i\}$ all of which commute with the observable measured by M_1. If we now set M_2 to OFF, it measures the identity, which also commutes with all of the $\{O_i\}$. Thus the present example is insufficient to illustrate whether the theory is contextual or not, which is presumably the goal.

[84] They also claim that the probability distribution of the underlying Brownian motion is physical. This isn't the case, as we have seen.

In the case of the non-linear equation - Eqn. (4.82) - they claim that parameter dependence is significantly weakened by providing a proof that the physical probability is effectively parameter independent.

Regardless of whether this proof is correct or not, this strikes us as peculiar. For one, there is only *one* reduction model.[85] In the case of QMUPL, the original (linear) equation of motion is Eqn. (4.54). Using this equation, we compute the evolution of the normalized states, and the corresponding equation of motion turns out to be nonlinear. But the model is the same. When we then switch to the physical measure through a Girsanov transformation, the equation is entirely unchanged - the underlying process is just expressed in terms of another measure. Lastly, the ensemble of physical states is in either case weighted by the physical probability. Thus, even if one holds the view that the linear and nonlinear equations correspond to distinct physical models, an effective parameter independence of the physical probability would imply effective parameter independence for *both* models - not just the linear one. The authors may be thinking of the ensemble expressed by

$$\rho = \mathbb{E}^{\mathbb{R}}[|\psi\rangle\langle\psi|] \tag{4.233}$$

from which one may possibly conclude that parameter independence of the physical probability does not apply to the linear model. But the ensemble above contains unnormalized states - normalizing them gives us back the physical probability, and so we see that the argument applies equally well in this case.

In any case, we believe their proof of parameter independence of the physical probability is incorrect. Let us see why. Consider the state $|\psi\rangle$ of some composite system $S = S_1 + S_2$. The systems are isolated, such that the evolution operator is given by $U = U_1 \otimes U_2$. The physical probability for getting a state $|\psi_{w_1 \& w_2}(t)\rangle$ is given by $\mathbb{R}[w_1(t)\ \&\ w_2(t)]\||\psi_{w_1 \& w_2}(t)\rangle\|^2$. If the marginal probability with respect to S_2 is equal to the physical probability computed just for S_1, we can conclude that the probability is parameter independent. Now the $\mathbb{R}$-Wiener processes are spatially independent, so the raw probability factorizes. If also $|\psi\rangle$ is a product state, then

$$\int \mathcal{D}[w_2]\mathbb{P}[w_1\ \&\ w_2] = \int \mathcal{D}[w_2]\mathbb{R}[w_1\ \&\ w_2]\||\psi_{w_1\ \&\ w_2}(t)\rangle\|^2 \tag{4.234}$$

$$= \mathbb{R}[w_1]\||\psi_{w_1}(t)\rangle\|^2 \int \mathcal{D}[w_2]\mathbb{R}[w_2]\||\psi_{w_2}(t)\rangle\|^2 \tag{4.235}$$

$$= \mathbb{R}[w_1]\||\psi_{w_1}(t)\rangle\|^2 \int d\mathbb{P}[w_2] \tag{4.236}$$

$$= \mathbb{P}[w_1] \tag{4.237}$$

On the other hand, $|\psi\rangle$ may well be entangled, in which case we do not expect the above to hold. However, Butterfield et al. claim that the following holds:

$$\int \mathcal{D}[w_2]\mathbb{R}[w_1\ \&\ w_2]\||\psi_{w_1\ \&\ w_2}(t)\rangle\|^2 = \mathbb{R}[w_1]\||\psi_{w_1}\rangle\|^2 \int \mathcal{D}[w_2]\mathbb{R}[w_2]\frac{\||\psi_{w_1\ \&\ w_2}\rangle\|^2}{\||\psi_{w_1}\rangle\|^2} \tag{4.238}$$

$$\stackrel{?}{=} \mathbb{R}[w_1]\||\psi_{w_1}\rangle\|^2 \tag{4.239}$$

[85] This point has also been commented on by T. Norsen, in a paper discussing the article of Butterfield et al. [111]. Apart from this, the rest of his paper is devoted to showing that the conclusion as to whether reduction models violate parameter or outcome independence is intimately related with how we interpret the ontological status of the noise field. Norsen concludes that reduction models violate parameter independence if the noise field is taken to be physically real, otherwise they violate outcome independence. In the latter case one would hope that the noise field may be related to something that *is* physically real, in analogy to how we think about electromagnetic fields and their corresponding potentials.

where the last equality is arrived at on the grounds that the norm squared is conserved in the mean. We think this is quite worrying for two reasons. One, we recall that norm conservation of some state $|\psi\rangle$ was guaranteed because its norm was $\mathbb{R}$-martingale. But even if two processes are martingale with respect to the same distribution, their product need not be. A regular Wiener process serves to illustrate: if W_t^2 is to be martingale, then we require $\mathbb{E}[d(W_t^2)|f_s] = 0$, where $s \leq t$ and f_s is the natural filtration. Computing it, we find

$$\mathbb{E}[d(W_t^2)|f_s] = \mathbb{E}[2W_t dW_t + [dW]_t|f_s] = 2W_s\mathbb{E}[dW_t] + dt = dt \tag{4.240}$$

Second, we wanted to investigate entangled states as the marginal probability is trivially computed for product states. But then no pure state can be assigned to either subsystem, which is exactly what Butterfield et al. do. This part is easily fixed, however. The requirement of normalization for the physical probability - Eqn. (4.43) - generalizes to the requirement

$$\mathbb{E}^{\mathbb{R}}[\operatorname{Tr}\rho_r] = 1 \tag{4.241}$$

Returning to the proof, the analogous requirement is that

$$\int \mathcal{D}[w_2]\mathbb{R}[w_2]\frac{\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1} = 1 \tag{4.242}$$

Where ρ refers to the density matrix of the composite system S_1+S_2 and $\rho_1 = \operatorname{Tr}_2 \int \mathcal{D}[w_2]\mathbb{R}[w_2]\rho$ is the reduced density matrix of S_1. We have dropped the subscript r as we want to indicate reduced density matrices, but it should be kept in mind that for this calculation all density matrices are raw. Though perhaps obvious we also point out that $\operatorname{Tr}\rho$ is a stochastic process - strictly speaking, we should write $(\operatorname{Tr}\rho)_t$ to make it clear. But again we omit it to avoid clutter. To show the that the above requirement is satisfied, it suffices to show that the variation in the conditional expectation of $\operatorname{Tr}\rho/\operatorname{Tr}\rho_1$ vanishes with respect to the natural filtration generated by the Brownian motion coupled to S_2. Again the conditional expectation reduces to a regular expectation as the states are measurable with respect to their natural filtration, and the Wiener increments do not depend on past history. Using the generalization of Eqn. (4.51) to mixed states, denoting $(\operatorname{Tr}\rho)^n = \operatorname{Tr}^n\rho$ for notational convenience and defining $\int \mathcal{D}[w_2]\mathbb{R}[w_2] \equiv \mathbb{E}_2$, we have

$$\begin{aligned}
\mathbb{E}_2\left[d\frac{\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1}\right] &= \mathbb{E}_2\left[\frac{d\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1} - \frac{\operatorname{Tr}\rho}{(\operatorname{Tr}\rho_1)^2}d\operatorname{Tr}\rho_1 + \frac{\operatorname{Tr}\rho}{\operatorname{Tr}^3\rho_1}[d\operatorname{Tr}\rho_1]_t\right. && (4.243)\\
&\quad \left. + \left[\frac{d\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1}, \frac{\operatorname{Tr}\rho}{\operatorname{Tr}^2\rho_1}d\operatorname{Tr}\rho_1 + \frac{\operatorname{Tr}\rho}{\operatorname{Tr}^3\rho_1}[d\operatorname{Tr}\rho_1]_t\right]_t\right] && (4.244)\\
&= \frac{\operatorname{Tr}\rho}{\operatorname{Tr}^3\rho_1}[d\operatorname{Tr}\rho_1]_t + \mathbb{E}_2\left[\frac{d\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1}, \frac{\operatorname{Tr}\rho}{\operatorname{Tr}^2\rho_1}d\operatorname{Tr}\rho_1 + \frac{\operatorname{Tr}\rho}{\operatorname{Tr}^3\rho_1}[d\operatorname{Tr}\rho_1]_t\right]_t && (4.245)\\
&= 4\gamma dt\langle A_1\rangle^2\frac{\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1} + \mathbb{E}_2\left[\frac{d\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1}, \frac{\operatorname{Tr}\rho}{\operatorname{Tr}^2\rho_1}d\operatorname{Tr}\rho_1 + 4\gamma dt\langle A_1\rangle^2\frac{\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1}\right]_t && (4.246)\\
&= 4\gamma dt\langle A_1\rangle^2\frac{\operatorname{Tr}\rho}{\operatorname{Tr}\rho} + \mathbb{E}_2\left[\frac{d\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1}, \frac{\operatorname{Tr}\rho}{\operatorname{Tr}^2\rho_1}d\operatorname{Tr}\rho_1\right]_t && (4.247)\\
&= 4\gamma dt\langle A_1\rangle^2\frac{\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1} && (4.248)\\
&\quad + \mathbb{E}_2\left[2\left(\langle A_1\rangle dB_1 + \langle A_2\rangle dB_2\right)\frac{\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1}, \frac{\operatorname{Tr}\rho}{\operatorname{Tr}^2\rho_1}2\langle A_1\rangle dB_1\operatorname{Tr}\rho_1 + \mathcal{O}(dt)\right]_t && (4.249)\\
&= 4\gamma dt\langle A_1\rangle^2\frac{\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1}\left(1 + \frac{\operatorname{Tr}\rho}{\operatorname{Tr}\rho_1}\right) && (4.250)
\end{aligned}$$

Where we have used angle brackets to denote normalized averages, e.g.

$$\langle A_i \rangle = \frac{\operatorname{Tr} \rho_i A_i}{\operatorname{Tr} \rho_i} \tag{4.251}$$

We see now explicitly that Eqn. (4.242) is not satisfied, contradicting the claim of Butterfield et al. From our point of view, the continuous reduction models *do* exhibit parameter dependence. This is evident in the case of the non-linear equation - Eqn. (4.82) - as the evolution of the state depends on the non-local quantity $\langle A_i \rangle$. Thus the evolution of the state at one point A depends on parameters chosen at any spacelike separated point B.[86] But to rigorously show it, we would want to compute the evolution of the reduced density matrix $\operatorname{Tr}_2 \rho$ of a composite system $S_1 + S_2$, and show that the probabilities therein depend on parameters chosen in S_2.

Now, we have just seen that it is quite tricky to compute the marginal physical probability. In fact, if $|\phi\rangle$ is a state of the composite system with an underlying process obeying the corresponding physical probability, we must first relate the physical probability of the total system to the physical probabilities of the subsystems. We would do this through a Girsanov transformation, but this cannot even be done directly - the Radon-Nikodym derivative, as we saw, is not martingale. However, recall that the ensemble of physical states was given by

$$\rho = \mathbb{E}^{\mathbb{R}}[|\psi\rangle \langle\psi|] = \mathbb{E}^{\mathbb{P}}[|\phi\rangle \langle\phi|] \tag{4.252}$$

Now it is clear how we can avoid the issue of whether the physical probability factorizes or not - we just compute the differential of the left hand side in the equation above, after which we may trace out one subsystem and integrate out the corresponding raw probability. Recall also that the raw measure is invariant under time translation(see Eqn. (C.15)), so there is no term corresponding to variation in the measure. Using the linear equation once again, the variation in the density matrix of the total system is

$$d\rho = idt[\rho, H] + dB_i\{A_i, \rho\} - \frac{\gamma\, dt}{2}\big[A_i, [A_i, \rho]\big] \tag{4.253}$$

Before we move on, we need to introduce some notation. In the case of a composite system, we are free to average over the underlying stochasticity pertaining to either a single subsystem or to both. If the systems are spatially isolated, we might want to compute a reduced density matrix that is non-dispersive for that system but dispersive for the other, emulating an observer with locally complete knowledge. To indicate a state which is non-dispersive with respect to systems $i, j, \ldots$, we'll use a superscript $nd : i, j, \ldots$ on the state. Returning to the equation above and tracing out S_2, we get

$$d \operatorname{Tr}_2 \rho^{nd:1,2} = d\rho_1^{nd:1,2} = idt[\rho_1^{nd:1,2}, H_1] + 2dB_1 A_1 \rho_1^{nd:1,2} + 2dB_2 \operatorname{Tr}_2 A_2 \rho^{nd:1,2} - \frac{\gamma\, dt}{2}[A_1, [A_1, \rho_1^{nd:1,2}]] \tag{4.254}$$

From the third term, it is evident that the evolution of ρ_1 depends on parameters chosen in the other system - so this type of reduction model is indeed parameter dependent. But it is also evident that this parameter dependence cannot be used to transmit information superluminally. At the very least, to get the non-dispersive state for S_1 we should average out the nonlocal realizations relevant to S_2, e.g.

$$\rho_1^{nd:1} = \int \mathcal{D}[w_2]\mathbb{R}[w_2] d\rho_1^{nd:1,2} = idt[\rho_1^{nd:1}, H_1] + 2dB_1 A_1 \rho_1^{nd:1} - \frac{\gamma\, dt}{2}[A_1, [A_1, \rho_1^{nd:1}]] \tag{4.255}$$

[86] Note that this also implies contextuality, i.e. the outcome of a measurement at the point A depends on what we do with the measurement apparatus at B. Then we also avoid the BKS theorem, as expected.

Which is independent of any parameters chosen in S_2, whether or not we subsequently integrate out the remaining term responsible for non-dispersion in S_1. To conclude, we find that the reduction models considered *are* parameter dependent, as Bell's theorem requires. But it is an effectively useless type of parameter dependency, not in conflict with causality.

4.8 The future of reduction models

We have now discussed dynamical reduction models quite extensively, and we have seen them to satisfy numerous constraints; they conserve the norm of states, they satisfy causality, violate the Bell inequality and incorporate non-linear statevector dynamics while still preserving a linear evolution law for the density matrix. Even the issue of energy non-conservation can in principle be resolved by taking into account the dynamics of the stochastic "medium" with which quantum states interact - whatever it may be. And while this is quite impressive, several challenges remain. On the theoretical side, the origin of the noise must be explained. This ties closely into the relativistic versions of collapse models, which as of now are highly unconstrained precisely because the nature of the noise field is unknown. In the non-relativistic case, it would be desirable to give also the equations of motion for the stochastic medium, and so show energy conservation to be upheld even in that regime.

On the experimental side, the models obviously need to be verified(or falsified). This is a tricky endeavour for several reasons. One is inherent to the reduction models themselves; although upper bounds on the collapse parameter γ can easily be obtained, lower bounds are inherently subjective. That is, depending on how ardently you believe in reduction models, you might insist that the collapse parameter is lower than any given real number. Of course, the models lose predictive power at some point - if the parameter choice is such that a macroscopic superposition is allowed to persist for a billion years, then the model - though still not falsified - would be a hard sell. Pearle and Collett write that a reasonable allowed time of persistence may be taken equal to the human perception time - about 100 ms.[112] But then one may wonder why humans define the cutoff, i.e. why are crayfish[87] allowed to see macroscopic superpositions? In any case, even if crayfish can see superpositions one would still need to come up with other explanations for our own non-observation of them, largely removing the reason for constructing reduction models in the first place.

As for upper bounds on the collapse parameters; they are legion. Of these we will mention a few, the interested reader may find comprehensive reviews in refs. [75, 114, 115]. Incidentally most upper bounds are given in terms of the parameters of the CSL model, so for reference we'll quickly write down its fundamental equation.[15] We consider k particle species of mass m_k and spin s_k located at a point x, with corresponding creation and annihilation operators $a_k^\dagger(x,s)$ and $a_k(x,s)$. We define a smeared mass density operator

$$M(x) = \sum_{k,s_k} m_k \int dy\; g(y-x) a_k^\dagger(y,s) a_k(y,s) \tag{4.256}$$

Where the smearing function $g(x)$ is taken to be

$$g(x) = \left(\frac{\alpha}{2\pi}\right)^{3/2} \exp\left(-\frac{\alpha}{2}x^2\right) \tag{4.257}$$

[87] When threatened, a second nervous system activates in crayfish. In particular, alarming visual stimuli will cause the so-called motor giant neurons to fire, triggering an escape mechanism that takes less than 20 ms from activation to execution.[113] The perception time - being less than the reaction time - is then quite clearly much smaller than that of a human.

To get the equation for the time evolution of states under CSL we simply assume that $M(x)dx$ are the collapse operators A_i of the QMUPL model in the continuum limit. Inserting them in Eqn. (4.54) we get

$$d\left|\psi\right\rangle = \left[-iHdt + \frac{1}{m_0}\int dx\ M(x)dB(x) - \frac{\gamma}{2m_0^2}\int dx\ M^2(x)dt\right]\left|\psi\right\rangle \tag{4.258}$$

Where m_0 is a reference mass typically taken equal to one amu and α is a new constant of nature defining the length scale of the collapse process. The canonical choice of parameters is very similar to that of QMSL, except the "rate" parameter γ is taken one order of magnitude weaker, e.g. $\gamma = \lambda/10 = 10^{-17}\ \mathrm{s}^{-1}$.[88]

Perhaps the most obvious signature of reduction models is the decay of interference. As seen from Eqn. (4.143), the non-detection of decay after some time t_n yields an upper bound on the collapse parameter γ - and we see also that the magnitude of this bound depends crucially on the separation of the superposition and the system's mass.[89] Ideally one would like to have an experiment wherein the system is both massive and in a superposition state with large separation, but of course this has not yet been done. Consequently current experiments use either systems of large mass but small separation or systems of low mass but large separation. In the former category one has experiments using micromechanical oscillators, typically of mass 10^6 amu. Unfortunately the spatial extent of their vibrational states is very small, of the order of femtometer even when the oscillator is cooled to a temperature of order μK.[116] As such they are not yet able to yield strong bounds. In the latter category one has molecular interferometry experiments, where sizable superpositions *can* be achieved. The strongest bound so far obtained from this approach is $\gamma < 10^{-5}\ \mathrm{s}^{-1}$ in an experiment involving molecules of mass $m \approx 7000$ amu.[117] The primary difficulty in scaling these experiments to higher masses comes from decoherence; in particular the effect of thermal decoherence becomes more pronounced as the mass(and so number of internal degrees of freedom) is increased, shielding the effects of dynamical collapse.

Another experimental signature of reduction models is that of radiative emission and/or energy increase, as we discussed in Section 4.6. In particular Fu compared his result for the energy radiated by free electrons to experimental data for Germanium, yielding a seemingly good bound of $\gamma < 1.4 \cdot 10^{-11}\ \mathrm{s}^{-1}$.[94, 118] Unfortunately this bound depends crucially on the assumption that the valence electrons of Germanium are free. This is only approximately true; taking into account further corrections reduces the bound to $\gamma < 10^{-5}\ \mathrm{s}^{-1}$.[115] In a somewhat similar vein, S. Adler obtained an upper bound of $\gamma < 10^{-8}\ \mathrm{s}^{-1}$ by considering the heating of the intergalactic medium, assuming the only source of heat to be from collapse and the only cooling mechanism to be due to the expansion of the universe.[115] As for other bounds, they are all weaker than those we have mentioned so far. They include bounds obtained from supercurrent decay, proton decay, atomic excitation and so forth - for further discussion of these bounds, we refer again to the review articles cited at the beginning of this section.

[88] The reason for taking γ one order of magnitude lower than its QMSL counterpart is that the equation for the decay of off-diagonal density matrix elements then is the same for both CSL and QMSL.[15]

[89] Although we refer to an equation derived in the context of QMUPL, the corresponding equation for CSL is highly similar. As we are only giving a superficial discussion of this part, we use the QMUPL equation for brevity.

Chapter 5

Conclusion

In this thesis we have studied nonlinear quantum mechanics, primarily motivated by its potential to resolve the infamous measurement problem. Beginning with a study of deterministic NLQM in the context of Weinberg's framework, we found that it was especially the nonlinear evolution of density matrices that was problematic. In particular we found that this led to issues with superluminal(or otherworldly) communication, statistical interference and the description of composite systems. Of these problems, we found that all but the last could be resolved by insisting on different laws governing the evolution of proper and improper mixtures, as suggested by Jordan. We subsequently discovered that this in turn entails an epistemic interpretation of the quantum state, largely removing the purpose of NLQM from our point of view.[1] Even accepting epistemic states, the consideration of composite systems turned out to be deadly for NLQM, as proved by Jordan. We did however find a mistake in this proof, and showed it to hold only for deterministic evolution. In other words, stochastic NLQM was still on the table as a possibility for solving the measurement problem.

This then motivated us to study dynamical reduction models, wherein a stochastic nonlinear term is added to the Schrödinger equation. We gave a derivation of the form the extra non-unitary terms must have, and noted that the resultant equation is not norm-conserving. In developing a norm-conserving equation we corrected a derivation found in the literature, and gave it a slightly more rigorous underpinning. We then proceeded to discuss a variety of peculiar features resulting from stochastic evolution. Using the density matrix, we found that the time taken for a state to collapse is a random variable, and we numerically estimated its distribution. We saw that using the density matrix to quantify collapse can only be done under certain assumptions; if they are not satisfied we must resort to another method, e.g. by monitoring the variance in position. Our numerics could give important hints about the distribution of the collapse time quantified by variance - we leave it for the future. A related feature is the reversal of collapse, which we found to be restricted to the order of magnitude determined by the mean of the collapse time. As such one need not fear that outcomes of experiments are reversed, provided they are recorded at a timescale exceeding that of collapse. We mentioned that it is not clear whether these revivals are experimentally detectable in practice, even given the ability to directly observe collapse. We leave it as an open question for future work.

Furthermore the collapse process causes wavefunctions to flow toward the expected position, and we numerically determined the acceleration involved in the simple case of a double Gaussian wavefunction. We found it to be utterly negligible compared to the acceleration caused by

[1] Recall that there is nothing mysterious about collapse if the state is epistemic - it is simply an observer updating his knowledge.

stochastic fluctuations, which we subsequently computed. Using this we calculated the energy radiated by a free, charged particle and compared it to a similar result in the literature, finding good agreement. We also noted that reduction models causes energy to oscillate between kinetic and potential, at least in potentials that are at most quadratic in position. The effect is however only expected to be relevant whenever the collapse dynamics are at least approximately as strong as unitary evolution, a regime that has not yet been experimentally explored. We further discussed potential solutions given in the literature to the problem of energy non-conservation, noting the ever-increasing analogy between reduction models and open quantum systems. A potential solution of Bassi et al. was briefly presented, where dissipation is introduced phenomenologically at the statevector level. We remarked that this seems troubling because interaction often causes entanglement, in turn preventing us from assigning a pure quantum state to either subsystem - undermining the motivation for constructing reduction models in the first place. Whether energy transfer is possible whilst maintaining product states would then be of interest to know.

We then investigated what type of hidden-variable theories reduction models are. We found that they are indeed non-local and explicitly violate parameter independence. This dependency is however contained purely in a stochastic term which disappears when averaged over, such that it can never lead to acausal events by e.g. superluminal communication. As we ended our study of reduction models, we presented the current experimental situation. We found that the bounds obtained on the collapse parameters so far are fairly weak and not able to reasonably reject dynamical reduction models. As such, they remain a real possibility yet.

Appendix A

Must quantum dynamics be linear?

In the following we present an abbreviated version of T.F. Jordan's proof that quantum dynamics are linear.[44] It contains some mistakes, which we point out where they occur. Their implications are discussed in Section 3.16. Consequently, the proof is not correct, but it contains a very important point which we make use of.

The starting point is a proof by Wigner and Bargmann[47, 48]. They show that quantum dynamics must be linear if the following assumptions are true:

1. Pure states are mapped to pure states
2. $\psi(t_0) \longleftrightarrow \psi(t_1)$ is injective
3. Inner products are preserved: $|\langle\psi(t_0)|\phi(t_0)\rangle|^2 = |\langle\psi(t_1)|\phi(t_1)\rangle|^2$.

The goal is then to prove these assumptions. Jordan claims that these assumptions are proved if the map that generates the time evolution of density matrices is linear and injective. That is, if $\partial_t\rho = g_t(\rho)$, then g_t is assumed to be linear and injective. Jordan claims that this is sufficient to guarantee the existence of the inverse map g_t^{-1}.

Consider then two distinct density matrices ρ_1 and ρ_2. We denote their evolution by a mark, that is $\rho_1 \to \rho_1'$. Linearity implies that

$$\rho = p\rho_1 + (1-p)\rho_2 \qquad \longrightarrow \qquad \rho' = p\rho_1' + (1-p)\rho_2' \tag{A.1}$$

where $p \in [0,1]$. Given the assumption of injectivity, mixed states are mapped to mixed states. If the inverse exists, it is linear and injective.[1] Then the very same argument shows that pure state are mapped to pure states: apply g_t^{-1} to ρ' - if it is mixed, then $g_t^{-1}(\rho')$ will be too.

To prove the preservation of the inner product, Jordan considers the evolution of a general density matrix ρ:

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i| \qquad \longrightarrow \qquad \rho' = \sum p_i' |\psi_i'\rangle\langle\psi_i'| \tag{A.2}$$

[1] Which we may quickly show. Consider $g_t^{-1}(\rho_1' + \rho_2') = g_t^{-1}(g_t(\rho_1) + g_t(\rho_2)) = g_t^{-1}g_t(\rho_1 + \rho_2) = \rho_1 + \rho_2$. Injectivity of g_t^{-1} - if it exists - follows from the injectivity of g_t. If $\rho_1' \neq \rho_2'$, then there exist distinct ρ_1 and ρ_2 such that $\rho_1' = g_t(\rho_1)$ and $\rho_2' = g_t(\rho_2)$. However injectivity of g_t is not enough to guarantee that it has an inverse.

Jordan assumes that $\langle\psi_i|\psi_i\rangle = \langle\psi'_i|\psi'_i\rangle = 1$, and that the p_i and p'_i are non-negative and sum to unity - which is the same as requiring our map to be a positive and trace-preserving. Taking the trace of ρ'^2, we get

$$\mathrm{Tr}\big[(\rho')^2\big] = \sum_{ij} p'_i p'_j |\left\langle\psi'_i|\psi'_j\right\rangle|^2 \geq \sum_i p'^2_i = \mathrm{Tr}\big[\rho^2\big] \tag{A.3}$$

We may then repeat the argument for the inverse map g_t^{-1} and conclude that

$$\mathrm{Tr}\,\rho^2 = \mathrm{Tr}\,\rho'^2 \tag{A.4}$$

Continuing on, Jordan considers a general density matrix of the form

$$\rho = \frac{1}{2}\left|\psi\right\rangle\left\langle\psi\right| + \frac{1}{2}\left|\phi\right\rangle\left\langle\phi\right| \tag{A.5}$$

such that

$$\mathrm{Tr}\,\rho^2 = \frac{1}{2}\left(1 + |\langle\psi|\phi\rangle|^2\right) \tag{A.6}$$

Similarly,

$$\mathrm{Tr}\,\rho'^2 = \frac{1}{2}\left(1 + |\langle\psi'|\phi'\rangle|^2\right) \tag{A.7}$$

From which it follows that $|\left\langle\psi|\phi\right\rangle|^2 = |\left\langle\psi|\phi\right\rangle|$. We could have seen this immediately from Eqn. (A.4), since this implies that g_t must be unitary and consequently also an isometry. But linearity and injectivity of g_t does *not* imply that the inner product is preserved, nor does it imply that g_t is unitary.[2] As such, the question remains open as to whether the dynamics of quantum mechanics are fundamentally linear.

The final part of Jordan's proof concerns the linearity of the evolution map g_t. Indeed, if the above part of the proof were correct, we would then have a proof that quantum dynamics must be linear. As we have seen, this is not the case, but the proof that g_t must be linear is nonetheless both interesting and useful to us. Let us then see how Jordan proves it. Consider two isolated systems A and B, separated by a large distance - on opposite ends of the Milky Way, for example. Suppose the total state of the systems is given by either

$$\Lambda = p\rho_1\left|\alpha\right\rangle\left\langle\alpha\right| + (1-p)\rho_2\left|\beta\right\rangle\left\langle\beta\right| \tag{A.8}$$

or

$$\Gamma = (p\rho_1 + (1-p)\rho_2)\left[p\left|\alpha\right\rangle\left\langle\alpha\right| + (1-p)\left|\beta\right\rangle\left\langle\beta\right|\right] \tag{A.9}$$

where $\{\left|\alpha\right\rangle, \left|\beta\right\rangle\}$ is a basis for $\mathcal{H}(B)$. In either case, the reduced density matrix for system A is given by

$$\rho = p\rho_1 + (1-p)\rho_2 \tag{A.10}$$

Suppose the total system is in the state Λ, and suppose we perform a measurement of an observable Υ that distinguishes $\left|\alpha\right\rangle$ and $\left|\beta\right\rangle$ in system B. If we subsequently measure a time-independent observable Ω in system A, its expectation value evolves in time according to

$$\partial_t\langle\Omega\rangle = \begin{cases} \mathrm{Tr}[\Omega\partial_t\rho_1] & \text{If we measured } \alpha \\ \mathrm{Tr}[\Omega\partial_t\rho_2] & \text{If we measured } \beta \end{cases}$$

[2] As a simple example, consider $f(x) = kx$ with $k \neq 1$. It is a bijective linear map but it is not an isometry.

If the measurement of Ω is non-selective, the expectation value evolves according to

$$\partial_t \langle \Omega \rangle = \mathrm{Tr}\left[\Omega(p\partial_t\rho_1 + (1-p)\partial_t\rho_2)\right] \tag{A.11}$$

On the other hand, we should be able to compute the evolution of the reduced density matrix even if we lack knowledge about whether the measurement of Υ has been performed or not. In this case, the expectation value of Ω evolves in time according to

$$\partial_t \langle \Omega \rangle = \mathrm{Tr}\left[\Omega\partial_t\big(p\rho_1 + (1-p)\rho_2\big)\right] \tag{A.12}$$

If observers in the vicinity of system A and B are to agree on their evolution, Eqn. (A.12) must be equal to Eqn. (A.11). And given that the observable Υ is arbitrary, we conclude that

$$\partial_t \left(p\rho_1 + (1-p)\rho_2\right) = p\partial_t\rho_1 + (1-p)\partial_t\rho_2 \tag{A.13}$$

In other words, the time evolution of the density matrix must be *linear* if we assume the ability to describe an isolated system without considering the rest of the universe.

Appendix B

Probability spaces

A probability space is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$. Ω is the *sample space*, which consists of all possible outcomes of whatever experiment we are doing. The sample space is by itself not enough to build a probability theory - it needs further structure. Suppose for example that $\Omega = \{1, 2, 3, 4, 5, 6\}$. What is the probability of *not* getting $\omega = 1$? We cannot say, because $\{2, 3, 4, 5\} \notin \Omega$. This motivates the introduction of a so-called σ-algebra[1] $\mathcal{F}$, satisfying the following properties:

(i) $\Omega \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$.

(ii) Given $f_1 \in \mathcal{F}$ and $f_2 \in \mathcal{F}$, then $f_1 \cup f_2 \in \mathcal{F}$, $f_1 \cap f_2 \in \mathcal{F}$ and $f_1 \setminus f_2 \in \mathcal{F}$.

(iii) If I is a countable index set and $f_i \in \mathcal{F}$, then $\bigcup_{i \in I} f_i$ belongs to $\mathcal{F}$.

Given any two events $f_1 \in \mathcal{F}$ and $f_2 \in \mathcal{F}$ we may now ask whether both f_1 *and* f_2 occured, whether f_1 did *not* occur and so on, as the corresponding elements(often referred to as *events*) belong to $\mathcal{F}$. Typically $\mathcal{F}$ is taken to be the power set of Ω, e.g. consisting of all possible events. Lastly we need to assign numbers to these events, which is done by the probability measure $\mathbb{P}$. It satisfies the so-called Kolmogorov axioms:

(i) $\mathbb{P}(f) \in [0, 1]$ for any $f \in \mathcal{F}$.

(ii) $\mathbb{P}(\Omega) = 1$ and $P(\emptyset) = 0$.

(iii) If I is a countable index set and $f_i \cap f_j = \emptyset$ for all $i, j \in I, i \neq j$, then $\mathbb{P}\left(\bigcup_{i \in I} f_i\right) = \sum_{i \in I} \mathbb{P}(f_i)$.

Thus constructed, the probability space is the basis for probability theory. In the context of our discussion on dynamical reduction models, we will want to consider an extension of probability spaces to so-called *filtered* probability spaces. We will discuss them in Appendix C.

[1] Also called a σ-field.

Appendix C

Stochastic processes

Before introducing a collapse model, we need some knowledge about stochastic processes in order to properly think about them. This section will largely follow Breuer and Petruccione[41]. We build up some general theory and finish the section with a simulation of a Wiener process. Formally, a stochastic process is defined like so

Definition
Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a parameter space T, a stochastic process is a mapping

$$X_i : \Omega \times T \longrightarrow \mathbb{R}^n \tag{C.1}$$

Where i runs from 1 to n.

Typically, $T \subseteq \mathbb{R}$ and $t \in T$ plays the role of time. If we lock in a particular $\omega \in \Omega$, the map

$$X_i(\omega, \cdot) : T \longrightarrow \mathbb{R}^n \tag{C.2}$$

is called a *realization* of the stochastic process. Typically, this is not how one works with stochastic processes however. Instead, one usually has information in the form of correlations between a given realization of the process at different times. This takes the form of a joint probability distribution:

$$P(B_1, t_1; B_2, t_2; \ldots; B_m, t_m) \equiv \mathbb{P}(X(t_1) \in B_1, X(t_2) \in B_2, \ldots X(t_m) \in B_m) \tag{C.3}$$

where B_i are Borel sets on $\mathbb{R}^n$, the collection of which also forms a σ-algebra. Note that $\mathbb{P}(X(t_i) \in B_i) = \mathbb{P}(X^{-1}(B_i))$, i.e. we are using a little bit of sleight of hand here. The collection of all joint probability distribution of arbitrary length m, on all Borel sets and for all times is called the family of finite joint probability distributions. For a given X_t, they satisfy the *Kolmogorov consistency conditions*.

So far this is not helping us, because we still assume knowledge of how the process X acts on Ω. This is where the *Kolmogorov extension theorem* comes to the rescue. It guarantees us that, given a family of finite joint probability distributions satisfying the abovementioned consistency conditions, there exists a corresponding stochastic process X on some sample space Ω. Thus we may equivalently work with our stochastic process in terms of the joint probability distributions, which is what we do in practice.

C.1 Markov processes

A Markov process is essentially a stochastic process with acute memory loss. More formally, the condition for a process to be Markovian may be stated as follows

$$\mathbb{P}(X(t) \in B | X(t_m) = x_m, \ldots, X(t_1) = x_1) = \mathbb{P}(X(t) \in B | X(t_m) = x_m) \tag{C.4}$$

An analogous statement holds for the conditional probability densities, i.e.

$$p(x, t|x_m, t_m; \ldots; x_1, t_1) = p(x, t|x_m, t_m) = T(x, t|x_m, t_m) \tag{C.5}$$

We will refer to $T(x, t|x_m, t_m)$ as the propagator. The neat thing about Markov processes is that given the propagator and an initial density, the density at arbitrary time may be constructed:

$$p(x, t) = \int dx' \; T(x, t|x', t_0) p(x', t_0) \tag{C.6}$$

Similarly any given joint probability distribution may be generated with the help of the propagator. Importantly, Markov processes satisfy the Chapman-Kolmogorov equation:

$$T(x', t'|x, t) = \int dx'' \; T(x', t'|x'', t'') T(x'', t''|x, t) \tag{C.7}$$

where $t < t'' < t'$. Thus a Markov process can be completely specified by a propagator satisfying the CKE and an initial density $p(x, t_0)$. Finding solutions to the CKE is not an easy task, but it helps to cast it in differential form:

$$\partial_t T(x, t|x', t') = A(t) T(x, t|x', t') \tag{C.8}$$

Where $A(t)$ is the infinitesimal time evolution operator under the process, given as

$$A(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int dx' \; \left[T(x, t + \Delta t|x', t) - \delta(x' - x) \right] \tag{C.9}$$

For a given process, the behaviour of the propagator for infinitesimal times is usually easy to formulate. It may then be plugged into the differential CKE, which may be integrated directly to yield the propagator itself.

An example of this is provided by considering a jump process. To this end, we consider the transition rate defined as $W(x|x', t)\Delta t = T(x, t + \Delta t|x', t)$. The transition rate $W(x|x', t)$ is thus to be interpreted as the probability that the system in state x' jumps to state x at time t. Integrating this over all possible final states, we obtain the probability that the system performs a jump at time t:

$$\Gamma(x', t) = \int dx \; W(x|x', t) \tag{C.10}$$

The infinitesimal behaviour of the propagator is now evident:

$$T(x, t + \Delta t|x', t) = W(x|x', t)\Delta t + (1 - \Gamma)\Delta t \delta(x - x') + \mathcal{O}(\Delta t^2) \tag{C.11}$$

Inserting this into Eqn. (C.9) and Eqn. (C.8) we find the *master equation*:

$$\partial_t T(x, t|x', t') = \int dx'' \; \left[W(x|x'') T(x'', t|x', t') - W(x''|x, t) T(x, t|x', t') \right] \tag{C.12}$$

The master equation may then be used to derive the Fokker-Planck equation. To do so, one writes $W(x'|x,t) = f(x',y,t)$ where $y = x - x'$. Assuming that we are dealing with a diffusion process, the system should be most likely to flow to nearby states, i.e. we may take f to be sharply peaked around $y = 0$. Performing a series expansion we obtain the Fokker-Planck equation

$$\partial_t p(x,t) = -\partial_{x_i}\big[g_i(x,t)p(x,t)\big] + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\big[D_{ij}(x,t)p(x,t)\big] \tag{C.13}$$

where

$$g_i(x,t) \equiv \int dy\ y_i f(x,y,t) \qquad\qquad D_{ij}(x,t) \equiv \int dy\ y_i y_j f(x,y,t) \tag{C.14}$$

Setting $g = 0, D = 1$ and choosing $p(x,t_0) = \delta(x)$ we obtain the *Wiener process* $W(t)$, with propagator

$$T(w,t|w',t') = \frac{1}{\sqrt{2\pi(t-t')}} \exp\left(-\frac{(w-w')^2}{2(t-t')}\right) \tag{C.15}$$

Simulating the process is now relatively straight-forward. To make things easy, we discretize the time axis into intervals of equal length Δt. Given a value of the process at time t_k, $W(t_k) = w(t_k)$, the probability distribution from which we draw $W(t_k + \Delta k)$ will have mean $\mu = w(t_k)$ and variance Δt. The process may then be generated as follows:

$$W(t_{k+1}) = w(t_k) + \sqrt{\Delta t} \cdot X \tag{C.16}$$

where X is a random variable drawn from the standard normal distribution, i.e. zero mean and unit standard deviation. Note that we have made use of a somewhat cheap trick in multiplying in the desired standard deviation; it is not immediately evident that $W(t_{k+1})$ follows a normal distribution with variance Δt. This is the case however, for the following reason. We recall that a sum of (independent) normal variables follows a normal distribution.[1] Thus we know that $\sqrt{\Delta t} \cdot X$ follows a normal distribution. Furthermore, since $E[\sqrt{\Delta t}X] = \sqrt{\Delta t}E[X]$, the variance and mean of this distribution is given. Given that the mean and variance is sufficient to completely characterize a normal distribution, $W(t_{k+1})$ is normally distributed.

C.2 Filtered probability spaces

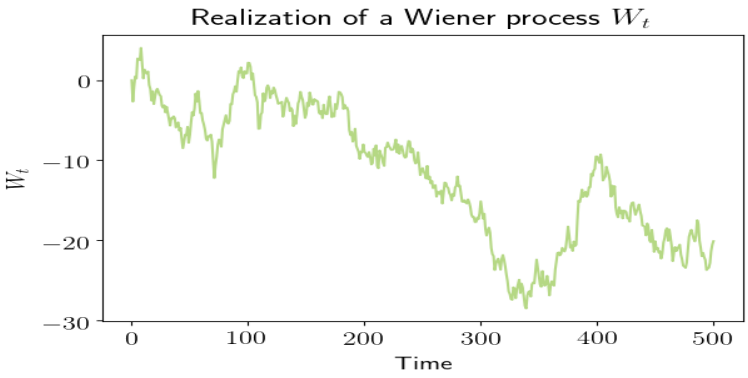
Central to stochastic processes is the idea that they gradually accumulate information. Consider a process X_t on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; expectation values taken with respect to X_t should presumably only depend on outcomes already attained by the process, or in other words only on the information available at the time. This information is typically represented by various σ-fields. For example, consider the throw of a die, with sample space given by $\Omega_1 = \{1,2,3,4,5,6\}$. The least possible information we can know about the die throw is simply that it occurs. We represent this by the σ-field $g_0 = \{\emptyset, \Omega_1\}$. Suppose we also know that the die turned up either even or odd. This increase in information is represented by a finer σ-field, namely $g_1 = \{\emptyset, \Omega_1, \{1,3,5\}, \{2,4,6\}\}$.

The information accumulated by the process X_t is similarly encoded by the σ-algebra it generates, denoted by $\sigma(\{X_t\})$ or just $\sigma(X_t)$. Formally, it is the *smallest* possible σ-algebra such that X_t remains measurable.[2] Practically, this means that $\sigma(X_t)$ contains no more information

[1] This may be quickly proven using e.g. the characteristic function.

[2] A function X_t between two measure spaces $(\Omega, \mathcal{F})$, $(\Delta, \mathcal{B})$ is measurable if $X_t^{-1}(b) \in \mathcal{F}$ for any $b \in \mathcal{B}$.

Figure C.1: Realization of a Wiener process. Note that although it pretends to be a continuous realization it is discrete, but interpolated between values of $w(t_k)$.



than it should. To illustrate, consider the case where X_t represents N dice thrown at times $t = 0, 1, \ldots, N$. We define X_t on the space $(\Omega_1^N, \mathcal{F}, \mathbb{P})$, where Ω_1 is as defined in the previous paragraph. At $t < 0$, our state of knowledge is again represented by the σ-field $g_0 = \{\emptyset, \Omega_1^N\}$. At $0 < t < 1$, we know the value of one die. Consequently, our state of knowledge is now represented by the field $g_1 = \{\emptyset, \Omega, \{1\} \times \Omega_1^{N-1}, \{2\} \times \Omega_1^{N-1}, \ldots, \{6\} \times \Omega_1^{N-1}\}$.[3] The remaining throws generate successively finer σ-fields, e.g. $g_0 \subset g_1 \cdots \subset g_{N-1} = \mathcal{F}$. As it turns out, the family $\mathbb{G} = \{g_i \mid i = 1, 2, \ldots, N-1\}$ is an example of a *filtration*. Formally, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration on it is any family of sub-σ-fields[4] of $\mathcal{F}$, provided they are strictly "growing", e.g.

$$\mathbb{G} = \{g_t\}_{t \in T},\ g_s \subset g_t \subset \mathcal{F},\ 0 \leq s < t < \max(T) \tag{C.17}$$

where T is an arbitrary index set. The filtration generated by a process is referred to as its *natural* filtration. A probability space with a filtration included is what constitutes a filtered probability space, e.g. $(\Omega, \mathcal{F}, \mathbb{G}, \mathbb{P})$. Having now constructed a framework for encoding varying levels of knowledge, we now turn to the issue of conditioning expectation values on this information. We begin by simply stating the definition. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a σ-field g generated by a partition[5] $(p_i)_{i \in I}$ of $\mathcal{F}$ and a random variable X_t, the expectation of X_t conditioned on g is defined to be

$$\mathbb{E}[X_t|g](\omega) = \sum_{i \in I} \mathbf{1}_{p_i}(\omega) \frac{1}{\mathbb{P}(p_i)} \int_{p_i} X_t d\mathbb{P} \tag{C.18}$$

where $\mathbf{1}_{p_i}$ is the indicator function for the set p_i and ω is an arbitrary element of Ω. We will typically suppress the argument for notational convenience. This conditional expectation is a generalization of the regular expectation value we are used to. In particular, when g provides no additional information(i.e. it is trivial, $g = \{\emptyset, \Omega\}$) it is clear that $\mathbb{E}[X|g] = \mathbb{E}[X]$. In general, the conditional expectation $\mathbb{E}[X|g]$ is a random variable and represents the best guess as to the value of X_t given the information encoded by g. Let us again give a simple example to illustrate.

[3] To stick with the previous example, we might have chosen an experiment wherein a particularly sadistic mathematician only reveals whether the die was odd or even; this would generate a field similar to the one we had.

[4] With possibly obvious meaning, e.g. if a, b are σ-fields and a is a sub-σ-field of b, then $a \subset b$.

[5] That is, $\mathcal{F} = \cup_{i \in I} p_i = \mathcal{F}$ and $p_i \cap p_j = \emptyset$ if $i \neq j$.

Consider again the throw of a single (fair) die, let X_t be the value of the die and suppose g encodes whether it was even or odd. Then

$$\mathbb{E}[X_t|g] = \mathbf{1}_{g_{\text{odd}}} \frac{1}{1/2}\left(\frac{1}{6}+\frac{3}{6}+\frac{5}{6}\right) + \mathbf{1}_{g_{\text{even}}} \frac{1}{1/2}\left(\frac{2}{6}+\frac{4}{6}+\frac{6}{6}\right) \tag{C.19}$$

$$= \mathbf{1}_{g_{\text{odd}}} 3 + \mathbf{1}_{g_{\text{even}}} 4 \tag{C.20}$$

Note that if we condition this result on the trivial σ-field, we recover $\mathbb{E}[X_t]$. If on the other hand we condition it on g one more time, it has no effect. These are manifestations of general properties of the conditional expectation; let us list a few:

(i) If X_t is g-measurable, then $\mathbb{E}[X_t|g] = X_t$. This is intuitively clear - if the event X_t is an element of g,[6] then we have information that it occured. More precisely, if X_t is g-measurable, then it is constant on the elements of the partition $(p_i)_{i\in I}$ generating g. Considering the action of $E[X_t|g]$ on arbitrary elements $\omega \in \Omega$, the result follows.

(ii) Let $g_1 \subset g_2 \subset \mathcal{F}$. Then $\mathbb{E}[\mathbb{E}[X_t|g_1]|g_2] = \mathbb{E}[X_t|g_1]$. This is immediately evident, as $\mathbb{E}[X_t|g_1]$ is g_1-measurable and consequently also g_2-measurable. Property (i) then implies the result. But it is perhaps not so intuitive - even though we condition on the finer algebra g_2 last, we do not get a better prediction for X_t.

(iii) If on the other hand we condition on the finer algebra first, then $\mathbb{E}[\mathbb{E}[X_t|g_2]|g_1] = \mathbb{E}[X_t|g_1]$. This is more intuitive - by conditioning on the coarser algebra last, we throw information away. Showing this property is a tad more work; see the book of Shreve for details.[79] The property is often referred to as the *tower rule*.

(iv) If g_i is *any* σ-algebra, then $\mathbb{E}[\mathbb{E}[X_t|g_i] = \mathbb{E}[X_t]$. This follows immediately from Eqn. (C.18).

(v) If Y_t is g-measurable. Then $\mathbb{E}[X_tY_t|g] = Y_t\mathbb{E}[X_t|g]$. This is also easily shown in a similar fashion to Property (i) - as Y_t is g-measurable, it is constant on the corresponding partition $(p_i)_{i\in I}$.

The conditional expectation has many other properties, but this will suffice for our purposes. Indeed, our original problem is now solved: given a process X_t, we ensure that expectations taken with respect to it use only information available at time s by conditioning on the natural filtration g_s, e.g. $\mathbb{E}[\cdot|g_s]$.[7] Note also that X_t is always g_t-measurable, consequently it is also g_w-measurable for $w \geq t$. By Property (i) above, this ensures that X_t does not depend on its future, since $\mathbb{E}[X_t|g_w] = X_t$. In other words, X_t is non-anticipating. This is an example of the more general concept of *adapted* processes.[8]

In fact, a certain kind of adapted process is of central importance to us in the context of reduction models; they are called *martingales*. Concretely, given a process X_t adapted to a filtration (g_s), we say that it is martingale if it satisfies

$$\mathbb{E}[X_t|g_s] = X_s \qquad\qquad 0 \leq s < t \tag{C.21}$$

and is finite, e.g. $\mathbb{E}[|X_t|] < \infty$. In words, if the best possible estimate of the future outcome of X_t is given by its present value, then the process is martingale. Note that this does not follow immediately from adaptability; even though X_t is (g_s)-adapted, it does not mean that X_t is g_s-measurable for $s < t$.

[6] We are being deliberately sloppy to avoid clutter, we mean more precisely that $X_t^{-1}(b) \in g$ for some set b in the image of X_t.

[7] Of course we could also condition on some other filtration, if additional information was made available through other means.

[8] Given a filtration (h_t) and a process Y_t, we say that Y_t is adapted to the filtration (h_t) if it is h_t-measurable.

Appendix D

Stochastic Calculus

Let us consider a stochastic process X_t which is, in some sense, the integral of a Wiener process W_t.

$$X_t = X_{t_0} + \int_{t_0}^{t} \mu(X_t, t)\ dt + \int_{t_0}^{t} \sigma(X_t, t)\ dW_t \tag{D.1}$$

Where μ and σ are arbitrary functions, typically referred to as the drift and diffusion coefficients respectively. The last term cannot be interpreted as a regular Riemann integral, as the measure dW_t is discontinuous. In fact, since it is also stochastic it generalizes the Riemann-Stieltjes integral. In trying to compute the integral, however, we encounter an ambiguity. Consider its discretization

$$\int_{t_0}^{t} \sigma(X_t, t)\ dW_t = \lim_{n\to\infty} \sum_{i=1}^{n} \sigma(X_\tau, \tau)(W_{t_{i+1}} - W_{t_i}) \tag{D.2}$$

where $\tau = (1-\lambda)t_{i+1} + \lambda t_i,\ \lambda \in [0,1]$. For a continuous measure, the choice of λ is irrelevant. But for a discontinuous measure like we have here, the value of the integral depends on λ. There are two common choices: if we pick $\lambda = 1$, we are led to what is called Ito calculus. If we pick $\lambda = 1/2$, we are led to Stratonovich calculus.[1]

To illustrate the difference between Ito and Stratonovich calculus, let us compute the stochastic

[1] The choice here is not arbitrary, but depends on the problem one is modelling. In finance for example, one sticks to Ito calculus as it is non-anticipating, i.e. to compute $f(t_i)$, we only need knowledge of $f(t_j), t_j < t_i$. In certain physical problems, Stratonovich may be preferred.

integral

$$\int_{t_0}^{t} W\ dW = \lim_{n\to\infty}\sum_{i=1}^{n} W((1-\lambda)t_{i+1}+\lambda t_i)\big(W_{t_{i+1}}-W_{t_i}\big) \tag{D.3}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}(W_{t_i}+(1-\lambda)(t_{i+1}-t_i)\dot{W}_{t_i})\big(W_{t_{i+1}}-W_{t_i}\big) \tag{D.4}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}(W_{t_i}+(1-\lambda)(W_{t_{i+1}}-W_{t_i}))\big(W_{t_{i+1}}-W_{t_i}\big) \tag{D.5}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}\big(W^2_{t_{i+1}}-W^2_{t_i}\big)-\left(\lambda-\frac{1}{2}\right)\big(W_{t_{i+1}}-W_{t_i}\big)^2 \tag{D.6}$$

$$= \frac{1}{2}W_t^2-\left(\lambda-\frac{1}{2}\right)\lim_{n\to\infty}\sum_{i=1}^{n}\big(W_{t_{i+1}}-W_{t_i}\big)^2 \tag{D.7}$$

$$= \frac{1}{2}W_t^2-\left(\lambda-\frac{1}{2}\right)t \tag{D.8}$$

The next to last line deserves some elaboration. We discussed Wiener processes briefly in Appendix C. In particular, using the propagator for the Wiener process(Eqn. (C.15)) we find the expectation and auto-correlation to be

$$E[W_t] = \int dw\ wT(w,t|0,0) = 0 \tag{D.9}$$

$$E[W_tW_{t'}] = \int dw_1dw_2\ w_1w_2T(w_2,t|w_1,t')T(w_1,t'|0,0) = \min(t,t') \tag{D.10}$$

Let us then consider the square of an increment $dW_t = W_{t+dt}-W_t$. Its expected value is

$$E[dW_t^2] = E[W^2_{t+dt}-2W_{t+dt}W_t+W_t^2] = t+dt-2t+t = dt \tag{D.11}$$

This isn't a big surprise since we already knew that $dW_t \sim N(0,dt)$. Let us also consider the variance:

$$\mathrm{Var}[dW_t^2] = E[dW_t^4]-E[dW_t^2]^2 = 3dt^2-dt^2 = 2dt^2 \tag{D.12}$$

Where the fourth power is again easily computed using the propagator. Generally, denoting $X_t = dW_t$, we have

$$E[X_t^{2n}] = \frac{1}{\sqrt{2\pi dt}}\int x^{2n}\exp\left[-\frac{x^2}{2dt}\right] = \frac{1}{\sqrt{2\pi dt}}\frac{d^n}{d(1/2dt)^n}\sqrt{2\pi dt} = (2n-1)!!dt^n \tag{D.13}$$

Passing to the continuum limit all higher moments vanish, and $dW_t^2 = E[dW_t^2] = dt$. Summing all these squared increments[2] we get the last line of the calculation above. Note also that this means that dW_t is of order $\sqrt{dt}$.

Returning to Eqn. (D.8), we see that it depends on λ as claimed. In particular, the choice $\lambda = 1/2$ gives us what we would expect from regular calculus. Thus it can be an advantage to use the Stratonovich formalism in computations, as one is less prone to make mistakes.

[2]The sum of squared increments is roughly equal to the quadratic variation $[W]_t$ of the process W_t. To be precise, the quadratic variation does not specify a particular partition of time, but only demands that the limit is taken in which the size of the largest interval goes to zero. Quadratic covariation $[X,Y]_t$ is analogously defined.

The stochastic process we have considered(Eqn. (D.1)) can also be cast in differential form, as a stochastic differential equation. To indicate whether we intend to take it in the Ito or Stratonovich sense, one typically uses different notation for the two cases:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t \qquad \text{Ito} \tag{D.14}$$

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t) \circ dW_t \qquad \text{Stratonovich} \tag{D.15}$$

Very often, SDE's are given in Ito form and it is then useful to know how to convert between the two. Consider the Ito process given by Eqn. (D.1). Switching to the Stratonovich interpretation only affects the last term, so we consider it in isolation. Furthermore, it is (perhaps) less confusing to consider the reverse direction, thus we discretize it according to Eqn. (D.2) in the Stratonovich interpretation. Suppressing arguments for notational convenience, we get

$$\int \sigma \circ dW = \lim_{n\to\infty} \sum_{i=1}^{n} \sigma\left[X\left(\frac{t_{i+1}+t_i}{2}\right), \frac{t_{i+1}-t_i}{2}\right](W_{t_{i+1}} - W_{t_i}) \tag{D.16}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\sigma + \frac{1}{2}\frac{d\sigma}{dt}dt + \frac{1}{2}\frac{d\sigma}{dX}dX\right]dW \tag{D.17}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\sigma dW + \frac{1}{2}\frac{d\sigma}{dX}dW(\mu dt + \sigma dW)\right] \tag{D.18}$$

$$= \int \sigma dW + \frac{\sigma}{2}\partial_X \sigma dt \tag{D.19}$$

Where we have discarded all terms of order higher than dt. Thus we see that we can convert an SDE from Ito to Stratonovich form(or vice versa) by subtracting(or adding) an appropriate drift term, namely $\sigma\partial_X\sigma dt/2$. In the next-to-last line, we have made use of the quadratic variation of the Wiener process, e.g. $[dW]_t = dt$.

As a simple example of the above conversion algorithm, consider the process $X_t = W_t^2 - t$. From above, we know that the corresponding Ito SDE is $dX_t = 2W_t dW_t$. Thus we identify $\sigma = 2W_t = 2\sqrt{X_t + t}$, and $dX_t = \sigma/2 d\sigma$. The drift term is then

$$\frac{\sigma}{2}\partial_X\sigma dt = \frac{\sigma}{2}\frac{2}{\sigma}dt = dt \tag{D.20}$$

So that the corresponding Stratonovich SDE is $dX_t = 2W_t \circ dW_t - dt$. As expected, it integrates using regular calculus rules to $X_t = W_t^2 - t$.

Using this conversion, we are able to do stochastic calculus with arbitrary functions given in Ito form. But for small calculations, this is an inconvenience, and it is simpler to use Ito calculus directly. Suppose we want the differential of some function $f(X_t, t)$ to first order, where X_t is assumed to satisfy Eqn. (D.14). It is *not* simply

$$df \neq \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX \tag{D.21}$$

because X_t contains dW, which we know is of order $\sqrt{dt}$. Thus we get

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}dX^2 \tag{D.22}$$

$$= \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial X} + \frac{1}{2}\sigma\frac{\partial^2 f}{\partial X^2}\right)dt + \sigma\frac{\partial f}{\partial X}dW \tag{D.23}$$

This result is known as *Ito's lemma*, and it is the stochastic equivalent of the chain rule. Note again that in these types of expansions, we take the notation dX^2 to imply the quadratic variation, e.g. $dX^2 = [dX]_t$. Using Ito's lemma, we can derive the corresponding product rule for two arbitrary functions of the stochastic process X_t. For notational convenience, we denote derivatives with respect to X_t by f'.

$$d(fg) = gf'dX + fg'dX + f'g'dX^2 + \frac{1}{2}gf''dX^2 + \frac{1}{2}fg''dX^2 \tag{D.24}$$

$$= gdf + fdg + f'g'dX^2 \tag{D.25}$$

$$= gdf + fdg + [df, dg]_t \tag{D.26}$$

Another tool we will need is the *stochastic exponential* of a process, $\mathcal{E}(X_t)$, where we assume that X_t is nice in the sense that its quadratic variation is continuously differentiable. Its usefulness arises when we consider SDE's of the type[3]

$$dY_t = Y_t dX_t \implies \frac{dY_t}{Y_t} = dX_t \tag{D.27}$$

We might expect the left hand side to integrate to $\ln Y_t$, but of course it does not. In fact, using Ito's lemma

$$d\ln Y_t = \frac{1}{Y_t}dY_t - \frac{1}{2Y_t^2}d[Y]_t = dX_t - \frac{1}{2}d[X]_t \tag{D.28}$$

So we see that, as always, there is an extra drift term hanging around that we must eliminate. Integrating and exponentiating, and assuming $X_0 = 0$, we get

$$Y_t = \exp\left[X_t - \frac{1}{2}[X_t]\right] \equiv \mathcal{E}(X_t) \tag{D.29}$$

Let us compute the differential of Y_t to make sure it satisfies Eqn. (D.27):

$$dY_t = Y_t d\left(X_t - \frac{1}{2}[X_t]\right) + \frac{1}{2}Y_t d\left[X_t - \frac{1}{2}[X_t]\right]_t \tag{D.30}$$

$$= Y_t dX_t \tag{D.31}$$

Where continuity of $d[X]_t$ ensures $[[X]_t]_t = 0$.

[3] The process satisfying the following SDE is an example of what is called geometric Brownian motion.

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