MHD and Disruptions

Ed Threlfall

November 2025

1 Introduction

2 Thursday 27 November 2025

2.1 The Lane-Emden equation in 2D - a simple free-boundary problem

All numerical results shown here are from Firedrake. For scripts see the public repo https://github.com/ethrelfall/MHD-and-Disruptions.

This is not an attempt to do anything clever, it's a starting point for solving a free-boundary problem ...

The Lane-Emden equation is used to find equilibria for stellar structures (attractive gravity vs. repulsive gas pressure). See wiki article for an introduction. The 2D case of this equation is chosen here as a simpler proxy for the Grad-Shafranov (GS) equation on the basis that both equations can be posed as a free-boundary problem. Matter density in the problem discussed here is very similar to current density in the GS equation.

The Lane-Emden equation in 2D (LE2D) is, in the standard 2D plane polars,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = \begin{cases} -\psi^n & \psi > 0 \\ 0 & \psi \leq 0 \end{cases} \tag{1}$$

n is called the polytropic index and it comes in from the equation of state - this is unimportant here; the only thing that matters is that there is a term on the right-hand side representing a density-like quantity.

Here this will be taken in non-circularly symmetric form and with external source j as

$$\nabla^2\psi = \begin{cases} -\psi^n + j & \psi > 0 \\ 0 + j & \psi \leq 0 \end{cases} \tag{2}$$

This will be solved by Picard iteration. Do an initial guess, solve, recompute the RHS and the Dirichlet BC (from the Green's function) based on the new solution, iterate. The matter boundary and the amount of matter in ψ will be controlled by using a form factor given by

$$\frac{1 + \tanh \frac{\psi}{\psi_0}}{2}, \tag{3}$$

where ψ_0 is a scale, in practice of order 1%, and also by constraining the total amount of matter using a Lagrange multiplier i.e. the equation to be solved is simply a constrained form of

$$\nabla^2\psi = -\psi^n \, \frac{1 + \tanh \frac{\psi}{\psi_0}}{2} + j. \tag{4}$$

Note that the Green's function in 2D, used to construct the boundary data, is simply

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}'|. \tag{5}$$

2.2 Analytic test - no external source

Script: `LaneEmden_iterative_constrained_analytic.py`.

Taking $n = 1$, $j = 0$, a circular-symmetric solution to Eq.4 is

$$\psi = \begin{cases} J_0(r) & r < r_0 \\ \frac{M}{2\pi} \ln \frac{r_0}{r} & r \geq r_0 \end{cases} \tag{6}$$

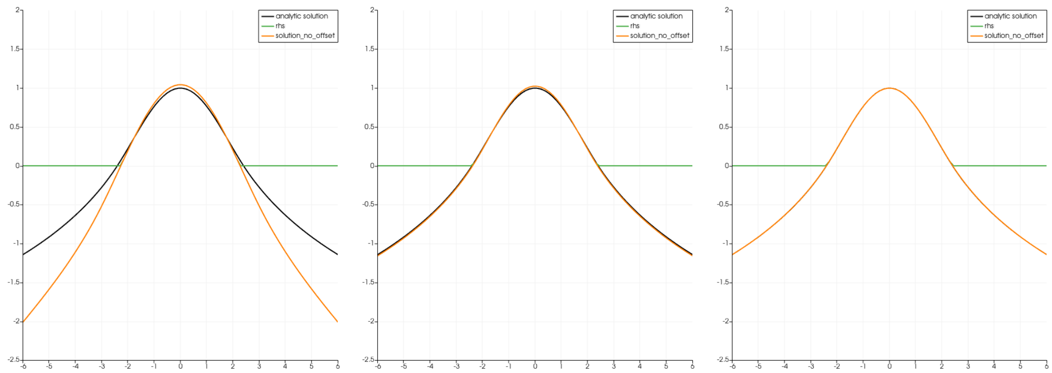

Figure 1: The radial profiles of ψ (orange), and the right-hand side (green), compared to the analytic solution (black), for the analytic test described in the text. The plots are after one, two, and 10 Picard iterations (L-R).

where r_0 is the first zero of the Bessel function $J_0(r)$, $r_0 \approx 2.4048$, and M is the total matter associated to the solution i.e.

$$M = 2\pi \int_0^{r_0} r J_0(r) dr \approx 7.8443 \tag{7}$$

The initial guess given to the script is, for no particular reason other than that this is a solution to the 3D $n = 1$ Lane-Emden equation,

$$\psi = \begin{cases} \sin r & r < \pi \\ \frac{\pi}{r} - 1 & r \geq \pi \end{cases} \tag{8}$$

Using the iterative procedure outlined above, the solution converges to the analytic answer in a handful of iterations (Fig.1).

This tests the Picard iteration but it does not test the implementation of external sources j, which will be tested in the next subsection.

2.3 Method of manufactured solutions test with external source

Script: `LaneEmden_iterative_constrained_mms.py`.

Again taking $n = 1$, the particular circular-symmetric solution chosen for the method of manufactured solutions (MMS) is

$$\psi = \begin{cases} 1 - \frac{r^2}{9} & r < 3 \\ \\ 2\ln\frac{3}{r} & r \geq 3 \end{cases} \tag{9}$$

This is associated to an external source $j = \frac{5-r^2}{9}$ defined in the region $r < 3$.

The total mass associated to the matter is

$$M_p = 2\pi \int_0^3 r\left(1 - \frac{r^2}{9}\right) dr = \frac{9\pi}{2} \tag{10}$$

The total mass associated to the matter and the source is, noting that $-\psi + j = -\frac{4}{9}$ for $r < 3$ (zero elsewhere),

$$M = 2\pi \int_0^3 r\frac{4}{9} dr = 4\pi \tag{11}$$

from which it is seen that the source contains a negative net amount of matter (the source function changes sign in its domain of support, at $r = \sqrt{5}$ - and do not forget that positive matter density corresponds to a negative right-hand-side, see also Newton's equation). Note also that the total matter density in the solution is not a smooth function.

The initial guess is as used in the preceding section.

Using the iterative procedure outlined above, the solution converges to the MMS answer in a handful of iterations (Fig.2), though convergence is not quite as rapid as in the no-source case.

2.4 Test with external source breaking circular symmetry

Script: `LaneEmden_iterative_constrained_2d_pointsources.py`.

This time a bunch of pointlike (actually narrow Gaussian) sources will be added, external to the matter region. It is found that a mixture of positive (attractive) and negative (repulsive)

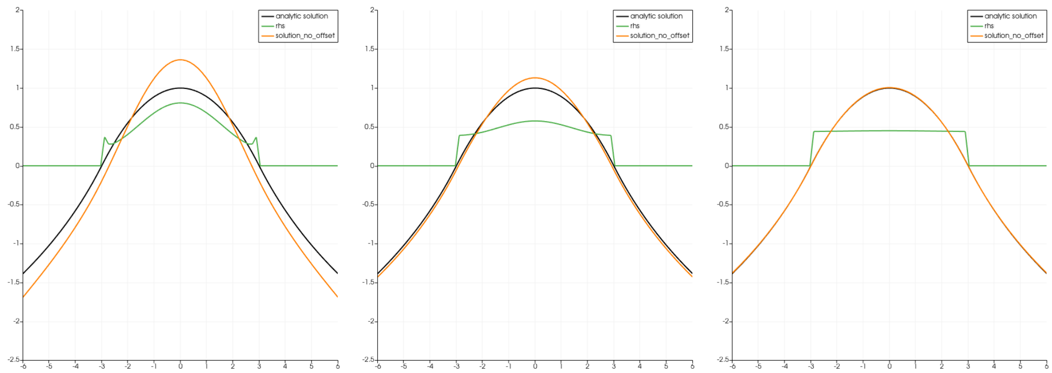

Figure 2: The radial profiles of ψ (orange), and the right-hand side (green), compared to the analytic solution (black), for the MMS test described in the text. The plots are after one, three, and 10 Picard iterations (L-R).

matter density leads to stable matter solutions (e.g. putting in too much positive density alone seemed to harm convergence; too much negative density creates an unstable case where the matter may be squeezed up against the boundary ...).

Eight alternating sign sources were placed in a ring shape surrounding the main plasma density (the shape can be seen in Fig.4).

The initial guess is as used in the preceding section; it is circularly-symmetric. The matter constraint was set to 7.8443 as in the analytic test.

The solution converges very quickly (Fig.3).

It is possible that a 2D MMS test could be constructed ...

2.5 Notes on the script

The examples shown here have the matter density coinciding with the sources. I do not see a problem with this, and the script would still work in the non-coincident case. Note in the GS case the source currents do not coincide with the plasma density.

When evaluating the boundary condition, the Green's function only needs to be worked out at points on the boundary. At present this is hacked in the code by looping over all points and then doing a simple geometrical test that a given point is on the boundary (trivial for a rectangular domain).

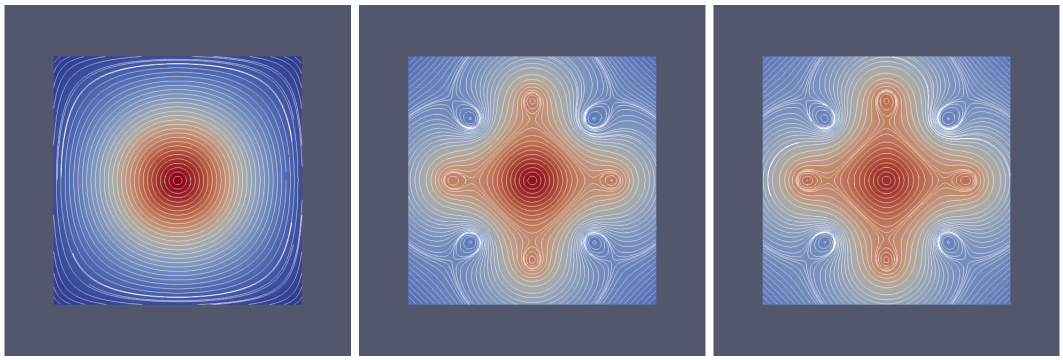

Figure 3: The 2d solution, ψ shown in the colourmap, for the ring of sources problem described in the text. The contours show the magnetic field (that would occur if the matter density is assumed to represent a current density). The plots are after one, two, and 13 Picard iterations (L-R).

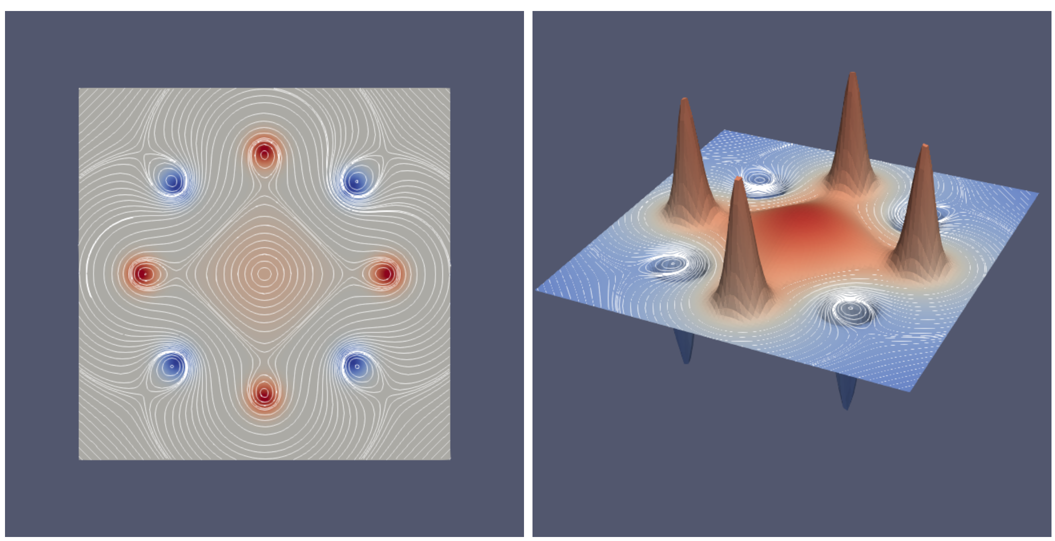

Figure 4: The 2d solution, equation right-hand-side shown in the colourmap, for the ring of sources problem described in the text. The amplitude of total matter is shown in the RHS plot, with the associated magnetic field lines.

When evaluating the boundary condition, the Green's function for the boundary points only ever needs to be computed once. Firedrake does seem to be smart enough to work this out for itself.

I do not know how to change the solver options for the constrained variation case - gives error ...

It is easy to convert the script to do the canonical G-S problem. Change the Laplacian weak form, insert the measure factor necessary to represent the geometry in the poloidal plane, change the Green's function to that associated to the poloidal plane (i.e. use the 'charged ring' solution with the complete elliptic integrals in it).

2.6 TODOs

Add 2D, non-circular-symmetric, MMS test

Test more-nonlinear cases, change polytropic index n to > 1; beware: for some n bounded solutions probably cease to exist (cf. 3D case described in wiki article).

Convert to actual GS problem