Interesting facts re two-stream instability
Ed Threlfall

1 Motivation

Some aspects of the 1D1V two-stream instability are studied: namely, dispersion relations and stability for spatially-homogeneous initial data, and a method for lifting (spatial) potential functions to full kinetic solutions.

The motivation for the former is to understand the stability of a selection of different initial conditions. In fact, the stability curve for some cases resembles the boundary between fixed-point solutions and strange attractors of the Lorenz model, a fact that is noted in the text.

The motivation for the latter is that the dynamics of the Vlasov-Poisson system are most neatly encapsulated in the potential function (in the sequel, $\Phi(x)$) because this is a function of space alone (contrast the DF $f(x,v)$ which depends on both components of phase space). This is also motivated by the fact that the nonlinear attractors of the two-stream instability have a potential function that is quite similar to a simple sinusoid. There is a connection to the dynamics of a rigid pendulum and the 'trapped' particle orbits are analogous to oscillations in which the total energy is bounded by the potential energy needed to reach the top of the pendulum swing.

2 Dispersion relations for linear perturbations

The Vlasov-Poisson system is, for particles in a neutralizing background,

$$\dot{f} + v\frac{\partial f}{\partial x} + E(x)\frac{\partial f}{\partial v} = 0; \tag{1}$$

$$\frac{d^2\Phi(x)}{dx^2} = \omega_P^2\left(\int_{-\infty}^{+\infty}\frac{f dv}{v_N} - 1\right); \tag{2}$$

$$E = -\frac{d\Phi}{dx}; \tag{3}$$

$$\tag{4}$$

where the constant v_N is chosen such that the domain contains no net charge. The constant ω_p specifies the strength of the coupling and is the only parameter in the problem (excepting initial data). Physically this translates into $\propto e\sqrt{n}$ for electron number density n for spatially-homogeneous solutions.

It is clear that any spatially-homogeneous solution $f(v)$ and $E = 0$ is a time-independent solution and one consideration is to check its stability by computing the dispersion relation (this formula applies for an x-independent background and a harmonic perturbation with wavenumber k - which is quantized if on a periodic spatial domain)

$$1 = \omega_p^2\int_{-\infty}^{+\infty}\frac{f(v)dv}{(\omega - kv)^2}. \tag{5}$$

This comes from linearizing the V-P system with e.g. $f = f_0 + \epsilon f_1$ about the zero-electric-field, x-uniform initial distribution; the equation before integrating by parts is

$$1 = -\frac{\omega_P^2}{k}\int_{-\infty}^{\infty}\frac{df_0(u)}{du}\frac{du}{\omega - u} \tag{6}$$

wh. $u = kv$ - note the integral is clearly the Hilbert transform of $\frac{df_0(u)}{du}$.

Note that it does not seem straightforward to obtain a linear eigenvalue problem of the sort one could solve with *Firedrake* (compare linearized Navier-Stokes) to obtain the growth rates ... the integral over v seems to prevent this, is there a way? And is there any sort of analogue of the Lorenz reduction of RB convection to ODEs? Chapman-Enskog?

2.1 Gaussian beams

This is done for the case of two counterpropagating Gaussian beams in https://github.com/ExCALIBUR-NEPTUNE/Documents/blob/main/reports/ukaea_reports/CD-EXCALIBUR-FMS0066-M4.1.pdf. There the initial f_0 is the bi-Gaussian (normed so that the whole system integrates over v to a density of 1) - for which total energy is $v_0^2 + \sigma^2$ (think of beam mean kinetic energy plus thermal energy and note the simple addition of these is due to Gaussianity) -

$$f_0(v) = \frac{1}{\sqrt{8\pi\sigma^2}}\left(e^{-\frac{(x-v_0)^2}{\sigma^2}} + e^{-\frac{(x+v_0)^2}{\sigma^2}}\right) \tag{7}$$

for which the dispersion integral can be done analytically to

$$1 = -\frac{\omega_P^2}{2\sigma^2 k^2}\left(2 - \sqrt{\pi}Ae^{-A^2}\left(\mathrm{erfi}(A) - i\right) - \sqrt{\pi}A'e^{-A'^2}\left(\mathrm{erfi}(A') - i\right)\right) \tag{8}$$

with $A \equiv \frac{\omega - kv_0}{\sqrt{2}\sigma k}$, $A' \equiv \frac{\omega + kv_0}{\sqrt{2}\sigma k}$.

(Note using a single Gaussian, or $v_0 = 0$ in the above, gives, in the small-k limit, the dispersion relation for a warm-plasma Langmuir wave $\omega^2 = \omega_P^2 + 3k^2\sigma^2$ - just use series expansion of the Dawson function.)

The boundary between stable initial data and unstable is found by inspecting where the sign of $\mathrm{Im}(\omega)$ changes. It is found that ω is purely imaginary over the range of interest, and setting $\omega = 0$ in the above gives the stability frontier (Fig.1)

$$\frac{\omega_P}{k\sigma} = \frac{1}{\sqrt{2xD_+(x) - 1}} \tag{9}$$

wh. $x \equiv \frac{v_0}{\sqrt{2}\sigma} \equiv \sqrt{\frac{m_e v_0^2}{2k_B T}}$ and the Dawson function is $D_+(x) \equiv \frac{\sqrt{\pi}}{2}e^{-x^2}\mathrm{erfi}(x)$. The initial data is stable if ω_P is below the value given by the RHS. (Note the cold-beam case can be obtained from the $\sigma \to 0$ limit, stable for $\omega_P < kv_0$.)

(As far as proving that the imaginary solution exists for arbitrarily large ω_P ... one could consider the limit $\omega_P \to 0$ and then one has for $\omega = i\gamma$ and $u \equiv \frac{\gamma}{\sqrt{2}\sigma k}$

$$1 + \sqrt{\pi}e^{u^2}\left(\mathrm{erf}(u) - 1\right) = 0 \tag{10}$$

wh. are the roots ... this is numerically nasty with the exponential of the square ...)

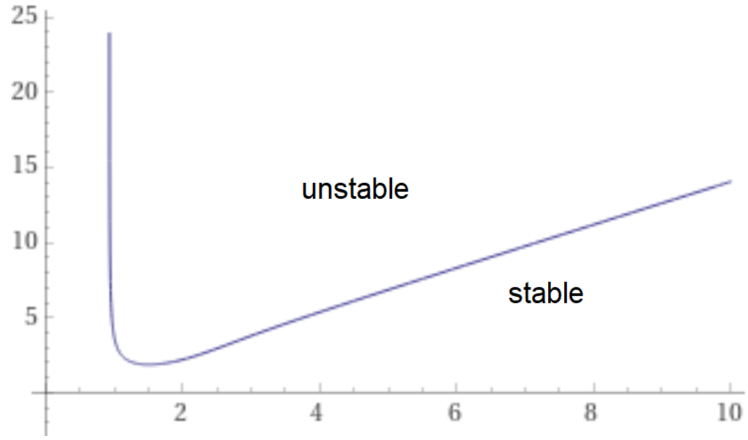

Figure 1: Stability frontier ('phase diagram') for the Gaussian initial condition two-stream instability: vertical axis is $\frac{\omega_P}{k\sigma}$ and horizontal is $\frac{v_0}{\sqrt{2}\sigma}$. Points below the curve represent stable initial conditions and overall scale is set by the Gaussian width, physically the temperature.

This tells that the system is always stable for x below the zero at $2xD_+(x) = 1 \to x \approx 0.92414$ (where $2xD_+(x) < 1$ so the quantity in the radical is negative) and that above this there is a stability boundary where the system is stable for small enough values of the normalized coupling $\frac{\omega_P}{k\sigma}$. This boundary slopes upwards i.e. for large enough normalized initial velocities, the system is more likely to be stable. The picture obtained here is consistent with the expectation for the $\omega_P \to 0$ limit, in which any initial data must be stable as there is no dynamics. This formula also predicts evenly-spaced instability for modes of increasing integer wave number i.e. the $n = 2$ mode requires twice the ω_P value to become unstable as does the $n = 1$ mode - hence it's apparent that the instability is a thing that occurs in the infra-red (meaning long wavelengths).

One should consider plotting this with the axes switched - then it resembles the phase diagrams of the Lorenz model (see Grossmann et al, *Extended phase diagram of the Lorenz model*) in (Ra, Pr) i.e. the coupling ω_P is analogous to the Rayleigh number and the velocity offset v_0 the Prandtl number (all scaled to the system temperature in the V-P case).

The above analysis gives good agreement for the stability bound of the system when simulated in *Nektar++*, predicting instability when $\omega_P > 2.9223$ in that case.

How can this stability conditional on the value of ω_P be reconciled with the stability criterion given by (Oliver - brother of Roger) Penrose's formula, which involves only the initial data and which does not include the value of ω_P? Probably the answer is that the Penrose formula applies in the non-periodic limit where there is no infra-red cut-off and in that case everything with a value of x above the zero at $2xD_+(x) = 1 \to x \approx 0.92414$ is unstable (indeed this condition seems to come out of doing the integral in the Penrose formula). The non-periodic case can also be accessed by plotting the graph for the periodic case and then rescaling the y axis by letting $k \to 0$, the stability curve will collapse to the x axis leaving the unstable region only for $x > 0.92414$.

This system is a sort of 'phase diagram' problem in the x, $\frac{\omega_P}{k\sigma}$ plane ... more parameters could be added to give a UQ problem with a greater number of inputs ... e.g. a $-\lambda v^4$ term in the exponential ...

Asymptotics: flipped axes one ($\frac{\omega_P}{k\sigma}$ along x-axis, like Ra, has $y = 0.92414$ as const asymptote and $x = \sqrt{2}y$. This is much like the analysis for the Lorenz model (see Grossmann et al paper) wh. (σ is Prandtl and $b \equiv \frac{8}{3}$)

$$\sigma_c = \frac{r}{2} - \frac{b+3}{2} \pm \frac{1}{2}\sqrt{r^2 - 2(3b+5)r + (b+3)^3} \tag{11}$$

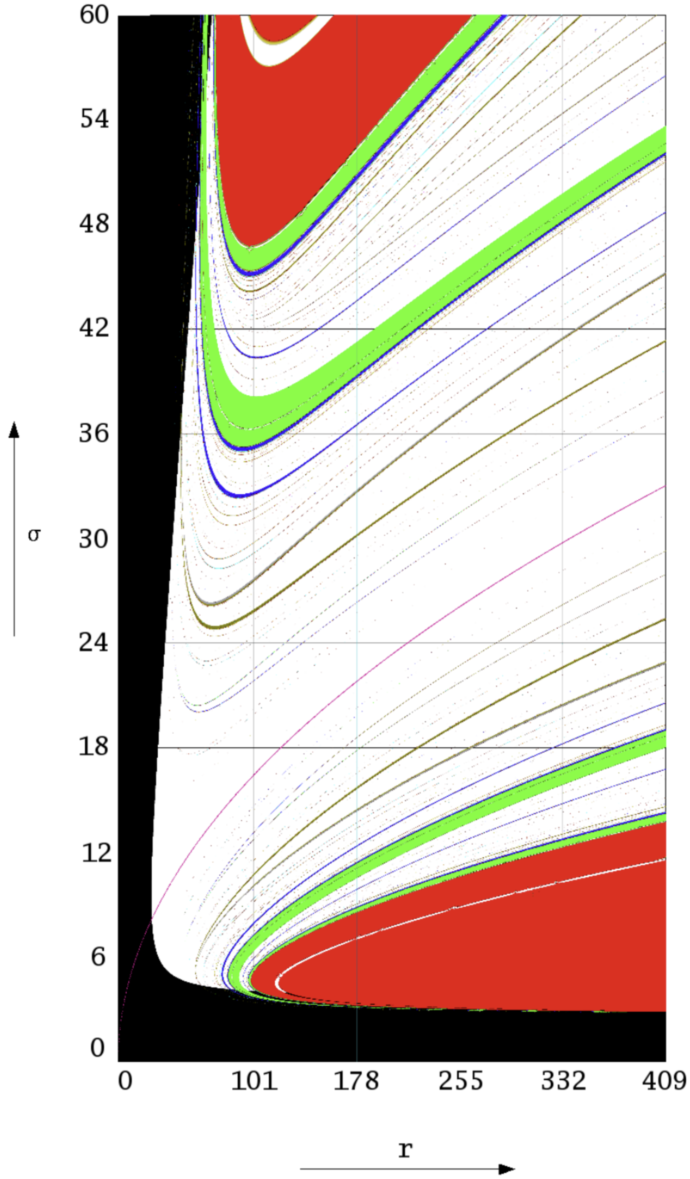

Figure 2: Phase diagram for the Lorenz model taken from paper by Grossmann; r is Rayleigh number and σ is Prandtl number. The black region is where the system settles to a fixed point - compare the stable region of the two-stream instability (flip the axes in Fig.1 and think of the fixed point solutions as corresponding to the stable initial beam conditions).

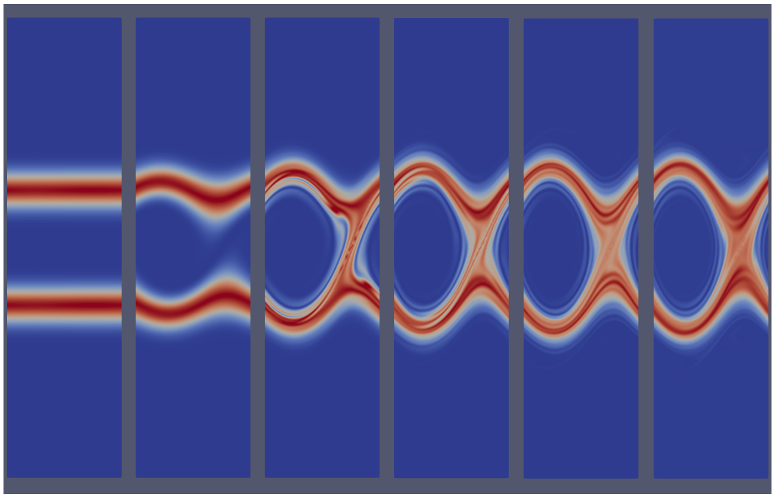

Figure 3: Time-evolution of the two-stream instability as a sequence of frames, taken from a *Nektar++* simulation.

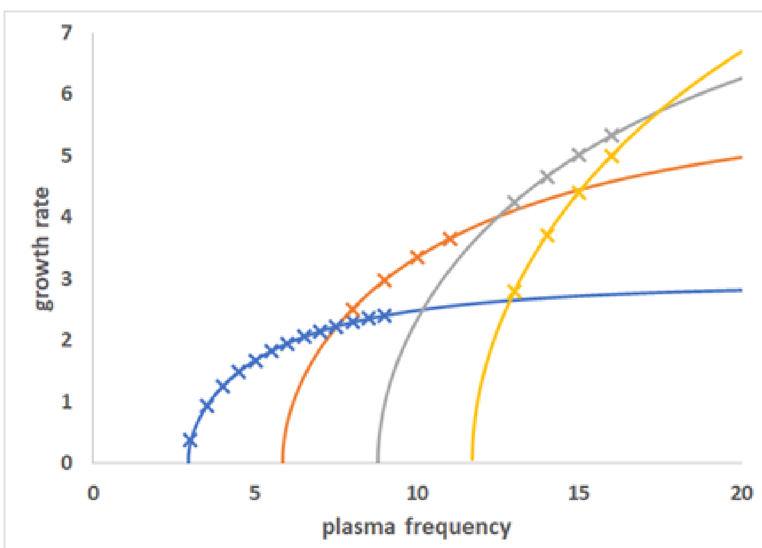

Figure 4: Dispersion relation for the two-stream instability for various values of k obtained from *Nektar++* code (crosses) compared to the analytic solution in Eq.8 (curves). As evident from Eq.9 the ascending spatial harmonics are evenly spaced in ω_P with spacing of 2.9223, which is the value of the stability frontier ω_P for $\sigma = \frac{1}{5}$ and $v_0 = 1$ as used in the *Nektar++* simulation.

with asymptotes $\sigma_u = r - 2(b+2)$ and $\sigma_l = b+1$.

It is worth noting that the Lorenz system obtained from RB convection is still decidedly nonlinear, whereas the linearization of Vlasov-Poisson done here is clearly linear (though it is hard to cast it as a straightforward linear PDE since there is still a velocity-space integral over $f_1(x,v)$).

2.2 Delta-function (i.e. cold) beams

This case uses non-differentiable functions so it is not straightforward to apply the Penrose criterion.

Cold beams: $f_0(v) = \frac{1}{2}\left(\delta(v - v_0) + \delta(v + v_0)\right)$. Dispersion relation is

$$z^2 = \frac{1}{2}\left(1 + 2y^2 \pm \sqrt{1 + 8y^2}\right) \tag{12}$$

wh. $z \equiv \frac{\omega}{\omega_P}$, $y \equiv \frac{kv_0}{\omega_P}$. I think the negative sign for the radical to be the correct choice. Considering stability bounds, setting $z = 0$ one finds a zero at $y = 1$ i.e. $\omega_P = kv_0$ and it seems the system is stable for $\omega_P < kv_0$. Clearly for $k = 0$ (continuum case) the system is always unstable.

2.3 Top hat function beams

Normalized to unity in total, two top hats centred on $\pm v_0$ and both width $= \sigma$. Again, not straightforward to apply the Penrose criterion.

$$z^2 = \frac{1}{2}\left(1 + 2y^2 + 2s^2 - \sqrt{1 + 8y^2 + 16s^2y^2}\right) \tag{13}$$

with $s \equiv \frac{k\sigma}{\omega_P}$.

One obtains the fact that this case is stable up to $\omega_P^2 = k^2(v_0^2 - \sigma^2)$ (solve quadratic and take positive root to ensure recovery of $s = 0$ limit) so again always unstable in the continuum and the width actually makes the system *more* unstable in the sense that if the top hats touch or overlap the system is always unstable even if periodic (weirdly, this means a single top hat centred on zero is always unstable and this is a function with no minimum!). The stability frontier can be expressed in the form

$$\frac{\omega_P}{kv_0} = \frac{\sqrt{x^2 - 1}}{x} \tag{14}$$

wh. $x \equiv \frac{v_0}{\sigma}$.

2.4 Cauchy (or Lorentzian, or Breit-Wigner) beams

This is a bit dodgy as the variance of the system is infinite (so infinite temperature?); the distribution is

$$f_0(v) = \frac{s}{2\pi}\left(\frac{1}{(v - v_0)^2 + s^2} + \frac{1}{(v + v_0)^2 + s^2}\right) \tag{15}$$

and in this case the integration can be done thanks to a formula obtained from Wolfram-alpha (standard contour integral I guess)

$$\int_{-\infty}^{\infty} \frac{dv}{(v^2+s^2)(\omega - kv)^2} = -\frac{\pi}{s(ks - i\omega)^2} \tag{16}$$

and doing a linear transformation on v. The resulting dispersion relation, with ω set zero to find the stability contour, gives

$$\frac{\omega_P}{kv_0} = \frac{x^2+1}{\sqrt{x^2-1}} \tag{17}$$

wh. as usual $x \equiv \frac{v_0}{s}$.

Asymptotics: $y = 1$ and $y = x$, trivial.

This stability contour is very much like that obtained in the Gaussian case, and indeed in the $k = 0$ limit (non-periodic) the system is stable for $s > v_0$. The latter concurs with the non-periodic Penrose formula analysis done on Fitzpatrick's web page. This case shows that Gaussianity is not required for stable solutions and indeed the v_0 limit represents a single Cauchy distribution centred on zero - so the dynamics here, unlike the collisional case, admit multiple stable solutions and not just Maxwellians.

2.5 Pöschl-Teller beams

This case needs the integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{sech}^2(v)dv}{\omega - kv)^2} \tag{18}$$

the result of which is not immediately obvious.

3 Distribution functions for given charge density or potential

To find stationary solutions, let $f(x, v) = f(\mathcal{H})$ where

$$\mathcal{H} = \frac{1}{2}v^2 + \Phi(x). \tag{19}$$

This automatically solves the Vlasov equation to $\dot{f} = 0$ and then the potential is evaluated by inverting the nonlinear Poisson equation.

Note that $\mathcal{H}$ is just the Hamiltonian for a unit mass particle moving in a potential $\Phi(x)$ i.e. a classical oscillator.

The remaining part of the problem is the *linear* equation for the charge density

$$\rho = -\omega_P^2 \left(\int f \frac{dv}{v_N} - 1 \right) \tag{20}$$

It is clear that the charge density function cannot have a nonzero $n = 0$ Fourier component, because there must be zero net charge. However, the potential $\Phi(x)$ may have a constant component as this is compatible with periodicity and it does not affect the dynamics.

Let us seek a solution where there is a maximum value of permitted energy. Thus define the constant Φ_0 to be this maximum and

$$\mathcal{E} \equiv \Phi_0 - \Phi - \frac{1}{2}v^2 \tag{21}$$

(note this automatically satisfies the Vlasov equation by the reasoning presented earlier) so that $\mathcal{E} > 0$ represents allowed values of v i.e. $f > 0$ for $\mathcal{E} > 0$ and $f = 0$ for $\mathcal{E} \leq 0$. Further define $\Psi \equiv \Phi_0 - \Phi$.

Now one has

$$v_N \left(1 - \frac{\rho(x)}{\omega_P^2}\right) = \int_{f>0} f dv = \int_0^\Psi \frac{f(\mathcal{E})d\mathcal{E}}{\sqrt{2(\Psi - \mathcal{E})}}. \tag{22}$$

One has here something of the form

$$g(\Psi) = \int f dv = \int_0^\Psi \frac{f(\mathcal{E})d\mathcal{E}}{\sqrt{(\Psi - \mathcal{E})}} \tag{23}$$

and it is known that this equation has a unique solution for f given by the Abel integral formula:

$$f(\mathcal{E}) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}} \int_0^{\mathcal{E}} \frac{g(t)dt}{\sqrt{\mathcal{E} - t}}. \tag{24}$$

Now ρ can be expressed in terms of Ψ using the Poisson equation (note change of sign)

$$\frac{d^2\Psi}{dx^2} = \rho. \tag{25}$$

Specialising to e.g. $\rho = \cos 2\pi x$ on $[0, 1]$ one has $\Psi = \Phi_0 - \frac{1}{4\pi^2}\cos 2\pi x$ and $\rho = 4\pi^2(\Phi_0 - \Psi)$.

Hence

$$\sqrt{2}v_N \left(1 + \frac{4\pi^2}{\omega_P^2}(\Psi - \Phi_0)\right) = \int f dv = \int_0^\Psi \frac{f(\mathcal{E})d\mathcal{E}}{\sqrt{(\Psi - \mathcal{E})}}. \tag{26}$$

Applying Abel's formula gives

$$f(\mathcal{E}) = \frac{\sqrt{2}v_N}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}} \int_0^{\mathcal{E}} \frac{\left(1 - \frac{4\pi^2\Phi_0}{\omega_P^2} + \frac{4\pi^2}{\omega_P^2}t\right)dt}{\sqrt{\mathcal{E} - t}}. \tag{27}$$

The necessary integrals are

$$\int_0^{\mathcal{E}} \frac{dt}{\sqrt{\mathcal{E} - t}} = 2\sqrt{\mathcal{E}}; \tag{28}$$

$$\int_0^{\mathcal{E}} \frac{tdt}{\sqrt{\mathcal{E} - t}} = \frac{4}{3}\mathcal{E}^{\frac{3}{2}}; \tag{29}$$

giving the DF as, for positive $\mathcal{E}$ (zero otherwise)

$$f(\mathcal{E}) = \frac{\mathrm{d}}{\mathrm{d}\mathcal{E}} \frac{\sqrt{8\mathcal{E}}v_N}{\pi} \left(1 - \frac{4\pi^2\Phi_0}{\omega_P^2} + \frac{8\pi^2\mathcal{E}}{3\omega_P^2}\right) \tag{30}$$

i.e.

$$f(\mathcal{E}) = \frac{\sqrt{2}v_N}{\pi} \left(\frac{1 - \frac{4\pi^2\Phi_0}{\omega_P^2}}{\sqrt{\mathcal{E}}} + \frac{8\pi^2\sqrt{\mathcal{E}}}{\omega_P^2}\right) \tag{31}$$

wh. $\mathcal{E} \equiv \Phi_0 - \Phi - \frac{1}{2}v^2 = \Phi_0 - \frac{1}{4\pi^2}\cos 2\pi x - \frac{1}{2}v^2$. Here Φ_0 is a maximum energy level (free parameter).

Note there is a singularity as $\mathcal{E} \to 0$ but it's an integrable one (cf. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$) ...

It is easy to substitute this solution back into the charge density equation to verify its correctness (as part of this, Φ_0 cancels out once the integration is done as it must as it does not appear in the charge density equation).

Note that $\omega_P \to \infty$ gives back the limit where the charge density does not depend on position (see this by evaluating the integral of f over velocity).

The case where $\Phi_0 = \frac{\omega_P^2}{4\pi^2}$ is special in that it avoids the singularity as $\mathcal{E} \to 0$.

If a lower limit is put on the energy as well by integrating from a rather than 0, all that happens is the relevant (differentiated) integrals are modified to

$$\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}} \int_a^{\mathcal{E}} \frac{dt}{\sqrt{\mathcal{E}-t}} = 2\sqrt{\mathcal{E}-a}; \tag{32}$$

$$\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}} \int_a^{\mathcal{E}} \frac{tdt}{\sqrt{\mathcal{E}-t}} = \frac{2\mathcal{E}-a}{\sqrt{\mathcal{E}-a}}; \tag{33}$$

This gives a singular (but normalizable) solution unless the parameters are chosen to remove it. If that be done then the solution is just $\propto \sqrt{\mathcal{E}-a}$ which is the same solution as before but with Φ_0 replaced by $\Phi_0 - a$.

In conclusion, solutions have been obtained but I have not managed to construct the 'eye' solutions observed in particle and continuum simulations of the two-stream instability.

How can the disconnected breadrack solutions be valid - should have no charge density where f is zero???

Some of this stuff is similar to the BGK paper though in that paper the Abel integral equation is not named as such.

Stability of these solutions could be analyzed as earlier ...

In a more realistic model, collisions (friction) would be expected to cause the distribution to relax to a collision-invariant distribution i.e. a Maxwellian centred on some velocity set by the net momentum of the initial data.