Fredholm Equations with Degenerated Kernels

Recall: $$K u - \lambda \cdot u = f$$

$$\Leftrightarrow \int_a^b k(x,y) \cdot u(y) \cdot dy - \lambda \cdot u(x) = f(x).$$

What about if $k(x,y) = \sum_{j=1}^{n} \alpha_j(x) \cdot \beta_j(y)$?

$$\Rightarrow \int_a^b \left(\sum_{j=1}^{n} \alpha_j(x) \cdot \beta_j(y) \right) \cdot u(y)\, dy - \lambda \cdot u(x) = f(x)$$

$$= \sum_{j=1}^{n} \alpha_j(x) \underbrace{\int_a^b \beta_j(y) \cdot u(y)\, dy}_{c_j} - \lambda \cdot u(x) = f(x) \qquad *$$

Consider that $\alpha_j(x)$ & $\beta_j(y)$ are independent.

$$\sum_{j=1}^{n} \alpha_j(x) \cdot c_j - \lambda \cdot u(x) = f(x) \qquad / (\beta_i, \cdot)$$

$$\sum_{j=1}^{n} (\alpha_j, \beta_i) \cdot c_j - \lambda \cdot \underbrace{(\beta_i, u)}_{c_i} = (f, \beta_i) = f_i$$

$$\Rightarrow (A - \lambda I) \cdot \vec{c} = \vec{f}$$

$$\vec{c} = (A - \lambda \cdot I)^{-1} \cdot \vec{f}$$

How do we get $u(x)$?

From $\circledast$ $\displaystyle\sum_{j=1}^{n} \alpha_j(x) \cdot \underbrace{\int_a^b \beta_j(y) \cdot u(y)\,dy}_{c_j} - \lambda \cdot u(x) = f(x)$

c_j ← known now!

$$\therefore \quad \boxed{u(x) = \frac{1}{\lambda}\left(\left[\sum_{j=1}^{n} \alpha_j(x) \cdot c_j\right] - f(x)\right)}$$

Th: Consider the integral equation $Ku - \lambda \cdot u = f$, where the kernel is given by $K(x,y) = \sum_{j=1}^{n} \alpha_j(x) \cdot \beta_j(y)$ & $\lambda \neq 0$. Let A be the matrix $A = ((\beta_i, \alpha_j))$. If λ is not an eigenvalue of A, then $Ku - \lambda \cdot u = f$ has a unique solution given by $u(x) = \frac{1}{\lambda}\left(\sum_{j=1}^{n} \alpha_j(x) \cdot c_j - f(x)\right)$.

Th: Consider the equation $Ku = f$ with separable kernel $K(x,y) = \sum_{j=1}^{n} \alpha_j(x) \cdot \beta_j(y)$. If f is not a linear combination of the α_j then there is no solution; if f is a linear combination of the α_j, then there are infinitely many solutions.

Th: The integral operator K with separable kernel $K(x,y) = \sum_{j=1}^{n} \alpha_j(x) \cdot \beta_j(y)$ has finitely many nonzero eigenvalues of finite multiplicity, and zero is an eigenvalue of infinite multiplicity

Hilbert – Schmidt theorem

Th: Consider the integral operator $\mathbb{K} \cdot u = \int_a^b K(x,y) \cdot u(y) \cdot dy$ where the kernel $K(x,y)$ is real, continuous, symmetric & not degenerated. Then $\mathbb{K}$ has infinitely many eigenvalues $\lambda_1, \lambda_2, \ldots,$ each with finite multiplicity an they can be ordered as:

$$0 < \ldots \leqslant |\lambda_n| \leqslant \ldots \leqslant |\lambda_2| \leq |\lambda_1|,$$

with $\lim_{n\to\infty} \lambda_n = 0$. Moreover, any square integrable function $f(x)$ can be expanded in terms of the set of orthonormal eigenfunctions $\phi_k(x)$ associated with the eigenvalues as:

$$f(x) = \sum_{k=1}^{\infty} f_k \cdot \phi_k(x),$$

where the series converges to $f(x)$ in $L^2[a,b]$, and the coefficients are given by

$$f_k = (f, \phi_k), \quad k = 1, 2, \ldots$$

Ex: Let $\mathbb{K} \cdot u - \mu \cdot u = f$, where $\mathbb{K}$ is not degenerated, real, continuous & symmetric.

Find $u(x)$:

Let $u(x)=\sum_{k=1}^{\infty} u_k \cdot \phi_k(x)$, $f(x)=\sum_{k=1}^{\infty} f_k \cdot \phi_k(x)$

$$u_k = (u, \phi_k) \quad , \quad f_k = (f, \phi_k)$$

$$\mathbb{K}\, u - \lambda \cdot u = f$$

$$\mathbb{K}\left(\sum_k u_k \cdot \phi_k\right) - \lambda \cdot \sum_k u_k \cdot \phi_k(x) = \sum_k f_k \, \phi_k(x)$$

$$\sum_{k=1}^{\infty} u_k \cdot \mathbb{K}(\phi_k) - \lambda \cdot \sum_k u_k \cdot \phi_k = \sum_k f_k \cdot \phi_k$$

but $\mathbb{K}(\phi_k) = \lambda_k \cdot \phi_k(x)$

$$\sum_k u_k \cdot \lambda_k \cdot \phi_k - \lambda \cdot \sum_k u_k \cdot \phi_k = \sum_k f_k \cdot \phi_k \quad \Big/ (\phi_i, \cdot)$$

$$u_i \cdot \lambda_i - u_i \cdot \lambda = f_i$$

$$u_i = \frac{f_i}{\lambda_i - \lambda}$$

$$\therefore \quad \boxed{u(x) = \sum_{k=1}^{\infty} \frac{\int_a^b f(y) \cdot \phi_k(y)\, dy}{\lambda_k - \lambda} \cdot \phi_k(x)}$$

but, $$u(x) = \sum_{k=1}^{\infty} \frac{\int_a^b f(y) \cdot \phi_k(y)\,dy}{\lambda_k - \lambda} \cdot \phi_k(x)$$

can be written as

$$u(x) = \int_a^b \underbrace{\left(\sum_{k=1}^{\infty} \frac{\phi_k(y) \cdot \phi_k(x)}{\lambda_k - \lambda} \right)}_{\text{Green's function}} \cdot f(y)\,dy$$

Recall:
$$K \cdot u - \lambda \cdot u = f$$
$$(K - \lambda \cdot I) \cdot u = f$$
$$u = (K - \lambda \cdot I)^{-1} \cdot f$$
$$\Updownarrow$$
$$u(x) = \int_a^b G(x,y) \cdot f(y)\,dy$$

Also, if $\lambda = \lambda_m$ & $f_m = (f, \phi_m) = 0$

$$\Rightarrow u(x) = c \cdot \phi_m(x) + \sum_{\substack{k=1 \\ k \neq m}}^{\infty} \frac{f_k}{\lambda_k - \lambda} \cdot \phi_k(x)$$

Green's function

What is it?

① **Mathematically:**

$$u(x) = \int_a^b \underbrace{K(x,y)}_{\text{the Kernel}} \cdot f(y)\,dy$$

② **Physically:**

It is the response of a system when a unit point source is applied to a system.

Ex: $\quad \underline{K} u - \lambda \cdot u = f$

① $\quad u(x) = \int_a^b G(x,y) \cdot f(y)\,dy$

② $\quad \underline{K} \cdot G - \lambda \cdot G = \delta(x,y)$

$$G = \int_a^b G(x,y) \cdot \delta(x,y)\,dy$$

$$= G(x,y)$$

$\delta(x,y) = 0, \quad x \neq y$

$$\int_a^b \delta(x,y)\,dx = 1$$

$$\int_a^b \delta(x,y) f(x)\,dx = \underline{\qquad}\ ?$$

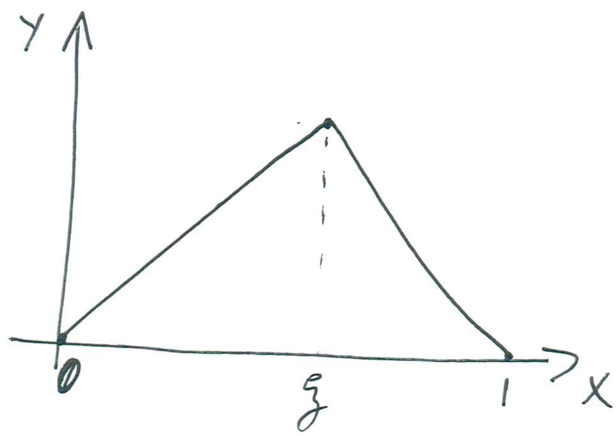
Rule of thumb: Green's function $\approx$ Inverse of differential operator

Ex: $-u''(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0$

$$-g'' = \delta(x, \xi) \Rightarrow g(x) = \tilde{a} \cdot x + \tilde{b}$$

$$g_1(x) = a \cdot x, \quad g(0) = 0, \quad x < \xi$$

$$g_2(x) = b \cdot (1-x), \quad g(1) = 0, \quad x > \xi$$



$$a \cdot \xi = b \cdot (1-\xi)$$

$$\Rightarrow -\int_{\xi-\varepsilon}^{\xi+\varepsilon} u''(x)\,dx = \int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x,\xi)\,dx$$

$$-(u'(\xi+\varepsilon) - u'(\xi-\varepsilon)) = 1$$

$$\lim_{\varepsilon \to 0} -(u'(\xi+\varepsilon) - u'(\xi-\varepsilon)) = -u'(\xi^+) + u'(\xi^-)$$

$$\therefore \quad u'(\xi^+) - u'(\xi^-) = -1$$

$$-b \quad - a \quad = -1 \Rightarrow a = 1 - b$$

$$\& \quad a \cdot \xi = (1-b) \cdot \xi = b \cdot (1-\xi)$$

$$\boxed{b = \xi}$$

$$\Rightarrow \quad a = 1 - \xi$$

$$\therefore \quad \boxed{a = 1 - \xi} \quad \& \quad \boxed{b = \xi}$$

$$\Rightarrow \quad g = (1-\xi)\cdot x, \quad x < \xi$$

$$g = (1-x)\cdot \xi, \quad x > \xi$$

$$\Rightarrow \quad \boxed{g(x,\xi) = (1-\xi)\cdot x \cdot H(\xi - x) + \xi \cdot (1-x) \cdot H(x-\xi)}$$

where

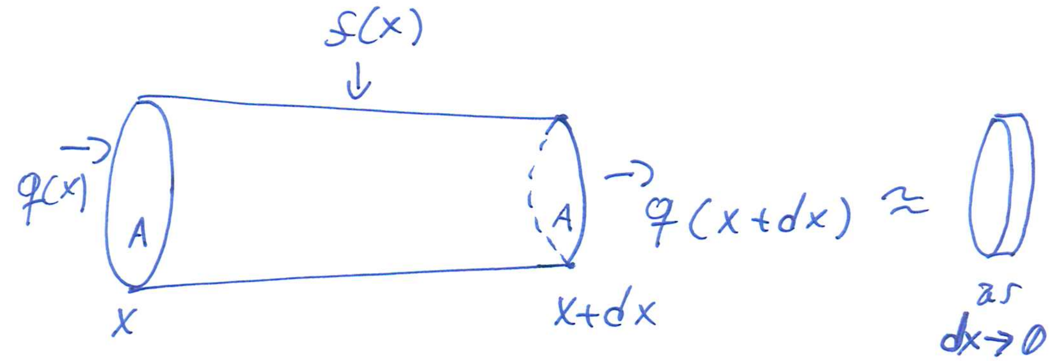
$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad u(x) = \int_0^1 g(x,y)\cdot f(y)\, dy$$

Notice that $\delta(x,y)$ is not a function, it is a distribution but this goes beyond the scope of this course.

Physical Interpretation

"steady state heat flow problem"



$u(x)$ = "temperature at cross section x" (1D problem)

$q(x)$ = "energy flux across the face at x" $\left[\frac{\text{Energy}}{\text{Area}\cdot\text{time}}\right]$

$f(x)$ = "distributed heat source over the length of the bar" $\left[\frac{\text{Energy}}{\text{Volume}\cdot\text{time}}\right]$

$q(x) > 0 \approx$ "flux moves to the right"

by Conservation

$$\underbrace{A\cdot q(x)}_{\text{inflow}} - \underbrace{A\cdot q(x+dx)}_{\text{outflow}} + \underbrace{f(x)\cdot A\cdot dx}_{\text{flow produced inside}} = \underbrace{0}_{\text{steady state}}$$

$$\Rightarrow \quad \frac{q(x+dx) - q(x)}{dx} = f(x)$$

as $dx \to 0$

$$q'(x) = f(x)$$

Assuming Fourier's heat conduction law (the flux is proportional to the negative temperature gradient)

$$q(x) = -k \cdot u'(x)$$

$$\Rightarrow \quad -k \cdot u''(x) = f(x)$$

Note: If this were not a steady state we get the heat equation with a source

$$u_t(x,t) = \underbrace{k \cdot u_{xx}(x,t)}_{\text{diffusion term}} + \underbrace{f(x)}_{\text{source}}$$

so, if $u(x,t)$ only depends on x

$\Rightarrow \quad u(x,t) = u(x)$

$\Rightarrow \quad -k \cdot u_{xx}(x) = -k \cdot u''(x) = f(x)$

So, let $k = 1$ &

$$-u'' = f(x), \quad 0 < x < 1$$

$$u(0) = u(1) = 0$$

What about if $f(x)$ is a "heat source" that only acts at point ξ?

Bonus: $\displaystyle\int_\Omega u\cdot dv = uv\Big|_{\partial\Omega} - \int_\Omega v\cdot du.$

(1) $\mathcal{L}u = -u'' = f \quad , \quad u|_{\partial\Omega} = 0$

(2) $\mathcal{L}G = -G'' = \delta \quad , \quad G|_{\partial\Omega} = 0$

(1) $\Rightarrow$ $$-\int_\Omega u''\cdot G\,dx = \int_\Omega f\cdot G\,dx$$

$$\underbrace{-G\cdot u'\Big|_{\partial\Omega}}_{0} + \int_\Omega u'\cdot G'\,dx = \int_\Omega f\cdot G\,dx$$

since $G|_{\partial\Omega} = 0$

$$\int_\Omega u'\cdot G'\,dx = \int_\Omega f\cdot G\cdot dx \qquad (3)$$

(2) $$-\int_\Omega G''\cdot u\,dx = \int_\Omega \delta(x,\xi)\cdot u(x)\,dx$$

$$\underbrace{-u\cdot G'\Big|_{\partial\Omega}}_{0} + \int_\Omega G'\cdot u'\,dx = u(\xi)$$

since $u|_{\partial\Omega} = 0$

$$\Rightarrow \int_\Omega u'\cdot G'\,dx = u(\xi) \qquad (4)$$

(4) $-$ (3) $\Rightarrow$ $$u(\xi) - \int_\Omega f\cdot G\,dx = \underbrace{\int_\Omega u' G'\,dx - \int_\Omega u'\cdot G'\,dx}_{0}$$

$$\therefore \boxed{u(x) = \int_\Omega f(y)\cdot G(y,x)\,dy}$$

Exercises:

(i) Discuss the solvability of the boundary value problem:

$$u'' + \pi^2 \cdot u = f(x), \quad u(0) = u(1) = 0$$
$$0 < x < 1$$

(ii) Consider the BVP

$$u'' - 2 \cdot x \cdot u' = f(x), \quad 0 < x < 1, \quad u(0) = u'(1) = 0$$

Find Green's function or explain why there isn't one.

(iii) Find the inverse of the differential operator $Lu = -(x^2 \cdot u')'$ on $1 < x < e$ subject to $u(1) = u(e) = 0$.