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$\mathcal{ML}$ measured lamination $\mathcal{MF}(S)$ space of all equivalence classes of measured foliations. $\mathcal{T}_g$ Teichmuller space of surface of genus g **?**

3 Théorie de Teichmuller

3.1 First definitions

Nous commencerons dans cette section par donner quelques définitions pour introduire la théorie des espaces de Teichmuller.

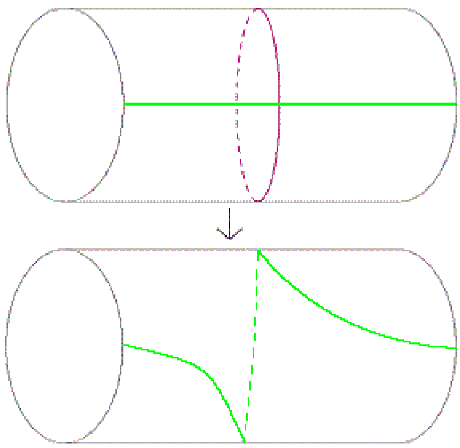
Définition 3.1. Espace de Teichmuller Soit S une surface de genre g, un marquage de S est un couple (X, f) formé d'une surface de Riemann X et d'un homéomorphisme préservant l'orientaion $f : S \to X$. Sur l'ensemble des marquages de S, nous pouvons faire une relation d'équivalence $(X_1, f_1) \sim (X_2, f_2)$ si il existe $\alpha : X_1 \to X_2$ tel que $f_2 \circ \alpha \circ f_1^{-1}$ soit un homéomorphisme de S préservant l'orientation et isotope à l'identité. L'espace des marquages quotienté par la relation s'appelle l'espace de Teichmuller et est noté $\mathcal{T}_g$

Remarque. Si $g \geq 2$, pour toute courbe simple fermée α de S, il existe une unique géodésique fermée de X librement isotope à $f(\alpha)$. Nous noterrons $L_\alpha(X)$ sa longeur hyperbolique et nous prenons la topologie la plus faible sur T_g qui rendent ces fonctions continues.

Définition 3.2. Espace des modules On appele groupe modulaire le groupe des homéomorphisme préservant l'oriention de S quotienté par ceux isotope à l'identité.Nous notterons ce groupe Mod_g. Il agit de façon discrète sur T_g et l'espace quotient est appelé espace des modules et est noté $\mathcal{M}_g$

Il est naturel de ce demander à quoi ressemble ces espaces.

Définition 3.3. Dehn twist Soit γ une courbe simple et fermée. Il existe un voisinage tubulaire de γ noté A homéomorphe à $[0;1] \times S^1$. On définit le Dehn's twist comme l'homéomorphisme qui vaut l'indentité hors de A et vaut $(t, s) \mapsto (t, e^{2i\pi t}s)$ sur A.



Remarque. Le théorème de Lickorisk affirme que le groupe modulaire est engendré par ces Dehn's twist et que plus précisément on peut choisir $2g + 1$ générateurs **?**.

Définition 3.4. Measured foliation Given a surface S and a finite set of points $P = (p_1, p_2, ...)$, given a open covering U_i on $S - P$, a collection of C^1 real function ν_i such than $\|d\nu_j\| = \|d\nu_i\|$ on $U_i \cap U_j$, and near each singular point p_s a coordinate neighborhood V with complex coordinate z such that $\|d\nu\| = \|Im(z^{\frac{k}{2}}dz)\|$ for some positive integer k called the degree of the singular point, leaves of the foliations are the graphs immersed S in along dv is constant. In addition if each boundary circle pf S is contained in a singular leaf, then ti is called a measurd foliation.

The height $h_\gamma(\|d\nu\|)$ of a (free homotopy class) of a loop γ on S is the infinimum in the homotopy class of the integral by $\|d\nu\|$

$$h_\gamma(\|d\nu\|) = inf_{\gamma\equiv\gamma'} \int_g amma\|d\nu\|$$

The topology on the measured lamination that we will use is the weakest that make the height functions continous.

Remarque. We won't actually study the set of measured lamination but the equivalence class of

$$h_\gamma(\|d\nu\|) = h_\gamma(\|d\mu\|), \text{ for each loop}\gamma \in S$$

. We can equivalently use Whitehead equivalence relation on singular foliations by collapsing critical intervals to points and taking isotopy of foliation.

Let $\mathcal{MF}(S)$ be the space of all equivalence classes of measured foliations.

Définition 3.5. Lamination Une lamination est un ensemble fermé qui est un union (non nécessairement finie) de géodésiques. Par chaque point x contenu dans λ il ne passe que une seul géodésique. Nous notterons cet espace $\mathcal{ML}(x)$

Définition 3.6. Geodesic currents Let $\mathcal{M}_\infty$ be the space of unordered pairs of distinct points in $\mathbb{S}^1$

$$\mathcal{M}_\infty := (z, w) \in \mathbb{S}^1 \times \mathbb{S}^1, z \neq w//(z, w) \equiv (w, z)$$

Let G be a discret torsion-free group in $PSL(2, \mathbb{R})$ such that $\mathbb{H}//G = S$ is a hyperbolic surface. A geodesic current μ on S is a G-invariant Radon measure on $\mathcal{M}_\infty$. We will note $\mathcal{GC}(S)$ the space of geodesic currents.

Remarque. $\mathcal{GC}(S)$ have a natural topology which is the weak $*$ convergence on continous functions.

Remarque. A mullticurve is a formel sum of geodesics $\gamma = \sum a_i\gamma_i$. The space of lamination is in some aspect the closure of the set of all multicurve

Définition 3.7. Intersection number Consider the square $\mathcal{M}^2_\infty := \mathcal{M}_\infty \times \mathcal{M}_\infty$. In this space we can consider the open subset $\mathcal{IM}^2_\infty$ corresponding to pair pairs of geodesics which have transversal intersections in $\mathbb{H}$. G act on $\mathcal{IM}^2_\infty$. If μ and ν are geodesic currents in $\mathcal{GC}(S)$, the product $\mu \times \nu$ define a G-invariant measure on $\mathcal{IM}^2_\infty$. Finally if we take the mass of the total space $\mathcal{IM}^2_\infty//G$, the reasult is called the intersection number, $i(\mu, \nu)$

Proposition 3.1.

$$i : \mathcal{GC}(S) \times \mathcal{GC}(S) \to \mathbb{R}_+$$

is continuous and bilinear

Remarque. If α and β are simple closed geodesics (dirac measure in $\mathcal{GC}(S)$), then the intersection number is the number of intersection between α and β. Actually, one can define intersection in this way, first on simple closed geodesic, then by bilinearity on multi-curves and finally by continuity on geodesic current.

Remarque. The topology on $\mathcal{ML}$ is the weakest that make $i(.,.)$ a continous function.

Définition 3.8. Différentielle quadratique Une différentielle quadratique est une section du carré de l'espace tangeant canonique à X. Il s'écrit localement comme $\phi = \phi(z)dz^2$

Remarque. Si $\phi(p) \neq 0$ on peut trouver une carte contenant p dans laquel $\phi = dz^2$. Ainsi ϕ détermine une métrique plate sur X et un feuilletage $\mathcal{F}$ correspondant aux lignes horizontales.

Une différentielle quadratique est dite intégrable si

$$\|\phi\| = \int_X |\phi| < \infty$$

Nous notterons $\mathcal{Q}(x)$ l'espace de Banach des différentielles quadratiques intégrables.

3.2 Flow on Teichmüller space

We will define the main object of this paper, earthquake flow.

Définition 3.9. The earthquake flow is family of maps defined for $t \in \mathbb{R}$

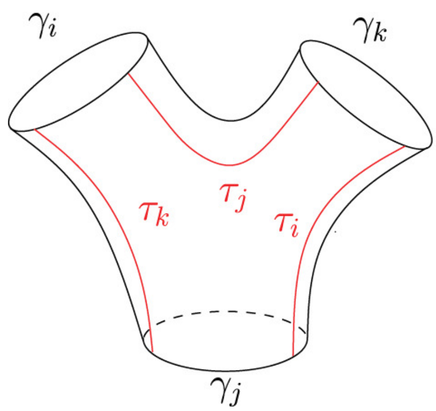
$$E_t : \mathcal{ML} \times \mathcal{T}_g \to \mathcal{ML} \times \mathcal{T}_g(\lambda, X) \mapsto (\lambda, E_{t\lambda}X)$$

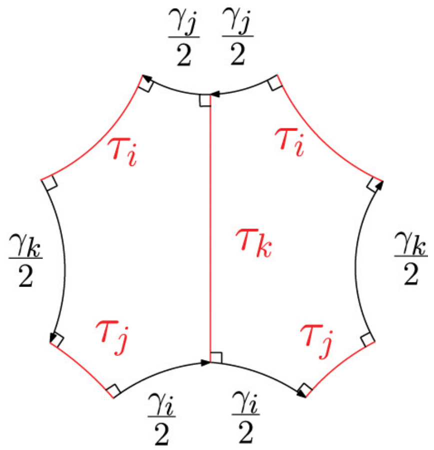
where E_λ is first defined on multi-curves $\gamma = \sum c_i\gamma_i$ by adding c_i to the twist coordinate of γ_i. As multi-curves are dense in lamination, we can show that it could be extend to the whole set $\mathcal{ML}$

3.3 Decomposition of hyperbolic surface

One way to construct all hyperbolic surface is to decompose them in elementary piece, that we will call pair of pant.

A hyperbolic geometric exercice show that a hexagone which side are geodesics and with right angles is determined by the lenght of three sides which are not consecutifs.





On the image above γ_i, γ_j and γ_k determined the hexagone. Then we can glued them to have a pair of pant.

Définition 3.10. A pair of pant is a hyperbolic surface with three geodesic boundaries and no ponctured.

Remarque. The pair of pant is uniquely determined by the lenght of the three boundarie geodesics.

Remarque. The lenght of one or more geodesic can go to zero and the boundaries become a ponctured.

We can now decompose, with the following theorem, all hyperbolic surfaces in a collection of pair of pants.

Théorème 3.2. *Let S be a surface of genus g and with n ponctured. There is a set of $3g-3+n$ simple closed curves $(\gamma_1, ..., \gamma_{3g-3+n})$ such that $S \Cup \gamma_i$ is a disjoint collection of pair of pants.*

Définition 3.11. Given a surface S and a pant decomposition $\gamma_1, ..., \gamma_{3g-3+n}$, we have a map

$$(S) \to (\mathbb{R}^{+3g-3+n}, \mathbb{R}^{3g-3+n}) X \mapsto (l_{\gamma_1}(X), ..., l_{\gamma_{3g-3+n}}(X), \tau_{\gamma_1}(X), ..., \tau_{\gamma_{3g-3+n}}(X))$$

This map is injective and is call the Fenchel-Nielsen cordinates.

4 Isomorphisme de Mirzhakani

The aim of this part is to demonstrate the followoing statement

Théorème 4.1. *There is a measurable conjugacy F between the earthquake flow $(\lambda, X) \mapsto (\lambda, E_{t\lambda}(X))$ on $\mathcal{ML} \times \mathcal{T}_g$ and the Teichmüller unipotent flow action of*

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

on the bundle $\mathcal{QD}$ of nonzero quadratic differentials over Teichmüller space $\mathcal{T}_g$.

$$\begin{array}{ccc} \mathcal{ML} \times \mathcal{T}_g & \xrightarrow{E_t} & \mathcal{ML} \times \mathcal{T}_g \\ F \downarrow & & \downarrow F \\ \mathcal{QD} & \xrightarrow[u_t]{} & \mathcal{QD} \end{array}$$

4.1 Tightening map

A first coresspondance, found by Thurston, exist between measured foliation and measured lamination. We will mostly follow the paper of Levitt **??**.

Théorème 4.2. *Let X be a closed orientable hyperbolic surface and $\mathcal{F}$ a foliation. There is a canonical geodesic lamination $\gamma(\mathcal{F})$ associated to $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{F}'$ are associated foliation then $\gamma(\mathcal{F}) = \gamma(\mathcal{F}')$. In the opposite direction given a géodisic lamination γ, one can find a foliation $\mathcal{F}$ such that $\gamma(\mathcal{F}) = \gamma$ and it's unique up to equivalence.*

Définition 4.1. A *tranverse curve* is a simple closed curve C which is never tangent to $\mathcal{F}$ and contain no singularity of $\mathcal{F}$.

Remarque. Since $\mathcal{F}$ contain only saddle singularities, C cannot be contractible, therefor C is isotopic to a geodesic

Lemme 4.3. *Let h be a leaf of $\tilde{\mathcal{F}}$. Each end of h converge to a point of S; the two point at infinity cannot be the same*

4.2 Correspondance between foliations and quadratic differentials

For a quadratic differential q, one can define two measured foliations, the horizontal $h(q)$ and the vertical $v(q)$ corresponding in locate coordinate to $Re(z)$ and $Im(z)$. This give a map two the pair of foliation but it is not the subjectif, we should restrict ro the image. Define $\Delta = (\alpha, \beta) : i(\alpha, \gamma) = i(\beta, \gamma) = 0,$ for some $\gamma \in \mathcal{MF}$. Δ contain the diagonal (α, α) and is kind of "fat" diagonal.

Lemme 4.4. *For any $q \in \mathcal{QD}$, $(h(q), v(q)) \notin \Delta$*

4.3 Shear Cordinate

5 Vitesse de mélange

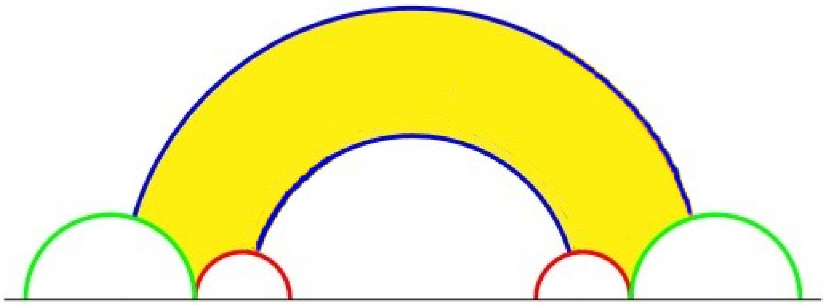
6 Example du tore à un trou

A torus can not have a hyperbolic structure, it has naturally a flat structure as the quotient $\mathbb{R}^2/\mathbb{Z}^2$. This changes when one study the one ponctured torus. It is a torus S where we choose a point p and remove it (or just marked it).

The construction of this object can be done in two manier at least. For the first construction, one have to choose a hyperbolic octogone where one side have length 0 and the two other one l, then we sew the border of two of this octogone which give a pair of pant. Finally we can glued with a twist τ to have the one ponctured torus.

A second construction is given by the representation. Given two hyperbolic isomorphism of $\mathbb{H}$ A and B with different fixed point on $\delta\mathbb{H}$ and with $H := ABA^{-1}B^{-1}$ the commutator should be a parabolic element.

A fundamental domaine is given by the following image:



Given two generators α and β of $\pi_1(S)$, two closed curves non homotopically trivial which intersect one, one can parametrize all other lamination. Indeed a given lamination $\lambda \in \mathcal{ML}$ is determined by the couple $(i(\alpha, \lambda), i(\beta, \lambda))$ where $i(.,.)$ is the geometric intersection number.