SORBONNE UNIVERSITÉ

MASTER THESIS

Teichmuller theory and Thurston Earthquake flow

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Contents

1 Introduction

Since Bernhard Riemann, mathematicians knew that a geometric hyperbolic surface can be described with only a finite number of parameters. Knowing this, for a given surface, we can be interested in the set of all geometries we can give it, modulo composition by map isotopic to the identity map, this set is called the Teichmuller space. Other problems rise shortly after this definition. How can we deform in a natural way a surface's geometry into another? What does it mean that a geometry is close to another? What are the natural boundaries of the Teichmuller space?

Oswald Teichmuller, a german mathematician, studied and answered this question in the year preceding the second World War. He created the first metric on this space by finding a solution to an extremal problem: between two hyperbolic geometry on the same surface is there a function which minimize the deformation ? The resulting theorem [21] proves not only the existence, but also the unicity of this function. It naturally induced a distance in the now called Teichmuller space by considering the logarithm of the deformation of the extremal function.

Figure 1: The mathematician Oswald Teichmuller

Thurston then added other important steps to this theory. He underlined the role of laminations. A lamination is a generalisation of simple closed curve. And he introduced the earthquake flow, in a course in 1976-1977 at Priceton, which plays an important role in Teichmuller theory . Kerckhoff used this tool to show the Nielsen realisation conjecture in 1983 [11] which states that every finite subgroup of the mapping class group have a fixed point in the Teichmuller space. This theory have been actively studied and therefore generated a lot of literature and contributions. One can refer to the following books [6], [9], [10] or course notes [16] [17] for a more detailed study of the theory.

Still, some questions remain open, raising further interest of the community in the topic of hyperbolic geometry. Among the main questions encountered, figures the asymptotic number of closed geodesics. To begin with, one can wonder what is the number $\pi(X, L)$ of closed geodesic on a hyperbolic surface X of length less than L. The answer was found in the 1940s and 1950s by Delsarte, Huber and Selberg and -due to its resemblance to the prime number theorem- is called the prime number theorem for hyperbolic surfaces. It states that:

$$\pi(X, L) \simeq e^L/L$$

as $L \to \infty$. The reader can refer to [4] for more details.

A much harder problem was to find the number, $\sigma(X, L)$, of simple (which don't intersect themselves) closed geodesic of length less than L on a hyperbolic surface X. It was found years later, in Mirzakhani's PhD [19], and we have

$$\sigma(X, L) \simeq C_X L^{6g-6}$$

As $L \to \infty$ where g is the genus of the surface X and C_X is a constant which depend of the geometry X.
To obtain such a result, Myriam Mirzhakhani conjugated the earthquake flow to the horocycle

flow. This step provides that the Earthquake flow is ergodic and allows us to use Birkoff theorem [[Ref?citation?]] to understand asymptotic quantities.

The question is now to give error terms to this quantity. In order to do that, we need to understand better the mixing rate of the earthquake flow. This flow is conjugate to the horocyclic flow which has a polynomial mixing rate. But as the conjugacy is only a measurable map it does not transport the rate of mixing. We should analyse the rate of mixing of the earthquake flow by other means. One research direction would be to consider natural functions on Teichmuller space such as the systole. The systole is the lenth of the shortest simple closed curve on the surface. This function behaves nicely along earthquake path, it is continuous, convex and we know its first derivative at the origin. Moreover in the case of the once punctured torus we can give a frame determined by the continued fraction of the slope.

In this master thesis, I will first give an introduction to Teichmuller theory and some useful and classical tools in this theory such as the collar lemma or a surface's decomposition in pair of pants. Then, we will review the proof of the Mirzhakani's conjugacy between the horocyclic flow and the earthquake flow. After doing so, we will focus on the mixing properties of the ergodic and horocyclic flow, and discuss their mixing rate. Finally we will discuss a special case: the once punctured torus, as it is one of the simplest example of hyperbolic surface.

2 Notations

$\mathcal{ML}$	measured lamination
$\mathcal{MF}(S)$	space of all equivalence classes of measured foliations.
$\mathcal{QD}$	bundle of nonzero quadratic differential.
$\mathcal{T}_g$	Teichmuller space of surface of genus g
$Mod(S)$	The modular group of a surface S
$\mathcal{P}^1\mathcal{M}_g$	Product of $\mathcal{T}_g$ and $\mathcal{MF}(S)$ quotiented by the modular group.
$\mathcal{GC}(S)$	the space of geodesic currents
$\mathbb{D}$	the Poincaré disk
$\mathbb{S}_\infty$	the unit circle seen as the boundary of $\mathbb{D}$
$\bar{\mathbb{D}}$	$\mathbb{D} \cup \mathbb{S}$
$\mathbf{S}$	set of all simple closed curve of a surface
$D_\gamma(\alpha)$	Dehn's twist of α around γ

3 Teichmuller Theory

3.1 First definitions

To begin we will in this section give the very definition to introduce the theory of Teichmuller space

Definition 3.1. Let S be a surface of genus g, a marking of S is a couple (X, f) made of a closed Riemann surface X and of one homeomorphism $f : S \to X$ which preserve the orientation. On the set of the marking S, we have a equivalence relation, $(X_1, f_1) \sim (X_2, f_2)$ if there exist $\alpha : X_1 \to X_2$ such that $f_2 \circ \alpha \circ f_1^{-1}$ be an homeomorphism of S preserving the orientation and isotope to the identity map.The set of the marking quotient by the previous relation is the *Teichmuller space* and is written $\mathcal{T}_g$.

Remark. If $g \geq 2$, for every closed curve α of S, there is only one closed geodesic of X freely isotope to $f(\alpha)$. Moreover if α is simple, so is the geodesic. We will note $l_\alpha(X)$ its hyperbolic length and we will take the weakest topologie on $\mathcal{T}_g$ which make this map continuous.

This give a map $L : \mathcal{T}_g \to \mathbb{R}^{\mathbf{S}}$, where $\mathbf{S}$ is the set of all simple closed curve of a given surface. We can ask if this map si injective, i.e. that a geometry is given by the length of the set of simple closed curve. The answer is yes and more precisely one can choose only $9g-9+3n$ curve so that this map is injective, as stated in Farb and Margalit (2011) Theorem 10.7. [6] This give the intuition that Teichmuller space can be describe only by using a finite set of parameter. We will see after that other coordinates which have nice propeties for other use.

An other curious but important fact about simple closed geodesic is that the sum of a function of theirs length is equal to a constant which does not depend on the genus or the geometry of the surface.

Theorem 3.1 (McShane's identity)**.** *Let X be a hyperbolic surface then*

$$\sum_{\gamma} \frac{1}{1+e^{l_\gamma(X)}} = \frac{1}{2}$$

Where the sum is taken over all simple closed geodesic.

This theorem was proven by McShane and therefore bears his name [18]. A good introduction with a proof using probablistic method on trees can be found in a paper by F. Labourie and S.P. Tan [12].

Definition 3.2. The *modular group* is the group of homeomorphisms of S which respect orientation quotiented by the subgroup of homeomorphisms isotopic to the identity map. We will call this group Mod_g It act discretely on the Teichmuller space $\mathcal{T}_g$ and the quotient is called the *moduli space* and is written $\mathcal{M}_g$.

The Teichmuller space is the universal covering of the modular space. In fact the Teichmuller space is homeomorph to a sphere and so is contractible and in particular, connected and simply connected. It is the universal covering of the moduli space.

We can ask ourselves how these spaces look and if we can give an easy representation of them. The fact is that the modular space is not manifold but an orbifold, that is a space which locally looks like a ball of a vector space quotiented by a finite group.

We will now describe objects that are natural to understand the Teichmuller space.

Definition 3.3. A *foliation* on a surface S is the collection of the following data: a finite set of points $P = (p_1, p_2, ...)$, given a open covering U_i on $S - P$, a collection of C^1 real function ν_i such than $\|d\nu_j\| = \|d\nu_i\|$ on $U_i \cap U_j$, and near each singular point p_s a coordinate neighborhood V with complex coordinate z such that $\|d\nu\| = \|Im(z^{\frac{k}{2}} dz)\|$ for some positive integer k called the degree of the singular point.

Leaves of the foliations are the graphs immersed in S along which dv is constant. In addition if the surface S have a boundary it is required that it is contained in a singular leaf.

The height $h_\gamma(\|d\nu\|)$ (of a free homotopy class) of a loop γ on S is the infimum in the homotopy class of the integral by $\|d\nu\|$

$$h_\gamma(\|d\nu\|) = inf_{\gamma \sim \gamma'} \int_{\gamma'} \|d\nu\|$$

The topology on the measured lamination that we will use is the weakest that makes the height functions continuous.

Remark. We won't actually study the set of measured foliation but the equivalence class of

$$h_\gamma(\|d\nu\|) = h_\gamma(\|d\mu\|), \text{ for each loop } \gamma \in S$$

. We can equivalently use Whitehead equivalence relation on singular foliations by collapsing critical intervals to points and taking isotopy of foliation.

Let $\mathcal{MF}(S)$ be the space of all equivalence classes of measured foliations.

Definition 3.4. A *lamination* λ is a closed set made of an union (non necessarily finite) of geodesics. For each point x in λ, passes only one geodesic of the lamination. We will write this space $\mathcal{ML}(x)$.

To understand the space of geodesic laminations it is sometimes easier to understand the "trace" of this space in the boundary circle of the universal cover of the surface. This observation gives the following definition.

Definition 3.5. Let $\mathcal{M}_\infty$ be the space of unordered pairs of distinct points in $\mathbb{S}^1$

$$\mathcal{M}_\infty := \{(z, w) \in \mathbb{S}^1 \times \mathbb{S}^1, z \neq w\} // (z, w) \equiv (w, z)$$

Let G be a discrete torsion-free group in $PSL(2, \mathbb{R})$ such that $\mathbb{H}//G = S$ is a hyperbolic surface. A *geodesic current* μ on S is a G-invariant Radon measure on $\mathcal{M}_\infty$. We will note $\mathcal{GC}(S)$ the space of geodesic currents, see [10]

Remark. $\mathcal{GC}(S)$ have a natural topology which is the weak $*$ convergence on continuous functions.

Example 3.1. Let $g \in G$ be an element that represent a simple closed geodesic on the surface S. The G-orbit of the fix point of g is a discrete set on which we can put Dirac measure. This links the geodesic currents to the measured lamination.

Remark. A multicurve is a formal sum of geodesics $\gamma = \sum a_i \gamma_i$. The space of lamination is in some aspect the closure of the set of all multicurve

Definition 3.6. Consider the square $\mathcal{M}_\infty^2 := \mathcal{M}_\infty \times \mathcal{M}_\infty$. In this space we can consider the open subset $\mathcal{IM}_\infty^2$ corresponding to pairs of geodesics which have transversal intersections in $\mathbb{H}$. G acts on $\mathcal{IM}_\infty^2$. If μ and ν are geodesic currents in $\mathcal{GC}(S)$, the product $\mu \times \nu$ defines a G-invariant measure on $\mathcal{IM}_\infty^2$. Finally if we take the mass of the total space $\mathcal{IM}_\infty^2//G$, the result is called the *intersection number*, $i(\mu, \nu)$

Proposition 3.2.

$$i : \mathcal{GC}(S) \times \mathcal{GC}(S) \to \mathbb{R}_+$$

is continuous and bilinear [3].

Remark. If α and β are simple closed geodesics (Dirac measure in $\mathcal{GC}(S)$), then the intersection number is the number of intersection between α and β. Actually, one can define intersection in this way, first on simple closed geodesic, then by bi-linearity on multi-curves and finally by continuity on geodesic current.

Remark. The topology on $\mathcal{ML}$ is the weakest that make $i(.,.)$ a continuous function.

Definition 3.7. A *quadratic differential* is a section of the square of the tangent space to X. It is locally as $\phi = \phi(z)dz^2$.

Remark. If $\phi(p) \neq 0$ we can find a map including p in which $\phi = dz^2$. Hence ϕ determine a flat metric on X and a foliation $\mathcal{F}$ corresponding to horizontal lines.

A quadratic differential is said to be integral if

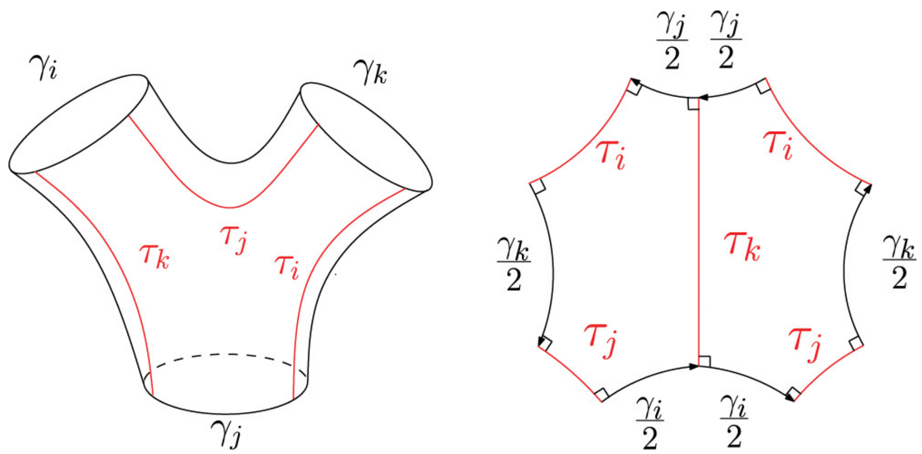
$$\|\phi\| = \int_X |\phi| < \infty$$

We will write $\mathcal{Q}(x)$ the Banach space of integral quadratic differentials.

3.2 Decomposition of hyperbolic surface

One way to construct all hyperbolic surface is to decompose them in elementary piece, that we will call pair of pant.

A hyperbolic geometric exercise shows that a hexagon which side are geodesics and with right angles is determined by the length of three sides which are not consecutive.



On the image above γ_i, γ_j and γ_k determined the hexagon. Then we can glue them to have a pair of pant.

Definition 3.8. A *pair of pant* is a hyperbolic surface with three geodesic boundaries and no punctures.

Remark. The pair of pant is uniquely determined by the length of the three boundary geodesics.

Proof. As a hyperbolic polygon is uniquely determined by three non consecutive sides, the surface with boundaries that we obtain by gluing two together is determined by the length of its three boundary geodesics. $\square$

Remark. The length of one or more geodesic can go to zero and the boundaries become a puncture.

We can now decompose, with the following theorem, all hyperbolic surfaces in a collection of pair of pants.

Theorem 3.3. *Let S be a surface of genus g and with n punctures. There is a set of $3g-3+n$ simple closed curves $(\gamma_1, ..., \gamma_{3g-3+n})$ such that $S \ \Cup \gamma_i$ is a disjoint collection of pair of pants.*

Definition 3.9. Given a surface S and a pant decomposition $\gamma_1, ..., \gamma_{3g-3+n}$, we have a map

$$(S) \to (\mathbb{R}^{+3g-3+n}, \mathbb{R}^{3g-3+n})X \mapsto (l_{\gamma_1}(X), ..., l_{\gamma_{3g-3+n}}(X), \tau_{\gamma_1}(X), ..., \tau_{\gamma_{3g-3+n}}(X))$$

This map is injective and is call the *Fenchel-Nielsen* coordinates.

Definition 3.10 (Dehn twist)**.** Let γ be a simple closed curve. There is a tubular neighbourhood of γ called A homeomorph to $[0;1] \times S^1$. A *Dehn's twist* around γ is the homeomorphism which is the identity out of A and is $(t,s) \mapsto (t, e^{2i\pi t}s)$ on A.

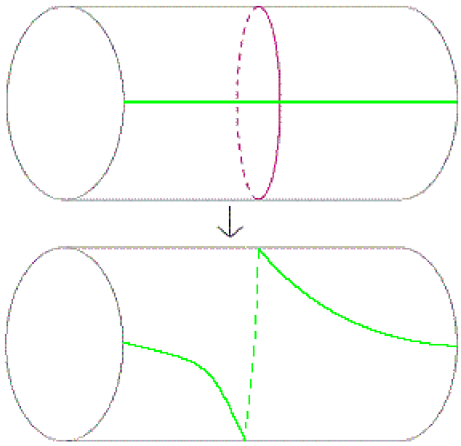


Figure 2: A Dehn twist

We can now give a set of generator of the modular group.

Remark. The Lickorisk theorem states that the modular group is generated by the Dehn's twist and more precisely that one can choose only $2g+1$ generators [14]. We will give here a easier version of this theorem.

Theorem 3.4. *There is a collection $\delta_1, ..., \delta 9g-9$ of simple closed curves such that $\mathcal{T}_g \to \mathbb{R}^{9g-9}$ is injective.*

Proof. Let us take $(\gamma_1, ..., \gamma_{9g-9})$ a decomposition in pair of pants, let $(\alpha_1, ..., \alpha_{9g-9})$ be a collection of simple closed curves such that $i(\gamma_i, \alpha_i) > 0$ and $i(\gamma_i, \alpha_j) = 0$ for $i \neq j$, finally we take $\beta_i = D_{\gamma_i}(\alpha_i)$. We want to show that the length of this collection of $9g-9$ curves determined the hyperbolic structure $X \in \mathcal{T}_g$. X already has the Fenchel-Nielsen coordinate, of the pant decomposition $(\gamma_1, ..., \gamma_{3g-3})$, $(l_{\gamma_1}(X), \tau_{\gamma_1}(X), ..., l_{\gamma_{3g-3}}(X), \tau_{\gamma_{3g-3}}(X))$ so we need only to show that the parameters $\tau_{\gamma_i}(X)$ are determined by the length of the collection. Up to a re-normalisation we can take $\tau_{\gamma_i}(X) = 0$ for every i. Now let's take $t = (t_1, ..., t_{3g-3}) \in \mathbb{R}^{3g-3} \backslash 0$. We will note X_t the hyperbolic geometry which has the same length as X and twist parameters t in the Fenchel-Nielsen coordinate of the pair of pants. So $X_0 = X$. We will consider the function $A(t) = l_{\alpha_1}(X_t)$ and $B(t) = l_{\beta_1}(X_t)$. This function depend only of t_1 as $i(\gamma_i, \alpha_j) = 0$ for

$i \neq j$. Moreover they are strictly convex and by definition we have $A(t_1 + l_(\gamma_1)(X)) = B(t_1)$. We will show that there is no $t_1 \neq 0$ such that $A(t_1) = A(0)$ and $B(t_1) = B(0)$ that is $A(t_1 + l_(\gamma_1)(X)) = A(l_(\gamma_1)(X))$. We will note $s = t_1$ and $L = l_(\gamma_1)(X)$ Suppose there is $s \neq 0$ such that $A(s) = A(0)$, we can take $s > 0$, the other case is symmetric. If $s < L$, then by convexity for every $t \in]0; s[$, $A(t) < A(0) = A(s)$ and A is strictly increasing after s so as $s < L < L + s$ we have $A(L) < A(L + s)$. If $s > L$, then $L < s < L + s$ and $A(L) < A(L + s)$. The final case $s = L$ is also impossible since we would have $A(0) = A(L) = A(2L)$. Finally we can make the same argument for the other twist parameters which conclude the proof. □

3.3 Flow on Teichmüller space

We will define the main object of this document, earthquake flow.

Definition 3.11. The *earthquake flow* is family of maps defined for $t \in \mathbb{R}$

$$\begin{array}{rrcl} E_t : & \mathcal{ML} \times \mathcal{T}_g & \to & \mathcal{ML} \times \mathcal{T}_g \\ & (\lambda, X) & \mapsto & (\lambda, E_{t\lambda} X) \end{array}$$

where $E_{t\lambda}$ is first defined if λ is a simple closed curve. In this case, we open the surface along lambda, then twist the left part of t unit and put it together again. Then if γ_1 and γ_2 are two curves that don't intersect, E_{γ_1} and E_{γ_2} commute. So we can define the earthquake map on weighted multicurves by twisting one curve after the other by amounts proportional to the weight. Finally as multicurves are dense in the set of lamination we can extend this map for any lamination [11].

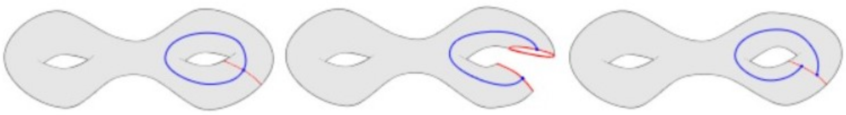


Figure 3: Effect of a twist on a transverse curve, image from [23]

Remark. If the lamination is just a simple closed curve γ then $E_{l_\gamma(X)}(X, \gamma)$ is just a Dehn twist around γ. Moreover if we take a decomposition in a pair of pant that contains γ, it is just a translation in the coordinate of the twist of γ.

One could give a precise proof that the extension of the earthquake map from the multicurves to the lamination is rigorous, that is if we take two sequence α_n and α'_n of multicurves which converge to the same lamination, then the sequences of earthquake map along this multicurves converge to the same map.

We will work on universal covering of the surface, the half-plane $\mathbb{H}$ If $v \in T^1\mathbb{H}$ we will call $\gamma(v)$ the geodesic passing by the base-point of v and with v as tangent vector at this point.

We have two useful lemmas to control the distortion of the earthquake map. We will not prove them, but their proofs are in [11].

Lemma 3.5. *Let l and l' be two geodesics and $x \in l$, $y \in l'$ two points at most ϵ apart. Then if v and v' are the two geodesic tangents to l and l' respectively at point x and y, we have $d(v, v') < C\epsilon$ for a universal constant C.*

Lemma 3.6. *Let v and v' be two vector such that $d(v, v') < \epsilon$, let denote $\gamma = \gamma(v)$ and $\gamma' = \gamma(v')$. Let $w \in T^1\mathbb{H}$, then*

1. $d(E_{t\gamma}w, E_{t\gamma'}w) \leq Kt\epsilon$

2. $d(E_{t\gamma}w, w) \leq Kt$

for all $t \leq T$ and for a constant K depending on T and on the distance between the base-point of v and w.

With these two lemma, we can describe what happens if we change a discrete lamination by a simple closed curve that average it.

Lemma 3.7. *Let $x, y \in \mathbb{H}$, $v \in T_1\mathbb{H}$ based at y, $\bar{A}$ the geodesic from x to y. Suppose γ is a discrete lamination with equal measure on each leaf whose intersection with $\bar{A}$ is included in a subarc A. Let l be a single geodesic intersecting A with angle equal to the average angle of the intersections of γ and A and with mass equal to $\mu = i(A, \gamma)$.*

If A has length less than δ then for every $T \in \mathbb{R}^+$ the distance between $E_{t\gamma}v$ and $E_{tl}v$ is less than $Kt\mu\delta$, for all $t \leq T$ and K is a constant depending only of $T\mu$ and $d(x, y)$.

Proof. Let denote $l_1, l_2, ..., l_n$ the leaves of γ. The image of $\bar{A}$ is a disconnected arc. The point x is connected to $E_{t\gamma}y$ by a staircase path going along component of $\bar{A}\backslash\gamma$ and subarc of l_i. Let denote the successive components $A_0, A_1, ..., A_n$ and δ_i the length of A_i. So the staircase path is moving by δ_0 along A_0 then by μ/n along l_1,and so on.

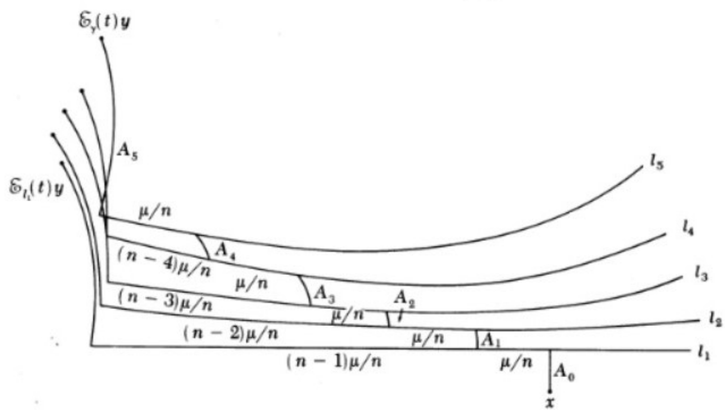


Figure 4: Image of the construction described, image from [11]

We now alter the path by replacing the shearing along l_n distance μ/n by a shearing along l_{n-1} distance $2\mu/n$. The change is less than $Kt\frac{\mu}{n}C\delta_{n-1}$ by lemma 3.5 and lemma 3.6. Then we change the shearing by a shearing along l_{n-2} of distance $3\mu/n$ and we continue until we shear a distance μ along l_1. The total change is less than $KtC\sum_{i=1}^{n-1}\frac{i\mu}{n}\delta_{n-i}$ which is less than $KCt\mu\delta$. We now pass from l_1 to l with lemma 3.5 and 3.6, and we obtain the new lemma. □

We need a final lemma to conclude of the well-founded definition of the earthquake flow. In this case we control the difference after earthquaking by two simple geodesic going through the same point of the arc.

Lemma 3.8. *Let x, y, v, and $\bar{A}$ be as above, if l and l' are geodesics of $\mathbb{H}$ with measure μ and μ' such that $l \cup \bar{A} = l' \cup \bar{A} = p \notin \{x, y\}$ and the difference between the vectors tangent of l and l' at p of length μ and μ' is less than ϵ.*

Then for any T, $d(E_{tl}v, E_{tl'}v) < Kt\epsilon$, for $t \leq T$ and K a constant which depends only on $d(x, y)$ and $T\mu$

Finally this give the theorem which control the distance between two earthquake paths.

Proposition 3.9. *Let $\nu \in \mathcal{ML}$ be a lamination and let x, y be in $\mathbb{H}$, A be the geodesic from x to y and $v \in T^1\mathbb{H}$ be based at y, and x and y do not lie on the atomic part of ν. Then for any ϵ, T, there is a neighbourhood U of ν in $\mathcal{ML}$ such that for all γ, $\bar{\gamma}$ weighted multicurve in U, $d(E_{t\gamma}v, E_{t\bar{\gamma}}v) < Kt\epsilon$, for all $t \leq T$, K a constant depending only on $d(x,y)$ and $Ti(\nu, A)$*

Corollary. *The earthquake flow is well defined along any lamination $\nu \in \mathcal{ML}$ and for all time t.*

Remark. The earthquake flow is an isometry outside the support of the lamination and is continuous outside the atomic part, i.e. the simple closed geodesics of the lamination.

Thurston show that given two point in the Teichmüller space, there is a lamination λ such that the earthquake flow from one point with respect to λ reach the other point (also in [11]).

We can ask ourselves what is an invariant measure of this flow.

Definition 3.12. The *Weil-Peterson form* is the form

$$\omega_{WP} = \sum dl_i \wedge d\tau_i$$

Where $(l_1, ..., \tau_1)$ are the Fenchel-Nielsen according to a pant decomposition.

This induces a measure μ_{WP}.

There is a finite measure ν_g in the Lebesgue measure class on $\mathcal{P}^1\mathcal{M}_g$ which is invariant under the earthquake flow. This measure projects to the volume form given by $B(X) \times \mu_{WP}$ on $\mathcal{M}_g$, where

$$B(X) = \mu_{Th}(\lambda \in \mathcal{ML}, l_\lambda(X) \leq 1)$$

There are two other important flows, the geodesic flow and the horocyclic flow. First there is a natural homeomorphism between $T^1\mathbb{H} \simeq SL_2(\mathbb{R})/SO_2(\mathbb{R})$, since $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ act simply transitively on it. This morphism can be chosen up to a conjugation via an other element of $SL_2(\mathbb{R})/SO_2(\mathbb{R})$. We we will be interested in a special kind of subgroup.

Definition 3.13. A *fuchsian group* Γ is a finitely generated and discrete subgroup of $SL_2(\mathbb{R})$. Then Γ acts discontinuously on $\mathbb{H}$.

A Hyperbolic surface can be represented as $\mathbb{H}/\Gamma = SL_2(\mathbb{R})/SO_2(\mathbb{R})/\Gamma$. If U is a one parameter subgroup of $SL_2\mathbb{R}$ it acts on the quotient.

There are two important example:

Definition 3.14. The *geodesic flow* is a flow on the Teichmuller space given by the action of the diagonal matrices

$$u_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

Definition 3.15. The *horocycle flow* is a flow on the bundle of non-zero quadratic differential, $\mathcal{QD}$, of the Teichmuller space given by the unipotent action of

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

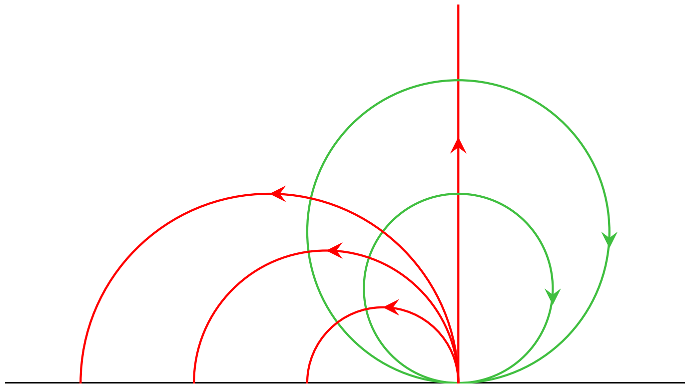


Figure 5: Representation of the horocycle flow, in green, and the geodesic flow, in red

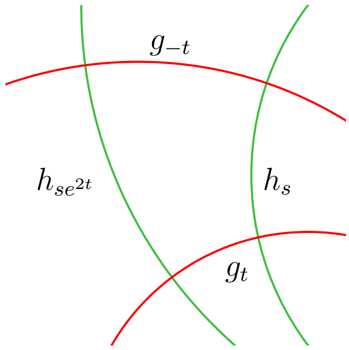


Figure 6: The conjugaison of the horocycle flow by the geodesic one.

The geodesic flow is also the flow that we obtain by following geodesic line on $\mathbb{H}$ and the horocycle is the flow we obtain by following curves which are everywhere orthogonal to the geodesic, which is the horizontal line and the circle tangent to the real line.

An important relation is how this two flows interact between each other

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} u_t = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} = \begin{pmatrix} 1 & se^{2t} \\ 0 & 1 \end{pmatrix}$$

So the the conjugation of the horocyclic flow by the geodesic one is still the horocyclic flow.

We give the definition of a third flow that we will use to demonstrate the ergodicity of the two previous one.

Definition 3.16. The *rotational flow* is a flow on the bundle of nonzero quadratic differential, $\mathcal{QD}$, of the Teichmuller space given by the action of

$$e_\theta = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}$$

We have the relation

$$h_s = e_\theta g_t e_{\pi+\theta}$$

3.4 The collaring theorem

We will now give a useful tool to give necessary condition on length of two intersecting geodesics.

The collar function $\eta :]0; \infty[\to]0; \infty[$ is defined as follow. We draw a segment of length $l > 0$ on a geodesic γ, then we project perpendicularly to the geodesic the end of this segment to infinity and draw the geodesic δ which have this endpoint. So we have $\eta(l) = d(\gamma, \delta)$.

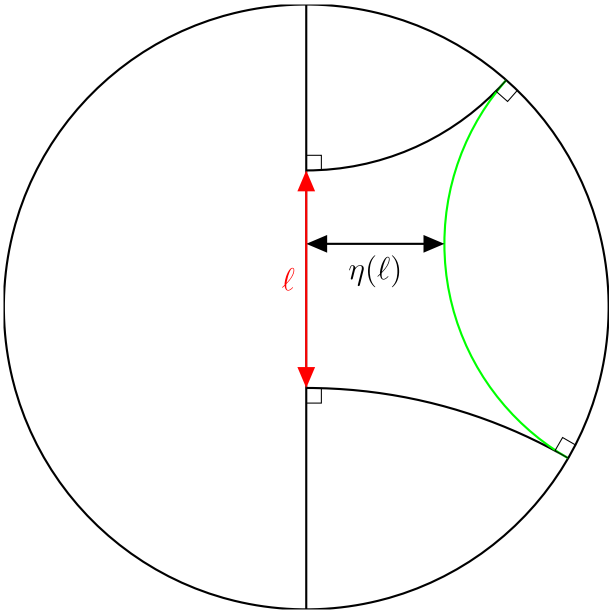


Figure 7: In red the segment of length l and in green the geodesic whose endpoint are orthogonal projection of the end of the segment.

It is an exercise to show that:

$$\eta(l) = \frac{1}{2} ln(\frac{cosh(l/2)+1}{cosh(l/2)-1})$$

This quantity will be the side of a long "tube" generated by a simple closed geodesic in the hyperbolic surface. We give a definition to make this a little more precise.

Definition 3.17. Let γ be a simple closed geodesic of length l on a hyperbolic surface X. If the δ-neighbourhood

$$A_\delta(\gamma) := \{x \in X | d(x,\gamma) < \delta\}$$

is isometric to the δ-neighbourhood of the unique simple closed geodesic on the cylinder of modulus $\frac{\pi}{l}$, we say that γ admit a *δ-collar* , or that $A_\delta(\gamma)$ is the δ-collar of γ.

We can now state a useful theorem.

Theorem 3.10. *Let X be a complete hyperbolic surface, and let $\Gamma := \gamma_1, \ldots$ be a collection of disjoint simple closed geodesic, each γ_i of length l_i. Then $A_{\eta(l_i)}(\gamma_i)$ are collars around the γ_i, and they are disjoint.*

Proof. Choose γ_1 and γ_2 and add other simple closed curve to have a maximal multicurve that includes both. Now cutting along this curve we have a set of pair of pants so we only have to show that the $\eta(l_i)$ neighbourhood of γ_i the boundaries of the pair of pants do not intersect each other. We cut the pair of pants along geodesics coming from a boundary C and meeting the two other boundaries A and B. We unfold this figure in the hyperbolic plane and name the side of the octagon following the figure below.

Since a' and b' have the common perpendicular C', they do not intersect and similarly for a'' and b''. The theorem follow easily by the definition of the function η.

□

There are some corollaries which follow from this theorem and are ready to use in many occasions.

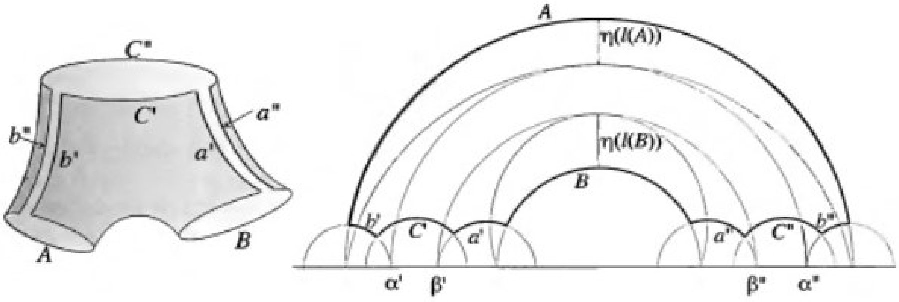


Figure 8: image from [9]

Corollary. *Let X be a hyperbolic surface, and γ_1, γ_2 two simple closed geodesics on X of lengths l_1 and l_2. If $l_2 < 2\eta(l_1)$, then either $\gamma_1 = \gamma_2$ or $\gamma \cap \gamma_2 = \emptyset$*

Proof. If $\gamma_1 \neq \gamma_2$ and $\gamma \cap \gamma_2 \neq \emptyset$ then γ_2 must cross the collar neighbourhood of γ_1 from one boundary to the other and so have length strictly superior than $2\eta(l_1)$. $\square$

Corollary. *Let X be a hyperbolic surface, and let γ_1, γ_2 be two simple closed geodesics with lengths $< ln(3 + 2\sqrt{2})$. Then either $\gamma_1 = \gamma_2$ or $\gamma_1 \cap \gamma_2 = \emptyset$.*

Corollary. *Let X be a complete hyperbolic surface, γ a simple closed geodesic on X of length l, and A_γ the collar around γ. Then any simple geodesic δ on X that enter A_γ either intersect γ or spirals towards γ.*

Proof. Suppose the geodesic δ enters A_γ. We can lift the situation in the universal cover of the hyperbolic disc, where $\tilde{\gamma}$ is a lift of γ can be a diameter of the circle. Then if $\tilde{\delta}$ do not intersect $\tilde{\gamma}$ and do not have the same point at infinity, then its two endpoint are on the same side of $\tilde{\gamma}$ in the disc. Now the translation along $\tilde{\gamma}$ of length $l_\gamma(X)$ is in the representation of $\pi_1(X)$. If $\tilde{\delta}$ intersect A_γ then by the definition of δ it will intersect with its image by the translation cited before and hence is not simple in X. $\square$

4 Isomorphisme of Mirzhakani

The aim of this part is to demonstrate the following statement

Theorem 4.1. *There is a measurable conjugacy F between the earthquake flow $(\lambda, X) \mapsto (\lambda, E_{t\lambda}(X))$ on $\mathcal{ML} \times \mathcal{T}_g$ and the Teichmüller unipotent flow action of*

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

on the bundle $\mathcal{QD}$ of nonzero quadratic differentials over Teichmüller space $\mathcal{T}_g$.

$$\begin{tikzcd} \mathcal{ML} \times \mathcal{T}_g \arrow[r, "E_t"] \arrow[d, "F"'] & \mathcal{ML} \times \mathcal{T}_g \arrow[d, "F"] \\ \mathcal{QD} \arrow[r, "u_t"'] & \mathcal{QD} \end{tikzcd}$$

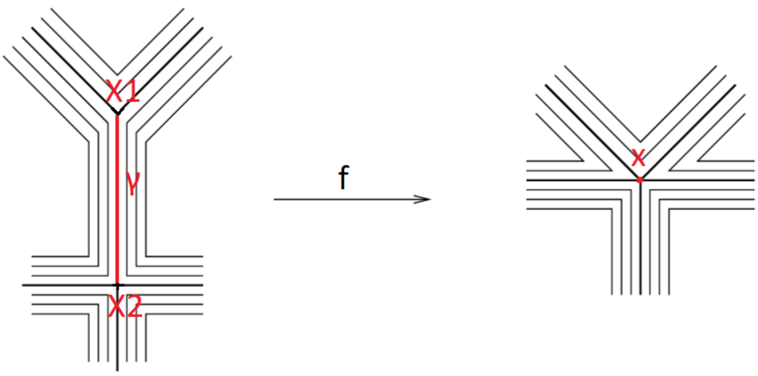
To do this we will decompose F between intermediates maps.

4.1 Tightening map

A first coresspondance, found by Thurston, exist between measured foliations and measured laminations. We will mostly follow the paper of Levitt [13].

Definition 4.1. We say that two foliations are equivalent if we can pass from one to the other by Whithead (see definition below) moves or isotopy (homeomorphism isotopic to the identity).

Definition 4.2. Given a measure foliation, an critical segment γ is an arc between two singularities along a leaf which is not a simple closed curve. There is a map f homotopic to the identy that collapse γ to a point x and is identity outside a neighboorhood of γ which contain no other singularity. Doing so we reduce the number of singularities of the foliation and if the extremities of γ are singularities of order k_1 and k_2, x is now a singularity of the new foliation of order $k_1 + k_2 - 2$. This action is called a *Whithead move.*



Theorem 4.2. *Let X be a closed orientable hyperbolic surface and $\mathcal{F}$ a foliation. There is a canonical geodesic lamination $\gamma(\mathcal{F})$ associated to $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{F}'$ are associated foliation then $\gamma(\mathcal{F}) = \gamma(\mathcal{F}')$. In the opposite direction given a geodisic lamination γ, one can find a foliation $\mathcal{F}$ such that $\gamma(\mathcal{F}) = \gamma$ and it's unique up to equivalence.*

Definition 4.3. A *tranverse curve* is a simple closed curve C which is never tangent to $\mathcal{F}$ and contain no singularity of $\mathcal{F}$.

Remark. Since $\mathcal{F}$ contain only saddle singularities, C cannot be contractible, therefor C is isotopic to a simple closed geodesic.

We will work on the universal cover of X, which is the Poincaré disc $\mathbb{D}$ with circle "at infinity" $\mathbb{S}_\infty$. We call $p : X \to \mathbb{D}$ the universal projection. $\tilde{\mathcal{F}}$ is $p^{-1}(\mathcal{F})$.

We will say that a foliation follow the (*) condition if the following is true:

If f_1 and f_2 are two compact homotopic leaves then all leaf in the open annulus between them is also compact.

Lemma 4.3. *Let h be a leaf of $\tilde{\mathcal{F}}$. Each end of h converge to a point of S; the two point at infinity cannot be the same.*

Proof. First, we should notice that the behavior of leaf at infinity do not change if we take an equivalent foliation. Indeed a homeomorphism ϕ on a compact fundamental domain isotopic to the identy can be extend to an homeomorphism $\tilde{\phi}$ on $\mathbb{D}$ such that $dist(x, \tilde{\phi}(x)) \leq K$. This implie that $\tilde{\phi}$ extend es the identity on the boundary $\mathbb{S}_\infty$.

Then given a leaf h of $\tilde{\mathcal{F}}$, we take a half leah h_0. If $p(h_0)$ is compact or spiral toward a compact leaf of $\mathcal{F}$ then the first part of the lemme is imediate.

Otherwise, $p(h)$ meet a transverse curve C infinitely often. With an isotopie we can take C to be a geodesic. Now h_0 can meet a connect component of $\tilde{C} = p^{-1}(C)$ only one time. Otherwise there will be a disk bound by an arc of $\tilde{C}$ and an arc of h_0, which is impossible considering the transversity of C and that $\mathcal{F}$ have no 1-type singularities.

Now every compact of $\mathbb{D}$ meet a finite number of connected components of $\tilde{C}$ so the limit set of h_0 must be on $\mathbb{S}_\infty$. This limit set is connected and non empty. Moreover it should not contains any end of a connected component of $\tilde{C}$. But the ends of connected components of $\tilde{C}$ are dense in $\mathbb{S}_\infty$ as $\tilde{C}$ is the image of a geodesic by $\pi_1(X)$. This show the first point of the lemma.

The second assertion is clear if $p(h)$ is compact or if it meet a transverse curve C at least twice since then every connected componnents of $\tilde{C}$ separate the end of h.

Otherwise $p(h)$ spiral toward two compact leaf f_1 and f_2. If $f_1 = f_2$ and the two end point of h are the same then there will be a singularity that would not be a saddle. $f_1 \neq f_2$ is impossible since $\mathcal{F}$ follow the condition (*). $\square$

We can now associate to every leaf h a geodesic $\gamma(g)$ by joining the endpoint. Then $\gamma(\tilde{\mathcal{F}}) = \cup_{h \in \mathcal{F}} \gamma(h)$ is a disjoint union of geodesic invariant by $\pi_1(X)$. We have to show that this set is closed to conclued that we have a lamination.

Lemma 4.4. *$\gamma(\tilde{\mathcal{F}})$ is closed in $\bar{\mathbb{D}}$*

Proof. Let $g_n = \gamma(h_n)$ be a sequence of geodesics in $\gamma(\tilde{\mathcal{F}})$ converging to a geodesic g. We want to show $g \in \gamma(\tilde{\mathcal{F}})$. We can suppose that all the g_n are distinct of g and are all on the same side.

Let L be the limit set in $\bar{\mathbb{D}}$. For all leaf m in $\tilde{\mathcal{F}}$, we call $\bar{m}$ the closure of m by adding the two end point in $\mathbb{S}_\infty$. Then L meet at least one connected component of $\bar{\mathbb{D}}$
$\bar{m}$. As the end points of all leaf of $\tilde{\mathcal{F}}$ is a dense subset of $\mathbb{S}_\infty$, L contain a leaf h. Taking a half-leaf h_0, we want to show that the end point is the same as one of g.

A first case is if there is a simple closed curve C transverse to $\mathcal{F}$ which meet $p(h_0)$ infinitely often. If h_0 does not converge to the corresponding point at infinity then there would be a connected component of $p^{-1}(C)$ that contains the point of infinity of h_0 but does not contain the point of infiny of h_n which is impossible for large n.

A second case is if $p(h_0)$ spirals toward a compact leaf, then closed leaf close to $p(h_0)$ also spirals toward the same compact leaf. Then h_0 converges to one of the points at infiniy of g which is a point at inifinity of h_n fot n large.

Finally if $p(h)$ is compact then $p(h_n)$ spirals toward it for large n, therefore $\gamma(h)$ and g have one point in common at infinity. If the second was different, by applying a transformation leaving $\gamma(h)$ invariant (but no g), we would separate h from the leaves h_n, and it is a contradiction.

□

Now we want to exhib an inverse construction which take a lamination λ and give a foliation μ. To do this we still consider $\tilde{\lambda}$ in the universal cover. We will suppose that every complementary region is a ideal polygon.

We can built a skeleton that it compose of edge between vertex and a choosen point in the center. After building the skeleton for every polygon we fill the complementary region which will be between four edge by line between the two vertex.

This map is often called the "collapsing" map and its inverse the "tightening" map.

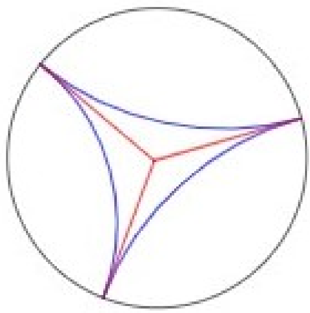


Figure 9: In red the skeleton of the ideal triangle in blue, Image from [22]

The measure we put on this foliation is uniform on every region between four edges of the skeleton.

4.2 Correspondance between foliations and quadratic differentials

For a quadratic differential q, one can define two measured foliations, the horizontal $h(q)$ and the vertical $v(q)$ corresponding in locate coordinate to $Re(z)$ and $Im(z)$.

Moreover the measure are

$$h_\nu = \int Im(\sqrt{q(z)}dz)$$

and

$$h\mu = \int Re(\sqrt{q(z)}dz)$$

This give a map two the pair of foliation but it is not the subjectif, we should restrict to the image. Define $\Delta = (\alpha, \beta) : i(\alpha, \gamma) = i(\beta, \gamma) = 0,$ for some $\gamma \in \mathcal{MF}$. Δ contain the diagonal (α, α) and is kind of "fat" diagonal.

Lemma 4.5. *For any $q \in \mathcal{QD}$, $(h(q), v(q)) \notin \Delta$*

Proof. Suppose that there is γ such as $i(h(q), \gamma) = i(v(q), \gamma) = 0$ for somme $\gamma \in \mathcal{ML}$. Let take a sequence of simple closed weighted curves γ_i converging to γ. By continuity of the intersection number with have that $i(h(q), \gamma_i) \to 0$. So there is a sequence of saddle connection is the same homotopie class of γ_i whose x-components is very small. We can have the same argument in the vertical direction and find a contradiction. $\square$

Theorem 4.6. *The map $q \mapsto (h(q), v(q))$ define a homeomorphism $\mathcal{QD} \to \mathcal{MF} \times \mathcal{MF} \backslash \Delta$*

Proof. We can describe the inverse map. If we take to measured foliation h and v we can tighten them into lamination (which we also call h and v) as in the previous section. This two laminations do not share any leaf, otherwise we would have $(h, v) \in \Delta$ by considering the leaf as a geodesic. The complementary region of $h \cup v$ are compact polygons, i.e. they do not have a vertex on the boundary of the disk. Now we can fill the polygon to obtain the quadratic differential with a singularity in each complementary region of order the number of side of the polygon. $\square$

More detail of the proof can be found in [5] Lemma 6.2 and [8]. The injectivity is discuss with more detail in [7] section 3.

4.3 Shear Cordinate

Finally there is a map that, given a hyperbolic structure X and a lamination λ create a measured foliation which is transverse to λ.

For simplicity we will ask that λ is a maximal lamination i.e. if $\tilde{\lambda}$ is the pre-image of λ in the universal cover $\mathbb{D}$, $\mathbb{D} \backslash \tilde{\lambda}$ is made of ideal triangle. We will first work in this triangles minus a region on the center, then give a measure in this foliation, and finnally show that ot is a homeomorphism.

So in one the ideal triangle, given two side we can draw an arc prependicular with the this two sides, which is the intersection of the ideal triangle with the circle tangent to the boundary of $\mathbb{D}$ in the common extremity the two arc chosen. Then by rescaling by a factor $r \in [0; 1]$ and doing the same procedure for the two other pair we get a foliation in the ideal triangle minus a locus in the center.

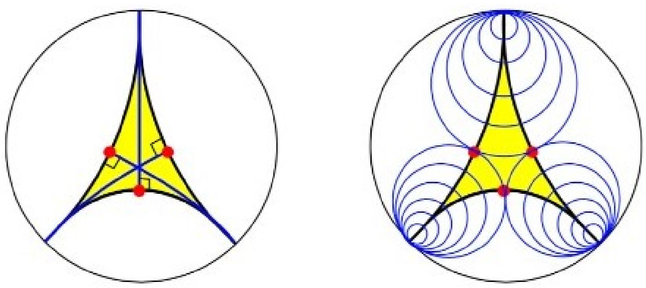


Figure 10: Image from [15]

Then we can find a full foliation by pitching the resulting.

We have a natural transverse measure to this foliation. For an arc in one ideal triangle of the lamination we can project it along the leaf of the foliation to asegment in the edge of the ideal triangle, the length of the arc will be the length of the segment. As the leaf are horocycle circle based on the same point, it will not depend of which side we choose to project. Then given an arbitrally arc we decompose it along the ideal triagle it meet.

We want to shwo that this construction is reversable, that is given $\mu \in \mathcal{MF}_\lambda$, the set of foliation transverse to the lamination, we can construct $X \in \mathcal{T}_g$ whose horocycle foliation is μ.

The idea is that, if we already know X, the lamination λ can be lift to $\tilde{lambda}$ which is invariant of Γ the fushian group of X. But we can built $\tilde{lambda}$ only with the information given by μ.

We will note $\tilde{\mu}$ the preimage of μ in $\mathbb{D}$. If we consider two triangles T_1 and T_2 that are complementary region of $\tilde{\lambda}$ and we suppose we take a segment A in a leaf of $\tilde{\mu}$ that goes to an edge of T_1 to T_2. We name v_1 and v_2 the two vectors with footprint in the edge and tangent to them. Then there is a Moebuis transformation S which take one to the other. With one more information, the "shear", we can place T_2 on $\mathbb{D}$, according to the position of T_1

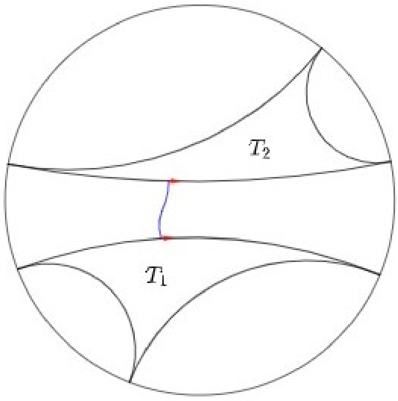


Figure 11: Image from [22]

Indeed given only a Moebuis transformation we still have a one parameter families of triangle T_2 with an edge generated by v_2. To fix this we trace two orthogeodesic comming from the vertex of the ideal triangles facing the considered edges and from the point of intersection in T_1 we follow a leaf of the foliation, then when we meet T_2 we have to move along the geodesic edge to find the other point of intersection. This lenght is the shear between T_1 and T_2

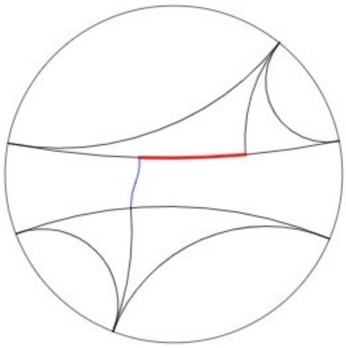


Figure 12: Image from [22]

Let I be the set of all triangles in $\mathbb{D}$ that A meet. For each $i \in I$ we can define v_i^+ and v_i^- the vectors tangent to the edge of the corresponding triangle at the intersection of the edges and A. Note that I is a countable totally ordered but non well ordered set. So if we take S_i the Moebuis transformation which take v_i^- to v_i^+, we have to give a meaning of the expression

$$\prod_{i \in I} S_i$$

Definition 4.4. Given a countable totally ordered set of indice I and element S_i in a Banach algebra, we say that $\prod_i S_i$ is *well defined* and equal to S if for any increasing chain

$$I_0 \subset I_1 \subset ... \subset I$$

with $\cup_k I_k = I$ we have $lim_{k\to\infty} \prod_{i\in I_k} S_i = S$.

Lemma 4.7. *For element s_i in a Banach algebra index by a countable totallyordered set, if $\sum \|s_i\| < \infty$, then $\prod(1+s_i)$ is well-defined.*

Proof. For $1 \leq m \leq n$, we have

$$\|\prod_{i=1}^{n}(1+s_i) - \prod_{i=1,i\neq m}^{n}(1+s_i)\| \leq \|s_m\|\|\prod_{i=1,i\neq m}^{n}(1+s_i)\| \leq \|s_m\|\prod_{i=1}^{n}(1+\|s_i\|)$$

Or with the assumption $\sum \|s_i\| < \infty$ we have that $\prod(1+\|s_i\|) \leq C < \infty$, so removing or adding $1+s_m$ produce a change bound by $\|s_m\|C$. $\square$

Now we want to apply this lemma to $S_i - Id$, with Id the identity matrice.

Lemma 4.8. *For the previous S_i, if we note $s_i = S_i - Id$ we have $\sum \|s_i\| < \infty$.*

Proof. Each S_i is conjugate to a horocycle transformation of time one. The conjugacy is made by a geodesic flow along the edges of the triangle. We can compute

$$\begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & e^{-t} \\ 0 & 0 \end{pmatrix}$$

So the norm of s_i is inversally corrolated to the amount of geodesic flow used in the conjugaison. Now we can parttion the indice set I into finely many subset (I_k) according to to which spike of the lamination the arc of A cross. Then for a spike the sum of $\|s_i\|$ where $i \in I_k$ is finite, indeed the distance between two neighbooring crossing is bound below by a constant and so the amount of time we should do the geodesic flow increase at most linerally and finnaly the norm of the s_i should decraese geometrically. $\square$

So we can conclude that there is an unique Moebuis transformation S equal to the meaningfull expression $\prod_i S_i$.

Now we can conclude the proof. There exist, without any hyperbolic structure X topological classes for $\tilde{\mu}$ and $\tilde{\lambda}$. We choose one arbitrary ideal triangle T_1 in the lamination. For every other triangle T_2, the Moeubuis transformation and the shear are data that can be compute only using the transverse measure of $\tilde{mu}$. So we can place T_2, and the other triangle. The closure of this set give the lamination $\tilde{\lambda}$. $\tilde{\lambda}$ will be preserve by a fushian group Γ and we will have $X = \mathbb{D}/\Gamma$.

4.4 Conjugaison between the horocyclic flow and the earthquake flow

We now have to prove that the map between $\mathcal{ML} \times \mathcal{T}_g$ and $\mathcal{QD}$ conjugate the earthquake flow and the horocyclic flow.

Lemma 4.9. *Denote by $Shear_X(T_1, T_2)$ the shear for two triangles joined by an arc A of the horocyclic foliation on the hyperbolic surface X. Then*

$$Shear_{E_{t\lambda}(X)}(T_1, T_2) = Shear_X(T_1, T_2) + t\lambda(A)$$

where $\lambda(A)$ denote the transverse measure of A and t is sufficiently small.

Proof. T_1 and T_2 are separeted by infinitely many leaves of $\tilde{\lambda}$. We want to undersant how T_2 moved relatively to T_1 by the action of the earthquake $E_{t\lambda}$. We can approximate the measured lamination between the two triangles by a discrete one. If the earthquake along a leaf of γ of $\tilde{\lambda}$ between T_1 and T_2 by an amount t, then this changes the shear coordinate between T_1 and T_2 by t. Indeed each arc of the horocycle foliation which have endpoint on γ see their endpoints translated by t. Similary if the earthquake moves finitelymany leave of λ with measure a_i, the shear changes by precisely $t \sum a_i$. So taking a limit, we have the lemma for an arbitrary measured foliation. $\square$

We will now show that the Mirzakhani's map conjugates the earthquake flow to the horocyclic flow. Let $\mathcal{ML}_0$ denote the set of maximal lamination and $\mathcal{QD}_0$ the locus of quadratic differentials with simple zeros and no horizontal saddle connection. We begin with an easy lemma.

Lemma 4.10. *An arc joining two singularities of a quadratic differential have in is isotopy class a path made of horizontal and vertical arc between two singularities of the quadratic differential.*

Corollary. *Supposeq$_t$ is a path in the space of quadratic differential such that for every t_0 and every path γ on q_{t_0} joining two singularities, the period $x_t + iy_t$ of γ satisfied*

$$\frac{d}{dt}|_{t=t_0} x_t = y_{t_0}, \frac{d}{dt}|_{t=t_0} y_t = 0$$

Then q_t is an orbit of the horocyclic flow.

Proof. We have

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ty \\ y \end{pmatrix}$$

And we can integrate the two equations to find the linear action. $\square$

We can now prove theorem 4.1.

Proof. We want to show that

$$q(\lambda, F_\lambda(E_{t\lambda}(X)))$$

is a horocyclic flow path. We pick an arbitrary time t_0 and look at the derivative of the path, called γ. The coordinate y_t is constant equal to $\lambda(\gamma)$ and the derivative of x_t is by lemma4.9 equal to y_t. We conclude by the previous lemma. $\square$

5 Mixing rate

5.1 Mixing proprities of the geodesic and horocyclic flows

We will first give the behavior of two flows we described before.

Theorem 5.1. *The ergodic flow and the horocycle flow are mixing.*

[16]

Proof. The step will be in four step, first we will show that the ergodic flow is ergodic, the horocyclic flow is ergodic, the ergodic flow is mixing, finally the horocycle flow is mixing.

Ergodicity flow is ergodic

We will look at the time average of a function $f \in L^2$ which is continuous and compactly support. It will be sufficient since this space is dense

$$F(x) = lim_{T\to\infty}\frac{1}{T}\int_0^T f(g_t x)dt$$

We want to show that F is almost everywhere constant.

Since F depend only of the geodesic $\gamma(a,b)$ which pass at x, if $a,b \in S_\infty$ are the two endpoints of γ, we have $F(a,b)$. Then as two geodesic with the same forward endpoint are asymptotic, F is a quantity that depend only of asymptotic average, we have that F do not depend of a. But we can reverse the argument to show that if we consider the inverse geodesic, $t \to \infty$, we have that F is also independent of b. Hence F is constant almost everywhere and the geodesic flow is ergodic.

Ergodicity of the horocycle flow

Now if we take $f \in L^2(X)$ a function invariant under the horocycle flow and of mean zero, we want to show that $f = 0$ almost everywhere. Let G^t, H^s and E^r correspond to the operators of the different flows. We have the relation

$$H^s = E^r G^t E^{\pi+r}$$

where $r \to 0$ as $t \to \infty$. Since $H^s f = f$,we have for any $T > 0$,

$$f = \frac{1}{T}\int_0^T E^r G^t E^{\pi+r} f dt$$

Thus for every $g \; inL^2(X)$ we have

$$< g, f >= \int_X g\frac{1}{T}\int_0^T E^r G^t E^{\pi+r}$$

As $r \to 0$ then $t \to \infty$ we can show by controlling the difference that

$$< g, f >= lim_{t\to\infty}\int_X g\frac{1}{T}\int_0^T G^t E^\pi f dt$$

Then as we have shown that the geodesic flow is ergodic, we have by Von Neumann ergodic theorem

$$< g, f >=< g, \int_X E^\pi f >= 0$$

This conclude the proof that the horocycle flow is ergodic.

Mixing of the geodesic flow

We have the relation

$$h^s g^t = g^t h^{sexp(2t)}$$

Let us take $f_0, f_1 \in C_0(X)$, we have for small s

$$< f_0, g^t f_1 > \equiv < h^{-s} f_0, g^t f_1 > = < f_0, h^s g^t f_1 > = < f_0, g^t h^{sexp(2t)} f_1 > = < g^{-t} f_0, h^{sexp(2t)} f_1 >$$

Now we have for small s,

$$h^{Sexp(2t)} f_1 \equiv \frac{1}{S} \int_0^S h^{exp(2t)s} f_1 ds = F_t$$

and for large t, as the horoclic flow is ergodic

$$F_t = \frac{1}{S} \int_0^S h^{exp(2t)s} f_1 ds \equiv \int_X f_1 = < f_1, 1 >$$

So we have

$$< f_0, g^t f_1 > \equiv < g^{-t} f_0, F_t > \equiv < g^{-t} f_0, 1 >< f_1, 1 > = < f_0, 1 >< f_1, 1 >$$

Which is the mixing of the geodesic flow.

Mixing of the horocyclic flow

We use again the relation $h^s = e^r g^t e^{\pi + r}$.

$$< h^s f_0, f_1 > = < g^t e^{\pi + r} f_0, e^{-r} f_1 > \equiv < g^t e^{\pi} f_0, f_1$$

for t large (and so r small). By the mixing propity of the geodesic flow, this quantity converges to $< f_0, 1 >< f_1, 1 >$, and the horocyclic flow is mixing.

□

Then we have this elementary corrolary

Corollary. *The Earthquake flow is also ergodic.*

Proof. With the conjugacie of Mirzharani, the earthquake flow is conjugated to the horocycle flow which is ergodic. This propriety is transmitted. □

5.2 Rate of mixing of this flows

Then we want to know at which rate the mixing of the geodesic and horocyclic flow happend. Ratner show in 1986 [20] that the geodesic flow is exponnetially mixing and the horocyclic flow has a polynomial rate of mixing. She used representation theory.

We will first give some definition before abording the main theorem.

Definition 5.1. Let H be a complexe separable Hilbert space, $U(H)$ the group of all unitary transformation of H onto itself and $T : G \to U(H)$ a unitary representation. We note $T(g) = T_g \in U(H)$.

An element $v \in H$ is called *a* $C^k - vector$ for T, $k = 0, 1, ..., \infty$ if $g \mapsto T_g(v)$ is a C^k-map from G to the Hilbert space.

Remark. The space of C^∞ vector is dense in H.

Definition 5.2. If v is a C^1-vector of H and X is in the Lie algebra of G, the *Lie derivative* $L_X v$ is

$$L_X v = lim_{t \to 0} \frac{T(exptX)v - v}{t}$$

Now if Γ is a fushian group, $SL_2(\mathbb{R})$ act on the hyperbolic surface with its measure (X, μ) as seen as $\mathbb{H}/\Gamma$. Then it will also act by a representation ρ on $L_0^2(X, \mu)$ the space of zero average function on the surface.

We want to study the decorellation induce by some subgroup of $SL_2(\mathbb{R})$. It will link with this quantity.

Definition 5.3. Let ϕ and ψ be zero-mean functions in L^2, the *matrix coefficient* $C_{\phi,\psi}$ is

$$g \mapsto | < \phi, \rho(g)\psi > |$$

Remark. As the horocyclic flow and the geodesic flow are mixing we have $C_{\phi,\psi}(e_t) \to 0$ and $C_{\phi,\psi}(h_s) \to 0$ for all functions ϕ and ψ.

We can decompose this representation as an integrale of irreductible representation

$$L_0^2(X, \mu) = \int H_\zeta d\nu(\zeta)$$

With ρ_ζ the representation oh H_ζ.

This set of representation decompose in three part.

1. The principal serie
2. The discrete serie
3. The complementary serie

This decomposition is given by the spectrum of the Casimir operator on each H_ζ. The discrete part of the spectrum is the discrete series, and if q is the iegenvalue, the part $q \geq 1/4$ is the principal series and $q < 1/4$ the complemetary series. This decomposition was studied by Bargmann in [2].

So we can decompose $\nu = \nu_p + \nu_d + \nu_c$.

Moreover we now that if ρ_ζ is in the complementary series, there exist $s = s(\zeta) \in]0; 1[$ such that the representation is isomorphic to the representation π_s on the Hilbert sapce

$$\mathcal{H}_s = \{f : \mathbb{R} \to \mathbb{C}, \|f\|^2 = \int_{\mathbb{R}\times\mathbb{R}} \frac{f(x)\bar{f(y)}}{|x_y|^{1-s}} dxdy < \infty\}$$

With the action

$$pi_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = \frac{1}{(cx+d)^{1+s}} f(\frac{ax+b}{cx+d0})$$

The representation rho is isolated from the trivial representation if and only if there exist $\epsilon > 0$ such that $s(\zeta) < 1 - \epsilon$ for ν_c almost every ζ.

The Lie algebra of $SL(2, \mathbb{R})$, is the vector space of 2×2 matrice with zero trace. A basis for this space is

$$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Which give the Casimir operator on C^2 vectors of $SL(2, \mathbb{R})$,

$$\Omega_T = (L_v^2 + L_Q^2 - L_W^2)/4$$

If T is irreductible then Ω_T is a scalar multiple of the identity, i.e.

$$\Omega_T v = \lambda v$$

for some $\lambda = \lambda(T) \in \mathbb{R}$ and all C^2-vectors $v \in H(T)$

If Γ is a lattice of $SL(2,\mathbb{R})$, Ω_T create an operator Δ on $\Gamma\backslash SL(2,\mathbb{R})$. We call $\Lambda(\Delta)$ its spectrum and

1. $A(\Gamma) = \Lambda(\Gamma) \cup]-1/4; 0[$
2. $B(\Gamma) = supA(\Gamma)$
3. $C(\Gamma) = -1 + \sqrt{1 + 4B(\Gamma)}$

We can now write the theorem about the decay of correlation.

Theorem 5.2. *Let Γ be a lattice in $SL(2,\mathbb{R})$, $M = \Gamma\backslash G$ and T be the regular representation of G on $L^2(M,\mu)$. Let $v, w \in K(T,p)$, $p > 0$, $< w, 1 >= 0$ and $B(t) =< v, w \circ g_t >$, $C(t) =< v, w \circ h_t >$. Then there exist $t_0 = t_0(\Gamma) > 0$ such that for all $|t| \geq t_0$ and some $E, F > 0$*

1. $|B(t)| \leq E(b(|t|))^{\alpha(p)}$
2. $|C(t)| \leq F(b(ln|t|))^{\alpha(p)}$

where

1. $b(t) = e^{\sigma(\Gamma)}$ *if* $A(\Gamma) \neq \emptyset$
2. $b(t) = e^{-t}$ *if* $A(\Gamma) = \emptyset$, $sup(\Lambda \cap]-\infty; -1/4[) < -1/4$ *and* $-1/4$ *is not an eigenvalue of* Ω
3. $b(t) = te^{-t}$ *if* $A(\Gamma) = \emptyset$, $sup(\Lambda \cap]-\infty; -1/4[) = -1/4$ *or* $-1/4$ *is an eigenvalue of* Ω

and $\alpha(p)$ is

1. 1 *if* $p \geq 3$
2. $\frac{2p}{2p+1}$ *if* $2 \leq p < 3$
3. $\frac{2p}{2p+3}$ *if* $1 \leq p < 2$
4. $\frac{p}{p+3}$ *if* $0 < p < 1$

On the other we can make the path backward, by learning information on the representation via the mixing rate of the flow it generates. This way is taken in the appendice B of [1].

Definition 5.4. Let G be a locally compact σ-compact group. A continuous unitary representation of G is said to have *almost invariant vectors* if for every $\epsilon > 0$ and for every compact subset $K \subset G$, there exists a unit vector V such that $\|g * v - v\| < \epsilon$ for all $g \in K$.

A unitary action which does not have almost invariant vectors is said to be *isolated from trivial representation.*

If G is a semi-simple Lie group, a representation which is isolated from trivial representation is alos said to have a spectral gap.

With this definition, we can write the following theorem.

Proposition 5.3. *Let consider a representation of $SL(2,\mathbb{R})$ by a measure presearving action of automorphisms of a probability space. Let ρ be the repesentation associate on H the space of the L^2 function with zero averages. Assume there is $\delta \in]0;1[$ and a dense subset of the subspace of $SO(2,\mathbb{R})$-invariant function $H' \subset H$ consisting of functions ϕ for which the correlations $< \phi, \rho(g_t) >$, $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, decay like $O(e^{-\delta t})$. Then ρ is isolated from the trivial representation.*

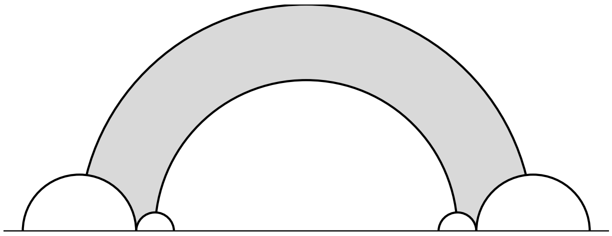


Figure 13: A fundamental domaine of a punctured torus

6 Example of the once ponctured torus

A torus can not have a hyperbolic structure, it has naturally a flat structure as the quotient $\mathbb{R}^2/\mathbb{Z}^2$. This changes when one study the one ponctured torus. It is a torus S where we choose a point p and remove it (or just marked it).

The construction of this object can be done in two manier at least. For the first construction, one have to choose a hyperbolic octogone where one side have length 0 and the two other one l, then we sew the border of two of this octogone which give a pair of pant. Finally we can glued with a twist τ to have the one ponctured torus.

A second construction is given by the representation. Given two hyperbolic isomorphism of $\mathbb{H}$ A and B with different fixed point on $\delta\mathbb{H}$ and with $H := ABA^{-1}B^{-1}$ the commutator should be a parabolic element.

A fundamental domaine is given by the following image:

Given two generators α and β of $\pi_1(S)$, two closed curves non homotopically trivial which intersect one, one can parametrize all other lamination. Indeed a given lamination $\lambda \in \mathcal{ML}$ is determined by the couple $(i(\alpha,\lambda), i(\beta,\lambda))$ where $i(.,.)$ is the geometric intersection number.

We have this useful lemma to estimate the length of the systole function in the Teichmuller space.

Lemma 6.1. *Pick γ a simple closed geodesic, and $X \in \mathcal{T}(S_{1,1})$, if X has Fenchel-Nielsen coordinate $(L, \frac{p}{q})$ with respect to γ, where $gcd(p,q) = 1$ and $\frac{p}{q} \in]0;1[$, then*

$$C_1(L)e^{\frac{-L}{2q}} < l_{sys}(X) < C_2(L)e^{\frac{-L}{2q}}$$

where $C_1(L)$, $C_2(L)$ both limits to 4 when p, q are fixed and L goes to ∞

Proof. Let $R(L)$ be the length of the shortest geodesic arc with endpoints on γ. We have

$$R(L) = 2log(coth(L/4)) = 2log(\frac{e^{l/2}+1}{e^{l/2}-1})$$

By the collar lemma 3.10. Then if we take α a simple closed curve that intersect γ, q times exactly, we obtain the following inequality with $a = l_\alpha(X)$

$$qR(a) < L < qR(a) + \frac{qa}{2}$$

Reorganizing the termes we have

$$e^{-a/4}tanh(a/4) < e^{-L/2q} < tanh(a/4)$$

As $a \to 0$ then $L \to \infty$

$$C_1(L)e^{\frac{-L}{2q}} < a < C_2(L)e^{\frac{-L}{2q}}$$

To complete the proof , we need to show that the length of α is shorter than any other géodesic closed curves, but with the collar lemma there is only one systole whose length goes to 0 and this is the case for α as $L \to \infty$.

If we don't don any approximation we have

$$2ln(\frac{1+e^{-L/2q}}{1-e^{-L/2q}}) < a$$

$\square$

Theorem 6.2. *Let $\nu(S_{1,1})$ be the finite measure on $\mathcal{P}^1\mathcal{M}(S_{1,1})$. Then*

$$\nu(S_{1,1})\{(X,\lambda) \in \mathcal{P}^1\mathcal{M}(S_{1,1}) | l_{sys}(X,\lambda) < \epsilon\} = O(\frac{\epsilon}{log\epsilon})$$

as $\epsilon \to 0$

Let fix $\epsilon = l_{sys}(X)$, and $T > 0$, then $\exists N = N(\epsilon, T)$ such that $\forall n \geq N$

$$|l_{sys}E_t(X,\lambda) - l_{sys}E_t(X, \frac{\gamma_n}{l_{\gamma_n}(X)})| < \epsilon$$

So we calculate T_n such that $\forall |t| < T_n$, $l_{sys}E_t(X, \frac{\gamma_n}{l_{\gamma_n}(X)}) < 2\epsilon$ and $T = liminf T_n$. If $T > 0$, $\exists N_1$ such that $\forall n \geq N_1$, $T_n \geq \frac{T}{2}$.

Then we set $N_2 = N(\epsilon, T/2)$ and we have $l_{sys}E_t(x,\lambda) \leq 2.5\epsilon$, $\forall 0 \leq t \leq T/2$

We have two useful tools to understand the length fonction along a earthquake path.

Lemma 6.3. *Let $X \in \mathcal{T}(S_g)$, and γ a curve which is part of a pant decomposition. χ_s is the twist of length s around γ, and b a closed curve with $i(b,\gamma) > 0$ then $s \mapsto l_{\chi_s}(b)$ is strictly convex.*

[6] proposition 10.8

Lemma 6.4. *If α is a closed curve, γ an other closed curve and λ a lamination.*

$$\begin{array}{rcl} \frac{dl_\alpha}{dt}(0) & = & \sum_{p_i \in \alpha\cap\gamma} cos(\theta_{p_i}) \\ \frac{dl_\alpha}{dt}(0) & = & \int_\alpha cos(\theta) d\theta \end{array}$$

[11] Corollary 3.3 and 3.4

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