Sorbonne Université

Master Thesis

Teichmuller theory and Thurston Earthquake flow

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Contents

1 Introduction

Since Bernhard Riemann, mathematician knew that only a finite number of parameter can describe a geometric surface. Considering this, we can be interested in the set of all geometry we can give to a given surface, modulo composition by map isotopic the identity, this set is called the Teichmuller space. Moreover other problem rise shortly after this definition. How can we deform in the natural way a geometry of a surface into an other? What does it mean that two geometries are closed one to each other? What are the natural boundary of the Teichmuller space?

Oswald Teichmuller, a german mathematician, gave answers to this question in the year preceding the second World war.He create the first metric on this space by finding a solution to an extremal problem: between two hyperbolic geometry on the same surface is there a function which minimize the deformation. the answer is not onmy yes, but this fonction is unique. It naturraly create a distance in the now called Teichmuller space by considering the logarithm of the deformation of the extremal function.

Figure 1: The mathematician Oswald Teichmuller

Thurston then add other important step to this theory. He underligne the role of lamination which are a generalisation of simple closed curve. And he create the earthquake flow which turn to play an important role in Teichmuller theory. Kerckhoff used this tool to show the Nielsen realisation conjecture in 1983 **??** which state that every finite subgroup of the mapping class group have a fixed point in the Teichmuller space.

An important question in Hyperbolic geometry is the asymptotic number of closed geodesic. To begin we can ask the number $\pi(X, L)$ of geodesic on a hyperbolic surface of length less than L. The answer was found by Delsarte, Huber and Selberg and is called the prime number theorem for hyperbolic surfaces (because of the ressemblance to the prime number theorem). It states that

$$\pi(X, L) \equiv e^L/L$$

as $L \to \infty$. A much harder question was to find the number, $\sigma(X, L)$, of simple (which don't intersect themselves) closed geodesic of length less than L on a hyperbolic surface X. It was find years later, in Mirzakhani's PhD, and we have

$$\sigma(X, L) \equiv C_X L^{6g-6}$$

As $L \to \infty$ where g is the genus of the surface X and C_X is a constant which depent of the geometry X. To do that Myriam Mirzhakhani conjugate the earthquake flow to the horocycle flow. This step give that the Earthquake flow is ergodic and allow us to use Birkoff theorem to understand asymptotic quantity.

The question is now to give error term to this quantity and to do that we need to understand better the earthquake flow.

In this master thesis, I will first give an introduction to Teichmuller theory and some useful tool for . Then we will review the proof of the Mirzhakani conhjugacie between the horocyclic flow and the earthquake flow. Finally we will discuss of a special case which can be the most simple example of hyperbolic surface, that is the once ponctured torus.

2 Notations

$\mathcal{ML}$	measured lamination
$\mathcal{MF}(S)$	space of all equivalence classes of measured foliations.
$\mathcal{QD}$	bundle of nonzero quadratic differential.
$\mathcal{T}_g$	Teichmuller space of surface of genus g
$Mod(S)$	The modular group of a surface S
$\mathcal{P}^1\mathcal{M}_g$	Product of $\mathcal{T}_g$ and $\mathcal{MF}(S)$ quotiented by the modular group.
$\mathcal{GC}(S)$	the space of geodesic currents
$\mathbb{D}$	the Poincaré disk
$\mathbb{S}$	the unit circle seen as the boundary of $\mathbb{D}$
$\bar{\mathbb{D}}$	$\mathbb{D} \cup \mathbb{S}$
$\mathbf{S}$	set of all simple closed curve of a surface
$D_\gamma(\alpha)$	Dehn's twist of α around γ

3 Teichmuller Theory

3.1 First definitions

To beggin we will in this section give the very definition to introduce the theory of Teichmuller space

Definition 3.1. Teichmuller space Let S be a surface of genus g, a marking of S is a couple (X, f) made of a closed Riemann surface X and of one homeomorphism $f : S \to X$ which preserve the orientation. On the set of the marking S, we have a equivalence relation, $(X_1, f_1) \sim (X_2, f_2)$ if there exist $\alpha : X_1 \to X_2$ such that $f_2 \circ \alpha \circ f_1^{-1}$ be an homeomorphism of S preserving the orientation and isotope to the identity map. The set of the marking quotient by the previous relation is the Teichmuller space and is written $\mathcal{T}_g$.

Remark. If $g \geq 2$, for every closed curve α of S, there is only one closed geodesic of X freely isotope to $f(\alpha)$. Moreover if α is simple, so is the geodesic. We will note $l_\alpha(X)$ its hyperbolic length and we will take the weakest topologie on T_g which make this map continuous.

This give a map $L : T_g \to \mathbb{R}^{\mathbf{S}}$, where $\mathbf{S}$ is the set of all simple closed curve of a given surface. We can ask if this map si injective, i.e. that a geometry is given by the length of the set of simple closed curve. The answer is yes and more precisely one can choose only $9g - 9 + 3n$ curve so that this map is injective [3] Theorem 10.7. This give the intuition that Teichmuller space can be only by using a finite set of parameter. We will see after that other coordinates which have nice propeties for other utilisations.

An other curious but important fact about simple closed geodesic is that the sum of a function of theirs length is equal to a constant which does not depend on the genus or the geometry of the surface.

Theorem 3.1. *Let X be a hyperbolic surface then*

$$\sum_{\gamma} \frac{1}{1 + e^{l_\gamma(X)}} = \frac{1}{2}$$

Where the sum is taken other all simple closed geodesic.

This theorem was prove by McShane and bear his name [12]. A good introduction with a proof using measure on tree can be found in this paper [7].

Definition 3.2. Modular Space The modular group is the group of homeomorphism of S which respect orientation quotiented by the one isotopic to the identity map. We will call this group Mod_g It act discretely on the Teichmuller space T_g and the quotient is called the moduli space and is written $\mathcal{M}_g$.

The Teichmuller space is the universal covering of the modular space. IN fact the Teichmuller space is homeomorph to a sphere and so is contractible and in particular, connected and simply connected. It is the universal covering of the moduli space.

We can ask ourselves how look this spaces and if we can give an easy representation of them. The fact is that the modular space is not manifold but an orbifold, that is a space which locally look like a ball of a vector space quotiented by a finite group.

A begginnig is to give a set of generator of the modular group.

Definition 3.3. Dehn twist Let γ be a simple closed curve. There is a tubular neighborhood of γ called A homeomorph to $[0; 1] \times S^1$. A Dehn's twist around γ is the homeomorphism which is the identity out of A and is $(t, s) \mapsto (t, e^{2i\pi t}s)$ on A.

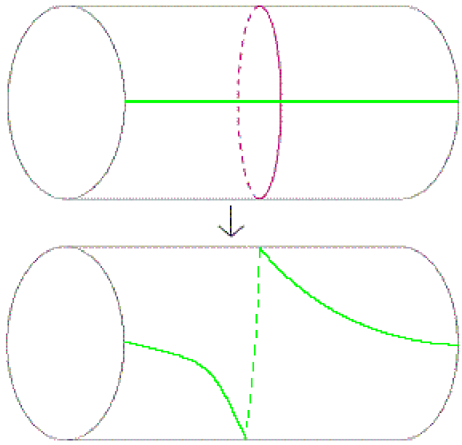


Figure 2: A Dehn twist

Remark. The Lickorisk theorem states that the modular group is generated by the Dehn's twist and more precisely that one can choose only $2g+1$ generators [9].

Definition 3.4. Measured foliation Given a surface S and a finite set of points $P = (p_1, p_2, ...)$, given a open covering U_i on $S - P$, a collection of C^1 real function ν_i such than $\|d\nu_j\| = \|d\nu_i\|$ on $U_i \cap U_j$, and near each singular point p_s a coordinate neighborhood V with complex coordinate z such that $\|d\nu\| = \|Im(z^{\frac{k}{2}} dz)\|$ for some positive integer k called the degree of the singular point, leaves of the foliations are the graphs immersed S in along dv is constant. In addition if each boundary circle pf S is contained in a singular leaf, then ti is called a measurd foliation.

The height $h_\gamma(\|d\nu\|)$ of a (free homotopy class) of a loop γ on S is the infinimum in the homotopy class of the integral by $\|d\nu\|$

$$h_\gamma(\|d\nu\|) = inf_{\gamma \equiv \gamma'} \int_g amma \|d\nu\|$$

The topology on the measured lamination that we will use is the weakest that make the height functions continous.

Remark. We won't actually study the set of measured lamination but the equivalence class of

$$h_\gamma(\|d\nu\|) = h_\gamma(\|d\mu\|), \text{ for each loop} \gamma \in S$$

. We can equivalently use Whitehead equivalence relation on singular foliations by collapsing critical intervals to points and taking isotopy of foliation.

Let $\mathcal{MF}(S)$ be the space of all equivalence classes of measured foliations.

Definition 3.5. Lamination A lamination is a closed set made of an union (non necessarely finite) of geodesic. For each point x in λ a lamination passed only one geodesic of the lamination. We will write this space $\mathcal{ML}(x)$.

Definition 3.6. Geodesic currents Let $\mathcal{M}_\infty$ be the space of unordered pairs of distinct points in $\mathbb{S}^1$

$$\mathcal{M}_\infty := (z, w) \in \mathbb{S}^1 \times \mathbb{S}^1, z \neq w//(z, w) \equiv (w, z)$$

Let G be a discret torsion-free group in $PSL(2, \mathbb{R})$ such that $\mathbb{H}//G = S$ is a hyperbolic surface. A geodesic current μ on S is a G-invariant Radon measure on $\mathcal{M}_\infty$. We will note $\mathcal{GC}(S)$ the space of geodesic currents, see [5]

Remark. $\mathcal{GC}(S)$ have a natural topology which is the weak $*$ convergence on continous functions.

Exemple 3.1. Let $g \in G$ be an element that represent a simple closed geodesic on the surface S. The G-orbit of the fix point of g is a discret set on which we can put diracs measure. This link the geodesic currents to the measured lamination.

Remark. A mullticurve is a formel sum of geodesics $\gamma = \sum a_i\gamma_i$. The space of lamination is in some aspect the closure of the set of all multicurve

Definition 3.7. Intersection number Consider the square $\mathcal{M}_\infty^2 := \mathcal{M}_\infty \times \mathcal{M}_\infty$. In this space we can consider the open subset $\mathcal{IM}_\infty^2$ corresponding to pair pairs of geodesics which have transversal intersections in $\mathbb{H}$. G act on $\mathcal{IM}_\infty^2$. If μ and ν are geodesic currents in $\mathcal{GC}(S)$, the product $\mu \times \nu$ define a G-invariant measure on $\mathcal{IM}_\infty^2$. Finally if we take the mass of the total space $\mathcal{IM}_\infty^2//G$, the reasult is called the intersection number, $i(\mu, \nu)$

Proposition 3.2.

$$i : \mathcal{GC}(S) \times \mathcal{GC}(S) \to \mathbb{R}_+$$

is continuous and bilinear [2].

Remark. If α and β are simple closed geodesics (dirac measure in $\mathcal{GC}(S)$), then the intersection number is the number of intersection between α and β. Actually, one can define intersection in this way, first on simple closed geodesic, then by bilinearity on multi-curves and finally by continuity on geodesic current.

Remark. The topology on $\mathcal{ML}$ is the weakest that make $i(.,.)$ a continous function.

Definition 3.8. A quadratic differential is a section of the square of the tangent space to X. It is locally as $\phi = \phi(z)dz^2$.

Remark. Si $\phi(p) \neq 0$ on peut trouver une carte contenant p dans laquel $\phi = dz^2$. Ainsi ϕ détermine une métrique plate sur X et un feuilletage $\mathcal{F}$ correspondant aux lignes horizontales.

Remark. If $\phi(p) \neq 0$ we can find a map including p in which $\phi = dz^2$. Hence ϕ determine a flat metric on X and a foliation $\mathcal{F}$ corresponding to horizontal lignes.

A quadratic differential is said to be integral if

$$\|\phi\| = \int_X |\phi| < \infty$$

We will write $\mathcal{Q}(x)$ the Banach space of integral quadratic differentials.

3.2 Flow on Teichmüller space

We will define the main object of this paper, earthquake flow.

Definition 3.9. The earthquake flow is family of maps defined for $t \in \mathbb{R}$

$$\begin{array}{rccc} E_t : & \mathcal{ML} \times \mathcal{T}_g & \to & \mathcal{ML} \times \mathcal{T}_g \\ & (\lambda, X) & \mapsto & (\lambda, E_{t\lambda}X) \end{array}$$

where $E_{t\lambda}$ is first defined if λ is a simple closed curve. In this case, we open the surface along lambda, then twist the left part of t unit and put it together again. Then if γ_1 and γ_2 are two curves that don't intersect, E_{γ_1} and E_{γ_2} commute. So we can define the earthquake map on weighted multicurves by twisting one curve after the other by amount proportional to the weight. Finally as multicurves are dense in the set of lamination we can extend this map for any lamination [6].

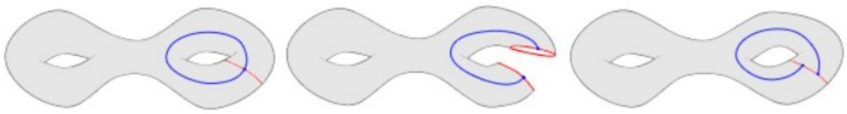


Figure 3: Effect of a twist on a transverse curve, image from [15]

Remark. If the lamination is just a simple closed curve γ then $E_{l_\gamma(X)}(X,\gamma)$ is just a Dehn twist aroud γ. Moreover if we take a decomposition in a pair of pant that contait γ, it is just a translation in the coordinate of the twist of γ.

One could give a precise proof that the extension of the earthquake map from the multicurves to the lamination is rigourous, that is if we take two sequence α_n and α'_n of multicurves which converge to the same lamination, then the sequences of earthquake map along this multicurves converge to the same map.

Remark. The earthquake flow is an isometry outside the support of the lamination and is continous outside the atomic part, i.e. the simple closed geodesics of the lamination.

Thurston show that given two point in the Teichmüller space, there is a lamination λ such that the earthquake flow from one point with respect to λ reach the other point (also in [6]).

We can ask ourselves what is an invariant measure of this flow.

Definition 3.10. The Weil-Peterson form is the the form

$$\omega_{WP} = \sum dl_i \wedge d\tau_i$$

Where $(l_1, ..., \tau_1)$ are the Fenchel-Nielsen according to a pant decomposition.

This give a measure μ_{WP}.

There is a finite measure ν_g in the Lebesgue measure class on $\mathcal{P}^1\mathcal{M}_g$ which is invariant under the earthquake flow. This measure projects to the volume form given by $B(X) \times \mu_{WP}$ on $\mathcal{M}_g$, where

$$B(X) = \mu_{Th}(\lambda \in \mathcal{ML}, l_\lambda(X) \leq 1)$$

There are two other important flows, the geodesic flow and the horocyclic flow. First there is a natural homeomorphism between $T^1\mathbb{H} \simeq PSL_2(\mathbb{R})$, since $PSL_2(\mathbb{R})$ act simply trasnsitively on it. This morphism can be choosen up to a conjugaison via an other element of $PSL_2(\mathbb{R})$. We we will be interested in a special kind of subgroup.

Definition 3.11. A fuchsian group Γ is a finitely generated and discrete subgroup of $PSL_2(\mathbb{R})$. Then Γ act discontinuously on $\mathbb{H}$.

Then a Hyperbolic surface can be represented as $PSL_2(\mathbb{R})/\Gamma$. If U is a one parameter subgroup of $PSL_2\mathbb{R}$ it act on the quotient.

There are two important exemple:

Definition 3.12. The geodesic flow is a flow on the Teichmuller space given by the action of the diagonal matrices

$$u_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

Definition 3.13. The horocycle flow is a flow on the bundle of nonzero quadratic differential, $\mathcal{QD}$, of the Teichmuller space given by the unipotent action of

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

The geodesic flow is also the flow that we obtain by following geodesic line on $\mathbb{H}$ and the horocycle is the flow we obtain by following curves which are everywhere orthogonal to the geodesic, which is the horizotal line and the circle tangent to the real line.

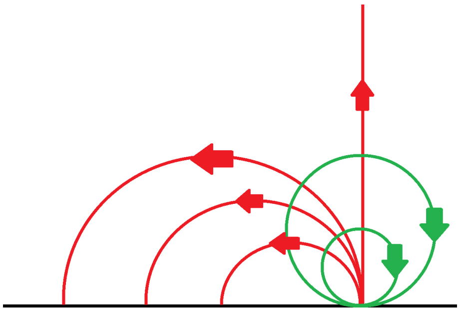


Figure 4: Representation of the horocycle flow, in green, and the geodesic flow, in red

An important relation is how this two flows interact between each other

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} u_t = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} = \begin{pmatrix} 1 & se^{2t} \\ 0 & 1 \end{pmatrix}$$

So the the conjugaison of the horocyclic flow by the geodesic one is still the horocyclic flow.

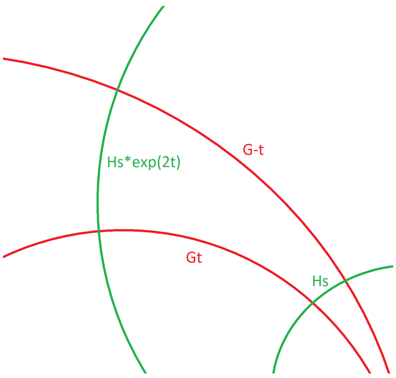


Figure 5: The conjugaison of the horocycle flow by the geodesic one.

3.3 Decomposition of hyperbolic surface

One way to construct all hyperbolic surface is to decompose them in elementary piece, that we will call pair of pant.

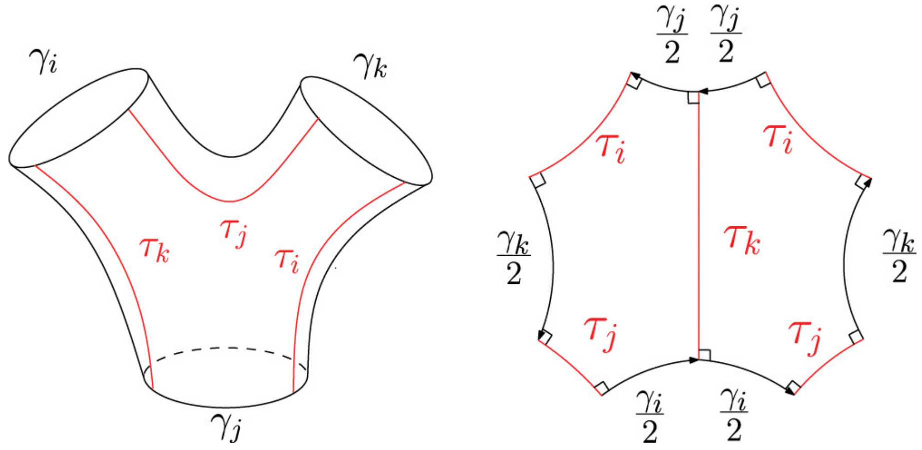
A hyperbolic geometric exercice show that a hexagone which side are geodesics and with right angles is determined by the lenght of three sides which are not consecutifs.

On the image above γ_i, γ_j and γ_k determined the hexagone. Then we can glued them to have a pair of pant.

Definition 3.14. A pair of pant is a hyperbolic surface with three geodesic boundaries and no ponctured.

Remark. The pair of pant is uniquely determined by the lenght of the three boundarie geodesics.

Remark. The lenght of one or more geodesic can go to zero and the boundaries become a ponctured.



We can now decompose, with the following theorem, all hyperbolic surfaces in a collection of pair of pants.

Theorem 3.3. *Let S be a surface of genus g and with n ponctured. There is a set of $3g-3+n$ simple closed curves $(\gamma_1, ..., \gamma_{3g-3+n})$ such that $S \ \Cup \gamma_i$ is a disjoint collection of pair of pants.*

Definition 3.15. Given a surface S and a pant decomposition $\gamma_1, ..., \gamma_{3g-3+n}$, we have a map

$$(S) \to (\mathbb{R}^{+3g-3+n}, \mathbb{R}^{3g-3+n})X \mapsto (l_{\gamma_1}(X), ..., l_{\gamma_{3g-3+n}}(X), \tau_{\gamma_1}(X), ..., \tau_{\gamma_{3g-3+n}}(X))$$

This map is injective and is call the Fenchel-Nielsen cordinates.

Theorem 3.4. *There is a collection $\delta_1, ..., \delta 9g-9$ of simple closed curves such that $\mathcal{T}_g \to \mathbb{R}^{9g-9}$ is injective.*

Proof. Let take $(\gamma_1, ..., \gamma_{9g-9})$ a decomposition in pair of pants, $(\alpha_1, ..., \alpha_{9g-9})$ be a collection of simple closed curves such that $i(\gamma_i, \alpha_i) > 0$ and $i(\gamma_i, \alpha_j) = 0$ for $i \neq j$, finnely we take $\beta_i = D_{\gamma_i}(\alpha_i)$. We want to show that the length of this collection of $9g-9$ curves determined the hyperbolic structure $X \in \mathcal{T}_g$. X already has the Fenchel-Nielsen coordinate, of the pant decomposition $(\gamma_1, ..., \gamma_{3g-3})$, $(l_{\gamma_1}(X), \tau_{\gamma_1}(X), ..., l_{\gamma_{3g-3}}(X), \tau_{\gamma_{3g-3}}(X))$ so we need only to show that the parameters $\tau_{\gamma_i}(X)$ are determined by the lenght of the collection. Up to a renormalisation we can take $\tau_{\gamma_i}(X) = 0$ for every i. Now let's take $t = (t_1, ..., t_{3g-3}) \in \mathbb{R}^{3g-3} \backslash 0$. We will note X_t the hyperbolic geometry which has the same length as X and twist parameters t in the Fenchel-Nielsen coordinate of the pair of pants. So $X_0 = X$. We will consider the function $A(t) = l_{\alpha_1}(X_t)$ and $B(t) = l_{\beta_1}(X_t)$. This function depend only of t_1 as $i(\gamma_i, \alpha_j) = 0$ for $i \neq j$. Moreover they are strictly convex and by definition we have $A(t_1 + l_(\gamma_1)(X)) = B(t_1)$. We will show that there is no $t_1 \neq 0$ such that $A(t_1) = A(0)$ and $B(t_1) = B(0)$ that is $A(t_1 + l_(\gamma_1)(X)) = A(l_(\gamma_1)(X))$. We will note $s = t_1$ and $L = l_(\gamma_1)(X)$ Suppose there is $s \neq 0$ such that $A(s) = A(0)$, we can take $s > 0$, the other case is symetric. If $s < L$, then by convexity for every $t \in]0; s[$, $A(t) < A(0) = A(s)$ and A is stricty increasing after s so as $s < L < L+s$ we have $A(L) < A(L+s)$. If $s > L$, then $L < s < L+s$ and $A(L) < A(L+s)$. The final case $s = L$ is also impossible since we would have $A(0) = A(L) = A(2L)$. Finally we can make the same argument for the other twist parameters which conclude the proof. $\square$

3.4 The collaring theorem

We will now give a useful tool to give necessary condition on length of two intersecting geodesics.

The collar function $\eta :]0; \infty[\to]0; \infty[$ is defined as follow. We draw a segment of lenght $l > 0$ on a geodesic γ, then we project perpendicullary to the geodesic the end of this segment to infinity and draw the geodesic δ which have this endpoint. So we have $\eta(l) = d(\gamma, \delta)$.

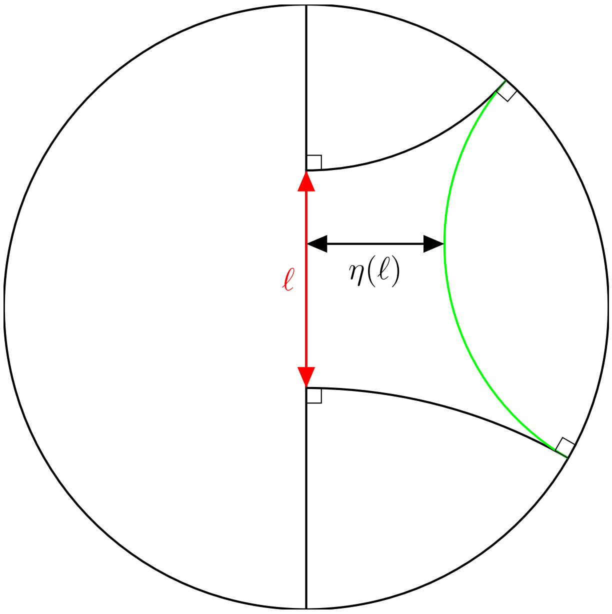


Figure 6: In red the segment of length l and in green the geodesic whose endpoint are orthogonal projection of the end of the segment.

It is an exercice to show that:

$$\eta(l) = \frac{1}{2} ln(\frac{cosh(l/2) + 1}{cosh(l/2) - 1})$$

This quantity will code a long in the "tube" generate by a simple closed geodesic. We give a definition to make this a little more precise.

Definition 3.16. Let γ be a simple closed geodesic of length l on a hyperbolic surface X. If the δ-neighborhood

$$A_\delta(\gamma) := \{x \in X | d(x, \gamma) < \delta\}$$

is isometric to the δ-neighbohood of the unique simple closed geodesic on the cylinder of modulus $\frac{\pi}{l}$, we say that γ admit a δ-collar , or that $A_\delta(\gamma)$ is the δ-collar of γ.

We can now state a useful theorem.

Theorem 3.5. *Let X be a complete hyperbolic surface, and let $\Gamma := \gamma_1, \ldots$ be a collection of disjoint simple closed geodesic, each γ_i of length l_i. Then $A_{\eta(l_i)}(\gamma_i)$ are collars around the γ_i, and they are disjoint.*

Proof. Choose γ_1 and γ_2 and add other simple closed curve to have a maximal multicurve that includes both. Now cutting along this curve we have a set of pair of pant so we only have to show that the $\eta(l_i)$ neighboorhood of γ_i the boundaries of the pair of pant do not intersect each other. We cut the pair of pant along geodesics comming from a boundary C and meeting the two other boundaries A and B. We unfold this figure in the hyperbolic plane and name the side of the octogone following the figure below.

Since a' and b' have the common perpendicular C', they do not intersect and similarly for a'' and b''. The theorem follow easly by the definition of the function η.

□

There are some corollaries which follow from this theorem and are ready to use in many occasions.

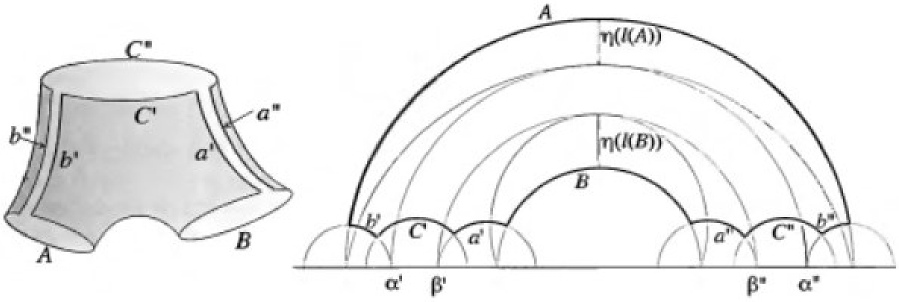


Figure 7: image from [4]

Corollaire. *Let X be a hyperbolic surface, and γ_1, γ_2 two simple closed geodesics on X of lengths l_1 and l_2. If $l_2 < 2\eta(l_1)$, then either $\gamma_1 = \gamma_2$ or $\gamma \cap \gamma_2 = \emptyset$*

Proof. If $\gamma_1 \neq \gamma_2$ and $\gamma \cap \gamma_2 \neq \emptyset$ then γ_2 must cross the collar neighboorhood of γ_1 from one boundary to the other and so have length strictly superior than $2\eta(l_1)$. □

Corollaire. *Let X be a hyperbolic surface, and let γ_1, γ_2 be two simple closed geodesics with lengths $< ln(3 + 2\sqrt{2})$. Then either $\gamma_1 = \gamma_2$ or $\gamma_1 \cap \gamma_2 = \emptyset$.*

Corollaire. *Let X be a complete hyperbolic surface, γ a simple closed geodesic on X of length l, and A_γ the collar around γ. Then any simple geodesic δ on X that enter A_γ either intersect γ or spirals towards γ.*

Proof. Suppose the geodesic δ enters A_γ. We can lift the situation in the universal cover of the hyperbolic disc, where $\tilde{\gamma}$ is a lift of γ can be a diameter of the circle. Then if $\tilde{\delta}$ do not intersect $\tilde{\gamma}$ and do not have the same point at infinity, then its two endpoint are on the same side of $\tilde{\gamma}$ in the disc. Now the translation along $\tilde{\gamma}$ of length $l_\gamma(X)$ is in the representation of $\pi_1(X)$. If $\tilde{\delta}$ intersect A_γ then by the definition of δ it will intersect with its image by the translation cited before and hence is not simple in X. □

4 Isomorphisme of Mirzhakani

The aim of this part is to demonstrate the followoing statement

Theorem 4.1. *There is a measurable conjugacy F between the earthquake flow $(\lambda, X) \mapsto (\lambda, E_{t\lambda}(X))$ on $\mathcal{ML} \times \mathcal{T}_g$ and the Teichmüller unipotent flow action of*

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

on the bundle $\mathcal{QD}$ of nonzero quadratic differentials over Teichmüller space $\mathcal{T}_g$.

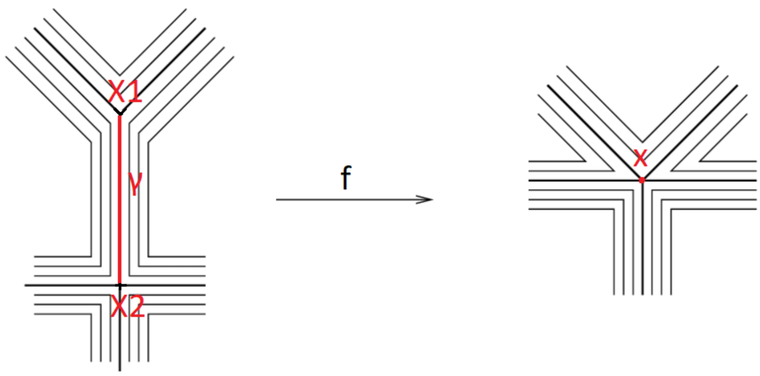
$$\begin{tikzcd} \mathcal{ML} \times \mathcal{T}_g \arrow[r, "E_t"] \arrow[d, "F"'] & \mathcal{ML} \times \mathcal{T}_g \arrow[d, "F"] \\ \mathcal{QD} \arrow[r, "u_t"'] & \mathcal{QD} \end{tikzcd}$$

4.1 Tightening map

A first coresspondance, found by Thurston, exist between measured foliation and measured lamination. We will mostly follow the paper of Levitt [8].

Definition 4.1. We say that two foliations are equivalent if we can pass from one to the other by Whithead (see definition below) moves or isotopy (homeomorphism isotopic to the identity).

Definition 4.2. Given a measure foliation, an critical segment γ is an arc between two singularities along a leaf which is not a simple closed curve. There is a map f homotopic to the identy that collapse γ to a point x and is identity outside a neighboorhood of γ which contain no other singularity . Doing so we reduce the number of singularities of the foliation and if the extremoties of γ are singularities of order k_1 and k_2, x is now a singularity of the new foliation of order $k_1 + k_2 - 2$.



Theorem 4.2. *Let X be a closed orientable hyperbolic surface and $\mathcal{F}$ a foliation. There is a canonical geodesic lamination $\gamma(\mathcal{F})$ associated to $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{F}'$ are associated foliation then $\gamma(\mathcal{F}) = \gamma(\mathcal{F}')$. In the opposite direction given a geodisic lamination γ, one can find a foliation $\mathcal{F}$ such that $\gamma(\mathcal{F}) = \gamma$ and it's unique up to equivalence.*

Definition 4.3. A *tranverse curve* is a simple closed curve C which is never tangent to $\mathcal{F}$ and contain no singularity of $\mathcal{F}$.

Remark. Since $\mathcal{F}$ contain only saddle singularities, C cannot be contractible, therefor C is isotopic to a simple closed geodesic.

We will work on the universal cover of X, which is the Poincaré disc $\mathbb{D}$ with circle "at infinity" $\mathbb{S}$. We call $p : X \to \mathbb{D}$ the universal projection. $\tilde{\mathcal{F}}$ is $p^{-1}(\mathcal{F})$.

We will say that a foliation follow the (*) condition if the following is true:

If f_1 and f_2 are two compact homotopic leaves then all leaf in the open annulus between them is also compact.

Lemma 4.3. *Let h be a leaf of $\tilde{\mathcal{F}}$. Each end of h converge to a point of S; the two point at infinity cannot be the same*

Proof. First, we should notice that the behavior of leaf at infinity do not change if we take an equivalent foliation. Indeed a homeomorphism ϕ on a compact fundamental domain isotopic to the identy can be extend to an homeomorphism $\tilde{\phi}$ on $\mathbb{D}$ such that $dist(x, \tilde{\phi}(x)) \leq K$. This implie that $\tilde{\phi}$ extend es the identity on the boundary $\mathbb{S}$.

Then given a leaf h of $\tilde{\mathcal{F}}$, we take a half leah h_0. If $p(h_0)$ is compact or spiral toward a compact leaf of $\mathcal{F}$ then the first part of the lemme is imediate.

Otherwise, $p(h)$ meet a transverse curve C infinitely often. With an isotopie we can take C to be a geodesic. Now h_0 can meet a connect component of $\tilde{C} = p^{-1}(C)$ only one time. Otherwise there will be a disk bound by an arc of $\tilde{C}$ and an arc of h_0, which is impossible considering the transversity of C and that $\mathcal{F}$ have no 1-type singularities.

Now every compact of $\mathbb{D}$ meet a finite number of connected components of $\tilde{C}$ so the limit set of h_0 must be on $\mathbb{S}$. This limit set is connected and non empty. Moreover it should not contains any end of a connected component of $\tilde{C}$. But the ends of connected components of $\tilde{C}$ are dense in $\mathbb{S}$ as $\tilde{C}$ is the image of a geodesic by $\pi_1(X)$. This show the first point of the lemma.

The second assertion is clear if $p(h)$ is compact or if it meet a transverse curve C at least twice since then every connected componnents of $\tilde{C}$ separate the end of h.

Otherwise $p(h)$ spiral toward two compact leaf f_1 and f_2. If $f_1 = f_2$ and the two end point of h are the same then there will be a singularity that would not be a saddle. $f_1 \neq f_2$ is impossible since $\mathcal{F}$ follow the condition (*). □

We can now associate to every leaf h a geodesic $\gamma(g)$ by joining the endpoint. Then $\gamma(\tilde{\mathcal{F}}) = \cup_{h \in \mathcal{F}} \gamma(h)$ is a disjoint union of geodesic invariant by $\pi_1(X)$. We have to show that this set is closed to conclued that we have a lamination.

Lemma 4.4. *$\gamma(\tilde{\mathcal{F}})$ is closed in $\bar{\mathbb{D}}$*

Proof. Let $g_n = \gamma(h_n)$ be a sequence of geodesics in $\gamma(\tilde{\mathcal{F}})$ converging to a geodesic g. We want to show $g \in \gamma(\tilde{\mathcal{F}})$. We can suppose that all the g_n are distinct of g and are all on the same side.

Let L be the limit set in $\bar{\mathbb{D}}$. For all leaf m in $\tilde{\mathcal{F}}$, we call $\bar{m}$ the closure of m by adding the two end point in $\mathbb{S}$. Then L meet at least one connected component of $\bar{\mathbb{D}}$
$\bar{m}$. As the end points of all leaf of $\tilde{\mathcal{F}}$ is a dense subset of $\mathbb{S}$, L contain a leaf h. Taking a half-leaf h_0, we want to show that the end point is the same as one of g.

A first case is if there is a simple closed curve C transverse to $\mathcal{F}$ which meet $p(h_0)$ infinitely often. If h_0 does not converge to the corresponding point at infinity then there would be a connected component of $p^{-1}(C)$ that contains the point of infinity of h_0 but does not contain the point of infiny of h_n which is impossible for large n.

A second case is if $p(h_0)$ spirals toward a compact leaf, then closed leaf close to $p(h_0)$ also spirals toward the same compact leaf. Then h_0 converges to one of the points at infiniy of g which is a point at inifinity of h_n fot n large.

Finally if $p(h)$ is compact then $p(h_n)$ spirals toward it for large n, therefore $\gamma(h)$ and g have one point in common at infinity. If the second was different, by applying a transformation leaving $\gamma(h)$ invariant (but no g), we would separate h from the leaves h_n, and it is a contradiction. $\square$

Now we want to exhib an inverse construction which take a lamination λ and give a foliation μ. To do this we still consider $\tilde{\lambda}$ in the universal cover. We will suppose that every complementary region is a ideal polygon.

We can built a skeleton that it compose of edge between vertex and a choosen point in the center. After building the skeleton for every polygon we fill the complementary region which will be between four edge by line between the two vertex.

This map is often called the "collapsing" map and its inverse the "tightening" map.

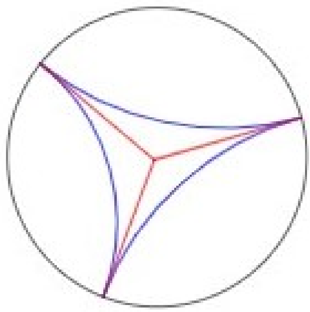


Figure 8: In red the skeleton of the ideal triangle in blue, Image from [14]

The measure we put on this foliation is uniform on every region between four edges of the skeleton.

4.2 Correspondance between foliations and quadratic differentials

For a quadratic differential q, one can define two measured foliations, the horizontal $h(q)$ and the vertical $v(q)$ corresponding in locate coordinate to $Re(z)$ and $Im(z)$. This give a map two the pair of foliation but it is not the subjectif, we should restrict ro the image. Define $\Delta = (\alpha, \beta) : i(\alpha, \gamma) = i(\beta, \gamma) = 0$, for some $\gamma \in \mathcal{MF}$. Δ contain the diagonal (α, α) and is kind of "fat" diagonal.

Lemma 4.5. *For any $q \in \mathcal{QD}$, $(h(q), v(q)) \notin \Delta$*

Theorem 4.6. *The map $q \mapsto (h(q), v(q))$ define a homeomorphism $\mathcal{QD} \to \mathcal{MF} \times \mathcal{MF} \backslash \Delta$*

4.3 Shear Cordinate

Finally there is a map that, given a hyperbolic structure X and a lamination λ create a measured foliation which is transverse to λ.

For simplicity we will ask that λ is a maximal lamination i.e. if $\tilde{\lambda}$ is the pre-image of λ in the universal cover $\mathbb{D}$, $\mathbb{D} \backslash \tilde{\lambda}$ is made of ideal triangle. We will first work in this triangles minus a region on the center, then give a measure in this foliation, and finnally show that ot is a homeomorphism.

So in one the ideal triangle, given two side we can draw an arc prependicular with the this two sides, which is the intersection of the ideal triangle with the circle tangent to the boundary of $\mathbb{D}$ in the common extremity the two arc chosen. Then by rescaling by a factor $r \in [0;1]$ and doing the same procedure for the two other pair we get a foliation in the ideal triangle minus a locus in the center.

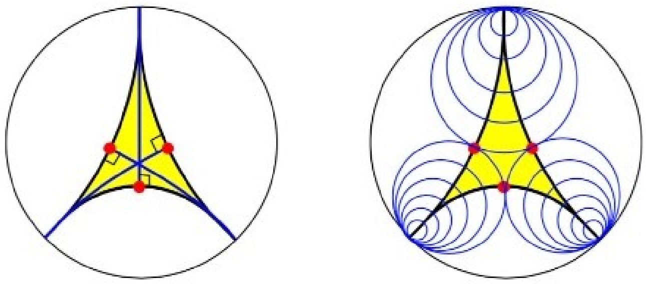


Figure 9: Image from [10]

Then we can find a full foliation by pitching the resulting.

We have a natural transverse measure to this foliation. For an arc in one ideal triangle of the lamination we can project it along the leaf of the foliation to asegment in the edge of the ideal triangle, the length of the arc will be the length of the segment. As the leaf are horocycle circle based on the same point, it will not depend of which side we choose to project. Then given an arbitrally arc we decompose it along the ideal triagle it meet.

We want to shwo that this construction is reversable, that is given $\mu \in \mathcal{MF}_\lambda$, the set of foliation transverse to the lamination, we can construct $X \in \mathcal{T}_g$ whose horocycle foliation is μ.

The idea is that, if we already know X, the lamination λ can be lift to $\widetilde{lambda}$ which is invariant of Γ the fushian group of X. But we can built $\widetilde{lambda}$ only with the information given by μ.

We will note $\tilde{\mu}$ the preimage of μ in $\mathbb{D}$. If we consider two triangles T_1 and T_2 that are complementary region of $\tilde{\lambda}$ and we suppose we take a segment A in a leaf of $\tilde{\mu}$ that goes to an edge of T_1 to T_2. We name v_1 and v_2 the two vectors with footprint in the edge and tangent to them. Then there is a Moebuis transformation S which take one to the other. With one more information, the "shear", we can place T_2 on $\mathbb{D}$, according to the position of T_1

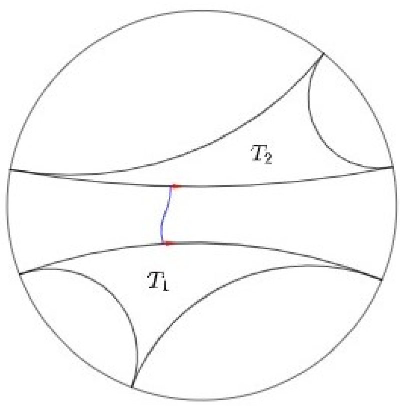


Figure 10: Image from [14]

Indeed given only a Moebuis transformation we still have a one parameter families of triangle T_2 with an edge generated by v_2. To fix this we trace two orthogeodesic comming from the vertex of the ideal triangles facing the considered edges and from the point of intersection in

T_1 we follow a leaf of the foliation, then when we meet T_2 we have to move along the geodesic edge to find the other point of intersection. This lenght is the shear between T_1 and T_2

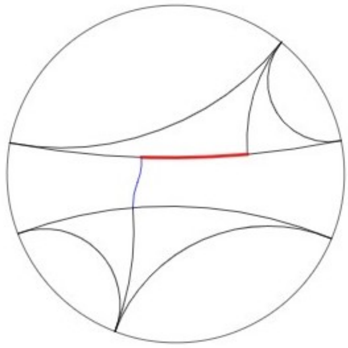


Figure 11: Image from [14]

Let I be the set of all triangles in $\mathbb{D}$ that A meet. For each $i \in I$ we can define v_i^+ and v_i^- the vectors tangent to the edge of the corresponding triangle at the intersection of the edges and A. Note that I is a countable totally ordered but non well ordered set. So if we take S_i the Moebuis transformation which take v_i^- to v_i^+, we have to give a meaning of the expression

$$\prod_{i \in I} S_i$$

Definition 4.4. Given a countable totally ordered set of indice I and element S_i in a Banach algebra, we say that $\prod_i S_i$ is well defined and equal to S if for any increasing chain

$$I_0 \subset I_1 \subset ... \subset I$$

with $\cup_k I_k = I$ we have $lim_{k \to \infty} \prod_{i \in I_k} S_i = S$.

Lemma 4.7. *For element s_i in a Banach algebra index by a countable totallyordered set, if $\sum \|s_i\| < \infty$, then $\prod(1 + s_i)$ is well-defined.*

Proof. For $1 \leq m \leq n$, we have

$$\| \prod_{i=1}^{n} (1 + s_i) - \prod_{i=1, i \neq m}^{n} (1 + s_i) \| \leq \|s_m\| \| \prod_{i=1, i \neq m}^{n} (1 + s_i) \| \leq \|s_m\| \prod_{i=1}^{n} (1 + \|s_i\|)$$

Or with the assumption $\sum \|s_i\| < \infty$ we have that $\prod(1 + \|s_i\|) \leq C < \infty$, so removing or adding $1 + s_m$ produce a change bound by $\|s_m\| C$. □

Now we want to apply this lemma to $S_i - Id$, with Id the identity matrice.

Lemma 4.8. *For the previous S_i, if we note $s_i = S_i - Id$ we have $\sum \|s_i\| < \infty$.*

Proof. Each S_i is conjugate to a horocycle transformation of time one. The conjugacy is made by a geodesic flow along the edges of the triangle. We can compute

$$\begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & e^{-t} \\ 0 & 0 \end{pmatrix}$$

So the norm of s_i is inversally corrolated to the amount of geodesic flow used in the conjugaison. Now we can parttion the indice set I into finely many subset (I_k) according to to which spike of the lamination the arc of A cross. Then for a spike the sum of $\|s_i\|$ where $i \in I_k$ is finite, indeed the distance between two neighbooring crossing is bound below by a constant and so the amount of time we should do the geodesic flow increase at most linerally and finnaly the norm of the s_i should decraese geometrically. $\square$

So we can conclude that there is an unique Moebuis transformation S equal to the meaningfull expression $\prod_i S_i$.

Now we can conclude the proof. There exist, without any hyperbolic structure X topological classes for $\tilde{\mu}$ and $\tilde{\lambda}$. We choose one arbitrary ideal triangle T_1 in the lamination. For every other triangle T_2, the Moeubuis transformation and the shear are data that can be compute only using the transverse measure of $\tilde{mu}$. So we can place T_2, and the other triangle. The closure of this set give the lamination $\tilde{\lambda}$. $\tilde{\lambda}$ will be preserve by a fushian group Γ and we will have $X = \mathbb{D}/\Gamma$.

5 Mixing rate

We will first give the behavior of two flows we described before.

Theorem 5.1. *The ergodic flow and the horocycle flow are mixing.*

[11]

Proof. The step will be in four step, first we will show that the ergodic flow is ergodic, the horocyclic flow is ergodic, the ergodic flow is mixing, finally the horocycle flow is mixing.

Ergodicity flow is ergodic

We will look at the time average of a function $f \in L^2$ which is continuous and compactly support. It will be sufficient since this space is dense

$$F(x) = lim_{T\to\infty}\frac{1}{T}\int_0^T f(g_t x)dt$$

We want to show that F is almost everywhere constant.

Since F depend only of the geodesic $\gamma(a,b)$ which pass at x, if $a, b \in S_\infty$ are the two endpoints of γ, we have $F(a,b)$. Then as two geodesic with the same forward endpoint are asymptotic, F is a quantity that depend only of asymptotic average, we have that F do not depend of a. But we can reverse the argument to show that if we consider the inverse geodesic, $t \to \infty$, we have that F is also independent of b. Hence F is constant almost everywhere and the geodesic flow is ergodic.

Ergodicity of the horocycle flow

Now if we take $f \in L^2(X)$ a function invariant under the horocycle flow and of mean zero, we want to show that $f = 0$ almost everywhere. Let G^t, H^s and E^r correspond to the operators of the different flows. We have the relation

$$H^s = E^r G^t E^{\pi+r}$$

where $r \to 0$ as $t \to \infty$. Since $H^s f = f$,we have for any $T > 0$,

$$f = \frac{1}{T}\int_0^T E^r G^t E^{\pi+r} f dt$$

Thus for every $g\ inL^2(X)$ we have

$$< g, f >= \int_X g\frac{1}{T}\int_0^T E^r G^t E^{\pi+r}$$

As $r \to 0$ then $t \to \infty$ we can show by controlling the difference that

$$< g, f >= lim_{t\to\infty}\int_X g\frac{1}{T}\int_0^T G^t E^\pi f dt$$

Then as we have shown that the geodesic flow is ergodic, we have by Von Neumann ergodic theorem

$$< g, f >=< g, \int_X E^\pi f >= 0$$

This conclude the proof that the horocycle flow is ergodic.

Mixing of the geodesic flow

We have the relation
$$h^s g^t = g^t h^{sexp(2t)}$$
Let us take $f_0, f_1 \in C_0(X)$, we have for small s
$$< f_0, g^t f_1 > \equiv < h^{-s} f_0, g^t f_1 > = < f_0, h^s g^t f_1 > = < f_0, g^t h^{sexp(2t)} f_1 > = < g^{-t} f_0, h^{sexp(2t)} f_1 >$$
Now we have for small s,
$$h^{Sexp(2t)} f_1 \equiv \frac{1}{S} \int_0^S h^{exp(2t)s} f_1 ds = F_t$$
and for large t, as the horoclic flow is ergodic
$$F_t = \frac{1}{S} \int_0^S h^{exp(2t)s} f_1 ds \equiv \int_X f_1 = < f_1, 1 >$$
So we have
$$< f_0, g^t f_1 > \equiv < g^{-t} f_0, F_t > \equiv < g^{-t} f_0, 1 >< f_1, 1 > = < f_0, 1 >< f_1, 1 >$$
Which is the mixing of the geodesic flow.

Mixing of the horocyclic flow

We use again the relation $h^s = e^r g^t e^{\pi + r}$.
$$< h^s f_0, f_1 > = < g^t e^{\pi + r} f_0, e^{-r} f_1 > \equiv < g^t e^{\pi} f_0, f_1$$
for t large (and so r small). By the mixing propity of the geodesic flow, this quantity converges to $< f_0, 1 >< f_1, 1 >$, and the horocyclic flow is mixing.

□

Then we have this elementary corrolary

Corollaire. *The Earthquake flow is also ergodic.*

Proof. With the conjugacie of Mirzharani, the earthquake flow is conjugated to the horocycle flow which is ergodic. This propriety is transmitted. □

We will now give an introduction to specral gap theory which permit to estimate the rate of mixing of somme flow.

Definition 5.1. Let G be a locally compact σ-compact group. A continuous unitary representation of G is said to have *almost invariant vectors* if for every $\epsilon > 0$ and for every compact subset $K \subset G$, there exists a unit vector V such that $\|g * v - v\| < \epsilon$ for all $g \in K$.

A unitary action which does not have almost invariant vectors is said to be *isolated from trivial representation.*

If G is a semi-simple Lie group, a representation which is isolated from trivial representation is alos said to have a spectral gap.

Proposition 5.2. *Let consider a representation of $SL(2, \mathbb{R})$ by a measure presearving action of automorphisms of a probability space. Let ρ be the repesentation associate on H the space of the L^2 function with zero averages. Assume there is $\delta \in]0; 1[$ and a dense subset of the subspace of $SO(2, \mathbb{R})$-invariant function $H' \subset H$ consisting of functions ϕ for which the correlations $< \phi, \rho(g_t) >$, $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, decay like $O(e^{-\delta t})$. Then ρ is isolated from the trivial representation. [1]*

[13]

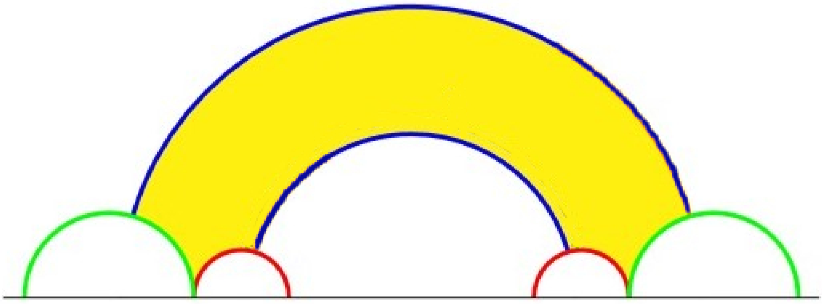
6 Example of the once ponctured torus

A torus can not have a hyperbolic structure, it has naturally a flat structure as the quotient $\mathbb{R}^2/\mathbb{Z}^2$. This changes when one study the one ponctured torus. It is a torus S where we choose a point p and remove it (or just marked it).

The construction of this object can be done in two manier at least. For the first construction, one have to choose a hyperbolic octogone where one side have length 0 and the two other one l, then we sew the border of two of this octogone which give a pair of pant. Finally we can glued with a twist τ to have the one ponctured torus.

A second construction is given by the representation. Given two hyperbolic isomorphism of $\mathbb{H}$ A and B with different fixed point on $\delta\mathbb{H}$ and with $H := ABA^{-1}B^{-1}$ the commutator should be a parabolic element.

A fundamental domaine is given by the following image:



Given two generators α and β of $\pi_1(S)$, two closed curves non homotopically trivial which intersect one, one can parametrize all other lamination. Indeed a given lamination $\lambda \in \mathcal{ML}$ is determined by the couple $(i(\alpha,\lambda), i(\beta,\lambda))$ where $i(.,.)$ is the geometric intersection number.

We have this useful lemma to estimate the length of the systole function in the Teichmuller space.

Lemma 6.1. *Pick γ a simple closed geodesic, and $X \in \mathcal{T}(S_{1,1})$, if X has Fenchel-Nielsen coordinate $(L, \frac{p}{q})$ with respect to γ, where $gcd(p,q) = 1$ and $\frac{p}{q} \in]0;1[$, then*

$$C_1(L)e^{\frac{-L}{2q}} < l_{sys}(X) < C_2(L)e^{\frac{-L}{2q}}$$

where $C_1(L)$, $C_2(L)$ both limits to 4 when p, q are fixed and L goes to ∞

Proof. Let $R(L)$ be the length of the shortest geodesic arc with endpoints on γ. We have

$$R(L) = 2log(coth(L/4)) = 2log(\frac{e^{l/2}+1}{e^{l/2}-1})$$

By the collar lemma. Then if we take α a simple closed curve that intersect γ, q times exactly, we obtain the following inequality with $a = l_\alpha(X)$

$$qR(a) < L < qR(a) + \frac{qa}{2}$$

Reorganizing the termes we have

$$e^{-a/4}tanh(a/4) < e^{-L/2q} < tanh(a/4)$$

As $a \to 0$ then $L \to \infty$

$$C_1(L)e^{\frac{-L}{2q}} < a < C_2(L)e^{\frac{-L}{2q}}$$

To complete the proof , we need to show that the length of α is shorter than any other géodesic closed curves, but with the collar lemma there is only one systole whose length goes to 0 and this is the case for α as $L \to \infty$.

If we don't don any approximation we have

$$2ln(\frac{1+e^{-L/2q}}{1-e^{-L/2q}}) < a$$

□

Theorem 6.2. *Let $\nu(S_{1,1})$ be the finite measure on $\mathcal{P}^1\mathcal{M}(S_{1,1})$. Then*

$$\nu(S_{1,1})\{(X,\lambda) \in \mathcal{P}^1\mathcal{M}(S_{1,1}) | l_{sys}(X,\lambda) < \epsilon\} = O(\frac{\epsilon}{log\epsilon})$$

as $\epsilon \to 0$

Let fix $\epsilon = l_{sys}(X)$, and $T > 0$, then $\exists N = N(\epsilon, T)$ such that $\forall n \geq N$

$$|l_{sys}E_t(X,\lambda) - l_{sys}E_t(X, \frac{\gamma_n}{l_{\gamma_n}(X)})| < \epsilon$$

So we calculate T_n such that $\forall |t| < T_n$, $l_{sys}E_t(X, \frac{\gamma_n}{l_{\gamma_n}(X)}) < 2\epsilon$ and $T = liminf T_n$. If $T > 0$, $\exists N_1$ such that $\forall n \geq N_1$, $T_n \geq \frac{T}{2}$.

Then we set $N_2 = N(\epsilon, T/2)$ and we have $l_{sys}E_t(x,\lambda) \leq 2.5\epsilon$, $\forall 0 \leq t \leq T/2$

We have two useful tools to understand the length fonction along a earthquake path.

Lemma 6.3. *Let $X \in \mathcal{T}(S_g)$, and γ a curve which is part of a pant decomposition. χ_s is the twist of length s around γ, and b a closed curve with $i(b,\gamma) > 0$ then $s \mapsto l_{\chi_s}(b)$ is strictly convex.*

[3] proposition 10.8

Lemma 6.4. *If α is a closed curve, γ an other closed curve and λ a lamination.*

$$\begin{aligned} \frac{dl_\alpha}{dt}(0) &= \sum_{p_i \in \alpha\cap\gamma} cos(\theta_{p_i}) \\ \frac{dl_\alpha}{dt}(0) &= \int_\alpha cos(\theta) d\theta \end{aligned}$$

[6] Corollary 3.3 and 3.4

References

[1] Artur Avila, Sebastien Gouezel, and Jean-Christophe Yoccoz. Exponential mixing for the Teichmuller flow. *arXiv Mathematics e-prints*, page math/0511614, November 2005.

[2] Francis Bonahon. The geometry of teichmüller space via geodesic currents. *Inventiones mathematicae*, 92(1):139–162, 1988.

[3] B. Farb and D. Margalit. *A Primer on Mapping Class Groups (PMS-49)*. Princeton Mathematical Series. Princeton University Press, 2011.

[4] John H. Hubbard. *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics*, volume 2 of *Surface Homeomorphisms and Rational Functions*. Matrix Editions, April 2016.

[5] M. Kapovich. *Hyperbolic Manifolds and Discrete Groups*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.

[6] Steven P. Kerckhoff. The nielsen realization problem. *Annals of Mathematics*, 117(2):235–265, 1983.

[7] François Labourie and Ser Peow Tan. The probabilistic nature of McShane's identity: planar tree coding of simple loops. 25 pages, December 2017.

[8] Gilbert Levitt. Foliations and laminations on hyperbolic surfaces. *Topology*, 22(2):119–135, 1983.

[9] W. B. R. Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. 1964.

[10] Bruno Martelli. An introduction to geometric topology. 2016.

[11] Curt Mcmullen. Hyperbolic manifolds , discrete groups and ergodic theory course notes. 1998.

[12] Greg Mcshane. Simple geodesics and a series constant over teichmüller space. *Invent. Math*, 132:607–632, 1995.

[13] Marina Ratner. The rate of mixing for geodesic and horocycle flows. *Ergodic Theory and Dynamical Systems*, 7(2):267–288, 1987.

[14] Alex Wright. Mirzakhani's work on earthquake flow. *arXiv preprint arXiv:1810.07571*, 2018.

[15] Alex Wright. A tour through mirzakhani's work on moduli spaces of riemann surfaces. *Bulletin of the American Mathematical Society*, 2020.