

# Computational Economics

## Lecture 1: An Intro to Dynamic (Stochastic) General Equilibrium Models

Etienne Briand

Concordia University

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# Motivation

**D(S)GE** models are disciplined way to think about the economy in which aggregate dynamics are consistent with individual decisions and constraints. In particular, they explicitly account for agents' expectations and forward-looking behavior.

DSGE are primarily used to study:

- Dynamic response to shocks.
- Propagation mechanisms.
- Counterfactuals and policy experiments

# A Simple Model

# Overview

As an intro to D(S)GE models, we study an optimal consumption–saving problem. The model is effectively a deterministic RBC framework in which labor input is set exogenously to one.

This environment remains the backbone of state-of-the-art macroeconomic models.

# Economic Environment

- Representative household period utility:  $U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma}$
- Discount factor:  $0 < \beta < 1$ .
- Aggregate production function:  $Y_t = K_{t-1}^\alpha$
- Law of motion for capital:  $K_t = (1 - \delta)K_{t-1} + I_t$
- Aggregate resource constraint:  $Y_t = C_t + I_t$

# Social Planner's Problem

# Social Planner's Problem

We solve the consumption–saving problem from the social planner's perspective. In a frictionless environment like the one above, the solution coincides with the decentralized equilibrium, where households and firms maximize taking prices as given.

The social planner maximizes the discounted sum of period utility, taking as given the aggregate production function, aggregate resource constraint and the capital's law of motion.

## Social Planner's Problem (cont'd)

Formally the social planner's problem solves:

$$\max_{\{C_t, K_t, I_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t E_t U(C_t) \quad (1)$$

subject to

$$Y_t = K_{t-1}^{\alpha} \quad (2)$$

$$K_t = (1 - \delta)K_{t-1} + I_t \quad (3)$$

$$Y_t = C_t + I_t \quad (4)$$

$$K_{-1} \text{ given.} \quad (5)$$

**Note:** There is no source of uncertainty in this version of the model, and thus the use of the expectation operator,  $E_t(\cdot)$  is not required.



## Social Planner's Problem (cont'd)

Combining the constraints, we get:

$$\max_{\{C_t, K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma} \quad (6)$$

subject to

$$K_{t-1}^{\alpha} = C_t + K_t - (1 - \delta)K_{t-1} \quad (7)$$

$$K_{-1} \text{ given.} \quad (8)$$

**Note:**  $C_t = K_{t-1}^{\alpha} - K_t + (1 - \delta)K_{t-1}$ , thus it would be possible to replace in the objective and solve for an unconstrained problem.

## Social Planner's Problem (cont'd)

We can write this problem as a Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma} + \lambda_t [K_{t-1}^\alpha - C_t - K_t + (1-\delta)K_{t-1}]. \quad (9)$$

# Optimality Conditions

Taking the derivatives of  $\mathcal{L}$  wrt to the controls,  $C_t$  and  $K_t$ , we get the FOCs:

$$C_t^{-\gamma} = \lambda_t \quad (10)$$

$$\lambda_t = \lambda_{t+1} \beta [\alpha K_t^{\alpha-1} + (1 - \delta)] . \quad (11)$$

The KKT are given by:

$$\lambda_t [K_{t-1}^{\alpha} - C_t - K_t + (1 - \delta)K_{t-1}] \geq 0 \quad (12)$$

$$\lambda_t \geq 0, [K_{t-1}^{\alpha} - C_t - K_t + (1 - \delta)K_{t-1}] \geq 0. \quad (13)$$

## Optimality Conditions (cont'd)

The aggregate resource constraint always holds at equality, thus  $\lambda_t > 0 \forall t$  and with  $K_{-1} > 0$ , we never encounter corner solutions.

Optimality implies:

$$C_t^{-\gamma} = \beta C_{t+1}^{-\gamma} [\alpha K_t^{\alpha-1} + (1 - \delta)] \quad [\text{Euler Eq.}]$$

with

$$K_{t-1}^{\alpha} = C_t + K_t - (1 - \delta)K_{t-1} \quad [\text{Aggr. resource constr.}]$$

# Equilibrium

# Definition

Formally, the definition of an equilibrium consists of a list of **objects** and **conditions** that these objects must satisfy.

**Definition:** Given an initial stock of capital  $K_{-1}$ , a sequential equilibrium to the consumption-saving problem is:

- An allocation  $\Omega := \{C_t, K_t\}_{t=0}^{\infty}$  for the social planner.

such that:

1. The allocation  $\Omega$  maximizes the discounted sum of period utilities.
2. The aggregate resource constraint holds at equality

$$K_{t-1}^{\alpha} = C_t + K_t - (1 - \delta)K_{t-1}.$$

# Characterization

The characterization of an equilibrium consists of listing all of the mathematical conditions necessary compute the social planner's allocation  $\Omega$ .

In the present case, the Euler equation and the aggregate resource constraint are sufficient.

# Steady-state



# The Steady-State

In a steady-state, all endogenous variables are constant over time and satisfy the model's equilibrium conditions.

We can compute the values for these variables by dropping the time index in the FOC and constraints:

$$K = \left( \frac{\beta^{-1} - (1 - \delta)}{\alpha} \right)^{\frac{1}{\alpha-1}} \quad (14)$$

$$I = \delta K \quad (15)$$

$$C = K^{\alpha} - I \quad (16)$$

$$\lambda = C^{-\gamma} \quad (17)$$

# Dynamics

# Solving for the Transition Dynamics

Our goal is solve for the dynamics associated with the pair of non-linear first-order difference equations formed by the Euler eq. and the resource constraint (obviously for a given  $K_{-1} \neq K$ ).

We can show that this economy has exactly **one stable manifold**, meaning that for a given  $K_{-1}$ , consumption and capital converge to their steady-state values only if  $C_t$  is chosen optimally  $\forall t$ .

# Shooting Algorithm

In other words, given  $K_{-1}$ , there exists a unique path  $\{C_t^*, K_t^*\}_{t=0}^T$  that satisfies optimality and converges to the steady state  $\{C, K\}$ .

The challenge is that  $C_0^*$  is not pinned down by the eq. cond.

We can solve for this path using a **shooting algorithm**:

1. Guess a value for  $C_0 \in [0, K_{-1}^\alpha]$ .
2. Given  $(C_0, K_{-1})$ , use the Euler equation and the aggregate resource constraint to compute  $(C_1, K_0)$ . Iterate forward up to  $t = T$ .
3. If  $K_T < K$ , set the guess for  $C_0$  as the lower bound of the interval; otherwise, set it as the upper bound.
4. Repeat steps 1–3 until convergence (i.e.,  $K_T = K$  and  $C_T = C$ ).

This procedure transforms a two-point boundary value problem into an initial-condition problem.

## Shooting Algorithm (cont'd)

The shooting algorithm is not practical: (i) it solves for a **single path** and (ii) it becomes computationally burdensome (infeasible) without **perfect foresight**.

It is therefore natural for us to seek a solution in the form of a policy function that:

- Maps the state  $K_{t-1}$  to the control  $C_t$  for all admissible values of the state.
- Accounts for uncertainty about future outcomes.