Computational Economics

Lecture 5: Linearization

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Motivation

Thus far, we solved economic models using dynamic programming. However, in a number of applications, this approach is impractical.

A common alternative is to linearize the model's equilibrium conditions around a particular point and solve the resulting system of equations.

Moving from dynamic programming to linearization involves a tradeoff: we go from a global to a local solution of the model, but save on computation time.

Note: In recent years, substantial progress has been made in mitigating the "curse of dimensionality," one of the main drawbacks of dynamic programming, through machine learning tools.

Linearization

General Idea

The general idea is to approximate the nonlinear system of equations that characterizes the equilibrium with a linear system, which can then be solved using linear algebra techniques.

The first step is to take a Taylor approximation around a specific point, typically the non-stochastic steady state.

Taylor Approximation

Recall the formula for a Taylor approximation of a function $f(\cdot)$ around a point a:

$$f(x) \approx f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If a is a vector, the first-order derivative $f^{(1)}(a)$ is the Jacobian, the second-order derivative $f^{(2)}(a)$ is the Hessian, and so on.

Taylor Approximation (cont'd)

In economics, nonlinear systems of equations are commonly approximated using variables expressed in log-deviations around the non-stochastic steady state.

Define

$$\hat{x}_t := \log(X_t/X) \quad \text{so that} \quad X_t = X\exp(\hat{x}_t).$$

If X_t is not too far from X, we have

$$\log(X_t/X) = \log\left(1 + \frac{X_t - X}{X}\right) \approx \frac{X_t - X}{X}.$$

Thus, $100 \times \hat{x}_t$ represents the percentage deviation of X_t from its non-stochastic steady-state value. This makes comparisons between variables independent of their respective level.

Taylor Approximation (cont'd)

Expressing X_t in terms of $\hat{x}_t$ and taking a first-order Taylor approximation around $\hat{x}_t = 0$ yields:

$$\begin{aligned} X_t &= X \exp\{\log(X_t/X)\} \\ X_t &= X \exp(\hat{x}_t) \\ X_t &\approx X(\exp(0) + \tfrac{\exp(0)}{1!}\hat{x}_t) \\ X_t &\approx X(1 + \hat{x}_t) \end{aligned}$$

This principle can be applied to linearize any nonlinear equation featuring X_t (and functions of X_t). Other equivalent methods exist.

Solving Systems of Linear Rational Expectations

Systems of LINRE

Log-linearizing the equilibrium conditions of an economic model yields a system of linear rational expectations equations. Such system can be written in matrix form as:

$$\Gamma_0 E_t[x_{t+1}] = +\Gamma_1 x_t + \Psi \varepsilon_t \tag{1}$$

We seek a solution that admits a VAR(1) representation:

$$x_t = \Theta_x x_{t-1} + \Theta_\varepsilon \varepsilon_t \tag{2}$$

where ε_t is a vector exogenous (structural) innovations.

Systems of LINRE (cont'd)

A solution of the form Equation (2) can be obtained by decoupling the original system in terms of predetermined and jump variables.

- Predetermined (state) variables are fixed for the next period such that $E_t[x_{t+1}] = x_{t+1}$ (e.g., capital).
- Jump variables are free to adjust to shocks at time $t+1$ (e.g., consumption, prices).

The key is to use rational expectations to guide the jump variables' dynamics and ensure a stable equilibrium.

Systems of LINRE (cont'd)

To illustrate the principle, we assume that the system Equation (1) can be rewritten as:

$$\begin{bmatrix} x_{1,t+1} \\ E_t x_{2,t+1} \end{bmatrix} = \tilde{\Gamma}_1 \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \tilde{\Psi} x_{3,t} \tag{3}$$

where

- x_1 are predetermined variables,
- x_2 are jump variables,
- x_3 are forcing variables (depending only on exogenous shocks).

Such system can be solve using the Blanchard-Khan method.

Systems of LINRE (cont'd)

The first step consists of factorizing the matrix $\tilde{\Gamma}_1$, and ordering the equation in increasing order in terms of their associated eigenvalues.

The Blanchard-Khan method does so with a Jordan decomposition

$$\Gamma_1 = \Lambda^{-1} J \Lambda,$$

with

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}.$$

J_1 contains the stable eigenvalues $|\lambda| \leq 1$ and J_2 the unstable eigenvalues $|\lambda| > 1$.

Systems of LINRE (cont'd)

Blanchard-Khan conditions:

- If the number of explosive eigenvalues is equal to the number of jump variables, the system is saddle-path stable and has a unique solution.
- If the number of explosive eigenvalues exceeds the number of jump variables the system is a source and no solution exists.
- If the number of stable eigenvalues exceeds the number of jump variables the system is a sink and an infinity of solutions exist.

Systems of LINRE (cont'd)

We are interested in solving systems that are saddle-path stable. To do so, first we premultiply Equation (3) by Λ and define

$$\tilde{x}_t = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}, \quad \begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$$

The system becomes block-diagonal:

$$\begin{bmatrix} \tilde{x}_{1,t+1} \\ E_t \tilde{x}_{2,t+1} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_{1,t} \\ \tilde{x}_{2,t} \end{bmatrix} + \begin{bmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{bmatrix} x_{3,t}$$

This transformation effectively "de-couples" the system, so that the jump variables depend only upon the unstable eigeinvalues contained in J_2.

Systems of LINRE (cont'd)

We solve for the jump variables forward by expressing the unstable block as:

$$\tilde{x}_{2,t} = J_2^{-1} E_t[\tilde{x}_{2,t+1}] - J_2^{-1} \tilde{\Psi}_2 x_{3,t}.$$

Next, we iterate forward:

$$\tilde{x}_{2,t} = -\sum_{i=0}^{\infty} J_2^{-(i+1)} \tilde{\Psi}_2 E_t[x_{3,t+i}].$$

Since J_2 contains explosive eigenvalues, the sum converges.

Systems of LINRE (cont'd)

$\tilde{x}_{2,t}$ can be mapped into the following expression for $x_{2,t}$

$$x_{2,t} = -\Lambda_{22}^{-1}\Lambda_{21}x_{1,t} - \Lambda_{22}^{-1}\sum_{i=0}^{\infty} J_2^{-(i+1)}\tilde{\Psi}_2 E_t[x_{3,t+i}].$$

Finally, to solve for the non-explosive portion of the system, we expand the upper portion of the original system

$$x_{1,t+1} = \tilde{\Gamma}_{1,11}\, x_{1,t} + \tilde{\Gamma}_{1,12}\, x_{2,t} + \Psi_1 x_{3,t}$$

and substitute the solution for $x_{2,t}$.

Systems of LINRE (cont'd)

The Blanchard-Khan method requires (i) the matrix Γ_0 to be invertible which is not always the case, and (ii) a cumbersome system reduction step
(i.e., writing the model in terms of a subset of variables uniquely determined).

Thus, we typically use an alternative method proposed by Chris Sims (`gensys.jl`) that is easier to apply. It only requires to cast Equation (1) in the form:

$$\Gamma_0 x_t = \Gamma_1 x_{t-1} + \Psi \varepsilon_t + \Pi \eta_t \tag{4}$$

Notice that the expectation operator is dropped and replaced by expectational errors η_t. Clearly, this allows x_t to contain variables that are not yet realized (e.g., expected consumption c_{t+1}).

Notes: For more details see DD Ch 2.

Real Business Cycle Model

RBC Model

To illustrate how to (i) linearize equilibrium conditions and (ii) cast them into Equation (4), we consider a variant of the RBC model studied in the previous lecture.

We solve for the decentralized equilibrium conditions under the sequential representation, assuming all conditions hold so that both the household and firm sectors can be treated as representative agents.

RBC Model (cont'd)

The representative household solves:

$$\max_{\{C_t, L_t, K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t E_t \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} - \varphi \frac{L_t^{1+\psi}}{1+\psi} \right\} \tag{5}$$

s.t.

$$C_t + K_t = W_t L_t + [R_t + (1-\delta)] K_{t-1} \tag{6}$$

$$K_{-1} \text{ given.} \tag{7}$$

where we replaced investment with the law of motion for capital in the period budget constraint.

RBC Model (cont'd)

The FOCs of the household's problem are:

$$\lambda_t = \beta E_t\{\lambda_{t+1}[R_{t+1} + (1-\delta)]\}$$

$$C_t^{-\gamma} = \lambda_t$$

$$\varphi L_t^{\psi} = \lambda_t W_t$$

where λ_t is the Lagrange multiplier on the period budget constraint. Combining them we get the household's optimality conditions:

$$C_t^{-\gamma} = \beta E_t\{C_{t+1}^{-\gamma}[R_{t+1} + (1-\delta)]\}$$

$$\varphi C_t^{\gamma} L_t^{\psi} = W_t$$

RBC Model (cont'd)

The representative firm solves a static problem (one in each period):

$$\max_{\{K_t^f, L_t^f\}} A_t (K_t^f)^{\alpha} (L_t^f)^{1-\alpha} - W_t L_t^f - R_t K_t^f \tag{8}$$

The FOCs of the firm's problem are:

$$\alpha A_t (K_t^f)^{\alpha-1} (L_t^f)^{1-\alpha} = R_t$$

$$(1-\alpha) A_t (K_t^f)^{\alpha} (L_t^f)^{-\alpha} = W_t$$

with

$$\log(A_t) = \rho \log(A_{t-1}) + \varepsilon_t \ , \ \varepsilon_t \sim N(0, \sigma^2)$$

.

RBC Model (cont'd)

Definition: Given an initial stock of capital K_{-1}, a sequential equilibirum of the RBC model is:

- An allocation $\Omega^h := \{C_t, L_t, K_t\}_{t=0}^{\infty}$ for the representative household.
- An allocation $\Omega^f := \{L_t^f, K_t^f\}_{t=0}^{\infty}$ for the representative firm.
- A set of prices $\Omega^p := \{R_t, W_t\}_{t=0}^{\infty}$.

such that:

1. The allocation Ω^h solve the household problem taking prices as given.
2. The allocation Ω^f solves the firm's problem taking prices as given.
3. Markets clear: (i) $Y_t = C_t + I_t$, (ii) $K_{t-1} = K_t^f$, (iii) $L_t = L_t^f$

RBC Model (cont'd)

Taking a log-linear approximation of the FOCs and remaining equilibrium conditions yields the following system:
(8 equations for 8 variables)

$$c_t = E_t[c_{t+1} - \frac{r_{t+1}}{\gamma}]$$

$$\gamma c_t + \psi l_t = w_t$$

$$a_t + (\alpha - 1)k_{t-1} + (1-\alpha)l_t = r_t$$

$$a_t + \alpha k_{t-1} + (-\alpha)l_t = w_t$$

$$y_t = a_t + \alpha k_{t-1} + (1-\alpha)l_t$$

$$y_t = \frac{C}{Y}c_t + \frac{I}{Y}i_t$$

$$k_t = (1-\delta)k_{t-1} + \delta i_t$$

$$a_t = \rho a_{t-1} + \varepsilon_t$$

The solution to that system characterizes the equilibrium.

RBC Model (cont'd)

In order to solve this system using `gensys.jl`, we must cast it in the form Equation (4). Define the vector

$$x_t = [c_t, i_t, k_t, y_t, l_t, w_t, r_t, a_t, c_{t+1}, r_{t+1}]'. \tag{9}$$

Notice that because we have dropped the expectations operator, we need to distinguish between c_t and c_{t+1}, r_t and r_{t+1}. This introduces two new variables into the system.

RBC Model (cont'd)

The matrices Γ_0, Γ_1, Ψ and Π are given by:

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1/\gamma \\ \gamma & 0 & 0 & 0 & \psi & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-\alpha) & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -(1-\alpha) & 0 & 0 & -1 & 0 & 0 \\ -C/Y & -I/Y & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\delta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-\alpha) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-\delta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma \\ 0 \\ 0 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$