Computational Economics

Lecture 2: Dynamic Programming I

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Motivation

In the previous lecture, we showed that the challenge of solving an economic model's dynamics essentially amounts to finding the optimal control(s) for any given state.

However, we have yet to see general and numerically stable methods to accomplish this. In this lecture, we introduce Dynamic Programming, a global approach to solving such problems.

The Principle of Optimality

Notation

We start by establishing some generic notation for dynamic optimization problems.

- Let X be the set of all possible values for the state variable x.
- Let $\Gamma : X \to X$ be the correspondence describing the feasibility constraints (i.e., $\forall\, x \in X$, $\Gamma(x)$ is the set of feasible values for x' if the current state is x).
- Let A be the graph of Γ such that $A := \{(x, x') \in X \times X : x' \in \Gamma(x)\}$.
- Let $F : A \to \mathcal{R}$ be the real-valued one-period return function.
- Let $\beta \geq 0$ be the discount factor.

Thus, the givens of the problem are X, Γ, F, β.

Sequential Representation

Such problems in economics are often expressed as maximizing over (infinte) sequences. We call this the sequential problem (SP) representation:

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \tag{1}$$

subject to

$$x_{t+1} \in \Gamma(x_t), \ \forall t \tag{2}$$

$$x_0 \in X \text{ given.} \tag{3}$$

Note: In general, solving the SP doesn't give policy functions for free. i.e., a mapping from the state to the control(s).

Sequential Representation (cont'd)

Let $\Pi(x_0) := \{\{x_t\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t)\ \forall t \geq 0\}$ denote the set of all feasible plans from any $x_0 \in X$.

Under the following assumptions, the SP is well-defined.

1. $\Gamma(x)$ is non-empty, for all $x \in X$.
2. $\forall\ x_0 \in X$ and $\{x_0, x_1, ...\}$, $\lim_{t\to\infty} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})\ \exists$.

That is the feasible set $\Pi(x_0)$ is non-empty $\forall\ x_0 \in X$ and the objective function F can be evaluated at any point in that set.

Recursive Representation

We can show that under assumptions 1-2, corresponding to the SP is a recursive or functional representation (FE) given by:

$$V(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta V(x')\} \tag{4}$$

with

$$x \in X. \tag{5}$$

We call $V(\cdot)$ a value function, and (4) a Bellman equation.

Note: It is common not to use time-indexed variables in the FE formulation, since the solution is time-invariant.

Recursive Representation (cont'd)

The solution to the FE is given by (i) the functional $V(\cdot)$, and (ii) a single-valued policy function which maps the current and next period state:

$$G(x) = \{x' \in \Gamma(x) : V(x) = F(x, x') + \beta V(x')\}. \tag{6}$$

Recursive Representation (cont'd)

The complete proof establishing the equivalence between the two representations is beyond the scope of this course (see SLP, Ch. 4).

The following points summarize the results:

- The solution $V(\cdot)$ to FE, evaluated at x_0, gives the same value for the supremum of the solution of the equivalent SP.
- Conversely, when x_0 is the initial state and $\{x_{t+1}\}_{t=0}^{\infty}$ attains the supremum in the SP, it must be that for any pair $\{x_t, x_{t+1}\}_{t=0}^{\infty}$ the FE is satisfied.

Richard Bellman called these ideas the Principle of Optimality.

Recursive Representation (cont'd)

The following result conveys the intuition behind why any feasible plan's value can be written recursively.

Lemma: For any $x_0 \in X$ and any feasible plan $\tilde{x} \in \Pi(x_0)$, let

$$\begin{aligned} v(\tilde{x}) &= \lim_{t\to\infty} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ &= F(x_0, x_1) + \beta \lim_{t\to\infty} \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_{t+1}) \\ &= F(x_0, x_1) + \beta v(\tilde{x}') \end{aligned}$$

Then if $\tilde{x}$ maximizes the SP, $v(\cdot) = V(\cdot)$. Conversely, if $\tilde{x}$ is generated by the solution to the FE, it solves the SP.

Recursive Representation (cont'd)

In most economic applications, bounded returns are either natural or an innocuous assumption. Suppose the following conditions hold:

3. X is a convex subset of $\mathcal{R}^n$ and the correspondence $\Gamma : X \to X$ is nonempty, compact-valued and continuous.

4. The function $F : A \to \mathcal{R}$ is bounded, continuous, and $0 < \beta < 1$.

Recursive Representation (cont'd)

Let the operator T on the set of continuous bounded functions $C(X)$, of which $V(\cdot)$ can be shown to be element, be defined as

$$(Tv)(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta v(x')\}. \tag{7}$$

Clearly that the solution to the FE must satisfy $V = TV$.

Recursive Representation (cont'd)

Under assumptions 1-4, we have the following results:

- $T : C(X) \to C(X)$ has a unique fixed-point and the corresponding functional $G(x)$ is non-empty and u.h.c.
- Under add. monotonicity assumptions on F and Γ, V is strictly increasing.
- Under add. convexity assumptions on F and Γ, V is strictly concave and G is continuous single-valued function.
- For $v^n = T^n v_0$, the associted sequence of policy functions $\{g_n\}$ converge uniformly to the optimal policy function G.
- If F is cont. differentiable on the interior of A, then so is V.
  (This result holds when the optimal policy lies in the interior; occasionally binding constraints can create kinks that break differentiability.)

Note: See SLP, Ch.4 for the proofs.

Consumption-Saving Problem

Sequential Reprensentation

Recall the consumption-saving problem from the previous lecture. Its SP is given by:

$$\max_{\{C_t, K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma} \tag{8}$$

subject to

$$K_{t-1}^{\alpha} = C_t + K_t - (1-\delta)K_{t-1} \tag{9}$$

$$K_{-1} \text{ given.} \tag{10}$$

Functional Reprensentation

We can reformulate it as an FE:

$$V(K) = \max_{K' \in \Gamma(K)} \{ \frac{C^{1-\gamma}-1}{1-\gamma} + \beta V(K') \} \tag{11}$$

with

$$\Gamma(K) = \{K' : 0 \leq K' \leq K^{\alpha} + (1-\delta)K\} \tag{12}$$

and

$$C := K^{\alpha} + (1-\delta)K - K'. \tag{13}$$

Notes: The same problem can be written with $\{C, K'\}$ as controls, with the resource constraint imposed via a Lagrange multiplier. However, we know the constraint to always be binding.

Functional Reprensentation

The first-order and the enveloppe conditions are given by:

$$-C^{-\gamma} + \beta V'(K') = 0 \qquad \text{[FOC wrt } K'\text{]}$$

$$V'(K) = C^{-\gamma}[\alpha K^{\alpha-1} + (1-\delta)] \qquad \text{[Envelope condition]}$$

Combining the FOC and (updated) envelope condition, we obtain:

$$C^{-\gamma} = \beta(C')^{-\gamma}[\alpha K'^{\alpha-1} + (1-\delta)] \qquad \text{[Euler eq.]}$$

Note: the solution coincides with that of the SP from lecture 1.

Envelope Theorem

Theorem: As we change parameters of the objective function $V(\cdot)$, changes in the optimizer do not contribute to the change in the objective function.

Example: $V'(K) = C^{-\gamma}\left[\alpha K^{\alpha-1} + (1-\delta) - \frac{\partial K'}{\partial K}\right] + \beta V'(K)\frac{\partial K'}{\partial K}$

$$\Leftrightarrow V'(K) = C^{-\gamma}\left[\alpha K^{\alpha-1} + (1-\delta)\right] + \frac{\partial K'}{\partial K} \underbrace{\left[\beta V'(K') - C^{-\gamma}\right]}_{= 0 \text{ by the FOC wrt } K'}$$

$$\Leftrightarrow V'(K) = C^{-\gamma}\left[\alpha K^{\alpha-1} + (1-\delta)\right].$$

Equilibrium

Defintion

Definition: A recursive equilibrium is:

- A value function $V(K)$ and its associated policy rule $G(K)$.

such that,

- $V(K)$ and $G(K)$ solve the the social planner's problem.

Characterization

The recursive equilibirum is again characerized by the Euler equation and the aggregate resource constraint for any state K in its set of possible values.