

# Computational Economics

## Lecture 2: Dynamic Programming I

Etienne Briand

Concordia University

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# Motivation

In the previous lecture, we showed that the challenge of solving an economic model's dynamics essentially amounts to finding the optimal control(s) for any given state.

However, we have yet to see general and numerically stable methods to accomplish this. In this lecture, we introduce **Dynamic Programming**, a global approach to solving such problems.

# The Principle of Optimality

# Notation

We start by establishing some generic notation for dynamic optimization problems.

- Let  $X$  be the set of all possible values for the **state variable**  $x$ .
- Let  $\Gamma : X \rightarrow X$  be the correspondence describing the **feasibility** constraints (i.e.,  $\forall x \in X$ ,  $\Gamma(x)$  is the set of feasible values for  $x'$  if the current state is  $x$ ).
- Let  $A$  be the graph of  $\Gamma$  such that  $A := \{(x, x') \in X \times X : x' \in \Gamma(x)\}$ .
- Let  $F : A \rightarrow \mathcal{R}$  be the real-valued one-period return function.
- Let  $\beta \geq 0$  be the discount factor.

Thus, the givens of the problem are  $X, \Gamma, F, \beta$ .

# Sequential Representation

Such problems in economics are often expressed as maximizing over (infinite) sequences. We call this the **sequential problem** (SP) representation:

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \quad (1)$$

subject to

$$x_{t+1} \in \Gamma(x_t), \forall t \quad (2)$$

$$x_0 \in X \text{ given.} \quad (3)$$

**Note:** In general, solving the SP doesn't give **policy functions** for free. i.e., a mapping from the state to the control(s).

## Sequential Representation (cont'd)

Let  $\Pi(x_0) := \{\{x_t\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t) \forall t \geq 0\}$  denote the set of all feasible plans from any  $x_0 \in X$ .

Under the following **assumptions**, the SP is well-defined.

1.  $\Gamma(x)$  is non-empty, for all  $x \in X$ .
2.  $\forall x_0 \in X$  and  $\{x_0, x_1, \dots\}$ ,  $\lim_{t \rightarrow \infty} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \exists$ .

That is the feasible set  $\Pi(x_0)$  is non-empty  $\forall x_0 \in X$  and the objective function  $F$  can be evaluated at any point in that set.

# Recursive Representation

We can show that under assumptions 1-2, corresponding to the SP is a recursive or **functional representation** (FE) given by:

$$V(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta V(x')\} \quad (4)$$

with

$$x \in X. \quad (5)$$

We call  $V(\cdot)$  a value function, and (4) a Bellman equation.

**Note:** It is common not to use time-indexed variables in the FE formulation, since the solution is time-invariant.

## Recursive Representation (cont'd)

The solution to the FE is given by (i) the functional  $V(\cdot)$ , and (ii) a single-valued **policy function** which maps the current and next period state:

$$G(x) = \{x' \in \Gamma(x) : V(x) = F(x, x') + \beta V(x')\}. \quad (6)$$



## Recursive Representation (cont'd)

The complete proof establishing the equivalence between the two representations is beyond the scope of this course (see SLP, Ch. 4).

The following points summarize the results:

- The solution  $V(\cdot)$  to FE, evaluated at  $x_0$ , gives the same value for the supremum of the solution of the equivalent SP.
- Conversely, when  $x_0$  is the initial state and  $\{x_{t+1}\}_{t=0}^{\infty}$  attains the supremum in the SP, it must be that for any pair  $\{x_t, x_{t+1}\}_{t=0}^{\infty}$  the FE is satisfied.

Richard Bellman called these ideas the **Principle of Optimality**.

## Recursive Representation (cont'd)

The following result conveys the intuition behind why any feasible plan's value can be written recursively.

**Lemma:** For any  $x_0 \in X$  and any feasible plan  $\tilde{x} \in \Pi(x_0)$ , let

$$\begin{aligned} v(\tilde{x}) &= \lim_{t \rightarrow \infty} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ &= F(x_0, x_1) + \beta \lim_{t \rightarrow \infty} \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_{t+2}) \\ &= F(x_0, x_1) + \beta v(\tilde{x}')$$

Then if  $\tilde{x}$  maximizes the SP,  $v(\cdot) = V(\cdot)$ . Conversely, if  $\tilde{x}$  is generated by the solution to the FE, it solves the SP.

## Recursive Representation (cont'd)

In most economic applications, **bounded returns** are either natural or an innocuous assumption. Suppose the following conditions hold:

3.  $X$  is a convex subset of  $\mathcal{R}^n$  and the correspondence  $\Gamma : X \rightarrow X$  is nonempty, compact-valued and continuous.
4. The function  $F : A \rightarrow \mathcal{R}$  is bounded, continuous, and  $0 < \beta < 1$ .

## Recursive Representation (cont'd)

Let the operator  $T$  on the set of continuous bounded functions  $C(X)$ , of which  $V(\cdot)$  can be shown to be element, be defined as

$$(Tv)(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta v(x')\}. \quad (7)$$

Clearly that the solution to the FE must satisfy  $V = TV$ .

## Recursive Representation (cont'd)

Under standard regularity assumptions (see SLP, Ch. 4), we have

1.  $T : C(S) \rightarrow C(S)$  has a unique fixed-point, with  $V(\cdot, z)$  strictly concave and increasing.
2. The policy function  $G(\cdot, z) : X \rightarrow X$  is continuous and single valued.
3. Off corners,  $V(\cdot, z)$  is continuously differentiable in  $x \in X$ .
4. For  $v^n = T^n v^0$ , the associated sequence of policy functions  $\{g^n\}$  converge uniformly to the optimal policy function  $G$ .

# Consumption-Saving Problem

# Sequential Representation

Recall the consumption-saving problem from the previous lecture.  
Its SP is given by:

$$\max_{\{C_t, K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma} \quad (8)$$

subject to

$$K_{t-1}^{\alpha} = C_t + K_t - (1 - \delta)K_{t-1} \quad (9)$$

$$K_{-1} \text{ given.} \quad (10)$$

# Functional Representation

We can reformulate it as an FE:

$$V(K) = \max_{K' \in \Gamma(K)} \left\{ \frac{C^{1-\gamma} - 1}{1-\gamma} + \beta V(K') \right\} \quad (11)$$

with

$$\Gamma(K) = \{K' : 0 \leq K' \leq K^\alpha + (1 - \delta)K\} \quad (12)$$

and

$$C := K^\alpha + (1 - \delta)K - K'. \quad (13)$$

**Notes:** The same problem can be written with  $\{C, K'\}$  as controls, with the resource constraint imposed via a Lagrange multiplier. However, we know the constraint to always be binding.



# Functional Representation

The first-order and the envelope conditions are given by:

$$-C^{-\gamma} + \beta V'(K') = 0 \quad [\text{FOC wrt } K']$$

$$V'(K) = C^{-\gamma}[\alpha K^{\alpha-1} + (1 - \delta)] \quad [\text{Envelope condition}]$$

Combining the FOC and (updated) envelope condition, we obtain:

$$C^{-\gamma} = \beta(C')^{-\gamma}[\alpha K'^{\alpha-1} + (1 - \delta)] \quad [\text{Euler eq.}]$$

**Note:** the solution coincides with that of the SP from lecture 1.

# Envelope Theorem

**Theorem:** As we change parameters of the objective function  $V(\cdot)$ , changes in the optimizer do not contribute to the change in the objective function.

**Example:**  $V'(K) = C^{-\gamma} [\alpha K^{\alpha-1} + (1 - \delta) - \frac{\partial K'}{\partial K}] + \beta V'(K) \frac{\partial K'}{\partial K}$

$$\Leftrightarrow V'(K) = C^{-\gamma} [\alpha K^{\alpha-1} + (1 - \delta)] + \frac{\partial K'}{\partial K} \underbrace{[\beta V'(K') - C^{-\gamma}]}_{= 0 \text{ by the FOC wrt } K'}$$

$$\Leftrightarrow V'(K) = C^{-\gamma} [\alpha K^{\alpha-1} + (1 - \delta)].$$

# Equilibrium

# Definition

**Definition:** A recursive equilibrium is:

- A value function  $V(K)$  and its associated policy rule  $G(K)$ .

such that,

- $V(K)$  and  $G(K)$  solve the the social planner's problem.

# Characterization

The recursive equilibrium is again characterized by the Euler equation and the aggregate resource constraint for any state  $K$  in its set of possible values.