Search Advertising: Numerical analysis

Model: demand function, consumer type, and profit

In[1]:=
```
Demand[theta_, p_, pbar_, eta_] := Piecewise[{{0, p > pbar}}, theta + p^(-eta)]
InverseDemand[theta_, q_, eta_] := Piecewise[{{0, theta > q}}, (q - theta)^(-1/eta)]
CdfTheta[theta_, gamma_] := Piecewise[{{0, theta < 0}, {1, theta > 1}}, theta^gamma]
PdfTheta[theta_, gamma_] = D[CdfTheta[theta, gamma], theta]
Profit[theta_, p_, pbar_, eta_] :=
 Piecewise[{{0, p > pbar }}, (p - 0.1) * Demand[theta, p, pbar, eta]]
DerivativeProfit[theta_, p_, pbar_, eta_] = D[Profit[theta, p, pbar, eta], p]
```

Out[4]=

$$\begin{cases} 0 & theta < 0 \\ gamma\ theta^{-1+gamma} & 0 < theta < 1 \\ 0 & theta > 1 \\ Indeterminate & True \end{cases}$$

Out[6]=

$$\begin{cases} 0 & p - pbar > 0 \\ \frac{eta\ (-0.1+p)\ theta}{p} - eta\ (-0.1+p)\ p^{-1-eta} \left(1. + p^{eta}\ theta\right) + p^{-eta} \left(1. + p^{eta}\ theta\right) & True \end{cases}$$

In[7]:=
```
ptheta = 1.0; pp = 0.2; ppbar = 1.0; peta = 2.0; pgamma = 0.5;
Demand[ptheta, pp, ppbar, peta]
InverseDemand[ptheta, Demand[ptheta, pp, ppbar, peta], peta]
Profit[ptheta, pp, ppbar, peta]
CdfTheta[0.5, pgamma]
```

Out[8]= 26.

Out[9]= 0.2

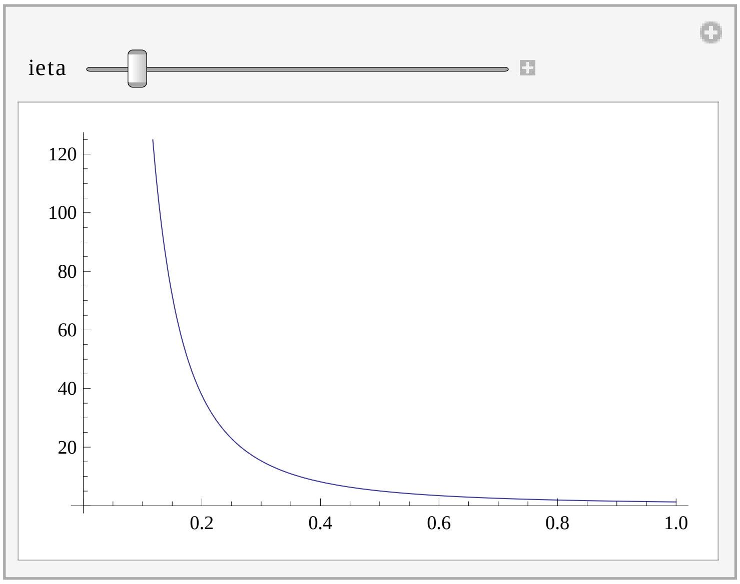
Out[10]= 2.6

Out[11]= 0.707107

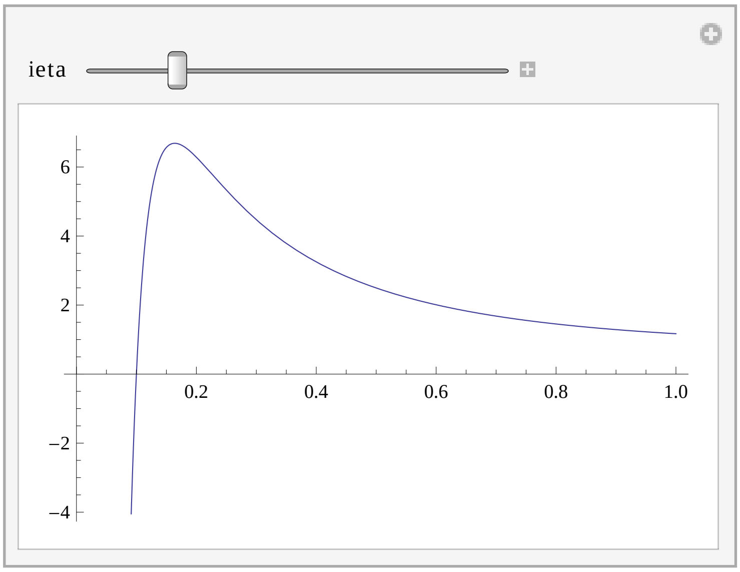
Remark: When eta increases, demand sensitivity increases => argmax of profit decreases... but max of profit increases (argmax closer to 0 in which demand goes to infinity)

In[81]:=
```
Manipulate[Plot[Demand[0.3, ip, ppbar, ieta], {ip, 0, 1}], {ieta, 2, 5}]
Manipulate[Plot[Profit[0.3, ip, ppbar, ieta], {ip, 0, 1}], {ieta, 2, 5}]
Manipulate[Plot[CdfTheta[itheta, igamma], {itheta, 0, 1}], {igamma, 0.5, 1.5}]
```

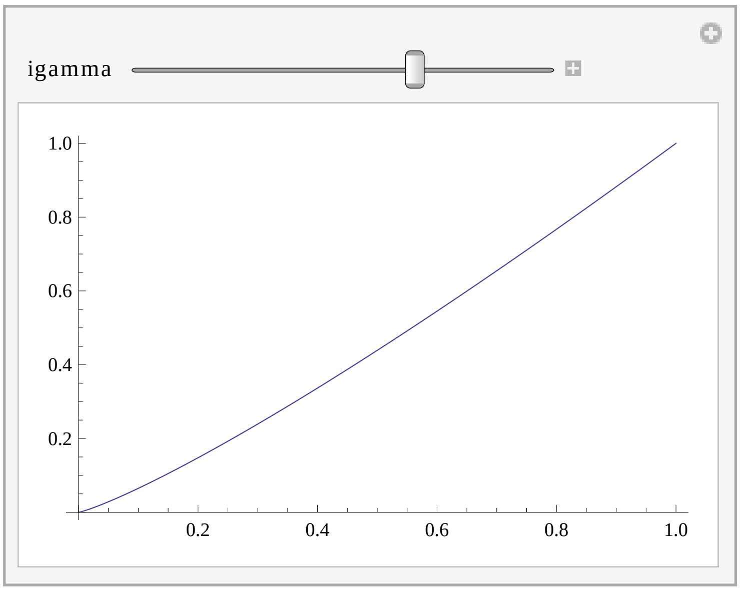
Out[81]=



Out[82]=



Out[83]=



First order condition (Privacy)

In[13]:=
```
FirstOrderCondition[p_, pbar_, eta_, gamma_] := Integrate[
  DerivativeProfit[theta, p, pbar, eta] * PdfTheta[theta, gamma] , {theta, 0, 1}]
FullSimplify[FirstOrderCondition[p, pbar, eta, gamma]]
```

Out[14]=
```
⎧ ⎡ 0.1 p^(-1.-1. eta) (eta+eta gamma+10. p-10. eta p+10. gamma p-10. eta gamma p+10. gamma p^(1.+eta))
⎪ ⎢ ----------------------------------------------------------------------------------------------------   Re[gamma] > 0.
⎪ ⎢                                          1.+gamma
⎨ ⎢ Integrate[0.1 gamma p^(-1.-1. eta) theta^(-1.+gamma)                                                      True        p - 1. pbar ≤
⎪ ⎢   (eta + 10. p - 10. eta p + 10. p^(1.+eta) theta), {theta, 0., 1.},
⎪ ⎣  Assumptions → p - 1. pbar ≤ 0. && Re[gamma] ≤ 0.]
⎩ 0.                                                                                                                      True
```

In[15]:=
```
pp = 0.2; ppbar = 1.0; peta = 2.0; pgamma = 0.5;
FirstOrderCondition[pp, ppbar, peta, pgamma]
```

Out[16]= 0.333333

Results: Profit, CS, Welfare (Privacy)

In[17]:=
```
ExpectedTheta[gamma_] := Integrate[theta * D[CdfTheta[theta, gamma], theta], {theta, 0, 1}]
ExpectedProfit[p_, pbar_, eta_, gamma_] :=
 Integrate[Profit[theta, p, pbar, eta] * D[CdfTheta[theta, gamma], theta], {theta, 0, 1}]
ExpectedConsumerSurplus[theta_, qstar_, eta_] :=
 Integrate[InverseDemand[theta, q, eta], {q, 0, qstar}]
FullSimplify[ExpectedProfit[p, pbar, eta, gamma]]
```

Out[20]=
```
⎧ ⎡ 0.1 p^(-1. eta) (-1.+10. p) (1.+gamma+gamma p^eta)
⎪ ⎢ -------------------------------------------------                    Re[gamma] > 0.
⎪ ⎢                    1.+gamma
⎨ ⎢ Integrate[0.1 gamma p^(-1. eta) (-1. + 10. p)                          True        p - 1. pbar ≤
⎪ ⎢   theta^(-1.+gamma) (1. + p^eta theta), {theta, 0., 1.},
⎪ ⎣  Assumptions → p - 1. pbar ≤ 0. && Re[gamma] ≤ 0.]
⎩ 0.                                                                                   True
```

In[21]:=
```
pp = 0.2; ppbar = 1.0; peta = 2.0; pgamma = 0.5;
ExpectedProfit[pp, ppbar, peta, pgamma] ==
 Profit[ExpectedTheta[pgamma], pp, ppbar, peta]
ExpectedConsumerSurplus[ExpectedTheta[pgamma],
 Demand[ExpectedTheta[pgamma], pp, ppbar, peta], peta]
```

Out[22]= True

Out[23]= 10.

Computation of equilibria: Tests (Privacy)

In[24]:=
```
ppbar = 1.0; peta = 2.0; pgamma = 0.5;
Quiet[NSolve[FirstOrderCondition[p, ppbar, peta, pgamma] == 0, p]]
```

Out[25]= {{p → 1. - 1. UnitStep^(-1)[0.]}, {p → -1.82451}, {p → 0.202779}, {p → 1.62174}}

Solving for various gamma (higher gamma: higher density of consumers towards gamma value 1 and thus higher profit)

In[26]:=
```
Quiet[For[igamma = 0.5, igamma ≤ 1.5, igamma += 0.5,
          arsol =  NSolve[FirstOrderCondition[p , ppbar, peta, igamma] == 0, p];
          sol = Select[p /. arsol, # ≥ 0 && # ≤  1 &][[1]];
          Print[{sol, ExpectedProfit[sol, ppbar, peta, igamma]}]]]
```

```
{0.202779, 2.53379}

{0.204261, 2.55104}

{0.205183, 2.56151}
```

Solving for various eta (higher eta: higher demand sensitivity incudes lower prices, but profit increases)

```
In[27]:= Quiet[For[ieta = 2.0, ieta ≤ 3.0, ieta += 0.5,
              arsol =  NSolve[FirstOrderCondition[p , ppbar, ieta, pgamma] == 0, p];
              sol = Select[p /. arsol, # ≥ 0 && # ≤ 1 &][[1]];
              Print[{sol, ExpectedProfit[sol, ppbar, ieta, pgamma]}]]]

{0.202779, 2.53379}

{0.16709, 5.90107}

{0.150085, 14.8315}
```

Order statistics (Disclosure)

```
In[28]:=
PdfOrderStat[x_, k_, n_, gamma_] := (n!) / ((k - 1) ! * (n - k) !) *
  CdfTheta[x, gamma] ^ (k - 1) * (1 - CdfTheta[x, gamma]) ^ (n - k) * PdfTheta[x, gamma]
CdfOrderStat[y_, k_, n_, gamma_] = Integrate[PdfOrderStat[x, k, n, gamma], {x, 0, y}]
```

Out[29]=
$$\int_0^y \frac{n!\left(1-\begin{cases}0 & x<0\\ 1 & x>1\\ x^{\text{gamma}} & \text{True}\end{cases}\right)^{-k+n}\left(\begin{cases}0 & x<0\\ 1 & x>1\\ x^{\text{gamma}} & \text{True}\end{cases}\right)^{-1+k}\left(\begin{cases}0 & x<0\\ \text{gamma}\,x^{-1+\text{gamma}} & 0<x<1\\ 0 & x>1\\ \text{Indeterminate} & \text{True}\end{cases}\right)}{(-1+k)!\,(-k+n)!}\,dx$$

```
In[30]:=

In[31]:=
pk = 3; pn = 4; pgamma = 0.5;
Print[{Integrate[PdfOrderStat[x, pk, pn, pgamma], {x, 0, 0.5}],
  CdfOrderStat[0.5, pk, pn, pgamma]}]
Print[{Integrate[PdfOrderStat[x, pk - 1, pn, pgamma], {x, 0, 0.5}],
  CdfOrderStat[0.5, pk - 1, pn, pgamma]}]

{0.664214, 0.664214}

{0.921573, 0.921573}

In[34]:= pk = 1; pn = 1;
Print[{CdfTheta[0.4, pgamma], CdfOrderStat[0.4, pk, pn, pgamma]}]
Print[{CdfOrderStat[0.4, pk + 1, pn + 1, pgamma], CdfOrderStat[0.4, pk, pn + 1, pgamma]}]

{0.632456, 0.632456}

{0.4, 0.864911}
```

Reading (General: Proba that the kth highest draw among n is below y)
With gamma as 0.5 and only one draw, the probability to draw a theta lower than 0.4 is c. 0.63.
With the same gamma and two draws: the proba that the highest of both (the 2nd of 2) is below 0.4 is 0.4 (<0.63) and the proba that the lowest (the 1st of 2) is below 0.4 is c. 0.86.

First order condition (Disclosure)

In[37]:=
```
FirstOrderConditionDisclosure[p_, pbar_, eta_, gamma_, k_, n_] :=
 Integrate[DerivativeProfit[theta, p, pbar, eta] *
   PdfTheta[theta, gamma] * CdfOrderStat[theta, k, n, gamma], {theta, 0, 1}]
FullSimplify[FirstOrderConditionDisclosure[p, pbar, eta, gamma, k, n]]
```

Out[38]=

$$\int_0^1 \left(\int_0^{theta} \frac{n!\left(1-\begin{cases}0 & x<0\\ 1 & x>1\\ x^{gamma} & True\end{cases}\right)^{-k+n}\left(\begin{cases}0 & x<0\\ 1 & x>1\\ x^{gamma} & True\end{cases}\right)^{-1+k}\left(\begin{cases}Indeterminate & x==0\,||\,x==1\\ gamma\,x^{-1+gamma} & 0<x<1\\ 0 & True\end{cases}\right)}{Gamma[k]\,Gamma[1-k+n]}\,dx \right)$$

$$\left(\begin{cases}0 & p>pbar\\ p^{-1-eta}\,(eta\,(0.1-1.\,p)+1.\,p)+theta & True\end{cases}\right)\left(\begin{cases}Indeterminate & theta==0\,||\,theta==1\\ gamma\,theta^{-1+gamma} & 0<theta<1\\ 0 & True\end{cases}\right) d\,theta$$

Results: Profit, CS, Welfare (Disclosure)

In[39]:=
```
ExpectedThetaOrder[k_, n_, gamma_] = Integrate[x * PdfOrderStat[x, k, n, gamma], {x, 0, 1}]
ExpectedProfitOrder[p_, pbar_, eta_, k_, n_, gamma_] =
 Integrate[Profit[x, p, pbar, eta] * PdfOrderStat[x, k, n, gamma], {x, 0, 1}]
```

Out[39]=

$$\int_0^1 \frac{x\,n!\left(1-\begin{cases}0 & x<0\\ 1 & x>1\\ x^{gamma} & True\end{cases}\right)^{-k+n}\left(\begin{cases}0 & x<0\\ 1 & x>1\\ x^{gamma} & True\end{cases}\right)^{-1+k}\left(\begin{cases}0 & x<0\\ gamma\,x^{-1+gamma} & 0<x<1\\ 0 & x>1\\ Indeterminate & True\end{cases}\right)}{(-1+k)!\,(-k+n)!}\,dx$$

Out[40]=

$$\int_0^1 \frac{1}{(-1+k)!\,(-k+n)!}\,n!\left(\begin{cases}0 & p>pbar\\ (-0.1+p)\left(\begin{cases}0 & p>pbar\\ p^{-eta}+x & True\end{cases}\right) & True\end{cases}\right)$$

$$\left(1-\begin{cases}0 & x<0\\ 1 & x>1\\ x^{gamma} & True\end{cases}\right)^{-k+n}\left(\begin{cases}0 & x<0\\ 1 & x>1\\ x^{gamma} & True\end{cases}\right)^{-1+k}\left(\begin{cases}0 & x<0\\ gamma\,x^{-1+gamma} & 0<x<1\\ 0 & x>1\\ Indeterminate & True\end{cases}\right) dx$$

In[41]:=
```
pk = 2; pn = 3; pgamma = 0.5; pp = 0.2; ppbar = 1.0; peta = 2.0;
ExpectedThetaOrder[pk, pn, pgamma]
ExpectedProfitOrder[pp, ppbar, peta, pk, pn, pgamma] ==
 Profit[ExpectedThetaOrder[pk, pn, pgamma], pp, ppbar, peta]
```

Out[42]= 0.3

Out[43]= True

Computation of equilibria: Tests (Disclosure)

In[44]:=
```
pp = 0.2; ppbar = 1.0; peta = 2.0; pgamma = 0.5; pk = 4; pn = 9;
FirstOrderConditionDisclosure[pp, ppbar, peta, pgamma, pk, pn]
FindRoot[FirstOrderConditionDisclosure[p, ppbar, peta, pgamma, pn, pn] == 0, {p, 0.1}]
FindRoot[FirstOrderConditionDisclosure[p, ppbar, peta, pgamma, pk, pn] == 0, {p, 0.1}]
```

Out[45]= 0.30303

Out[46]= {p → 0.207439}

Out[47]= {p → 0.204307}

Solving for various gamma (higher gamma: higher density of consumers towards gamma value 1 and thus higher profit):

In[48]:=
```
Quiet[For[igamma = 0.5, igamma ≤ 1.5, igamma += 0.5,
          sol =  p /. FindRoot[
      FirstOrderConditionDisclosure[p, ppbar, peta, igamma, pk, pn] == 0, {p, 0.1}];
          Print[{sol, ExpectedProfit[sol, ppbar, peta, igamma]}]]]
```

{0.204307, 2.53366}

{0.205957, 2.55089}

{0.206775, 2.56138}

Solving for various eta (higher eta: higher demand sensitivity... price decreases due to higher sensitivity)

In[49]:=
```
Quiet[For[ieta = 2.0, ieta ≤ 3.0, ieta += 0.5,
          sol =  p /. FindRoot[
      FirstOrderConditionDisclosure[p, ppbar, ieta, pgamma, pk, pn] == 0, {p, 0.1}];
          Print[{sol, ExpectedProfit[sol, ppbar, ieta, pgamma]}]]]
```

{0.204307, 2.53366}

{0.167312, 5.90105}

{0.150128, 14.8315}

Solving for various k, n

In[50]:=
```
pgamma = 0.5; peta = 2.0; ppbar = 1.0; pn = 10; ark = {10, 9, 8, 2, 1};
FindRoot[FirstOrderConditionDisclosure[p, ppbar, peta, pgamma, 10, 10] == 0, {p, 0.1}]
arpstardisclo =
 Table[p /. FindRoot[FirstOrderConditionDisclosure[p, ppbar, peta, pgamma, k, pn] == 0,
    {p, 0.1}], {k, ark}]
arprofitdisclo = Table[ExpectedProfitOrder[arpstardisclo[[i]], ppbar,
   peta, ark[[i]], pn, pgamma], {i, Range[Dimensions[ark][[1]]]}]
```

Out[51]= {p → 0.207567}

Out[52]= {0.207567, 0.206866, 0.206221, 0.203379, 0.203059}

Out[53]= {2.58632, 2.57011, 2.55566, 2.50401, 2.50099}

As the number of auctionned slots increases, proft/bid decreases... Disclosure is first profitable (with small k) then privacy is best.

Output for privacy

```
In[54]:= dim = Apply[Times, Dimensions[Table[0, {igamma, 0.5, 1.5, 0.5}, {ieta, 2.0, 3.0, 0.5}]]];
arparam = Range[dim];
arresult = Range[dim];
arprofit = Range[dim];
arcsurplus = Range[dim];
count = 0;
Quiet[Do[{count = count + 1,
   arsol = NSolve[FirstOrderCondition[p , ppbar, ieta, igamma] == 0 , p],
   sol = Select[p /. arsol, # ≥ 0 && # ≤ 1 &] ,
   arresult[[count]] = sol[[1]],
   arparam[[count]] = {igamma, ieta},
   arprofit[[count]] = ExpectedProfit[arresult[[count]], ppbar, ieta, igamma],
   etheta = ExpectedTheta[igamma],
   eqstar = Demand[etheta, sol[[1]], ppbar, ieta],
   arcsurplus[[count]] = ExpectedConsumerSurplus[etheta, eqstar, ieta]},
  {igamma, 0.5, 1.5, 0.5}, {ieta, 2.0, 3.0, 0.5}]]

In[61]:= Partition[arparam, Sqrt[dim]]
Partition[arresult, Sqrt[dim]]
Partition[arprofit, Sqrt[dim]]
Partition[arcsurplus, Sqrt[dim]]
Partition[arresult, Sqrt[dim]]

Out[61]= {{{0.5, 2.}, {0.5, 2.5}, {0.5, 3.}},
  {{1., 2.}, {1., 2.5}, {1., 3.}}, {{1.5, 2.}, {1.5, 2.5}, {1.5, 3.}}}

Out[62]= {{0.202779, 0.16709, 0.150085},
  {0.204261, 0.167305, 0.150127}, {0.205183, 0.167435, 0.150152}}

Out[63]= {{2.53379, 5.90107, 14.8315}, {2.55104, 5.91227, 14.8398}, {2.56151, 5.919, 14.8449}}

Out[64]= {{9.86294, 24.4018, 66.5916}, {9.79139, 24.3548, 66.5539}, {9.7474, 24.3265, 66.5313}}

Out[65]= {{0.202779, 0.16709, 0.150085},
  {0.204261, 0.167305, 0.150127}, {0.205183, 0.167435, 0.150152}}
```

Within row: increase in eta i.e. demand price sensitivity decreases optimal price but increases profit (and consumer surplus, which is ok if price decreases)
Between rows: increase in gamma i.e. shift of weight in customer valuations towards higher valuations increases price and profit (decreases consumer surplus)

Output for disclosure

```
In[66]:= pgamma = 0.5;
peta = 2.0;
ppbar = 1.0;
pn = 10;
ark = {10, 9, 8, 2, 1};
dimark = Dimensions[ark][[1]];
arresultdisclo = Range[dimark];
arprofitdisclo = Range[dimark];
arcsurplusdisclo = Range[dimark];
count = 0;
```

```
In[76]:= Quiet[Do[{count = count + 1,
     arsol = FindRoot[
       FirstOrderConditionDisclosure[p, ppbar, peta, pgamma, i, pn] == 0, {p, 0.1}],
     sol = Select[p /. arsol, # ≥ 0 && # ≤ 1 &] ,
     arresultdisclo[[count]] = sol[[1]],
     arprofitdisclo[[count]] =
      ExpectedProfitOrder[sol[[1]], ppbar, peta, i, pn, pgamma],
     etheta = ExpectedThetaOrder[i, pn, pgamma],
     eqstar = Demand[etheta, sol[[1]], ppbar, peta],
     arcsurplusdisclo[[count]] = ExpectedConsumerSurplus[etheta, eqstar, peta]},
    {i, ark}]]

In[77]:= ark
         arresultdisclo
         arprofitdisclo
         arcsurplusdisclo

Out[77]= {10, 9, 8, 2, 1}

Out[78]= {0.207567, 0.206866, 0.206221, 0.203379, 0.203059}

Out[79]= {2.58632, 2.57011, 2.55566, 2.50401, 2.50099}

Out[80]= {9.63544, 9.66807, 9.69831, 9.83384, 9.84934}
```