The Gamma Distribution
Properties, Proofs and Applications

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1 Historical Context

The Gamma Distribution is a distribution containing continuous random variables X with two non-negative parameters α and λ. It is written as

$$X \sim \mathcal{G}\text{amma}(\alpha, \lambda) \tag{1}$$

and its density function is

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \text{ for } x \in (0, \infty) \tag{2}$$

A gamma distributed random variable X is represented by

$$X \sim \Gamma(\alpha, \lambda) \tag{3}$$

This distribution gave rise to many other functions and distributions such as the Gamma Function, the Exponential Distribution, Erlang Distribution, Chi-Square Distribution. This Distribution was firstly presented by Leon Hard Euler, a Swiss mathematician and physician. As Euler gained popularity, the Gamma Distribution was further researched by important figures in the domain of mathematics such as Karl Weierstrass, Carl Friedrich Gauss, Charles Hermite, and many more [1], [2], [3] .

2   Distribution, Variables Defined

The gamma distribution is part of the two-parameters family of continuous probability distributions. Indeed, it may be parameterized with two different parameterizations[2] :

Parameterization 1:

$$\text{Shape: } \alpha > 0 \hspace{4cm} \text{Rate: } \lambda > 0 \tag{4}$$

Parameterization 2:

$$\text{Shape: } k > 0 \hspace{4cm} \text{Scale: } \theta > 0 \tag{5}$$

In the second parameterization, the scale parameter θ corresponds to the inverse of the rate parameter λ, allowing flexibility in the modelization of the distribution. Indeed, the application of the scale parameter is pertinent when one wants to use the value of the mean time between events in the probability density function, as opposed to the total time until a given number of events occurs.

That being said, in this document, as the two parameterizations only exist for the sake of convenience and are identical in their results, only the parameterization 1 presented in Equation 4 will be considered and used for proofs.

2.1   Probability density function INCOMPLETE

graph, integral of pdf=1

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \text{ for } x \in (0, \infty) \tag{6}$$

Proof. The probability density function of the gamma distribution will be proved using induction. First, take

$$W = \sum_{i=1}^{n} X_i \tag{7}$$

where W is a sum of n i.i.d. random variables following the exponential distribution $\mathcal{E}\text{xp}(\lambda)$. For $n = 1$: Then, with the induction that $n = \alpha$ works such that: Here is the proof that $n = \alpha + 1$ works such that: QED

2.2   Cumulative distribution function

The cumulative distribution function (CDF) $F(x)$ of the gamma distribution is defined as:

$$\int_0^x \frac{\lambda^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\lambda u}\, \mathrm{d}u \tag{8}$$

but this integral has no definite form, that is, the integrand has no antiderivative. Hence, this CDF is often represented in its simplified form using the lower incomplete gamma function [4] :

$$\gamma(\alpha, \lambda x) \equiv \int_0^x u^{\alpha-1} e^{-u}\, \mathrm{d}u \tag{9}$$

such that:

$$F(x) = \frac{\gamma(\alpha, \lambda x)}{\Gamma(\alpha)} \tag{10}$$

Interestingly, there is a special case to the CDF; when α is a positive integer, that is, when the distribution is an Erlang distribution (see subsection 7.1), the CDF follows the following series expansion [2] :

$$F(x) = 1 - \sum_{i=0}^{\alpha-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x} = e^{-\lambda x} \sum_{i=\alpha}^{\infty} \frac{(\lambda x)^i}{i!} \tag{11}$$

That being mentioned, generally, although lacking a closed form, the cumulative distribution function of the gamma distribution may be graphed using software:

Cumulative Distribution Function of Gamma Distribution
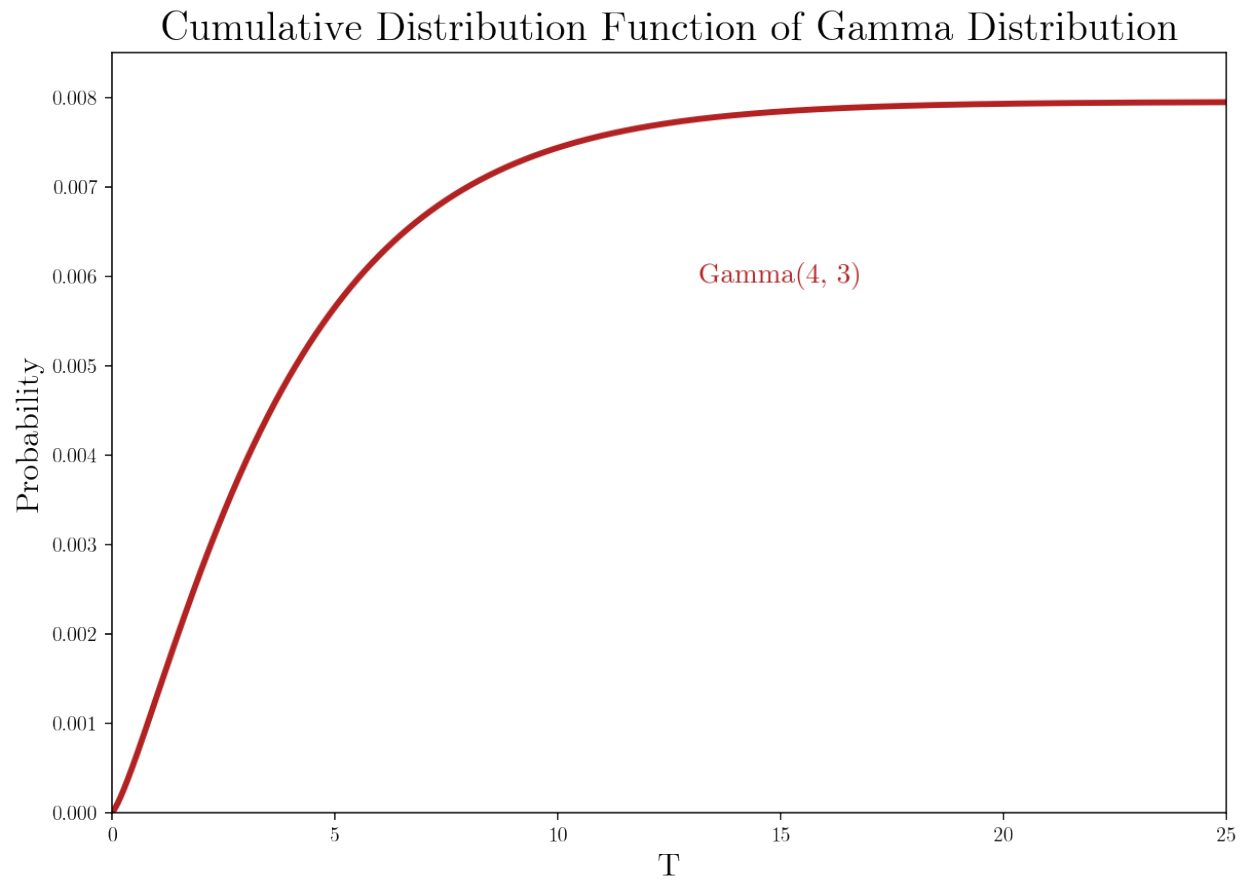
FIGURE 1: Graph featuring the plot of the cumulative distribution function of a gamma distribution.

3 Properties

Now that the fundamentals of the gamma distribution are set, a few properties stemming from the probability density function and the cumulative distribution function will be presented and proved. These properties, along with being extremely intriguing, are essential to multiple computations and statistical analyses.

3.1 Moment generating function INCOMPLETE

$$M(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha} \quad \text{for } t < \lambda \tag{12}$$

3.2   Expected Value

The expected value, also known as the theoretical mean, may be found for the gamma distribution using the following formula::

$$\mu = E(x) = \frac{\alpha}{\lambda} \tag{13}$$

Interestingly, there are multiple ways to prove this result. Three different proofs will be explored.

Proof 1:

The first technique uses the gamma distribution probability density function presented in subsection 2.1 as a means to get the expected value. In fact:

$$E(x) = \int_0^\infty x f(x)\, \mathrm{d}x = \frac{\alpha}{\lambda} \tag{14}$$

Proof.

$$
\begin{aligned}
E(x) &= \int_0^\infty x f(x)\, \mathrm{d}x \\
&= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^\alpha e^{-\lambda x}\, \mathrm{d}x \\
&= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{(\alpha+1)-1} e^{-\lambda x}\, \mathrm{d}x \\
&= \frac{\Gamma(\alpha+1)}{\lambda \Gamma(\alpha)} \int_0^\infty \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\lambda x}\, \mathrm{d}x \\
&= \frac{\alpha \Gamma(\alpha)}{\lambda \Gamma(\alpha)}(1) \\
&= \frac{\alpha}{\lambda}
\end{aligned}
\tag{15}
$$

QED

Proof 2:

Using the moment generating function obtained in subsection 2.1, the expected value of a gamma distribution may be found. That is:

$$E(x) = M'(0) = \frac{\alpha}{\lambda} \tag{16}$$

Proof.

$$
\begin{aligned}
M'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{\lambda - t}\right)^{\alpha} \\
&= \lambda\alpha\left(\frac{\lambda}{\lambda - t}\right)^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\lambda - t}\right) \\
&= \frac{\lambda\alpha}{(\lambda - t)^2}\left(\frac{\lambda}{\lambda - t}\right)^{\alpha-1} \\
M'(0) &= \frac{\lambda\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda}\right)^{\alpha-1} \\
&= \frac{\alpha}{\lambda}
\end{aligned}
\tag{17}
$$

QED

Proof 3:

A third way to demonstrate that $E(x) = \frac{\alpha}{\lambda}$, is by performing the following integral:

$$\int_0^{\infty} 1 - F(x)\,\mathrm{d}x \tag{18}$$

However, this technique is not possible in the case of the gamma distribution as there is no explicit formula for $F(x)$ as seen in subsection 2.2.

3.3   Variance INCOMPLETE

$$\text{Var}(x) = \frac{\alpha}{\lambda^2} \tag{19}$$

Proof. Using the moment generating function obtained in subsection 2.1, the variance value of a gamma distribution may be found.

$$E(x) = M''(0) \tag{20}$$

QED

3.4   Quantiles INCOMPLETE

Since we can generate the graph of the CDF, we should still explain what the quantiles are... at least visually using a graph

Usually, the quantile function is defined as the inverse of the cumulative distribution function. Since the gamma distribution has no closed form for its CDF (see subsection 2.2) and, therefore, the latter cannot be inverted, it has no quantile function. In other words, there is no explicit form for the quantile function since $q(p) = F^{-1}(p)$ but $F^{-1}(x)$ is not defined as $F(x)$ is an integral with no antiderivative.

3.5   Median proof

The median of a probability distribution is the value that separates the area under the curve of its distribution function in two equal parts. It follows that, in the case of the gamma distribution, the median is defined as the value m for which:

$$\frac{1}{2} = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^m x^{\alpha-1} e^{-\lambda x} \, \mathrm{d}x \tag{21}$$

However, it is interesting to note that the median of the gamma distribution has no simple closed form : due to its highly varying shape, it cannot be represented in an equation as a function of its parameters. Although approximations of the median can be performed for certain ranges of the shape parameter, this topic will not be explored in depth by this paper. Still, there is comfort in the fact that some special cases of the gamma distribution do have a median, such as the exponential distribution. Indeed, the median of a continuous random variable following the gamma distribution such that $X \sim \mathcal{G}\mathrm{amma}(1, \lambda) \equiv \mathcal{E}\mathrm{xp}(\lambda)$

may be obtained through a simple integration process:

$$
\begin{aligned}
\frac{1}{2} &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^m x^{\alpha-1} e^{-\lambda x} \, \mathrm{d}x \\
\Rightarrow \frac{1}{2} &= -e^{-\lambda m} + 1 \\
\Rightarrow \frac{1}{2} &= e^{-\lambda m} \\
\Rightarrow \ln(\frac{1}{2}) &= -\lambda m \\
\Rightarrow -\ln(2) &= -\lambda m \\
\Rightarrow m &= \frac{\ln(2)}{\lambda}
\end{aligned}
\tag{22}
$$

Hence, the median of an exponential distribution is defined as $\frac{\ln(2)}{\lambda}$.

3.6   Mode

The mode is defined as the value appearing the most through a certain set of data values, that is, the location of the highest peak on a probability density function plot. On Figure 2, the mode represents the value of T where the density is at its peak.
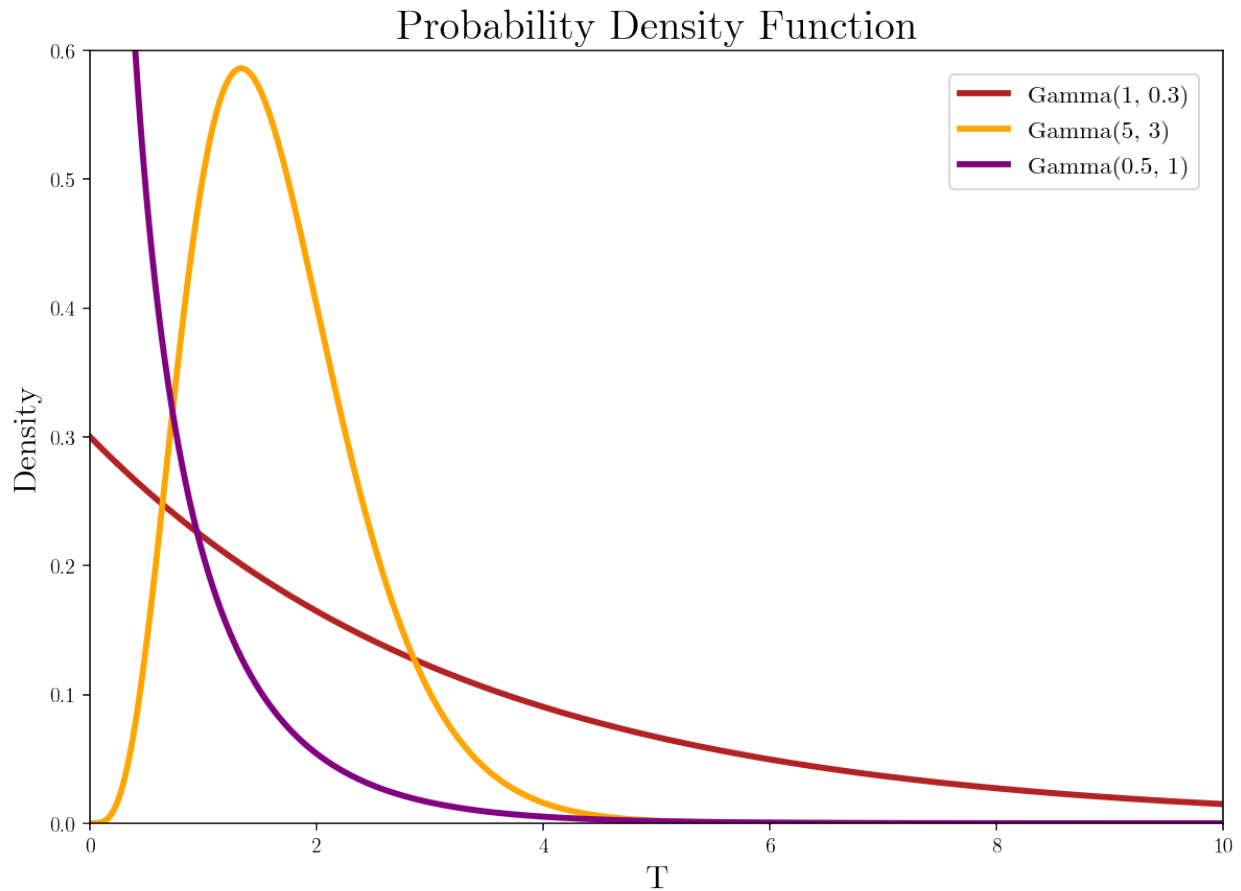
FIGURE 2: Graph featuring the probability density function of two distinct gamma distributions.

As may be observed in the red plot ($\mathcal{G}$amma$(1, 0.3)$), the highest value of the density on the curve is 0.3 at $T = 0$ which corresponds to the mode of this specific parameterization of the distribution. The mode of the density function $\mathcal{G}$amma$(1, 0.3)$ is thus 0.3. To be more precise and avoid the need for a graph, the following equation may be used to calculate the

mode of a gamma distribution:

$$\text{Mode} = \frac{(\alpha - 1)}{\lambda} \text{ for } \alpha \geq 1 \tag{23}$$

Ergo, for the orange plot ($\mathcal{G}$amma$(5, 3)$), the mode can be determined to be:

$$T = \frac{(5 - 1)}{3} = \frac{4}{3} \tag{24}$$

Accordingly, on Figure 2, the highest peak of the plot $\mathcal{G}$amma$(5, 3)$ is at $\frac{4}{3}$

3.6.1   Special Cases

There are two special cases concerning the mode of the gamma distribution: when the density of the PDF goes to infinity (the graph has a vertical asymptote) and when the parameter $\alpha = 1$.

Case 1:

The first special case is when the mode PDF never reaches its maximum value, that is, when there is a vertical asymptote on its plot. The purple plot ($\mathcal{G}$amma$(0.5, 1)$) on Figure 2 features this case. Indeed, if $f(x)$ is the probability density function, then:

$$\lim_{T \to 0^+} f(x) = \infty \tag{25}$$

Notice how this case arises when the condition stated in Equation 23 is broken, that is, that $\alpha \geq 1$. In fact, in the case of the purple plot, $\alpha = 0.5 < 1$

Case 2:

In a gamma distribution $\mathcal{G}$amma$(1, \lambda) \equiv \mathcal{E}xp(\lambda)$, the highest point of the plot of the PDF will be located at $T = 0$ and will correspond to a density value of λ. In Figure 2, the red plot

($\mathcal{G}$amma$(1, 0.3)$) is an example of this special case as it has $\alpha = 1$ as a parameter. Hence, the mode of this latter parameterization of the gamma distribution is of $\lambda = 0.3$ at $T = 0$.

4   Advanced Properties

4.1   Skewness INCOMPLETE

$$\text{Skewness} = \frac{2}{\sqrt{\alpha}} \tag{26}$$

4.2   Excess kurtosis INCOMPLETE

$$\text{Kurtosis} = \frac{6}{\alpha} \tag{27}$$

4.3   Information Entropy INCOMPLETE

$$\text{Entropy} = \alpha + \ln \lambda + \ln \Gamma(\alpha) + (1 - \alpha)\psi(\alpha) \tag{28}$$

4.4   Characteristic function INCOMPLETE

$$\text{CF} = \left(1 - \frac{it}{\lambda}\right)^{-\alpha} \tag{29}$$

4.5   Higher moments INCOMPLETE

The nth raw moment is given by:

$$E[X^n] = \frac{\Gamma(n + \alpha)}{\lambda^n \Gamma(\alpha)} \tag{30}$$

4.6   Methods of moments INCOMPLETE

$$\alpha = \frac{E(X)^2}{\text{Var}(X)}$$
$$\lambda = \frac{E(X)}{\text{Var}(X)} \tag{31}$$

5 The Gamma Function

Studied by Leonhard Euler (1707-1783), the gamma function is related to the gamma distribution. This extremely handy function allows the generalization of the factorial notation. As a matter of fact, whereas the factorial $\alpha!$ is restricted to $\alpha \in \mathbb{N}$, the gamma function $\Gamma(\alpha)$ allows the computation of fractional and negative numbers such that $\alpha \in \mathbb{Q}$. By abusing the notation, the gamma function may even be used to compute $\Gamma(\alpha)$ such that $\alpha \in \mathbb{R}$. The relationship between the gamma function and the factorial notation is as follows:

$$\Gamma(\alpha) = (\alpha - 1)! \text{ for } \alpha \in \mathbb{N} \tag{32}$$

5.1 Definition

First of all, the gamma function is defined as:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\, \mathrm{d}x \text{ for } \alpha \in \mathbb{R}^+ \tag{33}$$

Assuming that $\alpha > 1$, the closed form of the Gamma function may be found with integration by parts:

$u = x^{\alpha-1}$	$\mathrm{d}v = e^{-x}\,\mathrm{d}x$
$\mathrm{d}u = (\alpha - 1)x^{\alpha-2}\,\mathrm{d}x$	$v = -e^{-x}$

$$\begin{aligned}
\Gamma(\alpha) &= \int_0^\infty x^{\alpha-1}e^{-x}\, \mathrm{d}x \\
&= -\left[x^{\alpha-1}e^{-x}\right]_0^\infty + (\alpha - 1)\int_0^\infty x^{\alpha-2}e^{-x}\, \mathrm{d}x \\
&= 0 + (\alpha - 1)\Gamma(\alpha - 1) \\
&= (\alpha - 1)\Gamma(\alpha - 1)
\end{aligned} \tag{34}$$

Therefore,

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1) \text{ for } \alpha \in (1, \infty) \tag{35}$$

Furthermore, $\Gamma(1)$ is defined separately as being:

$$\Gamma(1) = \int_0^\infty e^{-x}\, \mathrm{d}x = \left[-e^{-x}\right]_0^\infty = 1 \tag{36}$$

5.1.1 Special case of $\Gamma(1/2)$ Gamma12_BNM_VC.pdf

An extremely interesting case of the gamma function is $\Gamma(1/2)$. In fact, this result allows the computation of fractional values of α such that $\alpha \in \mathbb{Q}$. This special case is as follows:

Claim 5.1.1.

$$\Gamma(1/2) = \sqrt{\pi}$$

Initial Setup

First, to prove Claim 5.1.1, let's start by setting up some equations in preparation for the proof.

$$I = \int_0^\infty e^{-x^2} \, \mathrm{d}x \tag{37}$$

From this result, one obtains that:

$$
\begin{aligned}
I^2 &= \left(\int_0^\infty e^{-x^2} \, \mathrm{d}x \right) \left(\int_0^\infty e^{-y^2} \, \mathrm{d}y \right) \\
&= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, \mathrm{d}x \, \mathrm{d}y
\end{aligned}
\tag{38}
$$

To solve this integral easily, one may use the following change of variables:

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

This change of variable requires the Jacobian of the transformation to be computed:

$$\mathcal{J} = \det \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r(\cos^2\theta + \sin^2\theta) = r \tag{39}$$

Using the result obtained in Equation 39, the integral in Equation 38 may be computed using polar coordinates:

$$
\begin{aligned}
I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{\frac{\pi}{2}} \int_0^\infty re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{\pi}{2} \int_0^\infty re^{-r^2}\,\mathrm{d}r \\
&= \frac{\pi}{2}\left[-\frac{e^{-r^2}}{2}\right]_0^\infty \\
&= \frac{\pi}{2}\frac{1}{2} \\
&= \frac{\pi}{4}
\end{aligned}
\tag{40}
$$

It then follows that Equation 37 may be solved:

$$
I = \sqrt{I^2} = \int_0^\infty e^{-x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}
\tag{41}
$$

Accordingly, Corollaries 5.1.1 and 5.1.2 logically follow:

Corollary 5.1.1.

$$
\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}
\tag{42}
$$

Corollary 5.1.2.

$$
\int_{-\infty}^\infty e^{-x^2/2}\,\mathrm{d}x = \sqrt{2\pi}
\tag{43}
$$

Let's prove these corollaries:

Proof. Consider this substitution of variables:

$$
z = \sqrt{2}x \qquad\qquad \mathrm{d}z = \sqrt{2}\,\mathrm{d}x
$$

Using this substitution, Corollaries 5.1.1 and 5.1.2 may be proven:

$$
\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x = \frac{1}{\sqrt{2}} \int_{-\infty}^\infty e^{-z^2/2}\,\mathrm{d}z = \sqrt{\pi}
\tag{44}
$$

$$
\int_{-\infty}^\infty e^{-z^2/2}\,\mathrm{d}z = \sqrt{2}\sqrt{\pi} = \sqrt{2\pi}
\tag{45}
$$

QED

The Proof

The proof for Claim 5.1.1 may now be articulated:

Proof. Consider, once again, this other substitution of variables:

$$x = \sqrt{z} \qquad\qquad \mathrm{d}x = \frac{1}{2\sqrt{z}}\,\mathrm{d}z = \frac{z^{-1/2}}{2}\,\mathrm{d}z$$

Using it, the following integral may be solved:

$$\begin{aligned}
\int_0^\infty e^{-x^2}\,\mathrm{d}x &= \frac{1}{2}\int_0^\infty z^{-1/2}e^{-z}\,\mathrm{d}z \\
&= \frac{1}{2}\Gamma(1/2) \\
&= \frac{\sqrt{\pi}}{2}
\end{aligned} \tag{46}$$

Conclusively, Claim 5.1.1 is proven:

$$\Gamma(1/2) = \sqrt{\pi} \tag{47}$$

$$\text{QED}$$

5.1.2 Negative values of α

As was mentioned earlier, by abusing this identity shown in Equation 35, one may extend the gamma function to negative non-integer values of α such that $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$. For example, to find $\Gamma(-9/2)$, one may compute:

$$
\begin{aligned}
\Gamma(1/2) &= (-1/2)\Gamma(-1/2) \\
&= (-1/2)(-3/2)\Gamma(-3/2) \\
&= (3/4)(-5/2)\Gamma(-5/2) \\
&= (-15/8)(-7/2)\Gamma(-7/2) \\
&= (105/16)(-9/2)\Gamma(-9/2) \\
&= (-945/32)\Gamma(-9/2)
\end{aligned}
\tag{48}
$$

$$
\therefore \Gamma(-9/2) = -\frac{32}{945}\Gamma(1/2) = -\frac{32}{945}\sqrt{\pi}
\tag{49}
$$

Hence, to calculate the value of a gamma function with a negative argument, one might modify a bit the result obtained in Equation 35 such that:

$$
\Gamma(\alpha) = \frac{\Gamma(\alpha + 1)}{\alpha} \text{ for } \alpha \in \mathbb{R}^- \setminus \mathbb{Z}
\tag{50}
$$

for example, take the same example of $\Gamma(-9/2)$

$$
\begin{aligned}
\Gamma(-9/2) &= \frac{\Gamma(-7/2)}{(-9/2)} \\
&= \frac{\Gamma(-5/2)}{(-9/2)(-7/2)} \\
&= \frac{\Gamma(-3/2)}{(63/4)(-5/2)} \\
&= \frac{\Gamma(-1/2)}{(-315/8)(-3/2)} \\
&= \frac{\Gamma(1/2)}{(945/16)(-1/2)} \\
&= \frac{\Gamma(1/2)}{(-945/32)}
\end{aligned}
\tag{51}
$$

$$
\therefore \Gamma(-9/2) = -\frac{32}{945}\Gamma(1/2) = -\frac{32}{945}\sqrt{\pi}
\tag{52}
$$

5.2 Relationship to Factorial Notation

Using a proof by induction, gamma functions may shown to be related to the factorial notation.

Proof. Let us prove Equation 32, that is, that:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, \mathrm{d}x = (n-1)! \text{ for } n \in \mathbb{N} \tag{53}$$

For $n = 1$:

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty x^{1-1} e^{-x} \, \mathrm{d}x \\
&= \int_0^\infty e^{-x} \, \mathrm{d}x \\
&= 1 \\
&= (1-1)! \\
&= 0!
\end{aligned} \tag{54}$$

Assume that the equation holds for $n = k$ such that:

$$\Gamma(k) = (k-1)! \tag{55}$$

Then, let us prove that the equation is valid for $n = k + 1$ using integration by parts:

$u = x^k$	$\mathrm{d}v = e^{-x} \, \mathrm{d}x$
$\mathrm{d}u = kx^{k-1} \, \mathrm{d}x$	$v = -e^{-x}$

$$\begin{aligned}
\Gamma(k+1) &= \int_0^\infty x^{(k+1)-1} e^{-x} \, \mathrm{d}x \\
&= \left[-x^k e^{-x} \right]_0^\infty + k \int_0^\infty x^{k-1} e^{-x} \, \mathrm{d}x \\
&= 0 + k\Gamma(k) \\
&= k(k-1)! \\
&= ((k+1)-1)! \\
&= k!
\end{aligned} \tag{56}$$

Therefore, Equation 32 is proven by mathematical induction.   QED

5.3   Conclusion

Using everything that was computed previously, the graph of $\Gamma(\alpha)$ may be plotted. It includes both positive, negative and fractional values of α. Notice how the asymptotes of the graph are every negative integer; the function, as was mentioned previously, is not defined for negative integers.
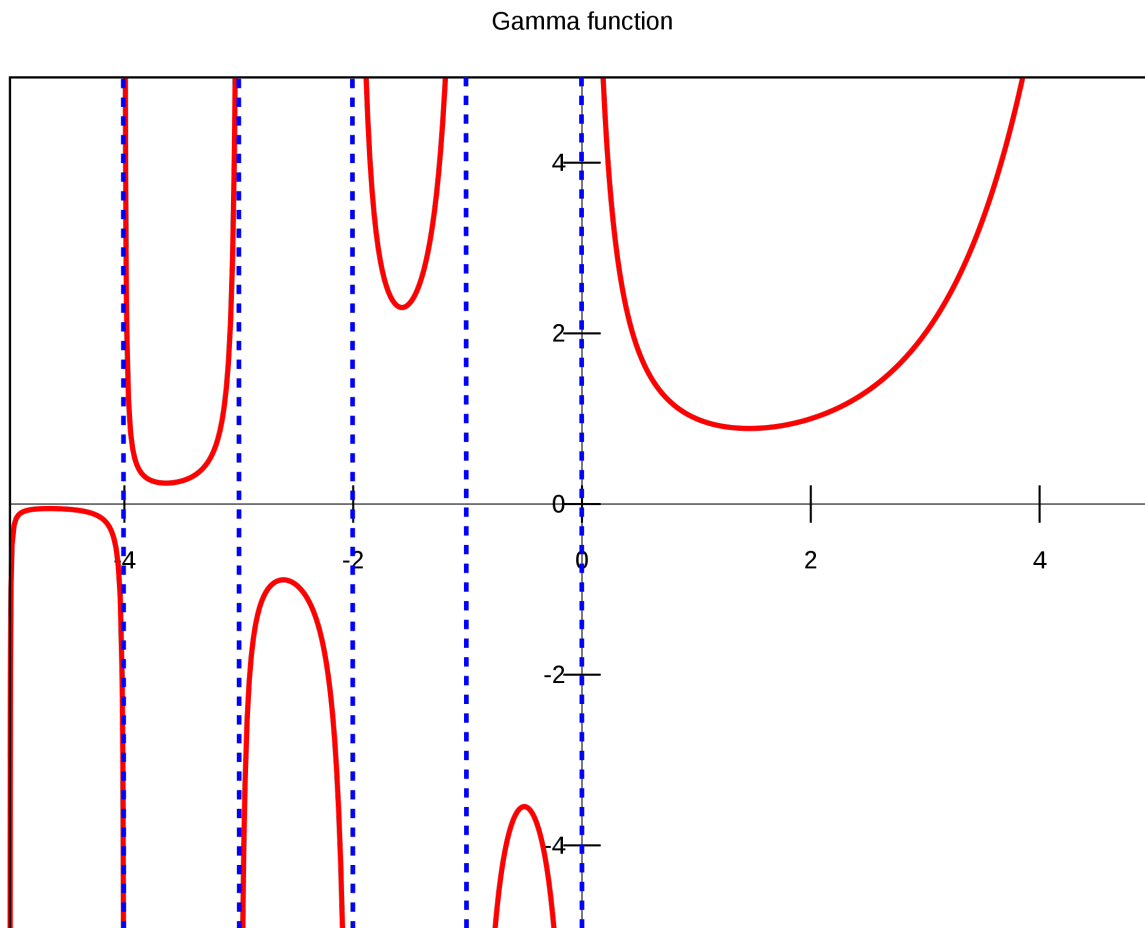

FIGURE 3: Graph featuring the plot of the function $\Gamma(\alpha)$ in red and the asymptotes of the function in dotted blue lines.

6   General Applications and Family

The gamma distribution is used in many fields, since it is directly related to the Erlang, normal and exponential distributions whose contributions extend to several disciplines. Here, we will present some of those applications.

6.1   Insurance Companies

First, the gamma distribution is of great use in the field of insurance services, given its direct relation to the exponential distribution. For example, an analyst could use this distribution to specify the amount of time a product lasts if one uses it at a constant average rate, thus modeling how reliable it is (and how much insurance should be charged for it). Then, the size of loan defaults and the cost of insurance claims are also often modeled according to a gamma distribution.

6.2   Natural Events Prediction

Then, the gamma distribution can also be used to model the amount of rainfall accumulated in a given reservoir. Indeed, this distribution fits positive data, represents rainfall distribution well and its two parameters -shape and scale- give it sufficient flexibility to fit various climates. [5]

6.3   Customer Satisfaction

Service time can also be modeled using the gamma distribution. For example, if one is waiting in line for a meal, the waiting time until one receives the long-awaited food can be modelled using an exponential distribution. Using the same principle, the Erlang distribution can allow one to determine the total length of a process, that is, the duration of a sequence of independent events. For instance, if a large number of people are waiting in line to be served,

the distribution of each of their individual waiting times (the sum of several independent exponentially distributed variables) will correspond to the time it takes for the employee to serve everyone in it. Therefore, the gamma distribution lies at the heart of what is called queuing theory: the mathematical study of the congestion of waiting lines. Since waiting lines are found in countless places such as banks, restaurants and hospitals, as well as on web servers or multistep manufacturing and distribution processes, the gamma distribution provides very useful applications in everyday life.

6.4 Call Centers

One of the most famous of those applications concerns phone queuing, on which A. K. Erlang famously worked. The Erlang distribution has indeed been developed in the goal to model the time in between incoming calls at a call center, along with the expected number of calls, thus allowing call centers to know what their staffing capacity should be depending on the time of day.

Beyond waiting lines, the Erlang distribution is often used by retailers to model the frequency of interpurchase times by consumers, which gives them an idea of how often a given consumer is expected to purchase a product from them and helps them control inventory and staffing.

6.5 Oncology

Finally, in the field of oncology, the age distribution of cancer incidence also follows a gamma distribution. Although the factors underlying cancer development are not yet fully understood, it has been hypothesized that cancers arise after several successive "driver events", that is, after some number of mutations occurs in a cell. Analyses of cancer statistics suggest that the incidence of the most prevalent cancer types with respect to the patients' age closely follows the gamma probability distribution or, more specifically, the Erlang distribution. This may be due to the fact that, more broadly, the Erlang distribution can be used

to model cell cycle time distribution[6] .

$$Y = A \cdot x^{k-1} \frac{e^{\frac{-x}{b}}}{b^k} \cdot \Gamma(k) \tag{57}$$

In this case, the shape parameter α predicts the number of carcinogenic driver events, whereas the shape parameter λ predicts the average time between those events for each cancer type. Using an additional amplitude parameter A, the maximal population susceptibility to a given type of cancer can even be predicted.[6] Given that experimental research on cancer development is crucial for the lives of many people, numerical references such as that provided by the gamma distribution are of paramount importance in our society. The gamma distribution can save lives, if it is used wisely!
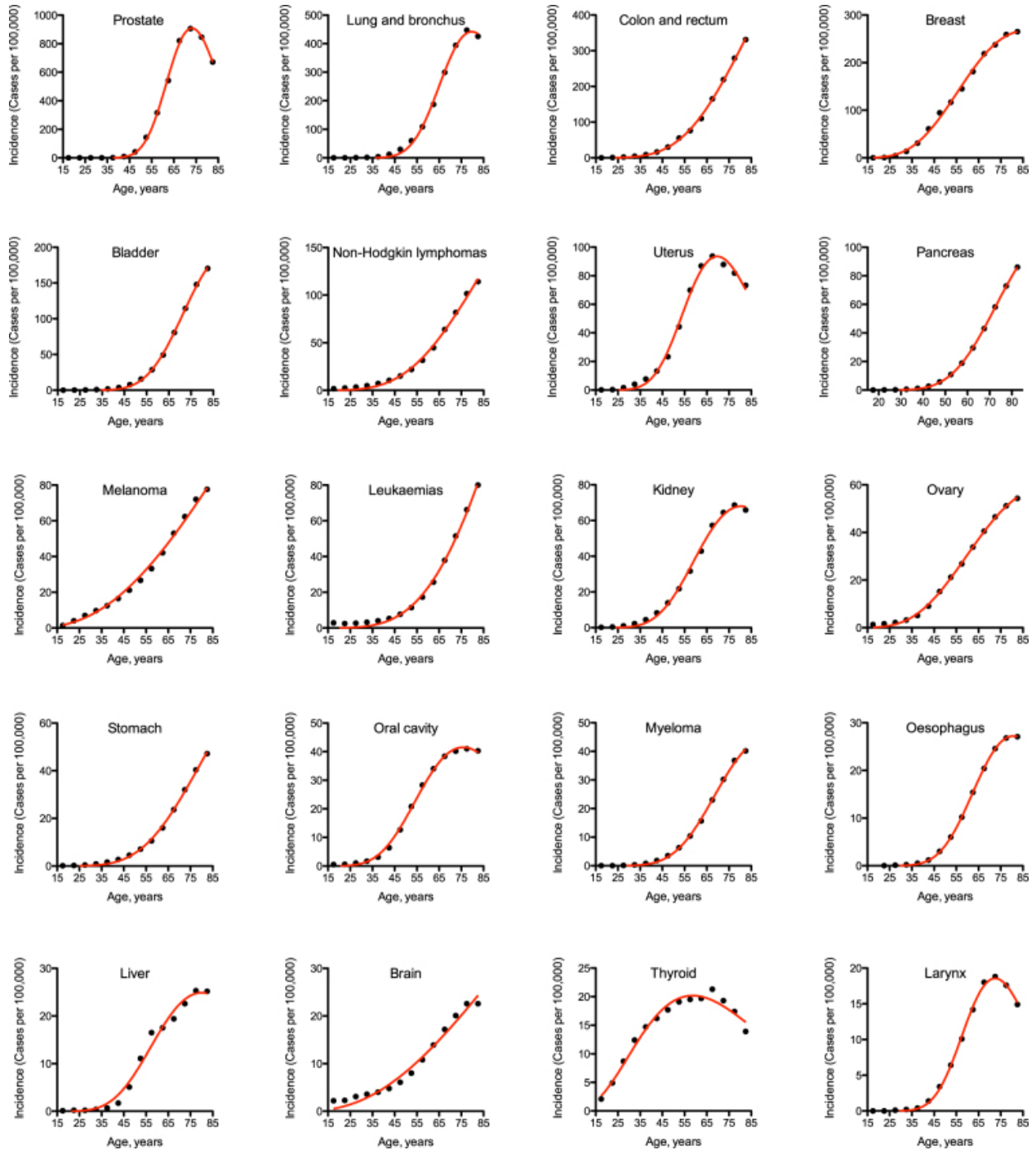
FIGURE 4: "The Erlang distribution approximates cancer incidence by age for 20 most prevalentcancer types. Dots indicate actual data for 5-year age intervals, curves indicate the PDF of the Erlang distribution fitted to the data [...]. The middle age of each age group is plotted. Cancer types are arranged in the order of decreasing incidence" [6] .

7   Special Cases of the Gamma Distribution

The gamma distribution has many common parameterizations. Hence, in order to make working with them easier, they were given a specific name such as the chi-square distribution, the exponential distribution and more. In this section, we will present some of these special cases and prove their relationship.

7.1   The Erlang Distribution

In real world applications, it is often useful to limit the number of occurrences to a positive integer (it may not be pertinent to determine, for example, the waiting time until 2.665 events take place). In this context, the special name "Erlang distribution" has been attributed to the specific gamma distribution in which the shape parameter α only takes positive integer values and where the rate parameter λ is a positive real number.

$$f_{X_\alpha}(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x} & \alpha \in \mathbb{N}; x \geq 0 \\ 0 & x < 0 \end{cases} \tag{58}$$

This distribution takes its name from A. K. Erlang, the man who initially popularized its use in order to examine the number of telephone calls which can be made at the same time to the operators of the switching stations, in the field of phone traffic engineering. Indeed, the Erlang distribution's original purpose was to measure the time between incoming calls, a statistic which can be used along with the expected duration of incoming calls to analyze phone traffic loads. His work has since been expanded to consider queuing systems in general, as well as the load on web servers, interpurchase times and cancer incidence (see section 6). [7]

7.2   The Exponential Distribution graph?, gamma_sum

The gamma distribution can model the elapsed time between random and independent events. However, what should one do when they want to strictly model the time to wait before a new event?

If a unique event is modeled, it follows that the shape parameter is limited to $\alpha = 1$. In such a case, the gamma distribution is said to follow an exponential distribution: more specifically, if $X \sim \mathcal{G}\text{amma}(1, \lambda)$, then it can also be said that $X \sim \mathcal{E}\text{xp}(\lambda)$. X thus follows an exponential distribution with rate parameter λ, where the rate parameter represents how quickly events occur. More concretely:

$$X \sim \mathcal{G}\text{amma}(1, \lambda) \equiv \mathcal{E}\text{xp}(\lambda) \tag{59}$$

MAYBE HERE INCLUDE GRAPH OF GAMMA DISTRIBUTION WITH SHAPE PARAMETER 1 COMPARED TO OTHER SHAPE PARAMETERS ? Indeed, using a simple substitution of the variables, one gets that:

$$
\begin{aligned}
f(x) &= \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x} \\
&= \frac{\lambda^1 x^0}{\Gamma(1)} e^{-\lambda x} \\
&= \lambda e^{-\lambda x}
\end{aligned}
\tag{60}
$$

$$\therefore X \sim \mathcal{G}\text{amma}(1, \lambda) \equiv X \sim \mathcal{E}\text{xp}(\lambda) \tag{61}$$

Accordingly, it can be demonstrated that the sum of exponential random variables is a gamma random variable. That is:

If X_1, X_2, $\ldots$, X_n are i.i.d random variables following $\mathcal{E}\text{xp}(\lambda)$, then $Y = \sum_{i=1}^{k} [X_i] \sim \mathcal{G}\text{amma}(k, \lambda)$

$$\tag{62}$$

Proof. **ADD HERE PROOF FOR THE DENSITY FUNCTION OF THE SUM OF EXPONENTIAL RANDOM VARIABLES (ETIENNE'S PART)**

$$XYZ \tag{63}$$

QED

7.3   The Chi-Squared Distribution graph

Then, the gamma distribution with parameters $\alpha = {}^{n}/_{2}$ and $\lambda = {}^{1}/_{2}$ is called the chi-squared distribution with n degrees of freedom such that

$$X \sim \mathcal{G}\text{ammma}({}^{n}/_{2}, {}^{1}/_{2}) \equiv \chi^2_n \tag{64}$$

Proof. First, to show that $\mathcal{G}\text{ammma}({}^{n}/_{2}, {}^{1}/_{2}) \equiv \chi^2_n$, let's prove that $\mathcal{G}\text{ammma}({}^{1}/_{2}, {}^{1}/_{2}) \equiv \chi^2_1$, a chi-squared distribution with one (1) degree of freedom, using a change of variables $Y = Z^2$ with $Z \sim \mathcal{N}(0, 1)$.

$$
\begin{aligned}
G(y) &= P(Y \leq y) \\[4pt]
&= P(Z^2 \leq y) \\[4pt]
&= P(-\sqrt{y} \leq z \leq \sqrt{y}) \\[4pt]
&= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) \\[8pt]
g(y) &= \frac{f(\sqrt{y})}{2\sqrt{y}} + \frac{f(-\sqrt{y})}{2\sqrt{y}} \\[8pt]
&= \frac{e^{-y/2}}{2\sqrt{2\pi y}} + \frac{e^{-y/2}}{2\sqrt{2\pi y}} \\[8pt]
&= \frac{\cancel{2}e^{-y/2}}{\cancel{2}\sqrt{2\pi y}} \\[8pt]
&= \frac{1}{\sqrt{2}\Gamma(\frac{1}{2})} y^{-1/2} e^{-y 1/2} \\[8pt]
&= \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma(\frac{1}{2})} y^{1/2-1} e^{-(1/2)y}
\end{aligned}
\tag{65}
$$

$$\therefore Y \sim \mathcal{G}\text{ammma}({}^{1}/_{2}, {}^{1}/_{2}) \equiv \chi^2_1 \tag{66}$$

One may generalize this result by using $Z_1, Z_2, \ldots, Z_n$ as i.i.d. random variables following $\mathcal{N}(0, 1)$ to find the distribution of

$$Y = \sum_{i=1}^{n} Z_i^2 \tag{67}$$

Hence, using the moment generating function,

$$
\begin{aligned}
M_{\sum_{i=1}^{n} Z_i^2}(t) &= \prod_{i=1}^{n} M_{Z_i^2}(t) \\
&= M(t)^n \\
&= \left(\frac{1/2}{1/2 - t} \right)^{n/2}
\end{aligned}
\tag{68}
$$

This last line is equal to the moment generating function of a random variable following the distribution $\mathcal{G}\mathrm{amma}(n/2, 1/2)$. Hence,

$$
\therefore Y \sim \mathcal{G}\mathrm{amma}(n/2, 1/2) \equiv \chi_n^2
\tag{69}
$$

$$\text{QED}$$

Using the result proved previously, the density function of the chi-squared distribution may be determined by substituting the parameters $\alpha = n/2$ and $\lambda = 1/2$ into the density function of the gamma distribution:

$$
\begin{aligned}
f(x) &= \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \\
&= \frac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{1}{2}x} \\
&= \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-x/2}
\end{aligned}
\tag{70}
$$

HERE INCLUDE GRAPH OF CHI SQUARE DISTRIBUTIONS

The chi square distribution is often used to describe the distribution of a sum of squared random variables, to test the fit of a distribution of data or to determine whether data series are independent or dependent.

7.4   The Normal Distribution

As was shown previously in subsection 7.3, the normal distribution is related to the gamma distribution as it may be used to prove that the latter is related to the chi-squared distribution. That is:

$$Z \sim \mathcal{N}\text{ormal}(0,1) \Rightarrow Z^2 \sim \mathcal{G}\text{amma}(1/2, 1/2) \equiv \chi_1^2 \tag{71}$$

Furthermore, as the gamma distribution $\mathcal{G}\text{amma}(\alpha, \lambda)$ may be viewed as a sum of α exponential distributions, then by the central limit theorem, the density function of the gamma distribution will be extremely similar to that of a normal distribution when α approaches ∞:

$$\mathcal{G}\text{amma}(\alpha, \lambda) \approx \mathcal{N}(\alpha/\lambda, \alpha/\lambda^2) \text{ for } \alpha \to \infty \tag{72}$$

The central limit theorem will not be proved as it is quite intuitive; the more data samples you have, the more centered around the expected value the probability density function graph will be. That being mentioned, here is a graph representing this concept by comparing the probability density function graph of a $\mathcal{G}\text{amma}(60, 1)$ to that of a $\mathcal{N}(60, 60)$. Because $\mathcal{G}\text{amma}(\alpha, \lambda)$ is in essence a sum of α i.i.d. random variables following a distribution $\mathcal{E}\text{xp}(\lambda)$, then it makes sense that having a distribution $\mathcal{G}\text{amma}(60, 1)$ is like having 60 data samples following a $\mathcal{E}\text{xp}(1)$ distribution which fits the central limit theorem. Here is the graph:
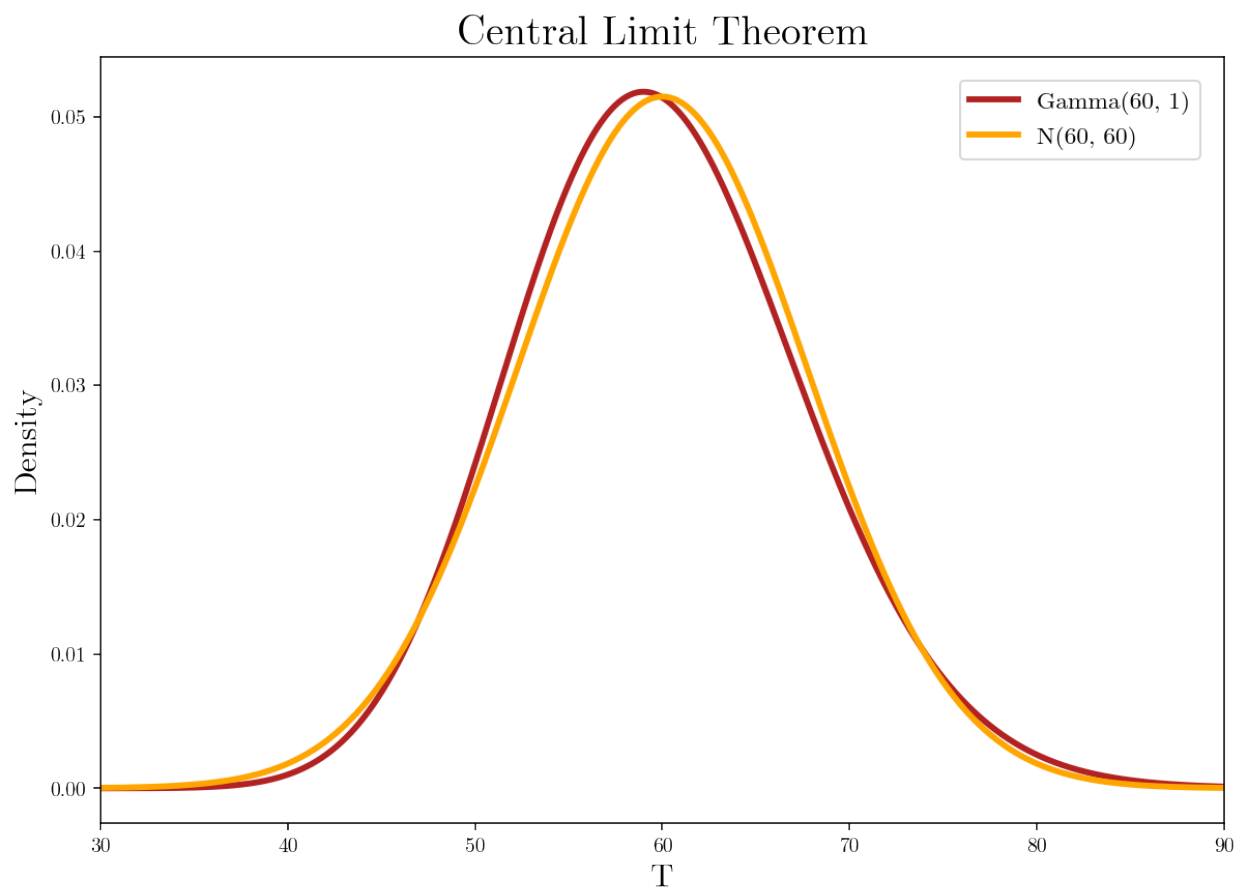
FIGURE 5: Graph showcasing the central limit theorem by comparing the probability density function graph of $\mathcal{G}$amma$(60, 1)$ to that of $\mathcal{N}(60, 60)$.

7.5   The Beta Distribution X/(X+Y)

start with proof of X/(X+Y) Text on the Beta distribution

7.6   The Wishart Distribution INCOMPLETE

what is Σ, what is $\Gamma_p(\alpha)$... explain The Wishart distribution, named in honor of the Scottish mathematician John Wishart who first formulated it in 1928, is a generalization to multiple dimensions of the gamma distribution: it is a family of distributions defined over symmetric positive matrices. Namely, if $X_1, ..., X_n$ are independent and form a $p \times n$ matrix $X = (X_1, ..., X_n)$, then the distribution of a $p \times n$ random matrix $M = XX' = \sum_{i=1}^{n} X_i X_i'$ is said to have a Wishart distribution with n degrees of freedom.

If $n \geq p$, such a matrix is denoted $\boldsymbol{M} \sim W_p(n, \Sigma)$ and has the following probability density function where $\Gamma_p(\alpha)$, is the multivariate gamma function.

$$f(\boldsymbol{M}) = \frac{1}{2^{np/2}\Gamma_p(\alpha)\,|\Sigma|^{n/2}}\,|\boldsymbol{M}|^{n-p-1/2}\,e^{-\frac{\text{trace}(\Sigma^{-1}\boldsymbol{M})}{2}} \tag{73}$$

Interesting similarities with the probability density function of the gamma distribution can clearly be noticed in the form of this equation. While this generalization of the gamma distribution to matrices is a rather intriguing topic that was important to mention, this paper will not explore the Wishart distribution in greater depth. [8]

A   Graphs - Python Code

A.1   Probability Density Function

Here is the code used to generate the probability density function graph of the gamma distribution in subsection 2.1:

```python
import math
import numpy as np                      # pip install numpy
import matplotlib.pyplot as plt         # pip install matplotlib

# Settings for matplotlib figure size, font and dpi
plt.rcParams["figure.figsize"] = [10, 7]
plt.rcParams["font.family"] = "CMU Serif"
plt.rcParams['figure.dpi'] = 140


def gamma_distribution_pdf(alpha_=4, lambda_=3):
    # Array with values from 0 to 25 with 10000 equal increments
    x = np.linspace(0, 25, 10000)

    # f(x) for X ~ Gamma(alpha_,lambda_)
    fx = (((1/lambda_)**alpha_)/math.gamma(alpha_)) * (x**(1/lambda_)) *
    np.exp(-(1/lambda_)*x)

    # Generating graph using matplotlib
    plt.title("Probability Density Function of Gamma Distribution", fontsize=20)
    plt.annotate(f"Gamma({alpha_}, {lambda_})", xy=(15, 0.0012), size=14, ha="center",
    va="center", color="firebrick")
    plt.xlabel("T", fontsize=16)
    plt.ylabel("Density", fontsize=16)
    plt.plot(x, fx, linewidth=3, color="firebrick")
    plt.xlim([0, 25])
    plt.ylim([0, 0.0015])
    plt.show()


gamma_distribution_pdf(4, 3)
```

A.2   Cumulative Distribution Function

Here is the code used to generate the cumulative distribution function graph of the gamma

distribution in subsection 2.2:

```python
import math
import itertools
import numpy as np                    # pip install numpy
import matplotlib.pyplot as plt       # pip install matplotlib

# Settings for matplotlib figure size, font and dpi
plt.rcParams["figure.figsize"] = [10, 7]
plt.rcParams["font.family"] = "CMU Serif"
plt.rcParams['figure.dpi'] = 140


def gamma_distribution_cdf(alpha_=4, lambda_=3):
    # Array with values from 0 to 25 with 10000 equal increments
    x = np.linspace(0, 25, 10000)

    # f(x) for X ~ Gamma(alpha_,lambda_)
    fx = (((1/lambda_)**alpha_)/math.gamma(alpha_)) * (x**(1/lambda_)) *
    np.exp(-(1/lambda_)*x)

    dx = 0.0025  # Same as gap between values in array x
    fxdx = fx * dx
    Fx = np.array(list(itertools.accumulate(fxdx)))

    # Generating graph using matplotlib
    plt.title("Distribution Function of Gamma Distribution", fontsize=20)
    plt.annotate(f"Gamma({alpha_}, {lambda_})", xy=(15, 0.006), size=14, ha="center",
    va="center", color="firebrick")
    plt.xlabel("T", fontsize=16)
    plt.ylabel("Probability", fontsize=16)
    plt.plot(x, Fx, linewidth=3, color="firebrick")
    plt.xlim([0, 25])
    plt.ylim([0, 0.0085])
    plt.show()


gamma_distribution_cdf(4, 3)
```

A.3   Central Limit Theorem

Here is the code used to generate the central limit theorem graph in subsection 7.4:

```python
import math
import numpy as np              # pip install numpy
import matplotlib.pyplot as plt # pip install matplotlib
import scipy as sp              # pip install scipy
from scipy import stats         # Used to generate gamma pdf array

# Settings for matplotlib figure size, font and dpi
plt.rcParams["figure.figsize"] = [10, 7]
plt.rcParams["font.family"] = "CMU Serif"
plt.rcParams['figure.dpi'] = 140

def central_limit_theorem(alpha_=60, lambda_=1, expected_=60, variance_=60):
    x = np.linspace(30, 90, 100000)
    y1 = sp.stats.gamma.pdf(x, alpha_, scale=1/lambda_)          # PDF of Gamma(alpha_,
    lambda_)
    y2 = sp.stats.norm.pdf(x, expected_, math.sqrt(variance_))  # PDF of N(expected_,
    variance_)

    # Generating graph using matplotlib
    plt.title("Central Limit Theorem", fontsize=20)
    plt.xlabel("T", fontsize=16)
    plt.ylabel("Density", fontsize=16)
    plt.plot(x, y1, label=f"Gamma({alpha_}, {lambda_})", linewidth=3, color="firebrick")
    plt.plot(x, y2, label=f"N{expected_, variance_}", linewidth=3, color="orange")
    plt.legend(bbox_to_anchor=(1, 1), loc="upper right", borderaxespad=1, fontsize=12)
    plt.xlim([30, 90])
    plt.show()

central_limit_theorem(60, 1, 60, 60)
```

B   Simulations - Python Code

B.1   Simulation 1 add ref

Comparison of the graph of $\mathcal{G}amma(\alpha + \beta, \lambda)$ and the histogram of a $X + Y$ of $X \sim$ $\mathcal{G}amma(\alpha, \lambda)$ and $Y \sim \mathcal{G}amma(\beta, \lambda)$ random variables.

```python
import numpy as np                        # pip install numpy
import matplotlib.pyplot as plt           # pip install matplotlib
import scipy as sp                        # pip install scipy
from scipy import stats                   # Used to generate gamma pdf array

# Settings for matplotlib figure size, font and dpi
plt.rcParams["figure.figsize"] = [10, 7]
plt.rcParams["font.family"] = "CMU Serif"
plt.rcParams['figure.dpi'] = 140


def histogram_gamma_sum(alpha_=1, beta_=3, lambda_=3):
    # x and y are arrays of 1000000 i.i.d. Gamma(alpha_,lambda_) r.v.
    x = np.random.gamma(alpha_, 1/lambda_, 1000000)
    y = np.random.gamma(beta_, 1/lambda_, 1000000)
    u = x + y  # r.v. of interest

    # Generating histogram using matplotlib
    plt.title(fr"(X+Y) where X$\sim$Gamma({alpha_}, {lambda_}) and Y$\sim$Gamma({beta_},
    {lambda_})", fontsize=20)
    plt.xlabel("T", fontsize=16)
    plt.ylabel("Density", fontsize=16)
    plt.hist(u, bins=10000, density=True, color="darkblue")
    plt.annotate("(X+Y)", xy=(3, 0.6), size=14, ha="center", va="center", color="darkblue")
    plt.ylim([0, 0.8])
    plt.xlim([0, 5])
    plt.show()


def gamma(alpha_=1, beta_=3, lambda_=3):
    x = np.linspace(0, 60, 1000000)
    y = sp.stats.gamma.pdf(x, (alpha_+beta_), scale=1/lambda_)

    # Generating graph using matplotlib
    plt.title("Probability Density Function of Gamma Distribution", fontsize=20)
    plt.xlabel("T", fontsize=16)
    plt.ylabel("Density", fontsize=16)
    plt.plot(x, y, linewidth=3, color="firebrick")
    plt.annotate(f"Gamma({alpha_}+{beta_}, {lambda_})", xy=(3, 0.6), size=14, ha="center",
    va="center", color="firebrick")
    plt.ylim([0, 0.8])
    plt.xlim([0, 5])
    plt.show()


histogram_gamma_sum(1, 3, 3)
gamma(1, 3, 3)
```

B.2   Simulation 2 add ref

Comparison of the graph of $\mathcal{B}\text{eta}(\alpha, \beta)$ and the histogram of a $\frac{X}{X+Y}$ of $X \sim \mathcal{G}\text{amma}(\alpha, \lambda)$

and $Y \sim \mathcal{G}\text{amma}(\beta, \lambda)$ random variables.

```python
0  import numpy as np                    # pip install numpy
1  import matplotlib.pyplot as plt       # pip install matplotlib
2  import scipy as sp                     # pip install scipy
3  from scipy import stats                # Used to generate gamma pdf array
4
5  # Settings for matplotlib figure size, font and dpi
6  plt.rcParams["figure.figsize"] = [10, 7]
7  plt.rcParams["font.family"] = "CMU Serif"
8  plt.rcParams['figure.dpi'] = 140
9
10
11 def histogram_beta(alpha_=2, beta_=5, lambda_=1):
12     # x and y are arrays of 1000000 i.i.d. Gamma(alpha_,lambda_) r.v.
13     x = np.random.gamma(alpha_, 1/lambda_, 1000000)
14     y = np.random.gamma(beta_, 1/lambda_, 1000000)
15     v = x/(x + y)   # r.v. of interest
16
17     # Generating histogram using matplotlib
18     plt.title(fr"X/(X+Y) where X$\sim$Gamma({alpha_}, {lambda_}) and Y$\sim$Gamma({beta_},
       {lambda_})", fontsize=20)
19     plt.xlabel("T", fontsize=16)
20     plt.ylabel("Density", fontsize=16)
21     plt.hist(v, bins=10000, density=True, color="darkblue")
22     plt.annotate(r"$\dfrac{X}{X+Y}$", xy=(0.6, 2.7), size=14, ha="center", va="center",
       color="darkblue")
23     plt.ylim([0, 3])
24     plt.xlim([0, 1])
25     plt.show()
26
27
28 def beta(alpha_=2, beta_=5):
29     x = np.linspace(0, 1, 10000)
30     y = sp.stats.beta.pdf(x, alpha_, beta_)
31
32     # Generating graph using matplotlib
33     plt.title("Probability Density Function of Beta Distribution", fontsize=20)
34     plt.xlabel("T", fontsize=16)
35     plt.ylabel("Density", fontsize=16)
36     plt.plot(x, y, linewidth=3, color="firebrick")
37     plt.annotate(f"Beta({alpha_}, {beta_})", xy=(0.6, 2.7), size=14, ha="center",
       va="center", color="firebrick")
38     plt.ylim([0, 3])
39     plt.xlim([0, 1])
40     plt.show()
41
42
43 histogram_beta(2, 5, 1)
44 beta(2, 5)
```

List of Figures

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