Estimating the probability tree of a Texas Hold'em hand

1 Introduction

When trying to estimate the probability tree of a Texas Hold'em hand, we face two big challenges. First, the tree is very big. Second, there is very little available poker data with complete information. This project attempts to find a remedy to the latter problem by proposing unbiased estimators for the probability tree using incomplete data.

The incompleteness of the data comes from the fact that some player's cards often remain unknown throughout a hand. When playing online poker, a player's hand is often saved on the player's hard drive in a format like the depicted in Figure 1. Unless all players make it to showdown, we do not have full information of the sequence of events that occurred during the hand. Hands from an observer's point of view can also be obtained from various sources. For these hands, we don't see the hole cards of any player that doesn't make it to showdown

```
Table "table_name", 2-max, Seat #1 is the button
Seat 1: player1 (500 in chips)
Seat 2: player2 (500 in chips)
player1: posts small blind 10
player2: posts big blind 20
*** HOLE CARDS ***
Dealt to player2 [6d 7d]
player1: raises 20 to 40
player2: calls 20
*** FLOP *** [7c 7s Kd]
player2: checks
player1: bets 46
player2: calls 46
*** TURN *** [7c 7s Kd] [Ts]
player2: checks
player1: bets 79
player2: raises 81 to 160
player1: folds
Uncalled bet (81) returned to player2
player2 collected 330 from pot
player2: doesn't show hand
*** SUMMARY ***
Total pot 330 | Rake 0 |
Board [7c 7s Kd Ts]
Seat 1: player1: (button) (small blind) folded on the Turn
Seat 2: player2 (big blind) collected (330)
```

Figure 1: This depicts a typical format in which a hand is recorded in online poker. This hand is seen from the perspective of player 2, so we see his hole cards. Player 1 folds before showdown, so we never get to see his cards.

```
Table "table_name", 2-max, Seat #1 is the button
Seat 1: player1 (500 in chips)
Seat 2: player2 (500 in chips)
player1: posts small blind 10
player2: posts big blind 20
*** HOLE CARDS ***
player1: raises 20 to 40
player2: calls 20
*** FLOP *** [7c 7s Kd]
player2: checks
player1: bets 46
player2: calls 46
*** TURN *** [7c 7s Kd] [Ts]
player2: checks
player1: bets 79
player2: raises 81 to 160
player1: folds
Uncalled bet (81) returned to player2
player2 collected 330 from pot
player2: doesn't show hand
*** SUMMARY ***
Total pot 330 | Rake 0
Board [7c 7s Kd Ts]
Seat 1: player1: (button) (small blind) folded on the Turn
Seat 2: player2 (big blind) collected (330)
```

Figure 2: This depicts a typical format in which a hand is recorded in online poker. This hand is seen from an observer's perspective, so we don't get to see either player's cards.

2 A Simplified Poker Hand

Consider a hand of No Limit Texas Hold'em where the following assumptions are made:

1. The action sequence at every stage of the hand (preflop, flop, turn, river) is limited to a maximum number turns. For example, if the limit of preflop raises is 5, a player facing a 5^{th} raise must either call or fold.

2. The set of possible raise sizes is finite. Raises are capped at an upper limit and their granularity is restricted.

3. No tells are picked up on an opposing player's cards during the hand, other than from their actions.

4. A player's decision making is not influenced by the stack sizes of the players at the table or by prior information he has on his opponents.

Assumptions 1 and 2 are not particularly restrictive and allow the action sequence to be expressed as a finite decision tree. Assumption 3 makes one players actions independent from other player's cards, conditional on his own cards and the prior action. This assumption is not very restrictive in online poker. Assumption 4 is restrictive and will be addressed later on.

Abstraction can be used as described in http://www.jogoremoto.pt/docs/extra/Q0wus1.pdf to greatly reduce the sample size of the game tree. Examples of abstraction include treating all raises as the same, regardless of sizes or bucketing similar cards and made hands together. Going forward, when we refer to any element of the game, we may be refering to an abstracted version of that element.

Notation and terminology

- The i^{th} turn refers to the i^{th} action taken in the hand, regardless of the stage of play.

- Define a null action as a placeholder action that has no impact on the play, taken by a player who does not have the option to play.

- For notational convenience, turn 1 always belongs to the player left of the dealer. For example, in a hand with 3 or more players, turn 1 is the small blind taking a null action with probability 1, turn 2 is the big blind taking a null action with probability 1, and the real first turn is turn 3.

- For notational convenience, in a hand with 3 or more players, once a player folds, we keep counting him in the action sequence, with all his his actions being null with probability 1 and his cards fixed to what they were prior to folding.

- Let K be the number of players playing a hand. We label players by their position with respect to the button: player 1 is left of the big blind, player 2 is left of player 1, and so on.

- Define c_k as the superset of all 7 card sequence such that the sequence of cards ultimately held by player k is a subset of c_k. For example, if player k folds on the flop with cards $\{As, Kd, 7s, 8c, Td\}$, c_k is the set of all 7 card sequences that start with $\{As, Kd, 7s, 8c, Td\}$.

- Define $c_{k,i}$ as the subset of the cards c_k held as of turn i. For example, if player k has hole cards $\{As, Kd\}$ and the flop turn and river are $\{7s, 8c, Td\}$, $\{Th\}$, and $\{Js\}$, and turn i happens at the flop stage, then $c_k = \{As, Kd, 7s, 8c, Td, Th, Js\}$ and $c_{k,i} = \{As, Kd, 7s, 8c, Td\}$

- $a_{i-1:i}$ denotes the i^{th} action taken, and $a_{i:j}$ denotes the sequence of action from turn i (exclusive) to j (inclusive).

- define $n(c_{k,i}, a_{0:i})$ as the number of nodes in the sample where player k holds cards $c_{k,i}$ as of turn i and the action sequence has been $a_{0:i}$

- $\mathbb{P}(a_{0:I}, c_1, ..., c_K)$ denotes the probability of observing the action sequence $a_{0:I}$ and players having cards c_K.

- $\mathbb{P}(c_1, ..., c_K)$ denotes the unconditional probability that players 1 to K will receive cards in $c_1, ..., c_K$, respectively, if dealt 2 hole cards each and 5 community cards. This probability has a closed form solution for any $c_1, ..., c_K$ but is probably easier to estimate with monte carlo simulation in most cases.

- Define the filtration $\mathcal{F}_i$ as the set of all events that have occurred up to turn i, inclusive (all cards and action sequence)

- Define $\mathcal{G}_{i,k}$ as the set of all events that have occurred up to turn i, inclusive, which are observable by player k (player k cards, community cards and action sequence)

Theorem 1
Consider an unbiased sample of independent hands from an unbiased sample of players for a Texas Hold'em game with K players. Given that the sample is abstracted in a way such that $n(c_k, a_{0:i}) > 0$ for all possible combinations of $a_{0:i}$ and c_k, the following is an unbiased estimator for the probability of observing any node:

$$\hat{p}(a_{0:I}, c_1, ..., c_K) = \prod_{i=1}^{I} \frac{n(c_{(i-1)\%K+1,i}, a_{0:i})}{n(c_{(i-1)\%K+1,i-1}, a_{0:i-1})} \cdot \mathbb{P}(c_1, ..., c_K)$$

proof:

$$\mathbb{E}[\hat{p}(a_{0:I}, c_1, ..., c_K)]$$

$$= \mathbb{E}[\prod_{i=1}^{I} \frac{n(c_{(i-1)\%K+1,i}, a_{0:i})}{n(c_{(i-1)\%K+1,i-1}, a_{0:i-1})} \cdot \mathbb{P}(c_1, ..., c_K)]$$

$$= \mathbb{E}[\prod_{k=1}^{i} \frac{n(c_{(i-1)\%K+1,i}, a_{0:i})}{n(c_{(i-1)\%K+1,i-1}, a_{0:i-1})}] \cdot \mathbb{P}(c_1, ..., c_K)$$

$$= \prod_{k=1}^{i} \mathbb{E}[\frac{n(c_{(i-1)\%K+1,i}, a_{0:i})}{n(c_{(i-1)\%K+1,i-1}, a_{0:i-1})}] \cdot \mathbb{P}(c_1, ..., c_K) \; (Independance\ from\ assumption\ 3)$$

$$= \prod_{k=1}^{i} \mathbb{P}(a_{i-1:i}|\mathcal{G}_{i-1,(i-1)\%K+1}) \cdot \mathbb{P}(c_1, ..., c_K)$$

$(The\ number\ of\ times\ player\ took\ an\ action\ divided\ by\ the\ number\ of\ times\ the\ action\ was\ possible\ is\ an\ unbiased\ estimate\ for\ the\ conditional\ probability\ of\ the\ action\ taking\ place)$

$$= \prod_{k=1}^{i} \mathbb{P}(a_{i-1:i}|\mathcal{F}_{i-1}) \cdot \mathbb{P}(c_1, ..., c_K)$$

$$= \prod_{i=1}^{I} \mathbb{P}(a_{i-1:i}|a_{0:1}, ..., a_{i-2:i-1}, c_{1,i-1}, ..., c_{K,i-1}) \cdot \mathbb{P}(c_1, ..., c_K)$$

$$= \prod_{i=1}^{I} \mathbb{P}(a_{i-1:i}|a_{0:1}, ..., a_{i-2:i-1}, c_1, ..., c_K) \cdot \mathbb{P}(c_1, ..., c_K)$$

$(Decision\ making\ is\ independant\ of\ the\ cards\ not\ yet\ observed)$

$$= \prod_{i=2}^{I} \mathbb{P}(a_{i-1:i}|a_{0:1}, ..., a_{i-1:i}, c_1, ..., c_k) \cdot \mathbb{P}(a_{0:1}|c_{1,0}, ..., c_{k,0}) \cdot \mathbb{P}(c_1, ..., c_K)$$

$$= \prod_{i=2}^{I} \mathbb{P}(a_{i-1:i}|a_{0:1}, ..., a_{i-1:i}, c_1, ..., c_k) \cdot \mathbb{P}(a_{0:1}, c_1, ..., c_K) \; (bayes\ theorem)$$

$$= \mathbb{P}(a_{0:1}, ..., a_{I-1:I}, c_1, ..., c_K) \; (recursive\ bayes\ theorem)$$

$$= \mathbb{P}(a_{0:I}, c_1, ..., c_K)$$

3 Removing assumption 4

We now relax assumption 4. That is, a player can adapt his strategy based on things such as the stack sizes and what he knows about the playing history of his opponents.

Notation and terminology

- Define s_k as the set of information known to player k at the start of the hand that may influence his decision making.
- Define $n(s_k)$ as the number of hands in the sample where player k holds prior information s_k.
- Define $n(c_{k,i}, a_{0:i}, s_k)$ as the number of nodes in the sample where player k holds cards $c_{k,i}$, the action sequence has been $a_{0:i}$ and the player held prior information s_k at the start of the hand.

Theorem 2
Consider an unbiased sample of independent hands from an unbiased sample of players for a Texas Hold'em game with K players. Given that the sample is abstracted in a way such that $n(c_k, a_{0:i}, s_k) > 0$ for all possible combinations of $a_{0:i}$, c_k and s_k, the following is an unbiased estimator for the probability of observing any node:

$$\hat{p}(a_{0:I}, c_1, ..., c_K, s_1, ..., s_K) = \prod_{i=1}^{I} \frac{n(c_{(i-1)\%K+1,i}, a_{0:i}, s_{(i-1)\%K+1})}{n(c_{(i-1)\%K+1,i-1}, a_{0:i-1}, s_{(i-1)\%K+1})} \cdot \prod_{k=1}^{K} \frac{n(s_k)}{n(k)} \cdot \mathbb{P}(c_1, ..., c_K)$$

proof:

$$\mathbb{E}[\hat{p}(a_{0:I}, c_1, ..., c_K, s_1, ..., s_K)]$$

$$= \mathbb{E}[\prod_{i=1}^{I} \frac{n(c_{(i-1)\%K+1,i}, a_{0:i}, s_{(i-1)\%K+1})}{n(c_{(i-1)\%K+1,i-1}, a_{0:i-1}, S_{(i-1)\%K+1})} \cdot \prod_{k=1}^{K} \frac{n(s_k)}{n(k)} \cdot \mathbb{P}(c_1, ..., c_K)]$$

$$= \mathbb{E}[\prod_{i=1}^{I} \frac{n(c_{(i-1)\%K+1,i}, a_{0:i}, s_{(i-1)\%K+1})}{n(c_{(i-1)\%K+1,i-1}, a_{0:i-1}, s_{(i-1)\%K+1})}] \cdot \mathbb{E}[\prod_{k=1}^{K} \frac{n(s_k)}{n(k)}] \cdot \mathbb{P}(c_1, ..., c_K) \ (\textit{conditional independance})$$

$$= \mathbb{E}[\prod_{i=1}^{I} \frac{n(c_{(i-1)\%K+1,i}, a_{0:i}, s_{(i-1)\%K+1})}{n(c_{(i-1)\%K+1,i-1}, a_{0:i-1}, s_{(i-1)\%K+1})}] \cdot \prod_{k=1}^{K} \mathbb{E}[\frac{n(s_k)}{n(k)}] \cdot \mathbb{P}(c_1, ..., c_K)$$

$$= \mathbb{E}[\prod_{i=1}^{I} \frac{n(c_{(i-1)\%K+1,i}, a_{0:i}, s_{(i-1)\%K+1})}{n(c_{(i-1)\%K+1,i-1}, a_{0:i-1}, s_{(i-1)\%K+1})}] \cdot \prod_{k=1}^{K} \mathbb{P}(s_k) \cdot \mathbb{P}(c_1, ..., c_K)$$

$$= \mathbb{E}[\prod_{i=1}^{I} \frac{n(c_{(i-1)\%K+1,i}, a_{0:i}, s_{(i-1)\%K+1})}{n(c_{(i-1)\%K+1,i-1}, a_{0:i-1}, s_{(i-1)\%K+1})}] \cdot \mathbb{P}(s_1, ..., s_K, c_1, ..., c_K)$$

$$= ...\ (similar\ to\ theorem\ 1)$$

$$= \prod_{i=1}^{I} \mathbb{P}(a_{i-1:i}|a_{0:1}, ..., a_{i-1:i}, c_1, ..., c_k, s_1, ..., s_K) \cdot \mathbb{P}(s_1, ..., s_K, c_1, ..., c_K)$$

$$= ...\ (similar\ to\ theorem\ 1)$$

$$= \mathbb{P}(a_{0:I}, c_1, ..., c_K, s_1, ..., s_K)$$

4 Supplementing With Observer Data

We now present a approach that could be used to improve the previous estimates with observer data.

Notation and terminology

- Consider the hand $h = \{a_{0:I}, c_1, ..., c_K, s_1, ..., s_K\}$. Define $h*$ as the superset of all hands that are indistinguishable from h from an observer's perspective.
- Define N as the number of hands in the observer data sample.
- Define $N(h*)$ as the number of hands $\in$ h* in the observer data sample.

The next theorem provides an improvement for the estimation of $\mathbb{P}(h*)$, which could result in an improvement in $\mathbb{P}(h)$. However, the estimation of $\mathbb{P}(h|h*)$ is not improved.

Theorem 3
Consider an unbiased sample of independent hands from an unbiased sample of players for a Texas Hold'em game with K players, abstracted in a way such that $n(c_k, a_{0:i}, s_k) > 0$ for all possible combinations of $a_{0:i}$, c_k and s_k. Also consider an unbiased sample of independent observer data, abstracted such that $N(h*) > 0$ for all $h*$. The following is an estimator for the probability of observing any node:

$$\hat{p*}(a_{0:I}, c_1, ..., c_K, s_1, ..., s_K) = p(h) = \frac{\hat{p}(h)}{\hat{p}(h*)} \cdot \frac{N(h*)}{N},$$

Where $\hat{p}$ is computed as in theorem 2

The proof is not trivial since $\hat{p}(h)$ and $\hat{p}(h*)$ are not independent, but I suspect that this estimator is unbiased.