Homework 2

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EARTHSS212, Winter 2024

Problem 1

a) $L[n+2] = L[n+1] + L[n]$, $L[0] = 2$, $L[1] = 1$

Then $L[2] = 3$, $L[3] = 4$, $L[4] = 7$.

b) $L[n] = A\left(\frac{1+\sqrt{5}}{2}\right)^n - B\left(\frac{1-\sqrt{5}}{2}\right)^n$.

We can try and solve for A and B as a system of equations where L[0] = 2, L[1] = 1.

$$L[0] = A\left(\frac{1+\sqrt{5}}{2}\right)^0 - B\left(\frac{1-\sqrt{5}}{2}\right)^0 = A - B = 2$$

$$L[1] = A\left(\frac{1+\sqrt{5}}{2}\right)^1 - B\left(\frac{1-\sqrt{5}}{2}\right)^1 = A\frac{1+\sqrt{5}}{2} - B\frac{1-\sqrt{5}}{2} = 1$$

In a matrix format, the system looks like:

$$\begin{bmatrix} 1 & -1 \\ \frac{1+\sqrt{5}}{2} & -\frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving the equation, we get: A = 1, B = -1.

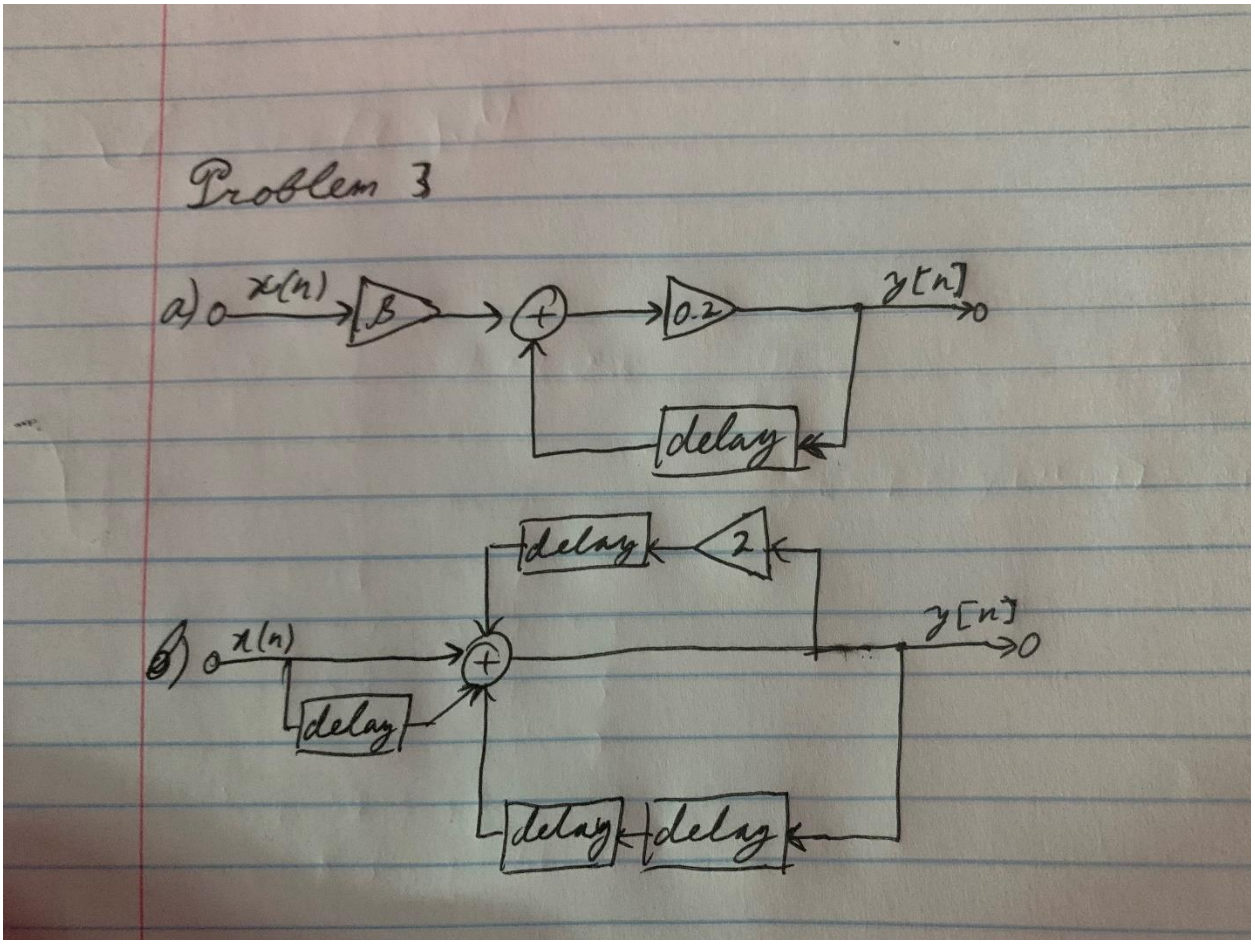
Thus, the closed form solution is $L[n] = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$.

Problem 2

a) $y[n] = x[n] + \alpha y[n-1]$

b) $y[n] = x[n] + y[n-1] + \alpha y[n-2]$

Problem 3



Problem 4

a) Only if the value of y[0] is provided.

b) If x[n] = 0 for n > N, then starting at n=N, the term $\beta x[n] = 0$ and $y[n] = 0.2y[n-1] = 0.2(n-N)y[N-1]$.

Problem 5

b) $S[n+2] = 2S[n+1] + 2S[n]$ is the difference equation. We can transform it in a following way:

$$S[n+2] - 2S[n+1] - 2S[n] = 0$$

$$Az^n - 2Az^{n-1} - 2Az^{n-2} = 0$$

$$z^2 - 2z - 2 = 0$$

The solution to this quadratic equation is $z = \frac{2 \pm \sqrt{12}}{4} = \frac{1 \pm \sqrt{3}}{2}$.

Thus, the solution to the difference equation will have the form $S[n] = A\left(\frac{1+\sqrt{3}}{2}\right)^n + B\left(\frac{1-\sqrt{3}}{2}\right)^n$.

We can apply It for cases S[0]=0, S[1]=1.

$$S[0] = A\left(\frac{1+\sqrt{3}}{2}\right)^0 + B\left(\frac{1-\sqrt{3}}{2}\right)^0 = A + B = 0\ (A = -B)$$

$$S[1] = A\left(\frac{1+\sqrt{3}}{2}\right)^1 + B\left(\frac{1-\sqrt{3}}{2}\right)^1 = A\left(\frac{1+\sqrt{3}}{2}\right)^1 - A\left(\frac{1-\sqrt{3}}{2}\right)^1 = \sqrt{3}A = 1$$

$$A = \frac{1}{\sqrt{3}}; B = -\frac{1}{\sqrt{3}}$$

The closed form solution: $S[n] = \frac{1}{\sqrt{3}}\left(\frac{1+\sqrt{3}}{2}\right)^n - \frac{1}{\sqrt{3}}\left(\frac{1-\sqrt{3}}{2}\right)^n$.

c) Upon calculating the values from the closed form solution and comparing them to the output of the code, there is a mismatch. The closed form solution must contain an error.

Problem 6

a) The Babylonian method: $x[n+1] = \frac{1}{2}(x[n] + \frac{c}{x[n]})$.

To confirm that this Is the valid method for calculating the square root of c, we can show that $(x[n+1])^2 \approx c$.

$$(x[n+1])^2 = \left(\frac{1}{2}\left(x[n] + \frac{c}{x[n]}\right)\right)^2 = \frac{1}{4}\left(x[n]^2 + \frac{2cx[n]}{x[n]} + \frac{c^2}{(x[n])^2}\right)$$
$$= \frac{1}{4}\left(x[n]^2 + 2c + \frac{c^2}{(x[n])^2}\right)$$

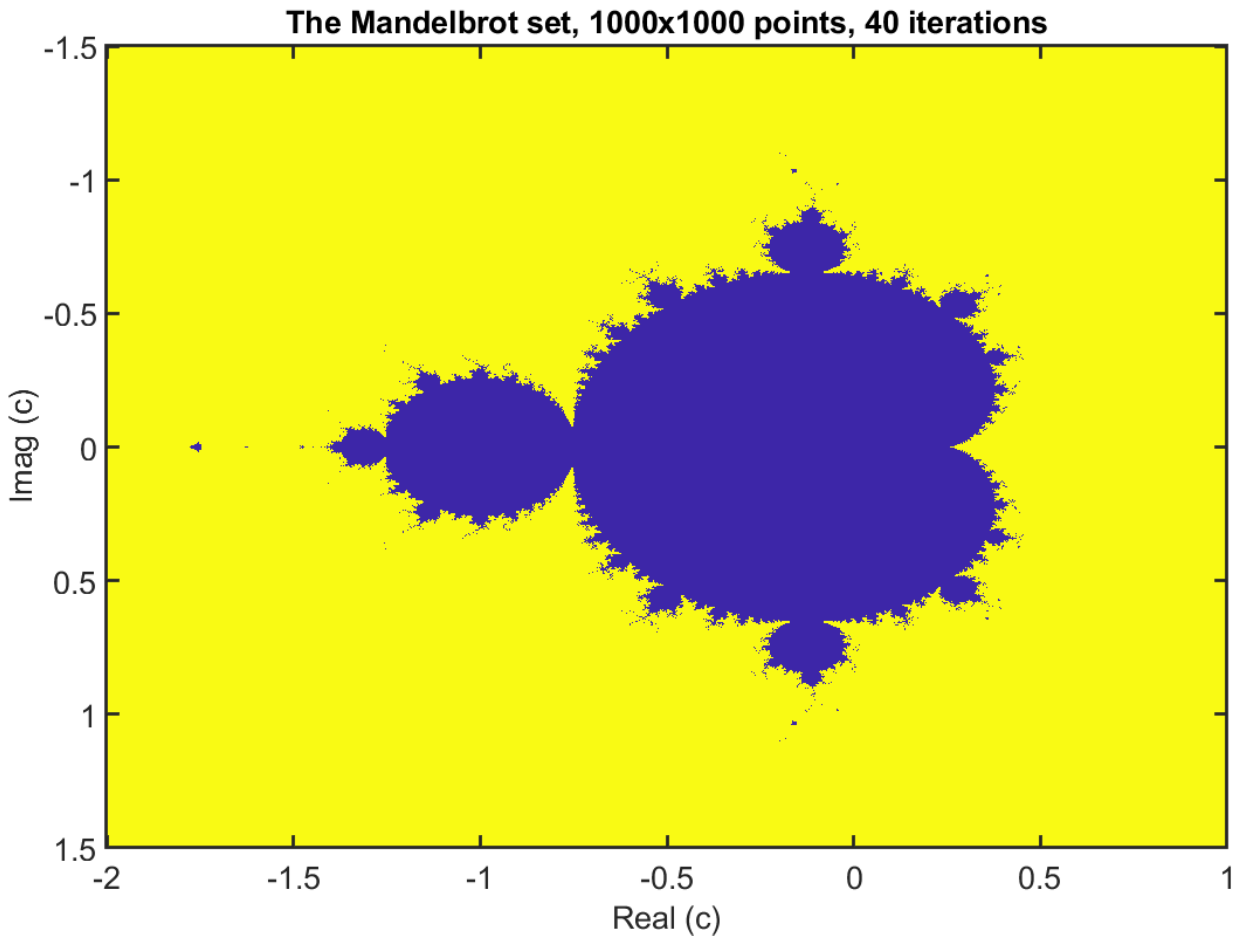
If we also remember that $x[n]$ is a cruder approximation of $\sqrt{c}$:

$$\frac{1}{4}\left(x[n]^2 + 2c + \frac{c^2}{(x[n])^2}\right) \approx \frac{1}{4}\left(\left(\sqrt{c}\right)^2 + 2c + \frac{c^2}{\left(\sqrt{c}\right)^2}\right) = \frac{1}{4}\left(c + 2c + \frac{c^2}{c}\right) = \frac{4c}{4} = c$$

With each iteration, $x[n+1]$ approaches the value of $\sqrt{c}$.

d) The Babylonian method converges for all values of $x[0]$. However, the initial guess cannot be 0, due to the $\frac{c}{x[n]}$ term and the resulting division by 0.

Problem 7



Problem 8

a) Implementing the code offered by Wille-E Coyote results in the infinite loop. Upon analyzing the formula $z(t+\Delta t) = z(t) - \Delta t(\sqrt{2g(z(t)-z_0)})$, we can see that at the very first time step, $z(t) - z_0 = 0$, resulting in $z(t+\Delta t) = z(t)$ for the first time step and every subsequent step after. While the energy is conserved, it is never converted into kinetic energy, making Wille-E hang in the air at the same height z_0 indefinitely – just like in the cartoon!

b) At the initial point: v(0) = z'(0) = 0, z(0) = h.

The governing equation: $m\frac{d^2z}{dt^2} = -mg$. Divide both sides by m : $\frac{d^2z}{dt^2} = -g$.

Expressing it as a system of differential equations:

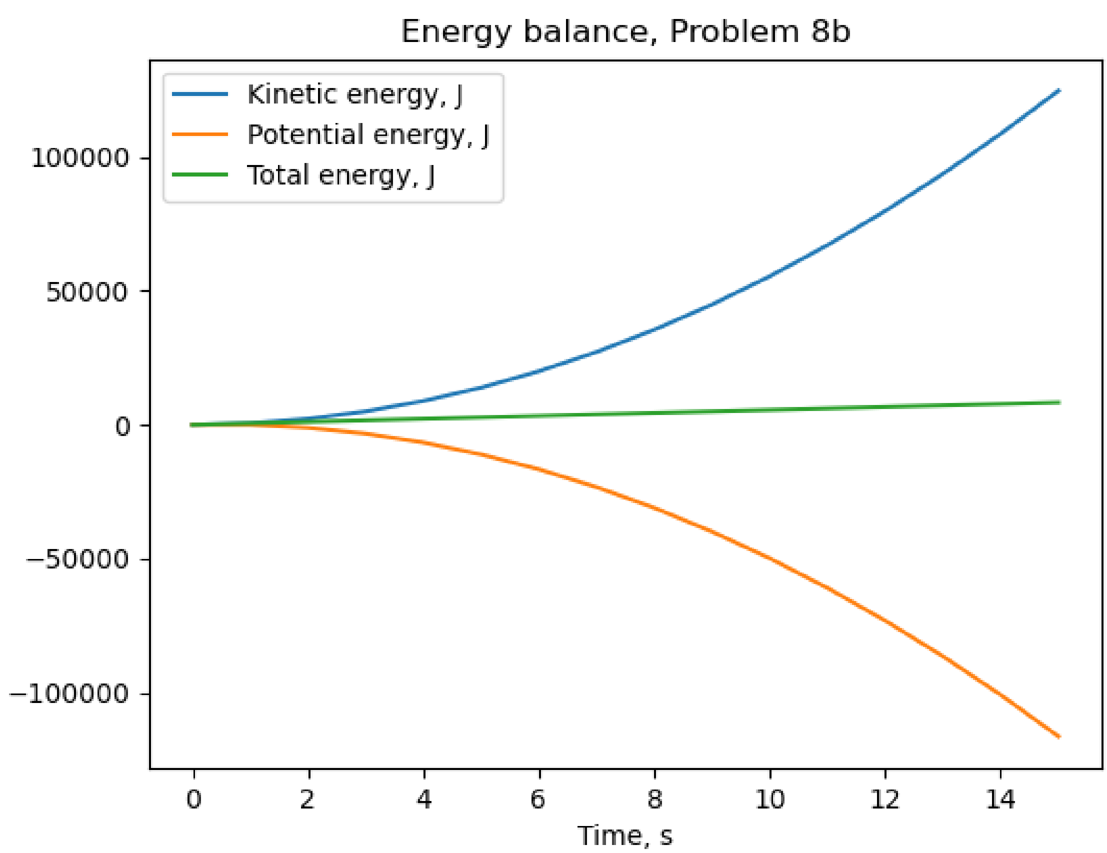
$$\begin{cases} \dfrac{d^2z}{dt^2} = z_t'' = \dfrac{dz'}{dt} = \dfrac{dv}{dt} = -g \\ \dfrac{dz}{dt} = v(t) \\ v(0) = 0, \qquad z(0) = h \end{cases}$$

Written as the Euler forward scheme:

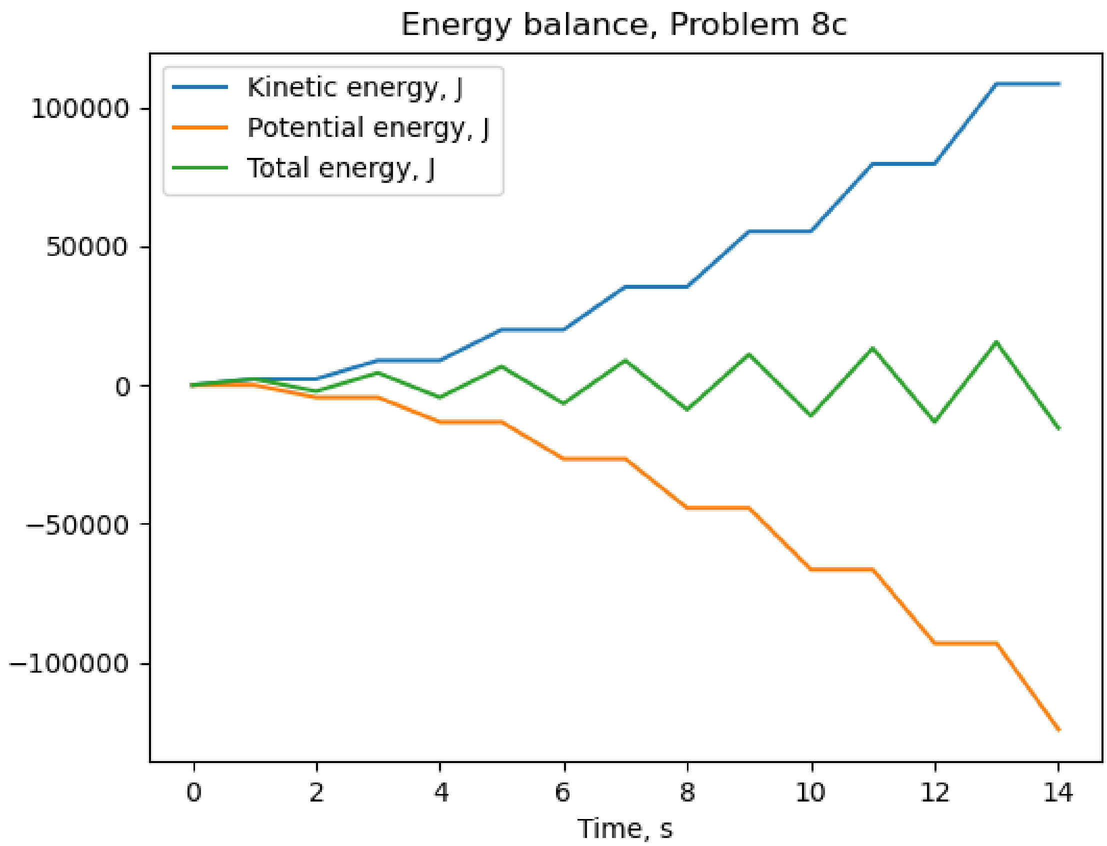
$$\begin{cases} v(t+1) = v(t) - g\Delta t \\ z(t+1) = z(t) + v(t)\Delta t \\ v(0) = 0, \qquad z(0) = h \end{cases}$$

The average mass of a southern coyote is 11.5 kg (needed as m in the energy formula).

The total energy remains the same at every point in time, confirming its conservation.



c) Here, the total energy oscillates around the initial value of the energy; I assume this to be the artefact from the way the algorithm is executed (there is a delay in calculating z), but if we average all values of total energy (provided in the code), the average comes up as the initial value of total energy, confirming that it is, indeed, conserved.



Code for different problems

Problem 5a,c: *problem5.py*

Problem 6b, c: *problem6.py*

Problem 7: *mandelbrot.m*

Problem 8a: *p8_1.py*

Problem 8b: *p8_2.py*

Problem 8c: *p8_3.py*