Homework 3

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Problem 1.

Points: (x_1, y_1) = (-4, 30), (x_2, y_2) = (0,2), (x_3, y_3) = (4,6).

Analytical solution:

If we make a system of linear equations, based on the model $y_i = ax_i^2 + bx_i + c + e$, $e = 0$:

$$\begin{cases} 16a - 4b + c = 30 \\ c = 2 \\ 16a + 4b + c = 6 \end{cases}$$

Thanks to the second point, we have the value of c right away, thus reducing the system to 2 equations:

$$\begin{cases} 16a - 4b = 28 \\ 16a + 4b = 4 \end{cases}$$

If we subtract the 2nd equation from the 1st : $-8b = 24$. We have $\underline{b = -3}$.

Substituting b in the 1st equation: $16a = 16$. We have $\underline{a = 1}$.

Projection:

Data $\boldsymbol{d} = [y_1\ y_2\ y_3]'$, projected on this model, where $\boldsymbol{a} = [x_1^2\ x_2^2\ x_3^2]'$, $\boldsymbol{b} = [x_1\ x_2\ x_3]'$, $\boldsymbol{c} = [1\ 1\ 1]'$.

Solution by projecting the data is coded in Python, file *proj.py*.

Newton's method:

$y_i = A(x_i - B)(x_i - C) = Ax_i^2 - ABx_i - ACx_i + ABC$, $i = \{1,2,3\}$.

$$F_i = Ax_i^2 - ABx_i - ACx_i + ABC - y_i = 0$$

$$\frac{\partial F_i}{\partial A} = (x_i^2 - Bx_i - Cx_i + BC)$$

$$\frac{\partial F_i}{\partial B} = -Ax_i + AC$$

$$\frac{\partial F_i}{\partial C} = -Ax_i + AB$$

Initial guess: $A = 1, B = 2, C = 3$.

The Newton's method and the plot is coded in Python, file *newton.py*.

Just from looking at the equation and seeing A as the multiplier for the rest of the equation, we can tell that the initial guess of $A = 0$ would result in $y_i = 0$ for all i, making this solution unviable.

Problem 2

The problem is solved in Python, file *p2.py*.

The dataset I used for my solution was sourced from National Oceanic and Atmospheric Administration (NOAA) and it represents the average annual precipitations in California for years 1985-2024.

Problem 3

4 plots:

Aa: CO_2 mixing ratio of 400 ppm, 2.0 cm of water added each week.

Ab: CO_2 mixing ratio of 400 ppm, 4.0 cm of water added each week.

Ba: CO_2 mixing ratio of 800 ppm, 2.0 cm of water added each week.

Bb: CO_2 mixing ratio of 800 ppm, 4.0 cm of water added each week.

16 seedlings: $i = \{1,2,\dots,16\}$.

The system of equations is: $d_i = \mu + \beta_1\Delta_{CO2,i} + \beta_2\Delta_{H2O,i} + \beta_3\Delta_{CO2,i}\Delta_{\mathrm{H2O,i}} + e_i$.

Each d_i is a data point; when we solve for β_j, j = {1,2,3}, we get the likelihood of the impact of extra CO_2 and water.

The matrix form is as follows:

$$\begin{matrix} d_1 \\ d_2 \\ \vdots \\ d_{16} \end{matrix} = \begin{pmatrix} 1 & \Delta_{CO2,1} & \Delta_{H2O,1} & \Delta_{CO2,1}\Delta_{\mathrm{H2O,1}} \\ 1 & \Delta_{CO2,2} & \Delta_{H2O,2} & \Delta_{CO2,2}\Delta_{\mathrm{H2O,2}} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \Delta_{CO2,16} & \Delta_{H2O,16} & \Delta_{CO2,16}\Delta_{\mathrm{H2O,16}} \end{pmatrix} \begin{matrix} \mu \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{matrix} + \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_{16} \end{matrix}$$

We will assign $d_1 \dots d_4$ to plot Aa, $d_5 \dots d_8$ to plot Ab, $d_9 \dots d_{12}$ to plot Ba, $d_{13} \dots d_{16}$ to plot Bb.

Thus,

$$\begin{matrix} d_1 \\ \vdots \\ d_4 \\ d_5 \\ \vdots \\ d_8 \\ d_9 \\ \vdots \\ d_{12} \\ d_{13} \\ \vdots \\ d_{16} \end{matrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} \mu \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{matrix} + \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_{16} \end{matrix}$$

where

$$\boldsymbol{d} = \begin{matrix} d_1 \\ \vdots \\ d_4 \\ d_5 \\ \vdots \\ d_8 \\ d_9 \\ \vdots \\ d_{12} \\ d_{13} \\ \vdots \\ d_{16} \end{matrix}, \boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix}, \boldsymbol{\beta} = \begin{matrix} \mu \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{matrix}, \boldsymbol{e} = \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_{16} \end{matrix}.$$

The $\boldsymbol{A}'\boldsymbol{A}$ matrix:

$$\boldsymbol{A}'\boldsymbol{A} = \begin{pmatrix} 1 \cdots 1 & 1 \cdots 1 & 1 \cdots & 1 \cdots 1 \\ 0 \cdots 0 & 0 \cdots 0 & 1 \cdots & 1 \cdots 1 \\ 0 \cdots 0 & 1 \cdots 1 & 0 \cdots & 0 \cdots 1 \\ 0 \cdots 0 & 0 \cdots 0 & 0 \cdots & 0 \cdots 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 16 & 8 & 8 & 4 \\ 8 & 8 & 4 & 4 \\ 8 & 4 & 8 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}.$$

Finally, the $\boldsymbol{A}'\boldsymbol{d}$ vector:

$$\boldsymbol{A}'\boldsymbol{d} = \begin{pmatrix} 1 \cdots 1 & 1 \cdots 1 & 1 \cdots & 1 & 1 \cdots 1 \\ 0 \cdots 0 & 0 \cdots 0 & 1 \cdots & 1 & 1 \cdots 1 \\ 0 \cdots 0 & 1 \cdots 1 & 0 \cdots & 0 & 1 \cdots 1 \\ 0 \cdots 0 & 0 \cdots 0 & 0 \cdots & 0 & 1 \cdots 1 \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_4 \\ d_5 \\ \vdots \\ d_8 \\ d_9 \\ \vdots \\ d_{12} \\ d_{13} \\ \vdots \\ d_{16} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{16} d_i \\ \sum_{i=9}^{16} d_i \\ \sum_{i=5}^{8} d_i + \sum_{i=13}^{16} d_i \\ \sum_{i=13}^{16} d_i \end{pmatrix}$$