# Homework 4

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## Code for the problems:

Problem 1: p1.m.

Problem 2: p2.m.

Problem 3: p3.m.

## Problem 1

We have data y that depends on time t and is believed to adhere to the model $y_i = at^2 + bt + c + e_i$.

We can express the model in the format $\boldsymbol{y} = \boldsymbol{A\beta} + \boldsymbol{e}$, where $A = \begin{pmatrix} 1 & t_1 & t_1^2 \\ \vdots & \ddots & \vdots \\ 1 & t_{10} & t_{10}^2 \end{pmatrix}$, $\beta = \begin{pmatrix} c \\ b \\ a \end{pmatrix}$, y and e are vectors of values $y_i$ and $e_i$, $i = 1..10$. $\boldsymbol{e}$ then can be expressed as $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{A\beta}$.

We can find the approximate values of a, b, and c by using the least square estimates from the data:

$$\hat{\beta} = \frac{A'A}{A'y}$$

We assume the prior probability as $prob(a,\ b,\ c,\ \sigma\ ) \propto \frac{1}{\sigma}$.

Given the normal distribution of the error, the likelihood is $prob(data|a,\ b,\ c,\ \sigma\ ) \propto \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2}\frac{y_1 - (at_1^2 + bt_1 + c)}{\sigma^2}\right) \times \dots \times \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2}\frac{y_{10} - (at_{10}^2 + bt_{10} + c)}{\sigma^2}\right) = \frac{1}{(2\pi)^5\sigma^{10}} exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{A\beta} - \boldsymbol{y})'(\boldsymbol{A\beta} - \boldsymbol{y})\right)$.

Posterior is $prob(\beta,\ \sigma\ |\ data) = \frac{prob(data|a,b,c,\sigma\ )\ prob(a,b,c,\sigma\ )}{prob(data)} \propto \frac{1}{\sigma^{11}} exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{A\beta} - \boldsymbol{y})'(\boldsymbol{A\beta} - \boldsymbol{y})\right)$.

Given that the variance of the posterior and the logarithm of the posterior is the same, we can take the logarithm to solve for $\sigma$: $L = \log\left(prob(\beta,\ \sigma\ |\ data)\right) = -\frac{1}{2\sigma^2}\boldsymbol{e}'\boldsymbol{e} - 11log\sigma.$

The most likely value of $\sigma$ is where the function reaches its highest value: $\frac{\partial L}{\partial \sigma} = \frac{1}{4}(\boldsymbol{A\beta} - \boldsymbol{y})'(\boldsymbol{A\beta} - \boldsymbol{y}) - 11\sigma^2 = 0$.

Solving for $\sigma$: $\sigma = \sqrt{\frac{0.25\ (\boldsymbol{A\beta} - \boldsymbol{y})'(\boldsymbol{A\beta} - \boldsymbol{y})}{11}}$ (we can solve assuming $\beta = \hat{\beta}$).

To assign the errorbars, we need to find the probability distribution function (PDF) and then mean and variance for each parameter. However, we first need to find the proportionality constant. We can find it by asserting the integral over all parameters to be equal to 1:

$$Z = \int_{0}^{\infty} \int_{-\infty}^{+\infty} \frac{1}{\sigma^{11}} \exp\left(-\frac{(A\beta - y)'(A\beta - y)}{2\sigma^2}\right) da\ db\ dc\ d\sigma = 1$$

Then the respective PDFs are:

$prob(\sigma|data) = \int_{-\infty}^{\infty} \frac{1}{Z} \frac{1}{\sigma^{11}} \exp\left(-\frac{(A\beta-y)'(A\beta-y)}{2\sigma^2}\right) da\ db\ dc;$

$prob(a|data) = \int_{-\infty}^{\infty} \frac{1}{Z} \frac{1}{\sigma^{11}} \exp\left(-\frac{(A\beta-y)'(A\beta-y)}{2\sigma^2}\right) db\ dc\ d\sigma;$

$prob(b|data) = \int_{-\infty}^{\infty} \frac{1}{Z} \frac{1}{\sigma^{11}} \exp\left(-\frac{(A\beta-y)'(A\beta-y)}{2\sigma^2}\right) da\ dc\ d\sigma;$

$prob(c|data) = \int_{-\infty}^{\infty} \frac{1}{Z} \frac{1}{\sigma^{11}} \exp\left(-\frac{(A\beta-y)'(A\beta-y)}{2\sigma^2}\right) da\ db\ d\sigma.$

And the mean and the variance can calculated as:

$\mu_\sigma = \int_0^\infty \sigma\ prob(\sigma|data)\ d\sigma,\ \sigma_\sigma = \int_{-\infty}^{\infty} (\sigma - \mu_\sigma)^2\ prob(\sigma|data)\ d\sigma;$

$\mu_a = \int_0^\infty a\ prob(a|data)\ d\sigma,\ \sigma_a = \int_{-\infty}^{\infty} (a - \mu_a)^2\ prob(a|data)\ da;$

$\mu_b = \int_0^\infty b\ prob(b|data)\ db,\ \sigma_b = \int_{-\infty}^{\infty} (b - \mu_b)^2\ prob(b|data)\ db;$

$\mu_c = \int_0^\infty c\ prob(c|data)\ dc,\ \sigma_c = \int_{-\infty}^{\infty} (c - \mu_c)^2\ prob(c|data)\ dc.$

The errorbars thus will be $\mu_i \pm \sigma_i$, $i$ is $a, b, c, \sigma$.

## Problem 2

We have data y that depends on x and is believed to adhere to the model $y_i = mx + b + e_i$.

The error for the first 2 measurements is provided, which allows us to solve for m and b:

$$\begin{cases} y_1 = mx_1 + b + e_1 \\ y_2 = mx_2 + b + e_2 \end{cases}$$

Plugging in the knowns, we get: m = 0.9909, y = - 0.566.

Then we plug in b and m and solve for e.

To find the error bars, we need to find the posterior distribution to integrate over:

$prob(m,\ b) \propto \frac{1}{\sigma}$.

$prob(data|\ m, b) \propto \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2}\frac{y_1-(mx_1+b)}{\sigma^2}\right) \times \ldots \times \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2}\frac{y_5-(mx_5+b)}{\sigma^2}\right) =$
$\frac{1}{\sqrt{2\pi}^5 \sigma^5} exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{A\beta} - \boldsymbol{y})'(\boldsymbol{A\beta} - \boldsymbol{y})\right).$

$prob(\beta,\ \sigma \mid data) \propto \frac{1}{\sigma^6} exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{A\beta}-\boldsymbol{y})'(\boldsymbol{A\beta}-\boldsymbol{y})\right)$.

Then we find the errorbars the same way we found them in Problem 1.

## Problem 3

We have 2-dimensional data y that depends on time t and is believed to adhere to the model $y_{i,j} = e_{i,j}\exp(-bt_j)$.

Since we work with log(e), we have: $e_{i,j} = \frac{y_{i,j}}{\exp(-bt_j)}$ and $\log(e_{i,j}) = bt + \log(y)$; this adheres to the $\boldsymbol{e} = \boldsymbol{y} + \boldsymbol{A\beta}$ format**.**

We can stretch out the model to make it 1-dimensional express the model in the format $\boldsymbol{y} = \boldsymbol{A\beta} + \boldsymbol{e}$, where $A = \begin{pmatrix} A_t \\ A_t \\ A_t \\ A_t \end{pmatrix}$, $A_t = \begin{pmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_5 \end{pmatrix}$, $\beta = \binom{b}{m}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$, $y_i = log\begin{pmatrix} y_{i,1} \\ \vdots \\ y_{i,5} \end{pmatrix}$ and $e = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}$, $e_i = \begin{pmatrix} e_{i1} \\ \vdots \\ e_{i5} \end{pmatrix}$.

$\boldsymbol{e}$ then can be expressed as $\log(\boldsymbol{e}) = \boldsymbol{y} + \boldsymbol{A\beta}$ , then we simply calculate $\boldsymbol{e} = e^{\log(\boldsymbol{e})}$.

Then we calculate the mean $\mu$ and the variance $\sigma$ from values of e.

To find the error bars, we need to find the posterior distribution to integrate over:

$prob(m,\ b) \propto \frac{1}{\sigma}$.

$prob(data|\ m, b) \propto \frac{1}{\sigma^{10}} exp\left(-\frac{1}{2\sigma^2}\boldsymbol{e}'\boldsymbol{e}\right)$.

$prob(\beta,\ \sigma \mid data) \propto \frac{1}{\sigma^6} exp\left(-\frac{1}{2\sigma^2}\boldsymbol{e}'\boldsymbol{e}\right)$.

Then we find the errorbars the same way we found them in Problem 1.