

## Homework 2

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### Problem 1

a)  $L[n + 2] = L[n + 1] + L[n]$ ,  $L[0] = 2$ ,  $L[1] = 1$

Then  $L[2] = 3$ ,  $L[3] = 4$ ,  $L[4] = 7$ .

b)  $L[n] = A \left( \frac{1+\sqrt{5}}{2} \right)^n - B \left( \frac{1-\sqrt{5}}{2} \right)^n$ .

We can try and solve for A and B as a system of equations where  $L[0] = 2$ ,  $L[1] = 1$ .

$$L[0] = A \left( \frac{1+\sqrt{5}}{2} \right)^0 - B \left( \frac{1-\sqrt{5}}{2} \right)^0 = A - B = 2$$

$$L[1] = A \left( \frac{1+\sqrt{5}}{2} \right)^1 - B \left( \frac{1-\sqrt{5}}{2} \right)^1 = A \frac{1+\sqrt{5}}{2} - B \frac{1-\sqrt{5}}{2} = 1$$

In a matrix format, the system looks like:

$$\begin{bmatrix} 1 & -1 \\ \frac{1+\sqrt{5}}{2} & -\frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving the equation, we get:  $A = 1$ ,  $B = -1$ .

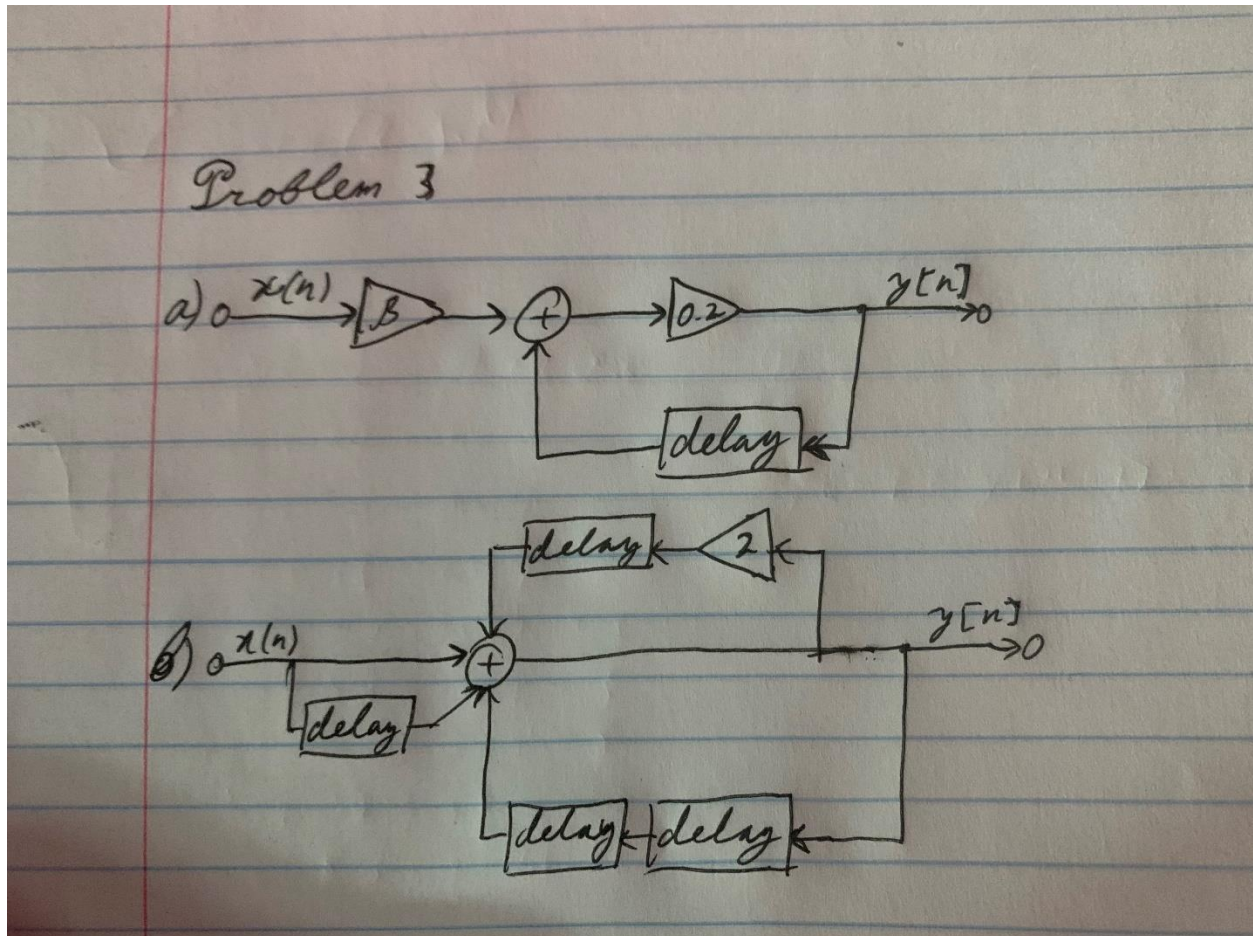
Thus, the closed form solution is  $L[n] = \left( \frac{1+\sqrt{5}}{2} \right)^n + \left( \frac{1-\sqrt{5}}{2} \right)^n$ .

### Problem 2

a)  $y[n] = x[n] + \alpha y[n - 1]$

b)  $y[n] = x[n] + y[n - 1] + \alpha y[n - 2]$

### Problem 3



### Problem 4

a) Only if the value of  $y[0]$  is provided.

b) If  $x[n] = 0$  for  $n > N$ , then starting at  $n=N$ , the term  $\beta x[n] = 0$  and  $y[n] = 0.2y[n-1] = 0.2(n-N)y[N-1]$ .

### Problem 5

b)  $S[n+2] = 2S[n+1] + 2S[n]$  is the difference equation. We can transform it in a following way:

$$S[n+2] - 2S[n+1] - 2S[n] = 0$$

$$Az^n - 2Az^{n-1} - 2Az^{n-2} = 0$$

$$z^2 - 2z - 2 = 0$$

The solution to this quadratic equation is  $z = \frac{2 \pm \sqrt{12}}{4} = \frac{1 \pm \sqrt{3}}{2}$ .

Thus, the solution to the difference equation will have the form  $S[n] = A \left( \frac{1+\sqrt{3}}{2} \right)^n + B \left( \frac{1-\sqrt{3}}{2} \right)^n$ .

We can apply it for cases  $S[0]=0$ ,  $S[1]=1$ .

$$S[0] = A \left( \frac{1+\sqrt{3}}{2} \right)^0 + B \left( \frac{1-\sqrt{3}}{2} \right)^0 = A + B = 0 \quad (A = -B)$$

$$S[1] = A \left( \frac{1+\sqrt{3}}{2} \right)^1 + B \left( \frac{1-\sqrt{3}}{2} \right)^1 = A \left( \frac{1+\sqrt{3}}{2} \right)^1 - A \left( \frac{1-\sqrt{3}}{2} \right)^1 = \sqrt{3}A = 1$$

$$A = \frac{1}{\sqrt{3}}; B = -\frac{1}{\sqrt{3}}$$

The closed form solution:  $S[n] = \frac{1}{\sqrt{3}} \left( \frac{1+\sqrt{3}}{2} \right)^n - \frac{1}{\sqrt{3}} \left( \frac{1-\sqrt{3}}{2} \right)^n$ .

c) Upon calculating the values from the closed form solution and comparing them to the output of the code, there is a mismatch. The closed form solution must contain an error.

## Problem 6

a) The Babylonian method:  $x[n+1] = \frac{1}{2} \left( x[n] + \frac{c}{x[n]} \right)$ .

To confirm that this is the valid method for calculating the square root of  $c$ , we can show that  $(x[n+1])^2 \approx c$ .

$$\begin{aligned} (x[n+1])^2 &= \left( \frac{1}{2} \left( x[n] + \frac{c}{x[n]} \right) \right)^2 = \frac{1}{4} \left( x[n]^2 + \frac{2cx[n]}{x[n]} + \frac{c^2}{(x[n])^2} \right) \\ &= \frac{1}{4} \left( x[n]^2 + 2c + \frac{c^2}{(x[n])^2} \right) \end{aligned}$$

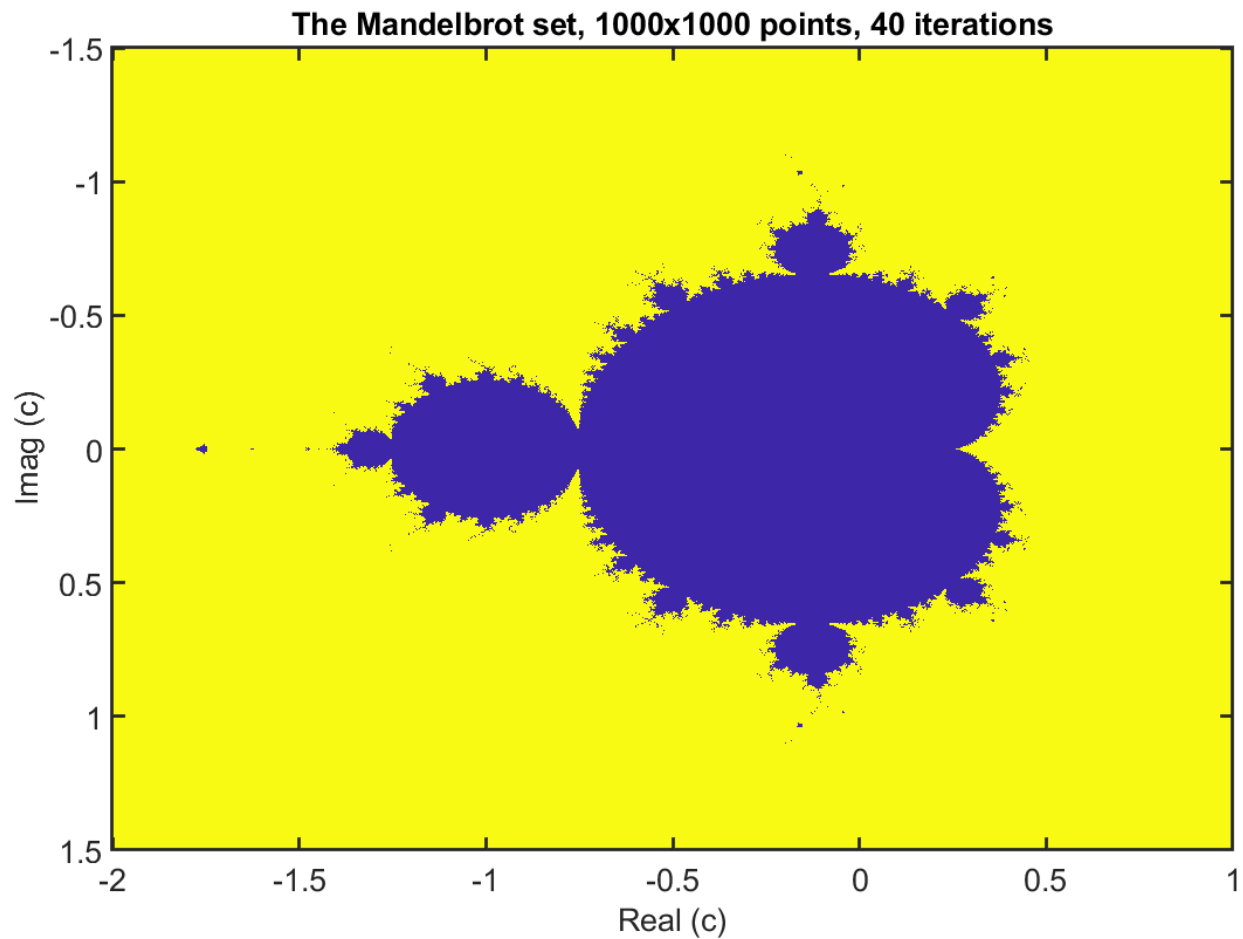
If we also remember that  $x[n]$  is a cruder approximation of  $\sqrt{c}$ :

$$\frac{1}{4} \left( x[n]^2 + 2c + \frac{c^2}{(x[n])^2} \right) \approx \frac{1}{4} \left( (\sqrt{c})^2 + 2c + \frac{c^2}{(\sqrt{c})^2} \right) = \frac{1}{4} \left( c + 2c + \frac{c^2}{c} \right) = \frac{4c}{4} = c$$

With each iteration,  $x[n+1]$  approaches the value of  $\sqrt{c}$ .

d) The Babylonian method converges for all values of  $x[0]$ . However, the initial guess cannot be 0, due to the  $\frac{c}{x[n]}$  term and the resulting division by 0.

### Problem 7



### Problem 8

a) Implementing the code offered by Wille-E Coyote results in the infinite loop. Upon analyzing the formula  $z(t + \Delta t) = z(t) - \Delta t(\sqrt{2g(z(t) - z_0)})$ , we can see that at the very first time step,  $z(t) - z_0 = 0$ , resulting in  $z(t + \Delta t) = z(t)$  for the first time step and every subsequent step after. While the energy is conserved, it is never converted into kinetic energy, making Wille-E hang in the air at the same height  $z_0$  indefinitely.

b) At the initial point:  $v(0) = z'(0) = 0$ ,  $z(0) = h$ .

The governing equation:  $m \frac{d^2 z}{dt^2} = -mg$ . Divide both sides by  $m$ :  $\frac{d^2 z}{dt^2} = -g$ .

Expressing it as a system of differential equations:

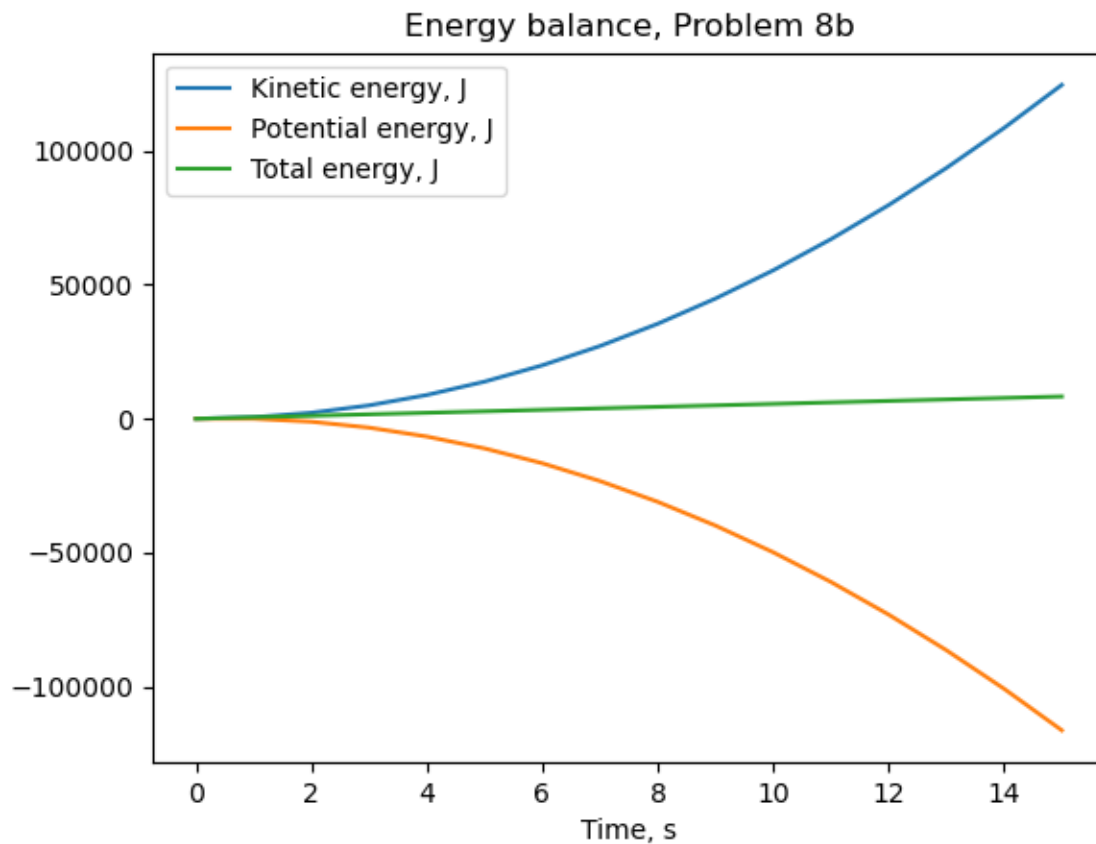
$$\begin{cases} \frac{d^2 z}{dt^2} = z_t'' = \frac{dz'}{dt} = \frac{dv}{dt} = -g \\ \frac{dz}{dt} = v(t) \\ v(0) = 0, \quad z(0) = h \end{cases}$$

Written as the Euler forward scheme:

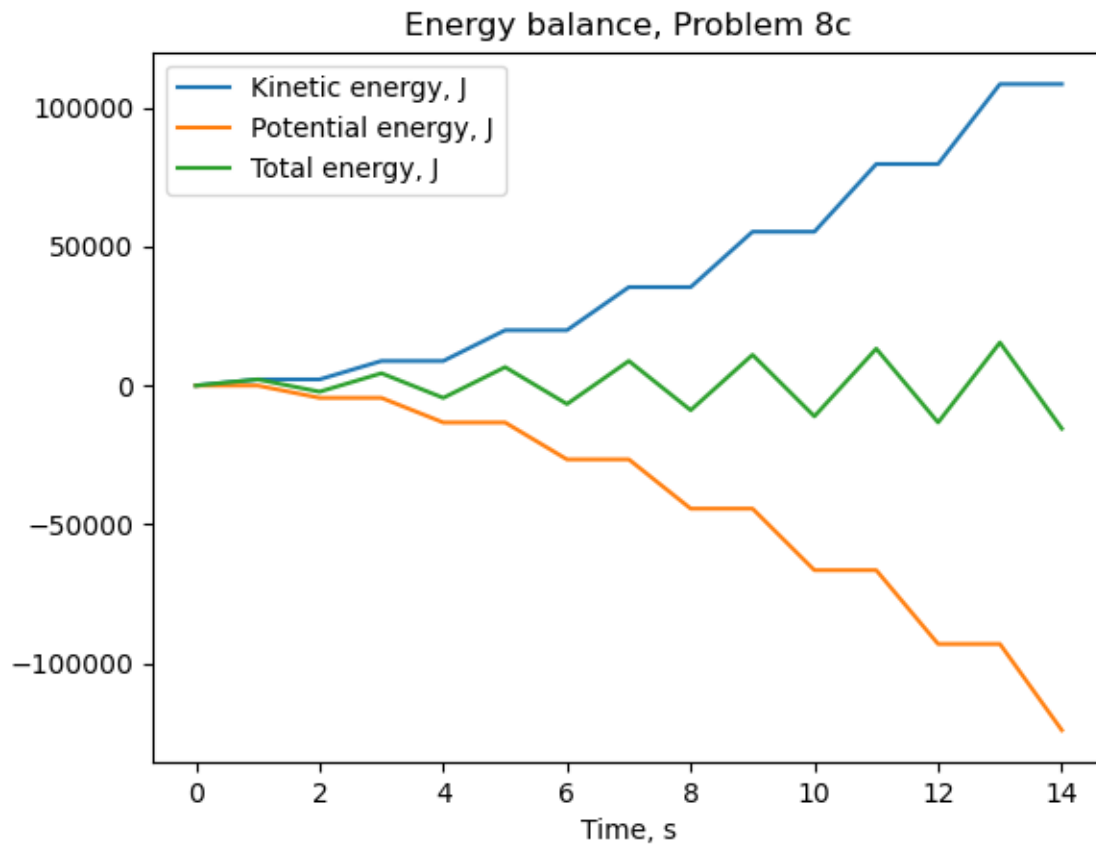
$$\begin{cases} v(t+1) = v(t) - g\Delta t \\ z(t+1) = z(t) + v(t)\Delta t \\ v(0) = 0, \quad z(0) = h \end{cases}$$

The average mass of a southern coyote is 11.5 kg (needed as  $m$  in the energy formula).

The total energy remains the same at every point in time, confirming its conservation.



c) Here, the total energy oscillates around the initial value of the energy; I assume this to be the artefact from the way the algorithm is executed (there is a delay in calculating  $z$ ), but if we average all values of total energy (provided in the code), the average comes up as the initial value of total energy, confirming that it is, indeed, conserved.



### Code for different problems

Problem 5a,c: *problem5.py*

Problem 6b, c: *problem6.py*

Problem 7: *mandelbrot.m*

Problem 8a: *p8\_1.py*

Problem 8b: *p8\_2.py*

Problem 8c: *p8\_3.py*