A Sequence of k^2 terms its relation with an Algorithm

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Abstract

This document contains my notes that I made when I attended a talk by Prof. Jaikumar Radhakrishnan, ICTS-TIFR, Bengalaru. The talk was titled 'The side effects of knowing an Algroritthm'.

1 Erdos-Szekeres Theorem

Before considering a k^2 term sequence we will look at the "Erdos-Szekeres Theorem" in this section. The theorem is as follows :

Theorem 1 (Erdos-Szekeres Theorem)**.** *Any sequence of distinct $k^2 + 1$ terms contains a k+1 increasing subsequence or a k+1 decreasing subsequence.*

I will not formally prove this theorem but give a justification.

Justification : Consider the sequence $a_1, a_2, \ldots, a_n$ of distinct elements. Assume that no k+1 increasing nor decreasing subsequence is present.
We will associate each element a_i of the series with the ordered pair (r_i, d_i) where r_i is the length of the longest increasing subsequence starting from a_i and d_i is the longest decreasing subsequence starting from a_i.
Now, in a (k x k)-grid associate each (r_i, d_i) with a block corresponding to each a_i (The horizontal axis will have r_i and vertical axis will have d_i.
Claim: Each block is associated with a single a_i.

If this were not the case, say a_i and a_j were at the same block where $i < j$, this implies that both a_i and a_j have a sequence of r_i increasing terms and d_i decreasing terms. As we assumed that the sequence has distinct terms **either** $a_i > a_j$ which would mean a_i could just be added before the decreasing sequence begining from a_j to get a d+1 term decreasing sequence from a_i.

OR if $a_j > a_i$ which would mean a_i could be added before a_j to get a r+1 term increasing sequence begining from a_i.
Contradicting the fact that a_i was placed at (r_i, d_i)

Therefor our grid looks like this,

c_i, r_i	$r_i = 1$	$r_i = 2$	$r_i = k$
$c_i = 1$	a_h	a_k	a_l
$c_i = 2$	a_p	a_q	a_n
$\ldots$			
$c_i = k$	a_q	a_t	a_s

We see that there are only k^2 blocks and we know that each block has a unique a_i but our sequence has $k^2 + 1$ terms. Thus, one element must be placed beyond the square but that would correspond to $r_i = k + 1$ or $d_i = k + 1$. From this we get that one block will have either r_i or d_i with the value of k+1. Which means that either a k+1 increasing or decreasing sequence can be found. (As r_i and d_i are the lengths of the longest increasing and decreasing subsequences from the given term respectively)

2 Sequences with k^2 Terms

In the above, theorem we dealt with $k^2 + 1$ term sequences and saw that any $k^2 + 1$ term sequence will have a k+1 increasing or decreasing subsequence, but what about k^2 term sequences ? we can see that this result is not true for a k^2 term sequence.

Although the above theorem is not true for k^2 term sequences. We do get to see some interesting properties if we consider k^2 term sequences without any k+1 increasing or decreasing subsequences. From now we will refer to these sequences as **Saturated Sequences.**

Before proceeding further, lets take a look at an Algorithm.

2.1 Algorithm to find subsequences

Given any sequence of distinct terms : $a_1, a_2, \ldots, a_n$

1. Place the first element in the first row and first column of an array.

2. Move to the next element, if the next element is less than a_1 place it in the same row at next column. Otherwise, place it at first column of next row.

3. Place all the following elements according to these rules:

a. Begin checking the last element of each row.

b. The first row that you find where the element is less than the last element of that row append the element to that row.

c. If the element could not be placed in any of the rows place it in the row following the last filled row.

Example : For sequence : 3,8,9,2,1,6,7,4,5. The first step: 3, followed by

Second Step:

3,

8, because $8 > 3$ doing this for entire series and making it into a table(A_s).

3	2	1
8	6	4
9	7	5

Claim : For a Staurated Sequence, this Algorithm will always yield a k by k square.

Justification : If a row were to have k+1 elements, it would mean that you have found a k+1 decreasing subsequence but by definition of Saturated Sequences this is not possible. So, each row must have at the most k elements. This implies columns should be each of k elements as the sequence has only k^2 terms.

This algorithm can also be used to prove the Erdos-Szekeres Theorem.

3 Interesting Properties of A_s

If we replace the elements of A_s with their index in the sequence we get another table. We will call this B_s. For the previously mentioned sequence B_s would be,

1	4	5
2	6	8
3	7	9

B_s can be thought of giving the time at which the elemets are placed in A_s.

There are many observations that are immediately seen.

Claim : Along a row the elements of B_s are increasing.

This makes sense because in the algorithm we place element from left to right. Hence, the index of element should increase.

Claim : Along a Column elements of B_s are increasing.

This conclusion comes from the fact that an element is placed in the next row only after comparing it with the last element of the row above it. This could be also said as elements of a column are placed from row 1 followed by the following rows.

Claim : For A_s The elements of a column are increasing.

We will prove this by induction on the number of columns j.

for j=1 and $1 \leq i \leq k+1$, $a_{i+1,j}$ is placed in row i+1 because $a_{i+1,j} > a_{i,j}$

We will assume that for all $j \leq k$ and $1 \leq i \leq k$,

$$a_{i,j} < a_{i+1,j}$$

Now inorder to place an element at $a_{i+1,j+1}$, We must have checked all previous rows. Using the fact the all previous elements in the column are filled the only way our element was placed at $i+1, j+1$ is if it was greater than the last element of row i at column j. Thus, $a_{i,j+1} < a_{i+1,j+1}$.

By Principle of Mathematical Induction ,for all $j \leq k$ and $1 \leq i \leq k$,

$$a_{i,j} < a_{i+1,j}$$

We can say that both A_s and B_s are both **monotonic arrays**, as in along a column and rows their elements are either increasing or decreasing.

4 A Formula for the number of Saturated Sequences

If we are given a saturated sequence, it is decribed by both A_s and B_s. and for every A_s and B_s we can find a saturated sequence.

There is a bijection between saturated sequences S and (A_s, B_s)

Now this the part that I do not have the proof of but the speaker stated: that the number of monotonic arrays comes from hook-length formula

$$\frac{(k^2)!}{\prod_{i=1}^{k}\prod_{j=1}^{k}(i+j-1)}$$

As both A_s and B_s are bijections, we can independently choose A_s and B_s so that,
we get the number of Saturated Sequences $= (\frac{(k^2)!}{\prod_{i=1}^{k}\prod_{j=1}^{k}(i+j-1)})^2$

This result is interesting as it is built upon an algroithm that is used to find subsequences in a given sequence, yet the speaker showed this connection which was quiet interesting.