

# Алгоритмы и модели вычислений.

## Задание 11: DFT

Сергей Володин, 272 гр.

задано 2014.04.17

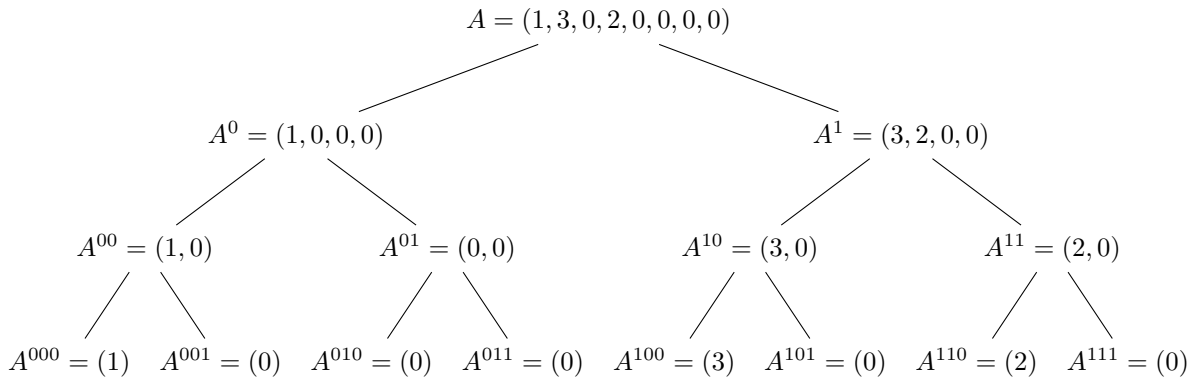
### Теория

(сюда будут ссылки)

1. Многочлен  $P_n(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} \longleftrightarrow (a_0, \dots, a_n) = P_n$  (порядок коэффициентов как на семинаре, а не как в задании). Считаем  $\exists l \in \mathbb{N} \cup \{0\} : n = 2^l$ .
2.  $\omega_n^k \stackrel{\text{def}}{=} e^{\frac{2\pi k}{n}i}$
3.  $\varphi(P) \stackrel{\text{def}}{=} (P_n(\omega_n^0), \dots, P_n(\omega_n^{n-1}))$  — дискретное преобразование Фурье
4.  $P_n^0 \stackrel{\text{def}}{=} (a_0, a_2, a_4, \dots)$ ,  $P_n^1 \stackrel{\text{def}}{=} (a_1, a_3, a_5, \dots) \Rightarrow$  свойство:  $P_n(x) = P_n^0(x^2) + x \cdot P_n^1(x^2)$ . Следствия :
  - (a)  $P_n(\omega_n^j) = P_n^0(\omega_{n/2}^j) + \omega_n^j P_n^1(\omega_{n/2}^j)$ ,  $0 \leq j < \frac{n}{2}$
  - (b)  $P_n(\omega_n^{\frac{n}{2}+j}) = P_n^0(\omega_{n/2}^j) - \omega_n^j P_n^1(\omega_{n/2}^j)$ ,  $0 \leq j < \frac{n}{2}$
5.  $n = 1 \Rightarrow \varphi(P_n) = \varphi((a_0)) = (a_0)$
6. Обозначаем  $\varphi(A) = \alpha$ , элементы кортежей как  $(a_0, \dots, a_{n-1})[i] = a_i$ .
7. Тогда  $4 \Rightarrow \begin{cases} \alpha[j] &= \alpha^0[j] + \omega_n^j \alpha^1[j] \\ \alpha[n/2 + j] &= \alpha^0[j] - \omega_n^j \alpha^1[j] \end{cases}$
8. Пусть  $A, B \in \mathbb{R}^{2n}$  — многочлены степени  $n-1$  (остальные коэффициенты — нули). Пусть  $C \in \mathbb{R}^{2n}$  — их произведение. Тогда  $\varphi(C) = \varphi(A) \times \varphi(B)$ , где  $\times$  — покомпонентное умножение кортежей. Действительно,  $\varphi(A)[i] = A(\omega_{2n}^i)$ ,  $\varphi(B)[i] = B(\omega_{2n}^i)$ , откуда  $\varphi(C)[i] = C(\omega_{2n}^i) = A(\omega_{2n}^i) \cdot B(\omega_{2n}^i) = \varphi(A)[i] \cdot \varphi(B)[i]$

### (каноническое) Задача 46

1.  $A = (1, 3, 0, 2, 0, 0, 0, 0)$ . Дерево вызовов:



- (a) Для  $A^{000}, A^{001}, \dots, A^{111}$  результат преобразования  $\alpha^{ijk} = A^{ijk}$  (см. 5)

- (b)  $\omega \stackrel{\text{def}}{=} e^{\frac{2\pi}{8}i} = \frac{1+i}{\sqrt{2}}$

- (c)  $\alpha^{00} = (\alpha^{000}[0] + \omega_2^0 \cdot \alpha^{001}[0], \alpha^{000}[0] - \omega_2^0 \alpha^{001}[0]) = |\omega_2^0 = 1 = \omega^0| = (1, 1)$

- (d)  $\alpha^{01} = (\alpha^{010}[0] + \omega_2^0 \cdot \alpha^{011}[0], \alpha^{010}[0] - \omega_2^0 \alpha^{011}[0]) = |\omega_2^0 = 1| = (0, 0)$

- (e)  $\alpha^{10} = (\alpha^{100}[0] + \omega_2^0 \cdot \alpha^{101}[0], \alpha^{100}[0] - \omega_2^0 \alpha^{101}[0]) = |\omega_2^0 = 1| = (3, 3)$

- (f)  $\alpha^{11} = (\alpha^{110}[0] + \omega_2^0 \cdot \alpha^{111}[0], \alpha^{110}[0] - \omega_2^0 \alpha^{111}[0]) = |\omega_2^0 = 1| = (2, 2)$

- (g)  $\alpha^0[0] = \alpha^{00}[0] + \underbrace{\omega_4^0}_{=1} \alpha^{01}[0] = 1$

- (h)  $\alpha^0[1] = \alpha^{00}[1] + \underbrace{\omega_4^1}_{=i} \alpha^{01}[1] = 1$

$$(i) \quad \alpha^0[2+0] = \alpha^{00}[0] - \underbrace{\omega_4^0}_{=1} \alpha^{01}[0] = 1$$

$$(j) \quad \alpha^0[2+1] = \alpha^{00}[1] - \underbrace{\omega_4^1}_{=i} \alpha^{01}[1] = 1$$

$$(k) \alpha^1[0] = \alpha^{10}[0] + \underbrace{\omega_4^0}_{=1} \alpha^{11}[0] = 5$$

$$(l) \alpha^1[1] = \alpha^{10}[1] + \underbrace{\omega_4^1}_{=i} \alpha^{11}[1] = 3 + 2i = 3 + 2\omega^2$$

$$(m) \alpha^1[2+0] = \alpha^{10}[0] - \underbrace{\omega_4^0}_{=1} \alpha^{11}[0] = 1$$

$$(n) \alpha^1[2+1] = \alpha^{10}[1] - \underbrace{\omega_4^1}_{=i} \alpha^{11}[1] = 3 - 2i = 3 - 2\omega^2$$

$$(o) \text{ Получаем } \alpha^0 = (1, 1, 1, 1), \alpha^1 = (5, 3 + 2i, 1, 3 - 2i) = (5, 3 + 2\omega^2, 1, 3 - 2\omega^2)$$

$$(p) \alpha[0] = \alpha^0[0] + \underbrace{\omega_8^0}_{=1} \alpha^1[0] = 6$$

$$(q) \alpha[1] = \alpha^0[1] + \underbrace{\omega_8^1}_{=\frac{1+i}{\sqrt{2}}} \alpha^1[1] = 1 + \frac{1+i}{\sqrt{2}}(3 + 2i) = 1 + \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i = 1 + \omega \cdot (3 + 2\omega^2) = 2\omega^3 + 3\omega + 1$$

$$(r) \alpha[2] = \alpha^0[2] + \underbrace{\omega_8^2}_{=i} \alpha^1[2] = 1 + i = 1 + \omega^2$$

$$(s) \alpha[3] = \alpha^0[3] + \underbrace{\omega_8^3}_{=\frac{-1+i}{\sqrt{2}}} \alpha^1[3] = 1 + \frac{-1+i}{\sqrt{2}}(3 - 2i) = 1 - \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i = 1 + \omega^3 \cdot (3 - 2\omega^2) = -2\omega^5 + 3\omega^3 + 1$$

$$(t) \alpha[4+0] = \alpha^0[0] - \underbrace{\omega_8^0}_{=1} \alpha^1[0] = -4$$

$$(u) \alpha[4+1] = \alpha^0[1] - \underbrace{\omega_8^1}_{=\frac{1+i}{\sqrt{2}}} \alpha^1[1] = 1 - \frac{1+i}{\sqrt{2}}(3 + 2i) = 1 - \frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}i = 1 - \omega \cdot (3 + 2\omega^2) = -2\omega^3 - 3\omega + 1$$

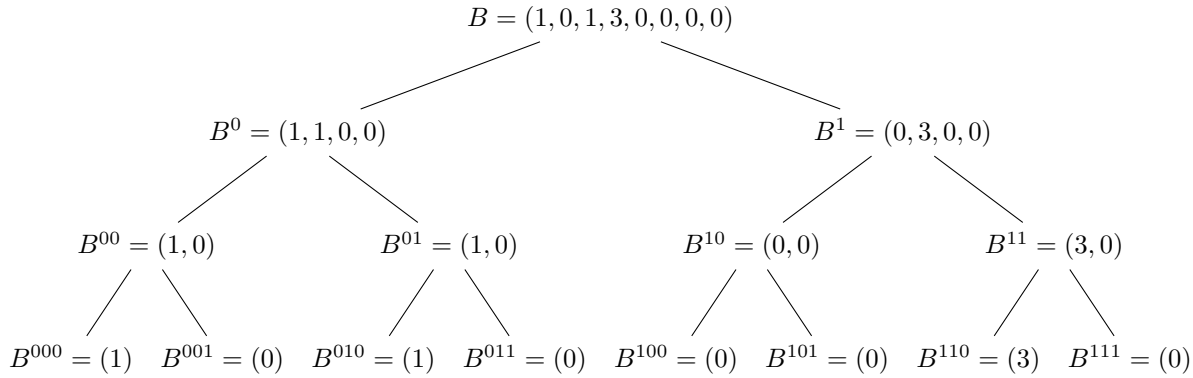
$$(v) \alpha[4+2] = \alpha^0[2] - \underbrace{\omega_8^2}_{=i} \alpha^1[2] = 1 - i = 1 - \omega^2$$

$$(w) \alpha[4+3] = \alpha^0[3] - \underbrace{\omega_8^3}_{=\frac{-1+i}{\sqrt{2}}} \alpha^1[3] = 1 - \frac{-1+i}{\sqrt{2}}(3 - 2i) = 1 + \frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}i = 1 - \omega^3 \cdot (3 - 2\omega^2) = 2\omega^5 - 3\omega^3 + 1$$

$$(x) \text{ Получаем } \alpha = (6, 1 + \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i, 1 + i, 1 - \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i, -4, 1 - \frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}i, 1 - i, 1 + \frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}i)$$

$$(y) \text{ Как многочлен от } \omega: \alpha = (6, 2\omega^3 + 3\omega + 1, 1 + \omega^2, -2\omega^5 + 3\omega^3 + 1, -4, -2\omega^3 - 3\omega + 1, 1 - \omega^2, 2\omega^5 - 3\omega^3 + 1)$$

2.  $B = (1, 0, 1, 3, 0, 0, 0, 0)$ . Дерево вызовов:



$$(a) \text{ Для } B^{000}, B^{001}, \dots, B^{111} \text{ результат преобразования } \beta^{ijk} = B^{ijk} \text{ (см. 5)}$$

$$(b) \beta^{00} = (\beta^{000}[0] + \omega_2^0 \cdot \beta^{001}[0], \beta^{000}[0] - \omega_2^0 \beta^{001}[0]) = |\omega_2^0 = 1 = \omega^0| = (1, 1)$$

$$(c) \beta^{01} = (\beta^{010}[0] + \omega_2^0 \cdot \beta^{011}[0], \beta^{010}[0] - \omega_2^0 \beta^{011}[0]) = |\omega_2^0 = 1| = (1, 1)$$

$$(d) \beta^{10} = (\beta^{100}[0] + \omega_2^0 \cdot \beta^{101}[0], \beta^{100}[0] - \omega_2^0 \beta^{101}[0]) = |\omega_2^0 = 1| = (0, 0)$$

$$(e) \beta^{11} = (\beta^{110}[0] + \omega_2^0 \cdot \beta^{111}[0], \beta^{110}[0] - \omega_2^0 \beta^{111}[0]) = |\omega_2^0 = 1| = (3, 3)$$

$$(f) \beta^0[0] = \beta^{00}[0] + \underbrace{\omega_4^0}_{=1} \beta^{01}[0] = 2$$

$$(g) \beta^0[1] = \beta^{00}[1] + \underbrace{\omega_4^1}_{=i} \beta^{01}[1] = 1 + i = 1 + \omega^2$$

$$(h) \beta^0[2+0] = \beta^{00}[0] - \underbrace{\omega_4^0}_{=1} \beta^{01}[0] = 0$$

$$(i) \beta^0[2+1] = \beta^{00}[1] - \underbrace{\omega_4^1}_{=i} \beta^{01}[1] = 1 - i = 1 - \omega^2$$

- (j)  $\beta^1[0] = \beta^{10}[0] + \underbrace{\omega_4^0}_{=1} \beta^{11}[0] = 3$
- (k)  $\beta^1[1] = \beta^{10}[1] + \underbrace{\omega_4^1}_{=i} \beta^{11}[1] = 3i = 3\omega^2$
- (l)  $\beta^1[2+0] = \beta^{10}[0] - \underbrace{\omega_4^0}_{=1} \beta^{11}[0] = -3$
- (m)  $\beta^1[2+1] = \beta^{10}[1] - \underbrace{\omega_4^1}_{=i} \beta^{11}[1] = -3i = -3\omega^2$
- (n) Получаем  $\beta^0 = (2, 1+i, 0, 1-i) = (2, 1+\omega^2, 0, 1-\omega^2)$ ,  $\beta^1 = (3, 3i, -3, -3i) = (3, 3\omega^2, -3, -3\omega^2)$
- (o)  $\beta[0] = \beta^0[0] + \underbrace{\omega_8^0}_{=1} \beta^1[0] = 5$
- (p)  $\beta[1] = \beta^0[1] + \underbrace{\omega_8^1}_{=\frac{1+i}{\sqrt{2}}} \beta^1[1] = 1+i+3i\frac{1+i}{\sqrt{2}} = 1 - \frac{3}{\sqrt{2}} + (1 + \frac{3}{\sqrt{2}})i = 1 + \omega^2 + \omega \cdot 3\omega^2 = 3\omega^3 + \omega^2 + 1$
- (q)  $\beta[2] = \beta^0[2] + \underbrace{\omega_8^2}_{=i} \beta^1[2] = -3i = -3\omega^2$
- (r)  $\beta[3] = \beta^0[3] + \underbrace{\omega_8^3}_{=\frac{-1+i}{\sqrt{2}}} \beta^1[3] = 1-i-3i\frac{-1+i}{\sqrt{2}} = 1 + \frac{3}{\sqrt{2}} - (1 - \frac{3}{\sqrt{2}})i = 1 - \omega^2 - \omega^3 \cdot 3\omega^2 = -3\omega^5 - \omega^2 + 1$
- (s)  $\beta[4] = \beta^0[0] - \underbrace{\omega_8^0}_{=1} \beta^1[0] = -1$
- (t)  $\beta[5] = \beta^0[1] - \underbrace{\omega_8^1}_{=\frac{1+i}{\sqrt{2}}} \beta^1[1] = 1+i-3i\frac{1+i}{\sqrt{2}} = 1 + \frac{3}{\sqrt{2}} + (1 - \frac{3}{\sqrt{2}})i = 1 + \omega^2 - \omega \cdot 3\omega^2 = -3\omega^3 + \omega^2 + 1$
- (u)  $\beta[6] = \beta^0[2] - \underbrace{\omega_8^2}_{=i} \beta^1[2] = 3i = 3\omega^2$
- (v)  $\beta[7] = \beta^0[3] - \underbrace{\omega_8^3}_{=\frac{-1+i}{\sqrt{2}}} \beta^1[3] = 1-i+3i\frac{-1+i}{\sqrt{2}} = 1 - \frac{3}{\sqrt{2}} - (1 + \frac{3}{\sqrt{2}})i = 1 - \omega^2 + \omega^3 \cdot 3\omega^2 = 3\omega^5 - \omega^2 + 1$
- (w) Получаем  $\beta = (5, 1 - \frac{3}{\sqrt{2}} + (1 + \frac{3}{\sqrt{2}})i, -3i, 1 + \frac{3}{\sqrt{2}} - (1 - \frac{3}{\sqrt{2}})i, -1, 1 + \frac{3}{\sqrt{2}} + (1 - \frac{3}{\sqrt{2}})i, 3i, 1 - \frac{3}{\sqrt{2}} - (1 + \frac{3}{\sqrt{2}})i)$
- (x) Как многочлен от  $\omega$ :  $\beta = (5, 3\omega^3 + \omega^2 + 1, -3\omega^2, -3\omega^5 - \omega^2 + 1, -1, -3\omega^3 + \omega^2 + 1, 3\omega^2, 3\omega^5 - \omega^2 + 1)$

3. Получаем

$$\begin{aligned} \alpha &= (6, \quad 2\omega^3 + 3\omega + 1, \quad 1 + \omega^2, \quad -2\omega^5 + 3\omega^3 + 1, \quad -4, \quad -2\omega^3 - 3\omega + 1, \quad 1 - \omega^2, \quad 2\omega^5 - 3\omega^3 + 1), \\ \beta &= (5, \quad 3\omega^3 + \omega^2 + 1, \quad -3\omega^2, \quad -3\omega^5 - \omega^2 + 1, \quad -1, \quad -3\omega^3 + \omega^2 + 1, \quad 3\omega^2, \quad 3\omega^5 - \omega^2 + 1), \end{aligned}$$

и по 8 получаем, что  $\varphi(C) \equiv \gamma = \alpha \times \beta = (30, 6\omega^6 + 2\omega^5 + 9\omega^4 + 8\omega^3 + \omega^2 + 3\omega + 1, -3\omega^4 - 3\omega^2, 6\omega^{10} - 9\omega^8 + 2\omega^7 - 8\omega^5 + 3\omega^3 - \omega^2 + 1, 4, 6\omega^6 - 2\omega^5 + 9\omega^4 - 8\omega^3 + \omega^2 - 3\omega + 1, -3\omega^4 + 3\omega^2, 6\omega^{10} - 9\omega^8 - 2\omega^7 + 8\omega^5 - 3\omega^3 - \omega^2 + 1)$ . Но  $\omega^8 = 1$ , поэтому

$$\gamma = \left\| \begin{array}{c} 30 \\ 6\omega^6 + 2\omega^5 + 9\omega^4 + 8\omega^3 + \omega^2 + 3\omega + 1 \\ -3\omega^4 - 3\omega^2 \\ 2\omega^7 - 8\omega^5 + 3\omega^3 + 5\omega^2 - 8 \\ 4 \\ 6\omega^6 - 2\omega^5 + 9\omega^4 - 8\omega^3 + \omega^2 - 3\omega + 1 \\ -3\omega^4 + 3\omega^2 \\ -2\omega^7 + 8\omega^5 - 3\omega^3 + 5\omega^2 - 8 \end{array} \right\|$$

(каноническое) **Задача 47**