Алгоритмы и модели вычислений.
Задание 11: DFT

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Теория

(сюда будут ссылки)

1. Многочлен $P_n(x) = a_0 + a_1 x + ... + a_{n-1}x^{n-1} \longleftrightarrow (a_0, ..., a_n) = P_n$ *(порядок коэффициентов как на семинаре, а не как в задании)*. Считаем $\exists l \in \mathbb{N} \cup \{0\} \colon n = 2^l$.

2. $\omega_n^k \overset{\text{def}}{=} e^{\frac{2\pi k}{n} i}$

3. $\varphi(P) \overset{\text{def}}{=} \big(P_n(\omega_n^0), ..., P_n(\omega_n^{n-1})\big)$ — дискретное преобразование Фурье

4. $P_n^0 \overset{\text{def}}{=} (a_0, a_2, a_4, ...)$, $P_n^1 \overset{\text{def}}{=} (a_1, a_3, a_5, ...) \Rightarrow$ свойство: $P_n(x) = P_n^0(x^2) + x \cdot P_n^1(x^2)$. Следствия :
   (a) $P_n(\omega_n^j) = P_n^0(\omega_{n/2}^j) + \omega_n^j P_n^1(\omega_{n/2}^j)$, $0 \leqslant j < \frac{n}{2}$
   (b) $P_n(\omega_n^{\frac{n}{2}+j}) = P_n^0(\omega_{n/2}^j) - \omega_n^j P_n^1(\omega_{n/2}^j)$, $0 \leqslant j < \frac{n}{2}$

5. $n = 1 \Rightarrow \varphi(P_n) = \varphi((a_0)) = (a_0)$

6. Обозначаем $\varphi(A) = \alpha$, элементы кортежей как $(a_0, ..., a_{n-1})[i] = a_i$.

7. Тогда $4 \Rightarrow \begin{cases} \alpha[j] & = \alpha^0[j] + \omega_n^j \alpha^1[j] \\ \alpha[n/2 + j] & = \alpha^0[j] - \omega_n^j \alpha^1[j] \end{cases}$

8. Пусть $A, B \in \mathbb{R}^{2n}$ — многочлены степени $n-1$ (остальные коэффициенты — нули). Пусть $C \in \mathbb{R}^{2n}$ — их произведение. Тогда $\varphi(C) = \varphi(A) \times \varphi(B)$, где $\times$ — покомпонентное умножение кортежей. Действительно, $\varphi(A)[i] = A(\omega_{2n}^i)$, $\varphi(B)[i] = B(\omega_{2n}^i)$, откуда $\varphi(C)[i] = C(\omega_{2n}^i) = A(\omega_{2n}^i) \cdot B(\omega_{2n}^i) = \varphi(A)[i] \cdot \varphi(B)[i]$

(каноническое) Задача 46

1. $A = (1, 3, 0, 2, 0, 0, 0, 0)$. Дерево вызовов:

$$A = (1, 3, 0, 2, 0, 0, 0, 0)$$

$$A^0 = (1, 0, 0, 0) \qquad A^1 = (3, 2, 0, 0)$$

$$A^{00} = (1, 0) \qquad A^{01} = (0, 0) \qquad A^{10} = (3, 0) \qquad A^{11} = (2, 0)$$

$$A^{000} = (1) \quad A^{001} = (0) \quad A^{010} = (0) \quad A^{011} = (0) \quad A^{100} = (3) \quad A^{101} = (0) \quad A^{110} = (2) \quad A^{111} = (0)$$

   (a) Для $A^{000}, A^{001}, ..., A^{111}$ результат преобразования $\alpha^{ijk} = A^{ijk}$ (см. 5)
   (b) $\omega \overset{\text{def}}{=} e^{\frac{2\pi}{8} i} = \frac{1+i}{\sqrt{2}}$
   (c) $\alpha^{00} = (\alpha^{000}[0] + \omega_2^0 \cdot \alpha^{001}[0], \alpha^{000}[0] - \omega_2^0 \alpha^{001}[0]) = |\omega_2^0 = 1 = \omega^0| = (1, 1)$
   (d) $\alpha^{01} = (\alpha^{010}[0] + \omega_2^0 \cdot \alpha^{011}[0], \alpha^{010}[0] - \omega_2^0 \alpha^{011}[0]) = |\omega_2^0 = 1| = (0, 0)$
   (e) $\alpha^{10} = (\alpha^{100}[0] + \omega_2^0 \cdot \alpha^{101}[0], \alpha^{100}[0] - \omega_2^0 \alpha^{101}[0]) = |\omega_2^0 = 1| = (3, 3)$
   (f) $\alpha^{11} = (\alpha^{110}[0] + \omega_2^0 \cdot \alpha^{111}[0], \alpha^{110}[0] - \omega_2^0 \alpha^{111}[0]) = |\omega_2^0 = 1| = (2, 2)$
   (g) $\alpha^0[0] = \alpha^{00}[0] + \underbrace{\omega_4^0}_{=1} \alpha^{01}[0] = 1$
   (h) $\alpha^0[1] = \alpha^{00}[1] + \underbrace{\omega_4^1}_{=i} \alpha^{01}[1] = 1$

(i) $\alpha^0[2+0] = \alpha^{00}[0] - \underbrace{\omega_4^0}_{=1} \alpha^{01}[0] = 1$

(j) $\alpha^0[2+1] = \alpha^{00}[1] - \underbrace{\omega_4^1}_{=i} \alpha^{01}[1] = 1$

(k) $\alpha^1[0] = \alpha^{10}[0] + \underbrace{\omega_4^0}_{=1} \alpha^{11}[0] = 5$

(l) $\alpha^1[1] = \alpha^{10}[1] + \underbrace{\omega_4^1}_{=i} \alpha^{11}[1] = 3 + 2i = 3 + 2\omega^2$

(m) $\alpha^1[2+0] = \alpha^{10}[0] - \underbrace{\omega_4^0}_{=1} \alpha^{11}[0] = 1$

(n) $\alpha^1[2+1] = \alpha^{10}[1] - \underbrace{\omega_4^1}_{=i} \alpha^{11}[1] = 3 - 2i = 3 - 2\omega^2$

(o) Получаем $\alpha^0 = (1,1,1,1)$, $\alpha^1 = (5, 3+2i, 1, 3-2i) = (5, 3+2\omega^2, 1, 3-2\omega^2)$

(p) $\alpha[0] = \alpha^0[0] + \underbrace{\omega_8^0}_{=1} \alpha^1[0] = 6$

(q) $\alpha[1] = \alpha^0[1] + \underbrace{\omega_8^1}_{=\frac{1+i}{\sqrt{2}}} \alpha^1[1] = 1 + \frac{1+i}{\sqrt{2}}(3+2i) = 1 + \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i = 1 + \omega \cdot (3 + 2\omega^2) = 2\omega^3 + 3\omega + 1$

(r) $\alpha[2] = \alpha^0[2] + \underbrace{\omega_8^2}_{=i} \alpha^1[2] = 1 + i = 1 + \omega^2$

(s) $\alpha[3] = \alpha^0[3] + \underbrace{\omega_8^3}_{=\frac{-1+i}{\sqrt{2}}} \alpha^1[3] = 1 + \frac{-1+i}{\sqrt{2}}(3-2i) = 1 - \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i = 1 + \omega^3 \cdot (3 - 2\omega^2) = -2\omega^5 + 3\omega^3 + 1$

(t) $\alpha[4+0] = \alpha^0[0] - \underbrace{\omega_8^0}_{=1} \alpha^1[0] = -4$

(u) $\alpha[4+1] = \alpha^0[1] - \underbrace{\omega_8^1}_{=\frac{1+i}{\sqrt{2}}} \alpha^1[1] = 1 - \frac{1+i}{\sqrt{2}}(3+2i) = 1 - \frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}i = 1 - \omega \cdot (3 + 2\omega^2) = -2\omega^3 - 3\omega + 1$

(v) $\alpha[4+2] = \alpha^0[2] - \underbrace{\omega_8^2}_{=i} \alpha^1[2] = 1 - i = 1 - \omega^2$

(w) $\alpha[4+3] = \alpha^0[3] - \underbrace{\omega_8^3}_{=\frac{-1+i}{\sqrt{2}}} \alpha^1[3] = 1 - \frac{-1+i}{\sqrt{2}}(3-2i) = 1 + \frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}i = 1 - \omega^3 \cdot (3 - 2\omega^2) = 2\omega^5 - 3\omega^3 + 1$

(x) Получаем $\alpha = (6, 1 + \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i, 1+i, 1 - \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i, -4, 1 - \frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}i, 1-i, 1 + \frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}i)$

(y) Как многочлен от ω: $\alpha = (6, 2\omega^3 + 3\omega + 1, 1 + \omega^2, -2\omega^5 + 3\omega^3 + 1, -4, -2\omega^3 - 3\omega + 1, 1 - \omega^2, 2\omega^5 - 3\omega^3 + 1)$

2. $B = (1,0,1,3,0,0,0,0)$. Дерево вызовов:

$B = (1,0,1,3,0,0,0,0)$

$B^0 = (1,1,0,0)$ $\qquad$ $B^1 = (0,3,0,0)$

$B^{00} = (1,0)$ $\qquad$ $B^{01} = (1,0)$ $\qquad$ $B^{10} = (0,0)$ $\qquad$ $B^{11} = (3,0)$

$B^{000} = (1)$ $\quad$ $B^{001} = (0)$ $\quad$ $B^{010} = (1)$ $\quad$ $B^{011} = (0)$ $\quad$ $B^{100} = (0)$ $\quad$ $B^{101} = (0)$ $\quad$ $B^{110} = (3)$ $\quad$ $B^{111} = (0)$

(a) Для $B^{000}, B^{001}, ..., B^{111}$ результат преобразования $\beta^{ijk} = B^{ijk}$ (см. 5)

(b) $\beta^{00} = (\beta^{000}[0] + \omega_2^0 \cdot \beta^{001}[0], \beta^{000}[0] - \omega_2^0 \beta^{001}[0]) = |\omega_2^0 = 1 = \omega^0| = (1,1)$

(c) $\beta^{01} = (\beta^{010}[0] + \omega_2^0 \cdot \beta^{011}[0], \beta^{010}[0] - \omega_2^0 \beta^{011}[0]) = |\omega_2^0 = 1| = (1,1)$

(d) $\beta^{10} = (\beta^{100}[0] + \omega_2^0 \cdot \beta^{101}[0], \beta^{100}[0] - \omega_2^0 \beta^{101}[0]) = |\omega_2^0 = 1| = (0,0)$

(e) $\beta^{11} = (\beta^{110}[0] + \omega_2^0 \cdot \beta^{111}[0], \beta^{110}[0] - \omega_2^0 \beta^{111}[0]) = |\omega_2^0 = 1| = (3,3)$

(f) $\beta^0[0] = \beta^{00}[0] + \underbrace{\omega_4^0}_{=1} \beta^{01}[0] = 2$

(g) $\beta^0[1] = \beta^{00}[1] + \underbrace{\omega_4^1}_{=i} \beta^{01}[1] = 1 + i = 1 + \omega^2$

(h) $\beta^0[2+0] = \beta^{00}[0] - \underbrace{\omega_4^0}_{=1} \beta^{01}[0] = 0$

(i) $\beta^0[2+1] = \beta^{00}[1] - \underbrace{\omega_4^1}_{=i} \beta^{01}[1] = 1 - i = 1 - \omega^2$

(j) $\beta^1[0] = \beta^{10}[0] + \underbrace{\omega_4^0}_{=1} \beta^{11}[0] = 3$

(k) $\beta^1[1] = \beta^{10}[1] + \underbrace{\omega_4^1}_{=i} \beta^{11}[1] = 3i = 3\omega^2$

(l) $\beta^1[2+0] = \beta^{10}[0] - \underbrace{\omega_4^0}_{=1} \beta^{11}[0] = -3$

(m) $\beta^1[2+1] = \beta^{10}[1] - \underbrace{\omega_4^1}_{=i} \beta^{11}[1] = -3i = -3\omega^2$

(n) Получаем $\beta^0 = (2, 1+i, 0, 1-i) = (2, 1+\omega^2, 0, 1-\omega^2)$, $\beta^1 = (3, 3i, -3, -3i) = (3, 3\omega^2, -3, -3\omega^2)$

(o) $\beta[0] = \beta^0[0] + \underbrace{\omega_8^0}_{=1} \beta^1[0] = 5$

(p) $\beta[1] = \beta^0[1] + \underbrace{\omega_8^1}_{=\frac{1+i}{\sqrt{2}}} \beta^1[1] = 1 + i + 3i\frac{1+i}{\sqrt{2}} = 1 - \frac{3}{\sqrt{2}} + (1 + \frac{3}{\sqrt{2}})i = 1 + \omega^2 + \omega \cdot 3\omega^2 = 3\omega^3 + \omega^2 + 1$

(q) $\beta[2] = \beta^0[2] + \underbrace{\omega_8^2}_{=i} \beta^1[2] = -3i = -3\omega^2$

(r) $\beta[3] = \beta^0[3] + \underbrace{\omega_8^3}_{=\frac{-1+i}{\sqrt{2}}} \beta^1[3] = 1 - i - 3i\frac{-1+i}{\sqrt{2}} = 1 + \frac{3}{\sqrt{2}} - (1 - \frac{3}{\sqrt{2}})i = 1 - \omega^2 - \omega^3 \cdot 3\omega^2 = -3\omega^5 - \omega^2 + 1$

(s) $\beta[4] = \beta^0[0] - \underbrace{\omega_8^0}_{=1} \beta^1[0] = -1$

(t) $\beta[5] = \beta^0[1] - \underbrace{\omega_8^1}_{=\frac{1+i}{\sqrt{2}}} \beta^1[1] = 1 + i - 3i\frac{1+i}{\sqrt{2}} = 1 + \frac{3}{\sqrt{2}} + (1 - \frac{3}{\sqrt{2}})i = 1 + \omega^2 - \omega \cdot 3\omega^2 = -3\omega^3 + \omega^2 + 1$

(u) $\beta[6] = \beta^0[2] - \underbrace{\omega_8^2}_{=i} \beta^1[2] = 3i = 3\omega^2$

(v) $\beta[7] = \beta^0[3] - \underbrace{\omega_8^3}_{=\frac{-1+i}{\sqrt{2}}} \beta^1[3] = 1 - i + 3i\frac{-1+i}{\sqrt{2}} = 1 - \frac{3}{\sqrt{2}} - (1 + \frac{3}{\sqrt{2}})i = 1 - \omega^2 + \omega^3 \cdot 3\omega^2 = 3\omega^5 - \omega^2 + 1$

(w) Получаем $\beta = (5, 1 - \frac{3}{\sqrt{2}} + (1 + \frac{3}{\sqrt{2}})i, -3i, 1 + \frac{3}{\sqrt{2}} - (1 - \frac{3}{\sqrt{2}})i, -1, 1 + \frac{3}{\sqrt{2}} + (1 - \frac{3}{\sqrt{2}})i, 3i, 1 - \frac{3}{\sqrt{2}} - (1 + \frac{3}{\sqrt{2}})i)$

(x) Как многочлен от ω: $\beta = (5, 3\omega^3 + \omega^2 + 1, -3\omega^2, -3\omega^5 - \omega^2 + 1, -1, -3\omega^3 + \omega^2 + 1, 3\omega^2, 3\omega^5 - \omega^2 + 1)$

3. Получаем

$$\begin{array}{rcl} \alpha & = & (6, 2\omega^3 + 3\omega + 1, 1 + \omega^2, -2\omega^5 + 3\omega^3 + 1, -4, -2\omega^3 - 3\omega + 1, 1 - \omega^2, 2\omega^5 - 3\omega^3 + 1), \\ \beta & = & (5, 3\omega^3 + \omega^2 + 1, -3\omega^2, -3\omega^5 - \omega^2 + 1, -1, -3\omega^3 + \omega^2 + 1, 3\omega^2, 3\omega^5 - \omega^2 + 1), \end{array}$$

и по 8 получаем, что $\varphi(C) \equiv \gamma = \alpha \times \beta = (30, t)$

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