Алгоритмы и модели вычислений.
Задание 11: DFT

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Теория

(сюда будут ссылки)

1. Многочлен $P_n(x) = a_0 + a_1 x + ... + a_{n-1}x^{n-1} \longleftrightarrow (a_0, ..., a_n) = P_n$ *(порядок коэффициентов как на семинаре, а не как в задании).* Считаем $\exists l \in \mathbb{N} \cup \{0\} \colon n = 2^l$.

2. $\omega_n^k \overset{\text{def}}{=} e^{\frac{2\pi k}{n} i}$

3. $\varphi(P) \overset{\text{def}}{=} \big(P_n(\omega_n^0), ..., P_n(\omega_n^{n-1})\big)$ — дискретное преобразование Фурье

4. $P_n^0 \overset{\text{def}}{=} (a_0, a_2, a_4, ...)$, $P_n^1 \overset{\text{def}}{=} (a_1, a_3, a_5, ...) \Rightarrow$ свойство: $P_n(x) = P_n^0(x^2) + x \cdot P_n^1(x^2)$. Следствия :
   (a) $P_n(\omega_n^j) = P_n^0(\omega_{n/2}^j) + \omega_n^j P_n^1(\omega_{n/2}^j)$, $0 \leqslant j < \frac{n}{2}$
   (b) $P_n(\omega_n^{\frac{n}{2}+j}) = P_n^0(\omega_{n/2}^j) - \omega_n^j P_n^1(\omega_{n/2}^j)$, $0 \leqslant j < \frac{n}{2}$

5. $n = 1 \Rightarrow \varphi(P_n) = \varphi((a_0)) = (a_0)$

6. Обозначаем $\varphi(A) = \alpha$, элементы кортежей как $(a_0, ..., a_{n-1})[i] = a_i$.

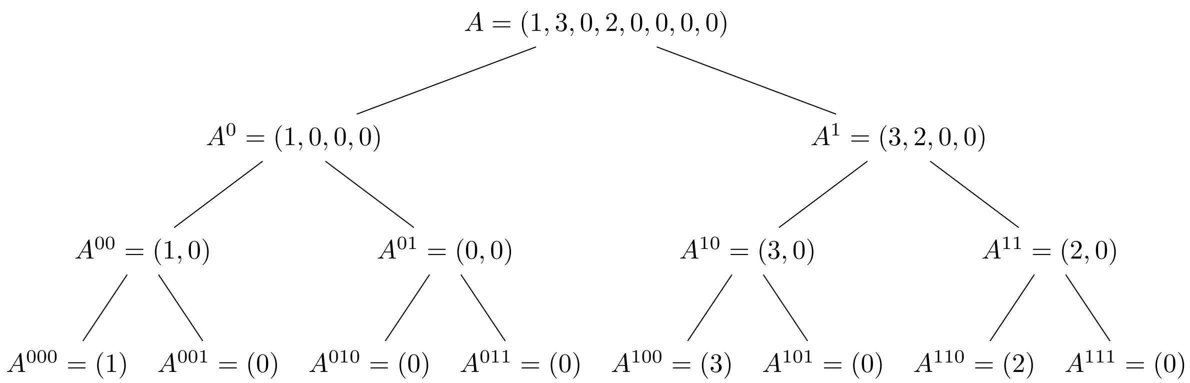
7. Тогда $4 \Rightarrow \begin{cases} \alpha[j] & = \alpha^0[j] + \omega_n^j \alpha^1[j] \\ \alpha[n/2 + j] & = \alpha^0[j] - \omega_n^j \alpha^1[j] \end{cases}$

8. Пусть $A, B \in \mathbb{R}^{2n}$ — многочлены степени $n - 1$ (остальные коэффициенты — нули). Пусть $C \in \mathbb{R}^{2n}$ — их произведение. Тогда $\varphi(C) = \varphi(A) \times \varphi(B)$, где $\times$ — покомпонентное умножение кортежей. Действительно, $\varphi(A)[i] = A(\omega_{2n}^i)$, $\varphi(B)[i] = B(\omega_{2n}^i)$, откуда $\varphi(C)[i] = C(\omega_{2n}^i) = A(\omega_{2n}^i) \cdot B(\omega_{2n}^i) = \varphi(A)[i] \cdot \varphi(B)[i]$

9.

(каноническое) Задача 46

1. $A = (1, 3, 0, 2, 0, 0, 0, 0)$. Дерево вызовов:



   (a) Для $A^{000}, A^{001}, ..., A^{111}$ результат преобразования $\alpha^{ijk} = A^{ijk}$ (см. 5)
   (b) $\omega \overset{\text{def}}{=} e^{\frac{2\pi}{8} i} = \frac{1+i}{\sqrt{2}}$
   (c) $\alpha^{00} = (\alpha^{000}[0] + \omega_2^0 \cdot \alpha^{001}[0], \alpha^{000}[0] - \omega_2^0 \alpha^{001}[0]) = |\omega_2^0 = 1 = \omega^0| = (1, 1)$
   (d) $\alpha^{01} = (\alpha^{010}[0] + \omega_2^0 \cdot \alpha^{011}[0], \alpha^{010}[0] - \omega_2^0 \alpha^{011}[0]) = |\omega_2^0 = 1| = (0, 0)$
   (e) $\alpha^{10} = (\alpha^{100}[0] + \omega_2^0 \cdot \alpha^{101}[0], \alpha^{100}[0] - \omega_2^0 \alpha^{101}[0]) = |\omega_2^0 = 1| = (3, 3)$
   (f) $\alpha^{11} = (\alpha^{110}[0] + \omega_2^0 \cdot \alpha^{111}[0], \alpha^{110}[0] - \omega_2^0 \alpha^{111}[0]) = |\omega_2^0 = 1| = (2, 2)$
   (g) $\alpha^0[0] = \alpha^{00}[0] + \underbrace{\omega_4^0}_{=1} \alpha^{01}[0] = 1$
   (h) $\alpha^0[1] = \alpha^{00}[1] + \underbrace{\omega_4^1}_{=i} \alpha^{01}[1] = 1$

(i) $\alpha^0[2+0] = \alpha^{00}[0] - \underbrace{\omega_4^0}_{=1} \alpha^{01}[0] = 1$

(j) $\alpha^0[2+1] = \alpha^{00}[1] - \underbrace{\omega_4^1}_{=i} \alpha^{01}[1] = 1$

(k) $\alpha^1[0] = \alpha^{10}[0] + \underbrace{\omega_4^0}_{=1} \alpha^{11}[0] = 5$

(l) $\alpha^1[1] = \alpha^{10}[1] + \underbrace{\omega_4^1}_{=i} \alpha^{11}[1] = 3 + 2i = 3 + 2\omega^2$

(m) $\alpha^1[2+0] = \alpha^{10}[0] - \underbrace{\omega_4^0}_{=1} \alpha^{11}[0] = 1$

(n) $\alpha^1[2+1] = \alpha^{10}[1] - \underbrace{\omega_4^1}_{=i} \alpha^{11}[1] = 3 - 2i = 3 - 2\omega^2$

(o) Получаем $\alpha^0 = (1,1,1,1)$, $\alpha^1 = (5, 3+2i, 1, 3-2i) = (5, 3+2\omega^2, 1, 3-2\omega^2)$

(p) $\alpha[0] = \alpha^0[0] + \underbrace{\omega_8^0}_{=1} \alpha^1[0] = 6$

(q) $\alpha[1] = \alpha^0[1] + \underbrace{\omega_8^1}_{=\frac{1+i}{\sqrt{2}}} \alpha^1[1] = 1 + \frac{1+i}{\sqrt{2}}(3+2i) = 1 + \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i = 1 + \omega \cdot (3 + 2\omega^2) = 2\omega^3 + 3\omega + 1$

(r) $\alpha[2] = \alpha^0[2] + \underbrace{\omega_8^2}_{=i} \alpha^1[2] = 1 + i = 1 + \omega^2$

(s) $\alpha[3] = \alpha^0[3] + \underbrace{\omega_8^3}_{=\frac{-1+i}{\sqrt{2}}} \alpha^1[3] = 1 + \frac{-1+i}{\sqrt{2}}(3-2i) = 1 - \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i = 1 + \omega^3 \cdot (3 - 2\omega^2) = -2\omega^5 + 3\omega^3 + 1$

(t) $\alpha[4+0] = \alpha^0[0] - \underbrace{\omega_8^0}_{=1} \alpha^1[0] = -4$

(u) $\alpha[4+1] = \alpha^0[1] - \underbrace{\omega_8^1}_{=\frac{1+i}{\sqrt{2}}} \alpha^1[1] = 1 - \frac{1+i}{\sqrt{2}}(3+2i) = 1 - \frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}i = 1 - \omega \cdot (3 + 2\omega^2) = -2\omega^3 - 3\omega + 1$

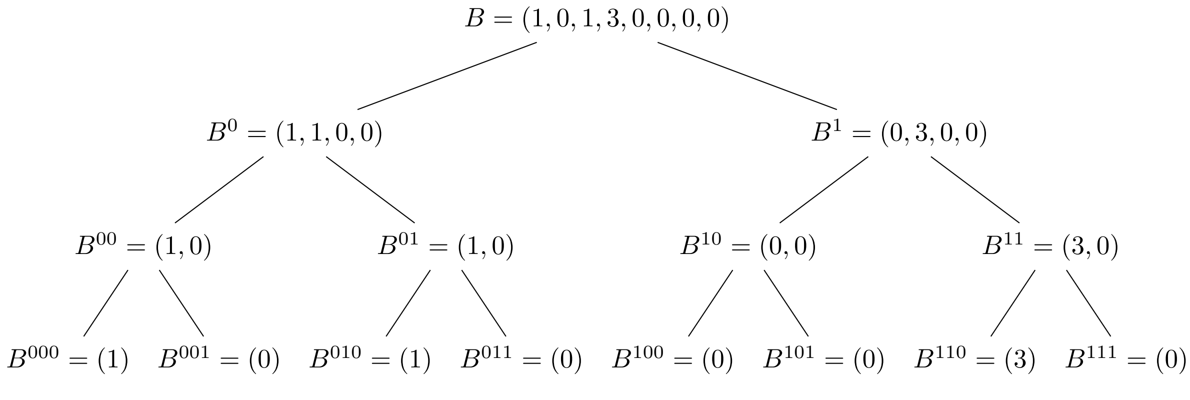
(v) $\alpha[4+2] = \alpha^0[2] - \underbrace{\omega_8^2}_{=i} \alpha^1[2] = 1 - i = 1 - \omega^2$

(w) $\alpha[4+3] = \alpha^0[3] - \underbrace{\omega_8^3}_{=\frac{-1+i}{\sqrt{2}}} \alpha^1[3] = 1 - \frac{-1+i}{\sqrt{2}}(3-2i) = 1 + \frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}i = 1 - \omega^3 \cdot (3 - 2\omega^2) = 2\omega^5 - 3\omega^3 + 1$

(x) Получаем $\alpha = (6, 1 + \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i, 1+i, 1 - \frac{1}{\sqrt{2}} + \frac{5}{\sqrt{2}}i, -4, 1 - \frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}i, 1-i, 1 + \frac{1}{\sqrt{2}} - \frac{5}{\sqrt{2}}i)$

(y) Как многочлен от ω: $\alpha = (6, 2\omega^3 + 3\omega + 1, 1 + \omega^2, -2\omega^5 + 3\omega^3 + 1, -4, -2\omega^3 - 3\omega + 1, 1 - \omega^2, 2\omega^5 - 3\omega^3 + 1)$

2. $B = (1,0,1,3,0,0,0,0)$. Дерево вызовов:



(a) Для $B^{000}, B^{001}, ..., B^{111}$ результат преобразования $\beta^{ijk} = B^{ijk}$ (см. 5)

(b) $\beta^{00} = (\beta^{000}[0] + \omega_2^0 \cdot \beta^{001}[0], \beta^{000}[0] - \omega_2^0 \beta^{001}[0]) = |\omega_2^0 = 1 = \omega^0| = (1,1)$

(c) $\beta^{01} = (\beta^{010}[0] + \omega_2^0 \cdot \beta^{011}[0], \beta^{010}[0] - \omega_2^0 \beta^{011}[0]) = |\omega_2^0 = 1| = (1,1)$

(d) $\beta^{10} = (\beta^{100}[0] + \omega_2^0 \cdot \beta^{101}[0], \beta^{100}[0] - \omega_2^0 \beta^{101}[0]) = |\omega_2^0 = 1| = (0,0)$

(e) $\beta^{11} = (\beta^{110}[0] + \omega_2^0 \cdot \beta^{111}[0], \beta^{110}[0] - \omega_2^0 \beta^{111}[0]) = |\omega_2^0 = 1| = (3,3)$

(f) $\beta^0[0] = \beta^{00}[0] + \underbrace{\omega_4^0}_{=1} \beta^{01}[0] = 2$

(g) $\beta^0[1] = \beta^{00}[1] + \underbrace{\omega_4^1}_{=i} \beta^{01}[1] = 1 + i = 1 + \omega^2$

(h) $\beta^0[2+0] = \beta^{00}[0] - \underbrace{\omega_4^0}_{=1} \beta^{01}[0] = 0$

(i) $\beta^0[2+1] = \beta^{00}[1] - \underbrace{\omega_4^1}_{=i} \beta^{01}[1] = 1 - i = 1 - \omega^2$

(j) $\beta^1[0] = \beta^{10}[0] + \underbrace{\omega_4^0}_{=1} \beta^{11}[0] = 3$

(k) $\beta^1[1] = \beta^{10}[1] + \underbrace{\omega_4^1}_{=i} \beta^{11}[1] = 3i = 3\omega^2$

(l) $\beta^1[2+0] = \beta^{10}[0] - \underbrace{\omega_4^0}_{=1} \beta^{11}[0] = -3$

(m) $\beta^1[2+1] = \beta^{10}[1] - \underbrace{\omega_4^1}_{=i} \beta^{11}[1] = -3i = -3\omega^2$

(n) Получаем $\beta^0 = (2, 1+i, 0, 1-i) = (2, 1+\omega^2, 0, 1-\omega^2)$, $\beta^1 = (3, 3i, -3, -3i) = (3, 3\omega^2, -3, -3\omega^2)$

(o) $\beta[0] = \beta^0[0] + \underbrace{\omega_8^0}_{=1} \beta^1[0] = 5$

(p) $\beta[1] = \beta^0[1] + \underbrace{\omega_8^1}_{=\frac{1+i}{\sqrt{2}}} \beta^1[1] = 1 + i + 3i\frac{1+i}{\sqrt{2}} = 1 - \frac{3}{\sqrt{2}} + (1 + \frac{3}{\sqrt{2}})i = 1 + \omega^2 + \omega \cdot 3\omega^2 = 3\omega^3 + \omega^2 + 1$

(q) $\beta[2] = \beta^0[2] + \underbrace{\omega_8^2}_{=i} \beta^1[2] = -3i = -3\omega^2$

(r) $\beta[3] = \beta^0[3] + \underbrace{\omega_8^3}_{=\frac{-1+i}{\sqrt{2}}} \beta^1[3] = 1 - i - 3i\frac{-1+i}{\sqrt{2}} = 1 + \frac{3}{\sqrt{2}} - (1 - \frac{3}{\sqrt{2}})i = 1 - \omega^2 - \omega^3 \cdot 3\omega^2 = -3\omega^5 - \omega^2 + 1$

(s) $\beta[4] = \beta^0[0] - \underbrace{\omega_8^0}_{=1} \beta^1[0] = -1$

(t) $\beta[5] = \beta^0[1] - \underbrace{\omega_8^1}_{=\frac{1+i}{\sqrt{2}}} \beta^1[1] = 1 + i - 3i\frac{1+i}{\sqrt{2}} = 1 + \frac{3}{\sqrt{2}} + (1 - \frac{3}{\sqrt{2}})i = 1 + \omega^2 - \omega \cdot 3\omega^2 = -3\omega^3 + \omega^2 + 1$

(u) $\beta[6] = \beta^0[2] - \underbrace{\omega_8^2}_{=i} \beta^1[2] = 3i = 3\omega^2$

(v) $\beta[7] = \beta^0[3] - \underbrace{\omega_8^3}_{=\frac{-1+i}{\sqrt{2}}} \beta^1[3] = 1 - i + 3i\frac{-1+i}{\sqrt{2}} = 1 - \frac{3}{\sqrt{2}} - (1 + \frac{3}{\sqrt{2}})i = 1 - \omega^2 + \omega^3 \cdot 3\omega^2 = 3\omega^5 - \omega^2 + 1$

(w) Получаем $\beta = (5, 1 - \frac{3}{\sqrt{2}} + (1 + \frac{3}{\sqrt{2}})i, -3i, 1 + \frac{3}{\sqrt{2}} - (1 - \frac{3}{\sqrt{2}})i, -1, 1 + \frac{3}{\sqrt{2}} + (1 - \frac{3}{\sqrt{2}})i, 3i, 1 - \frac{3}{\sqrt{2}} - (1 + \frac{3}{\sqrt{2}})i)$

(x) Как многочлен от ω: $\beta = (5, 3\omega^3 + \omega^2 + 1, -3\omega^2, -3\omega^5 - \omega^2 + 1, -1, -3\omega^3 + \omega^2 + 1, 3\omega^2, 3\omega^5 - \omega^2 + 1)$

3. Получаем

$$\begin{array}{rcllllllll} \alpha & = & (6, & 2\omega^3 + 3\omega + 1, & 1 + \omega^2, & -2\omega^5 + 3\omega^3 + 1, & -4, & -2\omega^3 - 3\omega + 1, & 1 - \omega^2, & 2\omega^5 - 3\omega^3 + 1), \\ \beta & = & (5, & 3\omega^3 + \omega^2 + 1, & -3\omega^2, & -3\omega^5 - \omega^2 + 1, & -1, & -3\omega^3 + \omega^2 + 1, & 3\omega^2, & 3\omega^5 - \omega^2 + 1), \end{array}$$

и по 8 получаем, что $\varphi(C) \equiv \gamma = \alpha \times \beta = (30, 6\omega^6 + 2\omega^5 + 9\omega^4 + 8\omega^3 + \omega^2 + 3\omega + 1, -3\omega^4 - 3\omega^2, 6\omega^{10} - 9\omega^8 + 2\omega^7 - 8\omega^5 + 3\omega^3 - \omega^2 + 1, 4, 6\omega^6 - 2\omega^5 + 9\omega^4 - 8\omega^3 + \omega^2 - 3\omega + 1, -3\omega^4 + 3\omega^2, 6\omega^{10} - 9\omega^8 - 2\omega^7 + 8\omega^5 - 3\omega^3 - \omega^2 + 1)$. Но $\omega^8 = 1$, поэтому

$$\gamma = \left\| \begin{matrix} 30 \\ 6\omega^6 + 2\omega^5 + 9\omega^4 + 8\omega^3 + \omega^2 + 3\omega + 1 \\ -3\omega^4 - 3\omega^2 \\ 2\omega^7 - 8\omega^5 + 3\omega^3 + 5\omega^2 - 8 \\ 4 \\ 6\omega^6 - 2\omega^5 + 9\omega^4 - 8\omega^3 + \omega^2 - 3\omega + 1 \\ -3\omega^4 + 3\omega^2 \\ -2\omega^7 + 8\omega^5 - 3\omega^3 + 5\omega^2 - 8 \end{matrix} \right\|$$

4. Выполним БПФ для $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7)$ (за $\gamma_0, ..., \gamma_7$ обозначены коэффициенты выше) Дерево вызовов:

$\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7)$

$\gamma^0 = (\gamma_0, \gamma_2, \gamma_4, \gamma_6)$ $\quad$ $\gamma^1 = (\gamma_1, \gamma_3, \gamma_5, \gamma_7)$

$\gamma^{00} = (\gamma_0, \gamma_4)$ $\quad$ $\gamma^{01} = (\gamma_2, \gamma_6)$ $\quad$ $\gamma^{10} = (\gamma_1, \gamma_5)$ $\quad$ $\gamma^{11} = (\gamma_3, \gamma_7)$

$\gamma^{000} = (\gamma_0)$ $\quad$ $\gamma^{001} = (\gamma_4)$ $\quad$ $\gamma^{010} = (\gamma_2)$ $\quad$ $\gamma^{011} = (\gamma_6)$ $\quad$ $\gamma^{100} = (\gamma_1)$ $\quad$ $\gamma^{101} = (\gamma_5)$ $\quad$ $\gamma^{110} = (\gamma_3)$ $\quad$ $\gamma^{111} = (\gamma_7)$

(a) $\Gamma \stackrel{\text{def}}{=} \varphi(\gamma)$

(b) Для $\gamma^{000}, \gamma^{001}, ..., \gamma^{111}$ результат преобразования $\Gamma^{ijk} = \gamma^{ijk}$ (см. 5)

(c) $\Gamma^{00} = (\Gamma^{000}[0] + \omega_2^0 \cdot \Gamma^{001}[0], \Gamma^{000}[0] - \omega_2^0 \Gamma^{001}[0]) = (\gamma_0 + \gamma_4, \gamma_0 - \gamma_4) = (34, 26)$

(d) $\Gamma^{01} = (\Gamma^{010}[0] + \omega_2^0 \cdot \Gamma^{011}[0], \Gamma^{010}[0] - \omega_2^0 \Gamma^{011}[0]) = (\gamma_2 + \gamma_6, \gamma_2 - \gamma_6) = (-6\omega^4, -6\omega^2)$

(e) $\Gamma^{10} = (\Gamma^{100}[0] + \omega_2^0 \cdot \Gamma^{101}[0], \Gamma^{100}[0] - \omega_2^0 \Gamma^{101}[0]) = (\gamma_1 + \gamma_5, \gamma_1 - \gamma_5) = (12\omega^6 + 18\omega^4 + 2\omega^2 + 2, 4\omega^5 + 16\omega^3 + 6\omega)$

(f) $\Gamma^{11} = (\Gamma^{110}[0] + \omega_2^0 \cdot \Gamma^{111}[0], \Gamma^{110}[0] - \omega_2^0 \Gamma^{111}[0]) = (\gamma_3 + \gamma_7, \gamma_3 - \gamma_7) = (10\omega^2 - 16, 4\omega^7 - 16\omega^5 + 6\omega^3)$

(g) $\Gamma^0[0] = \Gamma^{00}[0] + \underbrace{\omega_4^0}_{=1} \Gamma^{01}[0] = 34 - 6\omega^4$

(h) $\Gamma^0[1] = \Gamma^{00}[1] + \underbrace{\omega_4^1}_{=i} \Gamma^{01}[1] = 26 - 6\omega^4$ $(i = \omega^2)$

(i) $\Gamma^0[2+0] = \Gamma^{00}[0] - \underbrace{\omega_4^0}_{=1} \Gamma^{01}[0] = 34 + 6\omega^4$

(j) $\Gamma^0[2+1] = \Gamma^{00}[1] - \underbrace{\omega_4^1}_{=i} \Gamma^{01}[1] = 26 + 6\omega^4$

(k) $\Gamma^1[0] = \Gamma^{10}[0] + \underbrace{\omega_4^0}_{=1} \Gamma^{11}[0] = 12\omega^6 + 18\omega^4 + 12\omega^2 - 14$

(l) $\Gamma^1[1] = \Gamma^{10}[1] + \underbrace{\omega_4^1}_{=i} \Gamma^{11}[1] = 4\omega^9 - 16\omega^7 + 10\omega^5 + 16\omega^3 + 6\omega = |\omega^8 = 1| = -16\omega^7 + 10\omega^5 + 16\omega^3 + 10\omega$

(m) $\Gamma^1[2+0] = \Gamma^{10}[0] - \underbrace{\omega_4^0}_{=1} \Gamma^{11}[0] = 12\omega^6 + 18\omega^4 - 8\omega^2 + 18$

(n) $\Gamma^1[2+1] = \Gamma^{10}[1] - \underbrace{\omega_4^1}_{=i} \Gamma^{11}[1] = -4\omega^9 + 16\omega^7 - 2\omega^5 + 16\omega^3 + 6\omega = |\omega^8 = 1| = 16\omega^7 - 2\omega^5 + 16\omega^3 + 2\omega$

(o) $\Gamma[0] = \Gamma^0[0] + \underbrace{\omega_8^0}_{=1} \Gamma^1[0] = 12\omega^6 + 12\omega^4 + 12\omega^2 + 20 = 8$

(p) $\Gamma[1] = \Gamma^0[1] + \underbrace{\omega_8^1}_{=\omega} \Gamma^1[1] = -16\omega^8 + 10\omega^6 + 10\omega^4 + 10\omega^2 + 26 = |\omega^8 = 1| = 10\omega^6 + 10\omega^4 + 10\omega^2 + 10 = 0$

(q) $\Gamma[2] = \Gamma^0[2] + \underbrace{\omega_8^2}_{=\omega^2} \Gamma^1[2] = 12\omega^8 + 18\omega^6 - 2\omega^4 + 18\omega^2 + 34 = |\omega^8 = 1| = 18\omega^6 - 2\omega^4 + 18\omega^2 + 46 = 48$

(r) $\Gamma[3] = \Gamma^0[3] + \underbrace{\omega_8^3}_{=\omega^3} \Gamma^1[3] = 16\omega^{10} - 2\omega^8 + 16\omega^6 + 8\omega^4 + 26 = |\omega^8 = 1| = 16\omega^6 + 8\omega^4 + 16\omega^2 + 24 = 16$

(s) $\Gamma[4] = \Gamma^0[0] - \underbrace{\omega_8^0}_{=1} \Gamma^1[0] = -12\omega^6 - 24\omega^4 - 12\omega^2 + 48 = 72$

(t) $\Gamma[5] = \Gamma^0[1] - \underbrace{\omega_8^1}_{=\omega} \Gamma^1[1] = 16\omega^8 - 10\omega^6 - 22\omega^4 - 10\omega^2 + 26 = -10\omega^6 - 22\omega^4 - 10\omega^2 + 42 = 64$

(u) $\Gamma[6] = \Gamma^0[2] - \underbrace{\omega_8^2}_{=\omega^2} \Gamma^1[2] = -12\omega^8 - 18\omega^6 + 14\omega^4 - 18\omega^2 + 34 = -18\omega^6 + 14\omega^4 - 18\omega^2 + 22 = 8$

(v) $\Gamma[7] = \Gamma^0[3] - \underbrace{\omega_8^3}_{=\omega^3} \Gamma^1[3] = -16\omega^{10} + 2\omega^8 - 16\omega^6 + 4\omega^4 + 26 = -16\omega^6 + 4\omega^4 - 16\omega^2 + 28 = 24$

(w) Получаем $\Gamma = (8, 0, 48, 16, 72, 64, 8, 24)$

(x) $C = AB = \varphi^{-1}(\varphi(AB)) = \varphi^{-1}(\gamma) \stackrel{9}{=} \frac{1}{8}\varphi(\gamma) = \frac{\Gamma}{8} = (1, 0, 6, 2, 9, 8, 1, 3)$

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