A game of chance

ST449, Solution, Assignment 2, P4, 2018/19

1 P4 (max points 25)

Consider the following two-player game of chance. Two players, we refer to as players X and Y, have initial endowments of x and y tokens, respectively. Assume that $x, y > 0$. The game proceeds over rounds where in each round a dice is rolled. If the outcome of the dice is $1, 2$ or 3, player Y loses one token, and otherwise, if the outcome is 4, 5, or 6, player X loses a token. The game ends as soon one of the players runs out of tokens. The winner is the player who at the end of the game has at least one token left.

Answer the following questions:

P4-i (max points 5) What is the winning probability of player X for $x = 1$ and each value of y?

P4-ii (max points 5) What is the winning probability of player X for $y = 1$ and each value of x?

P4-iii (max points 20) What is the winning probability of player X for each value of x and y?

Note: You need to show derivations for your solutions.

2 Solution

Let $V(x, y)$ be the winning probability of player X by starting from initial value (x, y). Since the outcome is such that one of the players wins, we have that $1 - V(x, y)$ is the winning probability of player Y. By Bellman's equation, we have

$$V(x, y) = \frac{1}{2}V(x - 1, y) + \frac{1}{2}V(x, y - 1) \text{ for all } x, y > 0 \tag{1}$$

with boundary conditions $V(0, y) = 0$ for all $y \geq 0$ and $V(x, 0) = 1$ for all $x > 0$.

P4-i The answer is

$$V(1, y) = \frac{1}{2^y} \text{ for } y \geq 0.$$

By Bellman's equation (1) and boundary conditions $V(0, y) = 0$ for all $y \geq 0$, we have

$$V(1, y) = \frac{1}{2}V(1, y - 1), \text{ for } y > 0.$$

Hence,

$$V(1,y) = \frac{1}{2^y} V(1,0)$$

Combining with the boundary condition $V(1,0) = 1$, we have

$$V(1,y) = \frac{1}{2^y} \text{ for } y \geq 0$$

which corresponds to the asserted answer.

P4-ii The answer is

$$V(x,1) = 1 - \frac{1}{2^x} \text{ for } x \geq 0.$$

The answer follows directly from P4-i as $V(x,1) = 1 - V(1,x)$ and we already found out that $V(1,x) = 1/2^x$ for $x \geq 0$.

Alternatively, one may derive $V(x,1)$ directly from Bellman's equation (1) in a similar manner as we did in P4-i for $V(1,y)$.

P4-iii The solution is unique but can be expressed in different forms.

Solution 1 We claim that the solution is given by

$$V(x,y) = \sum_{i=1}^{x} \frac{1}{2^{x-i+y}} \binom{x-i+y-1}{x-i}. \tag{2}$$

Indeed, we can equivalently express (2) as

$$V(x,y) = \sum_{i=0}^{x-1} \frac{1}{2^{y+i}} \binom{y-1+i}{i}.$$

Proof by induction The claim can be established by induction over $n = x+y$ as follows. Base case: $n = 3$. By Bellman's equation (1), we have $V(1,2) = \frac{1}{2}V(0,2) + \frac{1}{2}V(1,1) = 0 + \frac{1}{2}\frac{1}{2} = \frac{1}{4}$. By (2), we have

$$V(1,2) = \frac{1}{4}$$

which completes the base case.

Induction step: assume that $V(x,y)$ is of the form (2) for all $x, y > 0$ such that $x + y \leq n$. We need to show that $V(x,y)$ is of the form (2) for all $y, x > 0$ such that $x + y = n + 1$.

Assume that $x, y > 0$ and $x + y = n + 1$. By the induction hypothesis $V(x-1,y)$ and $V(x,y-1)$ satisfy (2). Using this with Bellman's equation (1), we have

$$
\begin{aligned}
V(x,y) &= \frac{1}{2}V(x-1,y) + \frac{1}{2}V(x,y-1) \\
&= \frac{1}{2}\sum_{i=1}^{x-1} \frac{1}{2^{x-i+y-1}} \binom{x-i+y-2}{x-i-1} + \frac{1}{2}\sum_{i=1}^{x} \frac{1}{2^{x-i+y-1}} \binom{x-i+y-2}{x-i} \\
&= \sum_{i=2}^{x} \frac{1}{2^{x-i+y+1}} \binom{x-i+y-1}{x-i} + \sum_{i=1}^{x} \frac{1}{2^{x-i+y}} \binom{x-i+y-2}{x-i} \\
&= \sum_{i=1}^{x-1} \frac{1}{2^{x-i+y}} \left[\binom{x-i+y-2}{x-i-1} + \binom{x-i+y-2}{x-i} \right] + \frac{1}{2^y} \\
&= \sum_{i=1}^{x-1} \frac{1}{2^{x-i+y}} \binom{x-i+y-1}{x-i} + \frac{1}{2^y} \\
&= \sum_{i=1}^{x} \frac{1}{2^{x-i+y}} \binom{x-i+y-1}{x-i}
\end{aligned}
$$

which corresponds to the expression in (2).

Direct proof Note that $V(x,y)$ is the probability of the game ending in one of the following states $(i,0)$ for $i = 1,2,\ldots,x$, where i is the number of tokens held by player X and 0 is the number of tokens held by player Y at the end of the game.

For the game to end in state $(i,0)$, the game must reach state $(i,1)$ and in this state player Y must lose a token. To reach state $(i,1)$ from initial state (x,y), the game takes $x-i+y-1$ rounds, out of which player X loses $y-1$ tokens and player Y loses $x-i$ tokens. There are $\binom{x-i+y-1}{x-i}$ possible paths from (x,y) to $(i,0)$ and they are all equally likely, each having probability $1/2^{x-i+y-1}$. It follows that

$$
V(x,y) = \sum_{i=1}^{x} \frac{1}{2}\frac{1}{2^{x-i+y-1}} \binom{x-i+y-1}{x-i} = \sum_{i=1}^{x} \frac{1}{2^{x-i+y}} \binom{x-i+y-1}{x-i}
$$

which corresponds to the expression in (2).

Solution 2 If $x = y$, either player is equally likely to win, hence $V(x,x) = 1/2$ for all $x > 0$. It suffices to consider the case $x < y$ and the Bellman's equation with boundary conditions $V(0,y) = 0$ for all $y \geq 0$ and $V(x,x) = 1/2$ for all $x > 0$.

We claim that the solution is given by

$$
V(x,y) = \frac{1}{2}\sum_{i=1}^{x} \frac{1}{2^{(x-i)+(y-i)}} \frac{y-x}{(x-i)+(y-i)} \binom{(x-i)+(y-i)}{x-i}. \tag{3}
$$

Proof by induction The claim be established by induction over $n = x + y$ as follows. Base case: $n = 3$. By Bellman's equation (1), we have $V(1,2) = \frac{1}{2}V(0,2) + \frac{1}{2}V(1,1) = 0 + \frac{1}{2}\frac{1}{2} = \frac{1}{4}$. By (3), we have

$$
V(1,2) = \frac{1}{4}
$$

which completes the base case.

Induction step: assume that $V(x,y)$ has solution given by (3) for all $y > x > 0$ such that $x+y \leq n$. We next show that $V(x,y)$ has solution given by (3) for all $y > x > 0$ such that $x+y = n+1$.

Assume that $y > x > 0$ and $x+y = n+1$. By the induction hypothesis $V(x-1,y)$ and $V(x,y-1)$ are given by (3). Using this with Bellman's equation (1), we have

$$\begin{aligned}
V(x,y) &= \frac{1}{2}V(x-1,y) + \frac{1}{2}V(x,y-1) \\
&= \frac{1}{4}\sum_{i=1}^{x-1} \frac{1}{2^{(x-i)+(y-i)-1}} \frac{y-x+1}{(x-i)+(y-i)-1} \binom{(x-i)+(y-i)-1}{x-i-1} \\
&\quad + \frac{1}{4}\sum_{i=1}^{x} \frac{1}{2^{(x-i)+(y-i)-1}} \frac{y-x-1}{(x-i)+(y-i)-1} \binom{(x-i)+(y-i)-1}{x-i} \\
&= \frac{1}{2}\sum_{i=1}^{x-1} \frac{1}{2^{(x-i)+(y-i)}} \frac{y-x+1}{(x-i)+(y-i)-1} \frac{x-i}{(x-i)+(y-i)} \binom{(x-i)+(y-i)}{x-i} \\
&\quad + \sum_{i=1}^{x} \frac{1}{2^{(x-i)+(y-i)}} \frac{y-x-1}{(x-i)+(y-i)-1} \frac{y-i}{(x-i)+(y-i)} \binom{(x-i)+(y-i)}{x-i} \\
&= \frac{1}{2}\sum_{i=1}^{x} \frac{1}{2^{(x-i)+(y-i)}} \frac{1}{(x-i)+(y-i)} \frac{(y-x+1)(x-i)+(y-x-1)(y-i)}{(x-i)+(y-i)-1} \binom{(x-i)+(y-i)}{x-i} \\
&= \frac{1}{2}\sum_{i=1}^{x} \frac{1}{2^{(x-i)+(y-i)}} \frac{1}{(x-i)+(y-i)} \frac{(y-x)((x-i)+(y-i))+(x-y)}{(x-i)+(y-i)-1} \binom{(x-i)+(y-i)}{x-i} \\
&= \frac{1}{2}\sum_{i=1}^{x} \frac{1}{2^{(x-i)+(y-i)}} \frac{1}{(x-i)+(y-i)} \frac{(y-x)((x-i)+(y-i)-1)}{(x-i)+(y-i)-1} \binom{(x-i)+(y-i)}{x-i} \\
&= \frac{1}{2}\sum_{i=1}^{x} \frac{1}{2^{(x-i)+(y-i)}} \frac{y-x}{(x-i)+(y-i)} \binom{(x-i)+(y-i)}{x-i}
\end{aligned}$$

which corresponds to the expression in (3).

In the third equation of the last above math display, we use the elementary facts $\binom{a}{b} = \frac{a}{b}\binom{a-1}{b-1}$, for $a \geq b > 0$ and $\binom{a-1}{b} = \frac{a-b}{a}\binom{a}{b}$ for $a > b \geq 0$.

Direct proof We can establish (3) to be solution of (1) for $x < y$ directly as follows. Note that, by using Bellman's equations, $V(x,y)$ can be expanded as a weighted sum of boundary values $V(0,j) = 0$ for $j > 0$ and $V(i,i) = 1/2$ for $i > 0$. Since $V(0,j) = 0$ for all $j \geq 0$, we only need to consider the part of this sum for boundary values $V(i,i)$ for $i > 0$. For each element of this sum, corresponding to state (i,i), the weight is the probability of hitting state (i,i) by starting from state (x,y) and not hitting any of the states (k,k) for $i < k \leq x$.

Each path from (x,y) to (i,i) consists of $x-i$ transitions in x direction and $y-i$ steps in y direction. Hence, the probability of each such path is equal to

$$\frac{1}{2^{(x-i)+(y-i)}}.$$

The probability of hitting state (i,i) by starting from (x,y) and not hitting any of the states (k,k) for $i < k \leq x$ is equal to the product of $1/2^{(x-i)+(y-i)}$ and the number of paths from (x,y) to (i,i) that do not intersect any of the points (k, k) for $i < k \leq x$. Without this constraint, because each path contains $(x-i)+(y-i)$ steps with $x-i$ steps in the x direction, the total number of paths is indeed

$$\binom{(x-i)+(y-i)}{x-i}.$$

This needs to be multiplied by the fraction of paths that never intersect states (k, k) for $i < k \leq x$. This fraction follows by the Ballot theorem and is equal to

$$\frac{y-x}{(x-i)+(y-i)}.$$

The Ballot theorem is concerned with the following question: in an election where a candidate A receives p votes and a candidate B receives q votes with $p > q$, what is the probability that A will be strictly ahead of B throughout the vote count? The Ballot theorem states that the answer is $(p-q)/(p+q)$.

The Ballot theorem applies to our case with $(x-i)+(y-i)$ corresponding to the total number of votes given to candidates A and B, and $(x-i)$ and $(y-i)$ corresponding to the number of votes given to candidates B and A, respectively.