ST449 Artificial Intelligence and Deep Learning

Lecture 7

Dynamic programming and Monte Carlo methods

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Topics of this lecture

- Elementary solution methods:
    - Dynamic programming
    - Monte Carlo methods

- Next lecture: elementary solution methods cont’d
    - Temporal-difference learning

Dynamic programming (DP)

Dynamic Programming

- Dynamic Programming (DP): a collection of algorithms used to compute optimal policies given a prefect knowledge of the environment as a MDP

- DP algorithms are rarely used directly in practice

- However, they provide a foundation for other solution methods
  - Other methods can be seen as trying to achieve the same but with less computation and without assuming a perfect knowledge of the environment

Our focus: finite MDPs

- Environment modeled by a finite MDP: finite state and action sets S, $A(s), s \in S$

- Dynamics specified by the transition probabilities

$$P^a_{s,s'} = \mathbf{Pr}[s_{t+1} = s' \mid s_t = s, a_t = a]$$

and the immediate expected rewards for actions and state transitions:

$$R^a_{s,s'} = \mathbf{E}[r_{t+1} \mid a_t = a, s_t = s, s_{t+1} = s']$$

- DP ideas can be applied also to problems with continuous state and action sets
  - A common approach is to use quantization

Bellman optimality equations

- The optimal state value function V^* satisfies:

$$V^*(s) = \max_a \sum_{s' \in S} P_{s,s'}^a [R_{s,s'}^a + \gamma V^*(s')], \text{ for } s \in S$$

- The optimal action value function Q^* satisfies:

$$Q^*(s,a) = \sum_{s' \in S} P_{s,s'}^a [R_{s,s'}^a + \gamma \max_{a'} Q^*(s,a')], \text{ for } s \in S, a \in A(s)$$

Policy evaluation

- Policy evaluation: computation of the state value function V^{π} for a given policy π

- Also referred to as the prediction problem

- For any given policy π, the existence and uniqueness of V^{π} are guaranteed whenever either
  - The discount rate satisfies $\gamma < 1$, or
  - Eventual termination is guaranteed from all states under policy π

- V^{π} is a solution of a system of $|S|$ linear equations with $|S|$ unknowns

Iterative policy evaluation

- Iterative policy evaluation: an iterative solution method that outputs a sequence of approximate state-value functions $V_0, V_1, \ldots$
    - Initial value function V_0 is chosen arbitrarily subject to the constraint that at any terminal state it has value equal to 0
    - Iterative update rule:

$$V_{k+1}(s) = \mathbf{E}_\pi[r_{t+1} + \gamma V_k(s_{t+1}) \mid s_t = s]$$

$$= \sum_a \pi(s,a) \sum_{s'} P_{s,s'}^a [R_{s,s'}^a + \gamma V_k(s')] \text{ for } s \in S$$

- The limit point of V_k as $k \to \infty$ is V^π under the same conditions that guarantee the existence of V^π

Iterative policy evaluation cont'd

- The iterative policy evaluation in the previous slide is referred to as a full backup iterative method
  - In each time step, the state value function is updated based on the values of states evaluated in the previous time step

- An alternative backup method:
  - Updating the state value function at a state by using the most recent updates of the state value function at other states
  - Pseudo-code in the next slide

Iterative policy evaluation: pseudo-code

- **Input**: a policy π, θ (a small positive number)
- Initialization: $V(s) = 0$ for all $s \in S^+$

- **Repeat**:

  $\Delta \leftarrow 0$

  For each $s \in S$:

  $v \leftarrow V(s)$

  $V(s) \leftarrow \sum_a \pi(s,a) \sum_{s'} P_{s,s'}^a [R_{s,s'}^a + \gamma V(s')]$

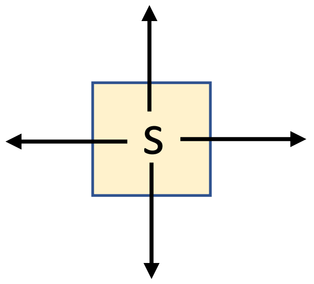
  $\Delta = \max\{\Delta, |v - V(s)|\}$

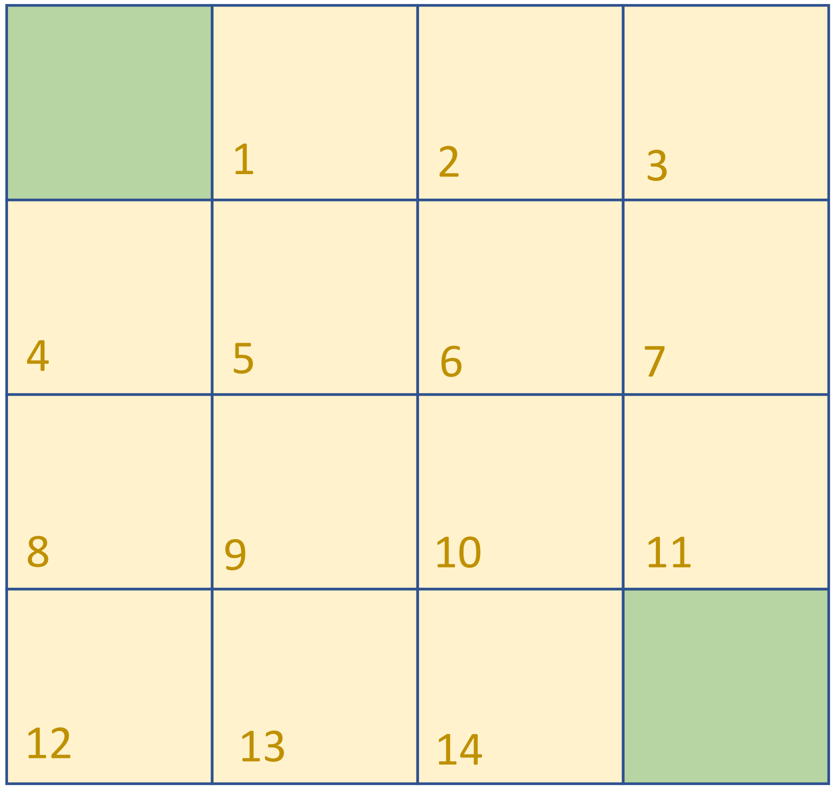
  until $\Delta < \theta$

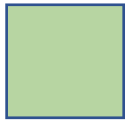
- **Output**: V (approximation of V^π)

$S^+ := S \cup S^0$ where S^0 is the set of terminal states

Example: GridWorld







terminal state

- Undiscounted, episodic task
- Action set for state s: $A(s) = \{\text{up}, \text{down}, \text{right}, \text{left}\}$
- Action that would take the agent off the grid leave the state unchanged
- Rewards: for each transition, the reward of value -1

GridWorld: values of equiprobable random policy

$V^\pi(s)$:

0	-14 1	-20 2	-22 3
-14 4	-18 5	-20 6	-20 7
-20 8	-20 9	-18 10	-14 11
-20 12	-20 13	-14 14	0

Q1: What is the value of $Q^\pi(11, \mathrm{down})$?

Q2: What is the value of $Q^\pi(7, \mathrm{down})$?

Value function evaluation

By symmetry:

0	v_1 1	v_2 2	v_3 3
v_1 4	v_5 5	v_6 6	v_2 7
v_2 8	v_6 9	v_5 10	v_1 11
v_3 12	v_2 13	v_1 14	0

Bellman optimality equation:

$$v_1 = \frac{1}{4}(-1 + v_2) + \frac{1}{4}(-1 + v_5) + \frac{1}{4}(-1 + 0) + \frac{1}{4}(-1 + v_1)$$

$$v_2 = \frac{1}{4}(-1 + v_3) + \frac{1}{4}(-1 + v_6) + \frac{1}{4}(-1 + v_1) + \frac{1}{4}(-1 + v_2)$$

$$v_3 = \frac{1}{4}(-1 + v_3) + \frac{1}{4}(-1 + v_2) + \frac{1}{4}(-1 + v_2) + \frac{1}{4}(-1 + v_3)$$

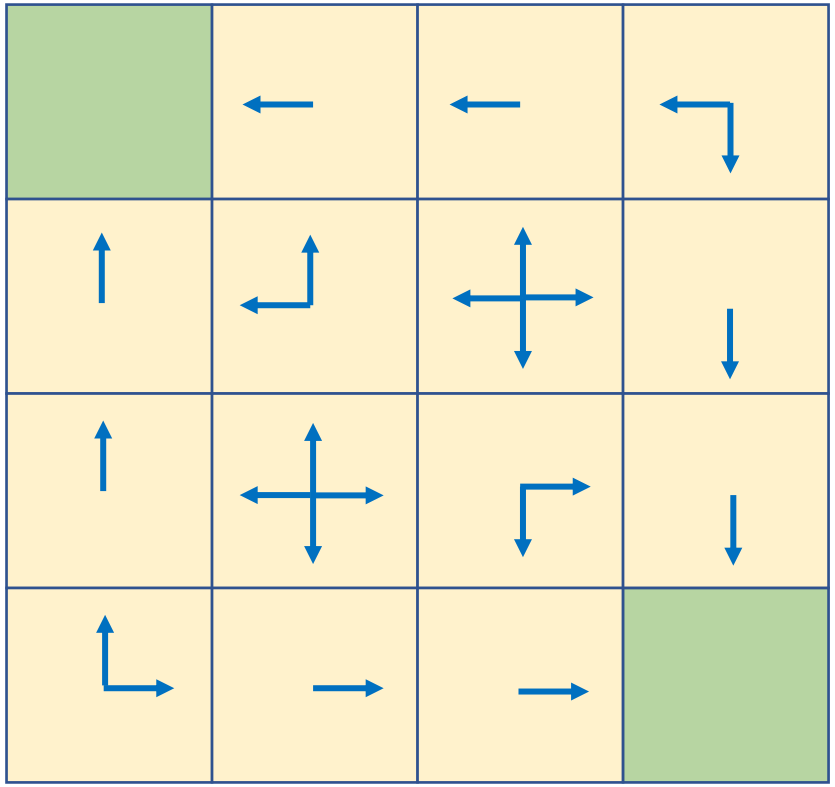
$$v_5 = \frac{1}{4}(-1 + v_6) + \frac{1}{4}(-1 + v_6) + \frac{1}{4}(-1 + v_1) + \frac{1}{4}(-1 + v_1)$$

$$v_6 = \frac{1}{4}(-1 + v_2) + \frac{1}{4}(-1 + v_5) + \frac{1}{4}(-1 + v_5) + \frac{1}{4}(-1 + v_2)$$

$$\Rightarrow \ (v_1, v_2, v_3, v_5, v_6) = (-14, -20, -22, -18, -20)$$

GridWorld: optimal value and policy

0	-1	-2	-3
-1	-2	-3	-2
-2	-3	-2	-1
-3	-2	-1	0



GridWorld: optimal value and policy (cont'd)

0	v_1 1	v_2 2	v_3 3
v_1 4	v_5 5	v_6 6	v_2 7
v_2 8	v_6 9	v_5 10	v_1 11
v_3 12	v_2 13	v_1 14	0

Bellman's optimality equations:

$$\boxed{1} \quad v_1 = \max\{-1 + v_2, -1 + v_5, -1 + 0, -1 + v_1\}$$

$$\boxed{2} \quad v_2 = \max\{-1 + v_3, -1 + v_6, -1 + v_1, -1 + v_2\}$$

$$\boxed{3} \quad v_3 = \max\{-1 + v_3, -1 + v_2, -1 + v_2, -1 + v_3\}$$

$$\boxed{5} \quad v_5 = \max\{-1 + v_6, -1 + v_6, -1 + v_1, -1 + v_1\}$$

$$\boxed{6} \quad v_6 = \max\{-1 + v_2, -1 + v_5, -1 + v_5, -1 + v_2\}$$

$$\boxed{1} \quad \Rightarrow v_1 = -1$$

$$\& \quad \boxed{2} \quad \Rightarrow v_2 = -2$$

…

Policy improvement

- Th policy improvement theorem: suppose that π and π' are two deterministic policies such that

$$Q^{\pi}\big(s, \pi'(s)\big) \geq V^{\pi}(s) \text{ for all } s \in S \qquad \text{(C)}$$

Then π' must be as good as, or better, than π, i.e.

$$V^{\pi'}(s) \geq V^{\pi}(s) \text{ for all } s \in S \qquad \text{(R)}$$

Moreover, if the inequality in (C) is strict for at least one state, then the inequality in (R) must be strict for at least one state

Proof of the policy improvement theorem

- $V^{\pi}(s) \quad \leq Q^{\pi}(s, \pi'(s))$

$$= \mathbf{E}_{\pi'}[r_{t+1} + \gamma V^{\pi}(s_{t+1}) \mid s_t = s]$$

$$\leq \mathbf{E}_{\pi'}\left[r_{t+1} + \gamma Q^{\pi}\left(s_{t+1}, \pi'(s_{t+1})\right) \mid s_t = s\right]$$

$$= \mathbf{E}_{\pi'}[r_{t+1} + \gamma r_{t+2} + \gamma^2 V^{\pi}(s_{t+2}) \mid s_t = s]$$

$$\vdots$$

$$= \mathbf{E}_{\pi'}[r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots \mid s_t = s]$$

$$\leq V^{\pi'}(s)$$

Greedy policy improvement

- Greedy policy π' for any given policy π is given by

$$\pi'(s) = \text{argmax}_a \, Q^{\pi}(s, a), \text{ for all } s \in S$$

- Greedy policy selects the best action in one step lookahead

- The greedy policy meets the conditions of the policy improvement theorem, thus it is as good as, or better than, the original policy

Greedy policy improvement cont'd

- Note that:

$$\pi'(s) = \mathrm{argmax}_a\, Q^\pi(s,a)$$

$$= \mathrm{argmax}_a \mathbf{E}[r_{t+1} + \gamma V^\pi(s_{t+1}) \mid s_t = s, a_t = a]$$

$$= \mathrm{argmax}_a \sum_{s'} P^a_{s,s'}[R^a_{s,s'} + \gamma V^\pi(s')]$$

- If π' is as good as but not better than π, then $V^\pi = V^{\pi'} \Rightarrow V^{\pi'}$ satisfies the Bellman optimality equation $\Rightarrow \pi$ and π' are optimal

- Policy improvement yields a strictly better policy, except when the original policy is already optimal

Policy iteration

- Policy iteration: an iterative method that alternates between

  (E) policy evaluation

  (I) policy improvement

- $\pi_0 \rightarrow V^{\pi_0} \rightarrow \pi_1 \rightarrow V^{\pi_1} \rightarrow \cdots \rightarrow \pi^* \rightarrow V^*$

- A finite MDP has a finite number of policies

  $\Rightarrow$ the policy iteration method must converge in a finite number of steps

Policy iteration: pseudo-code

Initialization: $V(s) \in \mathbf{R}$, $\pi(s) \in A(s)$ arbitrary for all $s \in S$

Repeat:

$\Delta \leftarrow 0$

For each $s \in S$:

$v \leftarrow V(s)$

$V(s) \leftarrow \sum_{s'} P_{s,s'}^{\pi(s)} [R_{s,s'}^{\pi(s)} + \gamma V(s')]$

$\Delta = \max\{\Delta, |v - V(s)|\}$

until $\Delta < \theta$

policy evaluation

policystable $\leftarrow$ True

For each $s \in S$:

$b \leftarrow \pi(s)$

$\pi(s) = \mathrm{argmax}_a \sum_{s'} P_{s,s'}^{a} [R_{s,s'}^{a} + \gamma V(s')]$

If $b \neq \pi(s)$ **then** policystable $\leftarrow$ False

If policystable, **then** return, **else** go to policy evaluation

policy improvement

Value iteration

- Drawbacks of the policy iteration method:
  - Each iteration requires executing policy evaluation which requires multiple iterations (sweeps through the state space)
  - This can be computationally inefficient

- Value iteration idea: evaluate only one step in the policy evaluation

$$V_{k+1}(s) = \max_a \mathbf{E}[r_{t+1} + \gamma V_k(s_{t+1}) | \, s_t = s, a_t = a]$$

$$= \max_a \sum_{s'} P^a_{s,s'} [R^a_{s,s'} + \gamma V_k(s')]$$

- The sequence $V_0, V_1, \ldots$ converges to V^* under the same conditions that guarantee the existence of V^*

Value iteration: pseudo-code

- Initialization: $V(s) = 0$ for all $s \in S^+$

- **Repeat**:

  $\Delta \leftarrow 0$

  For each $s \in S$:

  $v \leftarrow V(s)$

  $V(s) \leftarrow \max_a \sum_{s'} P^a_{s,s'}[R^a_{s,s'} + \gamma V(s')]$

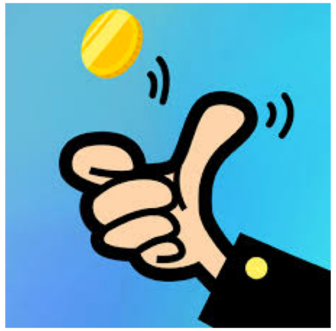
  $\Delta = \max\{\Delta, |v - V(s)|\}$

  until $\Delta < \theta$

- **Output**: deterministic policy π given by

$$\pi(s) = \text{argmax}_a \sum_{s'} P^a_{s,s'}[R^a_{s,s'} + \gamma V(s')] \text{ for all } s \in S$$

Example: Gambler's problem



- A gambler makes bets on the outcomes of a sequence of coin flips
  - The gambler must decide for each coin flip what portion of his capital to stake

- **If** outcome of the coin flip = heads **then**:

  The gambler wins as much money as he has staked on this flip

- **else**

  The gambler loses his stake

- The game ends when the gambler reaches his goal of $100 or loses by running out of money

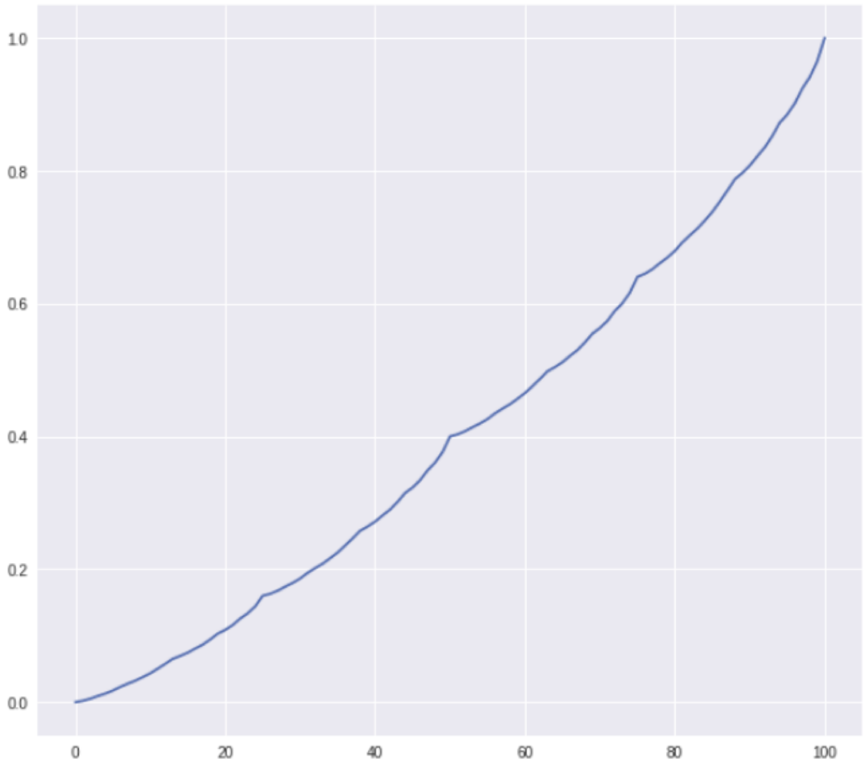
Gambler's problem cont'd

- Formulated as an undiscounted, episodic, finite MDP problem
- State set: $S = \{1,2,\ldots,99\}, S^+ = S \cup \{0,100\}$
- Action sets: $A(s) = \{1,2,\ldots,\min\{s,100-s\}\}$ for $s \in S$
- **Pr**[outcome of coin flip is heads] = p (known parameter)

- Exercise (seminar session):
  - Show value function for different iterations
  - Show the optimal policy

Gambler's problem cont'd

$V^\pi(s)$



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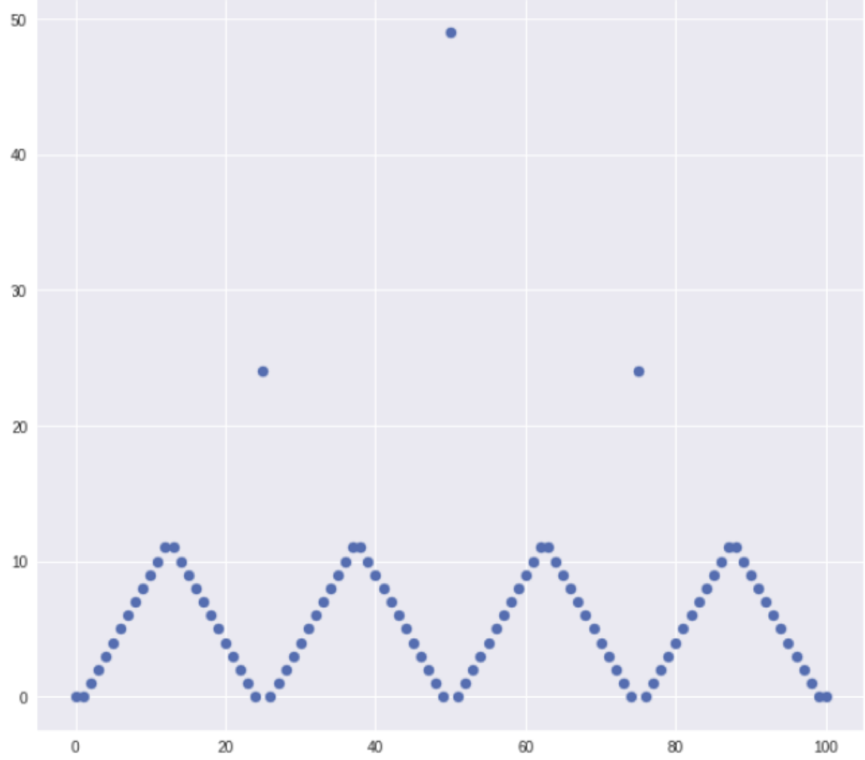
Optimal value function

$\pi(s)$



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Optimal policy

Asynchronous DP

- Drawback of standard DP method:
  - Requires operations over the entire state set of the MDP
  - For large state sets this can be computationally expensive

- Asynchronous DP: backups up values in any order using whatever values of other states are available
  - Convergence guaranteed provided each state is visited with a positive rate

- Asynchronous DP algorithms referred to as distributed DP algorithms
  - Implementation in multiprocessor systems with communication delays between processors

Monte Carlo methods

Monte Carlo methods

- Monte Carlo (MC) methods for reinforcement learning: learning methods for solving the RL problem based on averaging sample returns
  - Estimating value functions and discovering optimal policies
  - Not assuming a perfect model of the environment
  - Based only on experience: sample sequences of states, actions and rewards from online or simulated interactions with an environment
- Defined for episodic tasks to ensure well-defined returns
- Incremental methods in an episode-by-episode sense
  - Estimates of value functions and policies are updated only upon the completion of an episode
  - Different from step-by-step incremental methods (next lecture)

MC policy evaluation

- Suppose our goal is to estimate $V^{\pi}(s)$, the value of a state s for a given policy π, given a set of episodes obtained by following π and passing through state s
- Types of visits to states:
  - A visit to s: each occurrence of state s in an episode
  - First visit to s: each first occurrence of s in an episode
- Types of MC methods:
  - The every-visit MC method: $V^{\pi}(s)$ estimated by the average of the returns following **each visit** to s in a set of episodes
  - The first-visit MC method: $V^{\pi}(s)$ estimated by the average of the returns following **each first visit** to s in an episode of a set of episodes
- The every-visit and first-visit MC methods are similar but have different theoretical properties
  - Q: Can you think of a fundamental difference between the two methods?

The first-visit MC method: pseudo code

- **Initialization**:

  $\pi \leftarrow$ policy to be evaluated
  $V \leftarrow$ an arbitrary state-value function
  $\text{Returns}(s) \leftarrow$ an empty list, for all $s \in S$

- **Repeat**:

  Generate an episode using policy π

  For each distinct s appearing in the episode:

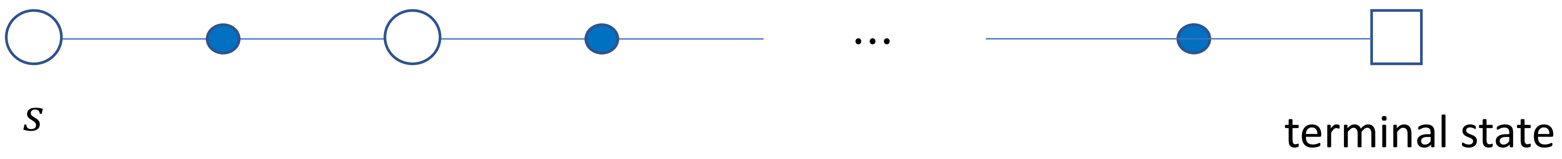
  $R \leftarrow$ return following the first occurrence of s

  Append R to $\text{Returns}(s)$

  $V(s) \leftarrow$ average($\text{Returns}(s)$)

Backup diagram

- Backup diagram for MC estimation of V^{π} shows states sampled in one episode



- Unlike to the DP backup diagram that shows only one-step transitions

Some pros and cons of MC methods

- Pros:
  - Estimating the value of a single state is independent of the number of states
  - One can generate many sample episodes starting from a given state to estimate the value of this state

- Cons:
  - Incremental updates on an episode-by-episode basis which introduces delays
  - Alternative methods allow for step-by-step incremental updates
    - Time-difference learning methods (discussed in next lecture)

MC estimation of action values

- Given a model, state values are sufficient to find a policy
  - Simply look one-step ahead and choose an action that leads to the best combination of reward and next state

- Without a model, state values are insufficient to find a policy
  - We need to explicitly estimate the value of each action to find a policy

- Primary goal of MC methods: estimate optimal action value function Q^*

- The estimates can be defined analogously to estimating the state value function
  - E.g. the first-visit MC method

MC estimation of action values (cont'd)

- Issue: many state-action pairs may never be visited under a policy
  - E.g. if π is a deterministic policy, only one action-state pair is observed for each distinct state
  - Need to maintain exploration !

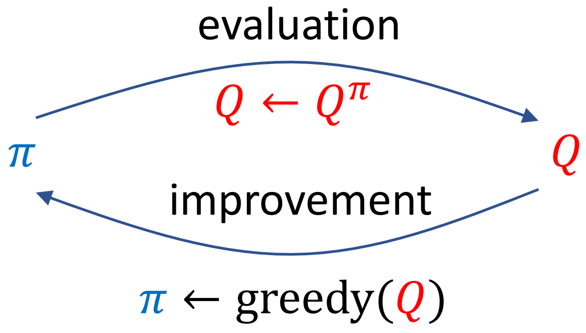
- Two approaches to ensuring continual exploration:
  - Exploring starts: the first step of each episode starts at a state-action pair and every such pair has non-zero probability of being selected at the start
  - Stochastic policies: use policies that ensure a non-zero probability of selecting each action from the set of available actions in each given state

MC control

- MC control: using MC estimation to approximate optimal policies

- Basic idea: use generalized policy iteration
  - Repeatedly alter the value function estimate to more closely approximate the value function of the current policy
  - Repeatedly improve the policy estimate wrt the current value function

- MC version of the standard policy iteration:



$$\pi_0 \longrightarrow Q^{\pi_0} \longrightarrow \pi_1 \longrightarrow Q^{\pi_1} \longrightarrow \quad \cdots \quad \longrightarrow \pi^* \longrightarrow Q^*$$

MC control with exploring starts

- **Initialization**: for all $s \in S$, $a \in A(s)$

  $Q(s, a) \leftarrow$ arbitrary

  $\pi(s) \leftarrow$ arbitrary

  $\mathrm{Returns}(s, a) \leftarrow$ empty list

- **Repeat**:

  Generate an episode using exploring starts and policy π

  For each pair (s, a) appearing in the episode:

  $R \leftarrow$ return following the first occurrence of (s, a)

  Append R to $\mathrm{Returns}(s, a)$

  $Q(s, a) \leftarrow \mathrm{average}\big(\mathrm{Returns}(s, a)\big)$

  For each s in the episode:

  $\pi(s) \leftarrow \mathrm{argmax}_a Q(s, a)$

On-policy vs off-policy control methods

- On-policy methods: attempt to evaluate or improve the policy that is used to make decisions

- Off-policy methods: attempt to evaluate a policy by observing episodes generated by using a different policy

Soft polices

- In on-policy control methods, policy π is usually a soft policy:

$$\pi(s, a) > 0 \text{ for all } s \in S \text{ and } a \in A(s)$$

- A policy π is said to be an ϵ-soft policy if

$$\pi(s, a) \geq \frac{\epsilon}{|A(s)|} \text{ for all } s \in S \text{ and } a \in A(s)$$

- ϵ-greedy policy: chose an action with maximum estimated action value with probability $1 - \epsilon$, and otherwise select an action at random
  - Any non-greedy action is selected with probability $\geq \epsilon/|A(s)|$
  - Greedy action is selected with probability $1 - \epsilon + \epsilon/|A(s)|$

An ϵ-soft on-policy MC control algorithm

- **Initialization**: for all $s \in S, a \in A(s)$
  $Q(s,a) \leftarrow$ arbitrary
  $\pi(s) \leftarrow$ arbitrary ϵ-soft policy
  $\text{Returns}(s,a) \leftarrow$ empty list
- **Repeat**:
  Generate an episode using policy π
  For each pair (s,a) appearing in the episode:
    $R \leftarrow$ return following the first occurrence of (s,a)
    Append R to $\text{Returns}(s,a)$
    $Q(s,a) \leftarrow \text{average}\big(\text{Returns}(s,a)\big)$

  For each s in the episode:
    $a^* \leftarrow \text{argmax}_a\, Q(s,a)$
    For each $a \in A(s)$:

$$\pi(s,a) \leftarrow \begin{cases} 1 - \epsilon + \epsilon\frac{1}{|A(s)|} & \text{if } a = a^* \\ \epsilon\frac{1}{|A(s)|} & \text{if } a \neq a^* \end{cases}$$

Improvement guarantees

- Suppose π is any ϵ-soft policy and π' is the ϵ-greedy policy wrt Q^{π}
- Fact: $Q^{\pi}\big(s, \pi'(s)\big) \geq V^{\pi}(s)$ for all $s \in S$
- By the policy improvement theorem: $V^{\pi'}(s) \geq V^{\pi}(s)$ for all $s \in S$

Off-policy MC control

- The policy used to generate behavior (behavior policy) may be different to the policy used for evaluation and improvement (estimation policy)

- An advantage of off-policy MC methods: the estimation policy may be deterministic (e.g. greedy) while the behavior policy can continue to sample all auctions

- The key point: evaluating one policy while following another

Evaluating one policy while following another

- Suppose episodes are generated by following policy π' while we want to estimate V^{π} or Q^{π} for a given policy π such that $\pi \neq \pi'$

- Requirement: $\pi(s, a) > 0 \Rightarrow \pi'(s, a) > 0$ for all $s \in S, a \in A(s)$

- How can we construct an estimate of $V^{\pi}(s)$ using returns, states, and actions observed under policy π'?

  - Use the importance sampling method described next

Importance sampling

- Let X be a random variable with distribution p
- Let q be a "proposal distribution" such that $q(x) > 0$ whenever $p(x) > 0$
- Goal: estimate $\mathbf{E}_{X\sim p}[f(X)]$ by using samples $z_1, z_2, \ldots, z_N$ drawn from q

- $\mathbf{E}_{X\sim p}[f(X)] = \sum_x f(x)p(x) = \sum_z f(z)\, w(z)q(z)$

  where $w(z) := p(z)/q(z)$

- Consider the estimator $\mu_N(f) = \frac{1}{N}\sum_{i=1}^{N} w(z_i)f(z_i)$

- Note: $\lim_{N\to\infty} \mu_N(f) = \mathbf{E}_{X\sim p}[f(X)]$

importance weights

Weighted importance sampling

- Note that $\mathbf{E}_{Z\sim q}[w(Z)] = 1$

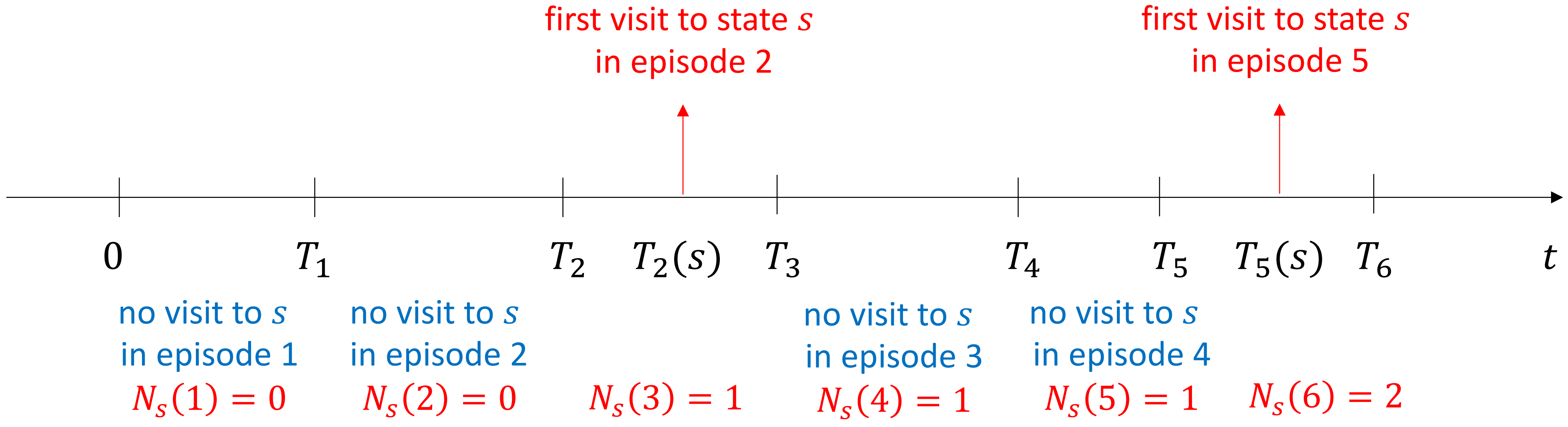
  and $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} w(z_i) = \mathbf{E}_{Z\sim q}[w(Z)] = 1$

- Hence, we can as well define the weighted importance sampling estimator:

$$\tilde{\mu}_N(f) = \frac{\sum_{i=1}^{N} w(z_i) f(z_i)}{\sum_{i=1}^{N} w(z_i)}$$

- Indeed, we have $\lim_{N\to\infty} \tilde{\mu}_N(f) = \mathbf{E}_{X\sim p}[f(X)]$

Application to estimating the value function



- N := the number of episodes
- $T_i(s)$:= first time step at which state s is visited in episode i, otherwise infinity
- T_i := the total number of steps in episode i
- $R_i(s)$:= return from the first visit to state s until the the end of episode i
- $N_s(N)$:= number of episodes with at least one visit of state s

Application to estimating the value function (cont'd)

- Value function estimator: $\hat{V}_N^{\pi'}(s) = \frac{1}{N_s(N)} \sum_{i=1}^{N} R_i(s) \mathbf{1}_{\{T_i(s)<T_i\}}$
- By the law of large numbers,

$$\lim_{N\to\infty} \hat{V}_N^{\pi'}(s) = \frac{\mathbf{E}_{\pi'}[R_1(s)\mathbf{1}_{\{T_1(s)<T_1\}}]}{\mathbf{Pr}_{\pi'}[T_1(s) < T_1]} = \mathbf{E}_{\pi'}[R_1(s) \mid T_1(s) < T_1]$$

- By the definition of the MDP environment:

$$\mathbf{E}_{\pi'}[R_1(s) \mid T_1(s) < T_1] = \frac{\sum_{(\boldsymbol{s},\boldsymbol{a})\in\Pi_s} \left(\prod_{k=t_1(s)}^{t_1-1} \pi'(s_k,a_k) P_{s_k,s_{k+1}}^{a_k}\right)\left(\sum_{k=t_1(s)}^{t_1-1} R_{s_k,s_{k+1}}^{a_k}\right)}{\sum_{(\boldsymbol{s},\boldsymbol{a})\in\Pi_s} \left(\prod_{k=t_1(s)}^{t_1-1} \pi'(s_k,a_k) P_{s_k,s_{k+1}}^{a_k}\right)}$$

where Π_s contains $\boldsymbol{s} = (s_{t_1(s)}, \dots, s_{t_1})$ and $\boldsymbol{a} = (a_{t_1(s)}, \dots, a_{t_1-1})$ such that $s_{t_1(s)} = s$

- But we want to compute the expected return from state s under policy π instead !

Weighted importance sampling estimator

- The weighted importance sampling estimator:

$$\hat{V}_N^{\pi}(s) = \frac{\sum_{i=1}^{N} \frac{p_i^{\pi}(s)}{p_i^{\pi'}(s)} R_i(s) \mathbf{1}_{\{T_i(s) < T_i\}}}{\sum_{i=1}^{N} \frac{p_i^{\pi}(s)}{p_i^{\pi'}(s)} \mathbf{1}_{\{T_i(s) < T_i\}}}$$

where

$$p_i^{\varphi}(s) := \prod_{k=T_i(s)}^{T_i - 1} \varphi(s_k, a_k) P_{s_k, s_{k+1}}^{a_k} \text{ for a policy } \varphi$$

Weighted importance sampling estimator (cont'd)

- $\lim_{N\to\infty} \hat{V}_N^{\pi}(s) = \dfrac{\mathbf{E}\left[\frac{p_1^{\pi}(s)}{p_1^{\pi'}(s)} R_1(s) \mid T_1(s)<T_1\right]}{\mathbf{E}\left[\frac{p_1^{\pi}(s)}{p_1^{\pi'}(s)} \mid T_1(s)<T_1\right]}$

- Note that:

$$\mathbf{E}\left[\frac{p_1^{\pi}(s)}{p_1^{\pi'}(s)} R_1(s) \mid T_1(s) < T_1\right]$$

$$= \sum_{(\boldsymbol{s},\boldsymbol{a})\in\Pi_s} \left(\prod_{k=t_1(s)}^{t_1-1} \pi'(s_k, a_k) P_{s_k,s_{k+1}}^{a_k}\right) \frac{\left(\prod_{k=t_1(s)}^{t_1-1} \pi(s_k,a_k) P_{s_k,s_{k+1}}^{a_k}\right)}{\left(\prod_{k=t_1(s)}^{t_1-1} \pi'(s_k,a_k) P_{s_k,s_{k+1}}^{a_k}\right)} \left(\sum_{k=t_1(s)}^{t_1-1} R_{s_k,s_{k+1}}^{a_k}\right)$$

$$= \sum_{(\boldsymbol{s},\boldsymbol{a})\in\Pi_s} \left(\prod_{k=t_1(s)}^{t_1-1} \pi(s_k, a_k) P_{s_k,s_{k+1}}^{a_k}\right) \left(\sum_{k=t_1(s)}^{t_1-1} R_{s_k,s_{k+1}}^{a_k}\right)$$

Weighted importance sampling estimator (cont'd)

- Similarly, we have

$$\mathbf{E}\left[\frac{p_1^{\pi}(s)}{p_1^{\pi'}(s)} \mid T_1(s) < T_1\right] = \sum_{(\boldsymbol{s},\boldsymbol{a})\in\Pi_s}\left(\prod_{k=t_1(s)}^{t_1-1} \pi(s_k, a_k) P_{s_k,s_{k+1}}^{a_k}\right)$$

- Hence

$$\lim_{N\to\infty} \hat{V}_N^{\pi}(s) = \frac{\sum_{(\boldsymbol{s},\boldsymbol{a})\in\Pi_s}\left(\prod_{k=t_1(s)}^{t_1-1} \pi(s_k,a_k) P_{s_k,s_{k+1}}^{a_k}\right)\left(\sum_{k=t_1(s)}^{t_1-1} R_{s_k,s_{k+1}}^{a_k}\right)}{\sum_{(\boldsymbol{s},\boldsymbol{a})\in\Pi_s}\left(\prod_{k=t_1(s)}^{t_1-1} \pi(s_k,a_k) P_{s_k,s_{k+1}}^{a_k}\right)}$$

as desired !

No need to know the environment

- The important point for the estimator $\hat{V}_N^{\pi}(s)$ is that it does not require knowledge of the environment

- The ratio of the weights depends only on the policies:

$$\frac{p_i^{\pi}(s)}{p_i^{\pi'}(s)} = \prod_{k=T_i(s)}^{T_i-1} \frac{\pi(s_k, a_k)}{\pi'(s_k, a_k)}$$

- The random return $R_i(s)$ is observable

Note alternative notation

- In the first edition of the book by Sutton and Barto, the value function estimator is defined equivalently but using different notation, which considers only the episodes in which state s is visited at least once (we only collect samples of the return for such episodes)

- The estimator $\hat{V}_N^{\pi}(s)$ can be written as

$$V(s) = \frac{\sum_{j=1}^{n_s} \frac{p_j(s)}{p'_j(s)} R_j(s)}{\sum_{j=1}^{n_s} \frac{p_j(s)}{p'_j(s)}}$$

  where n_s is the number of episodes for which state s is visited at least once, $p_j(s) = p_{i_s(j)}^{\pi}(s)$ and $p'_j(s) = p_{i_s(j)}^{\pi'}(s)$ where $i_s(j)$ is the index of the episode such that s occurred at least once in an episode for the j-th time

Recursive formulas

- Consider computing a weighted mean Q_n of values $G_1, G_2, \dots, G_n$ using positive valued weights $W_1, W_2, \dots, W_n$:

$$Q_n = \frac{\sum_{i=1}^{n} W_i G_i}{\sum_{i=1}^{n} W_i}$$

- Recursive implementation:

$$Q_{n+1} = Q_n + \frac{W_{n+1}}{C_{n+1}}(G_{n+1} - Q_n)$$

$$C_{n+1} = C_n + W_{n+1}$$

An off-policy MC prediction (policy evaluation)

- **Input**: an arbitrary **target policy** π
- **Initialization**: $Q(s,a)$ arbitrarily, $C(s,a) \leftarrow 0$

Repeat for each episode:

$\pi' \leftarrow$ any **behavior policy** such that $\pi'(s,a) > 0$ whenever $\pi(s,a) > 0$

Generate an episode using π': $s_0, a_0, r_1, s_1, a_1, r_2, \dots, s_{T-1}, a_{T-1}, r_T, s_T$

$G \leftarrow 0$

$W \leftarrow 1$

For each step $t = T-1, T-2, \dots, 0$:

$G \leftarrow \gamma G + r_{t+1}$

$C(s_t, a_t) \leftarrow C(s_t, a_t) + W$

$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \frac{W}{C(s_t,a_t)}(G - Q(s_t, a_t))$

$W \leftarrow W \frac{\pi(s_t,a_t)}{\pi'(s_t,a_t)}$

If $W = 0$ **then** exit the for loop

An off-policy MC control

- **Initialization**: $Q(s,a)$ arbitrarily, $C(s,a) \leftarrow 0$, $\pi(s) \leftarrow \arg\max_a Q(s,a)$

  Repeat for each episode:

  $\pi' \leftarrow$ any **soft policy (behavior policy)**
  Generate an episode using π': $s_0, a_0, r_1, s_1, a_1, r_2, \dots, s_{T-1}, a_{T-1}, r_T, s_T$
  $G \leftarrow 0$

  $W \leftarrow 1$
  For each step $t = T-1, T-2, \dots, 0$:

  $G \leftarrow \gamma G + r_{t+1}$
  $C(s_t, a_t) \leftarrow C(s_t, a_t) + W$
  $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \frac{W}{C(s_t,a_t)}\left(G - Q(s_t, a_t)\right)$

  $\pi(s_t) \leftarrow \arg\max_a Q(s_t, a)$

  If $a_t \neq \pi(s_t)$ **then** exit the for loop

  $W \leftarrow W \frac{1}{\pi'(s_t,a_t)}$

References

- R. S. Sutton and A. G. Barto, Reinforcement Learning: An Introduction, 1st edition, Chapters 4 and 5, 1998
- R. S. Sutton and A. G. Barto, Reinforcement Learning: An Introduction, 2nd edition, Chapter 5, 2018

Seminar exercises

- Iterative policy evaluation: Gridworld problem
- Value iteration: Gambler’s problem
- Monte Carlo prediction: Black Jack example
- Monte Carlo control: Black Jack example