L4 Quantum information and computing (QIC) 2020-21

Lecture 9: Two-qubits gates and quantum circuits

November 9, 2021

Aims of Lecture 9: To introduce the two-qubit gates (CNOT and SWAP) and simple quantum circuits.

Introduction: The two universal two-qubit gates are CNOT and $\sqrt{\mathsf{SWAP}}$.[1] We can express any two-qubit gates as a 4×4 matrix.

CNOT The CNOT gate performs a NOT operation on a Target qubit conditional on another qubit—the Control being in state $|1\rangle$. In matrix form

$$\mathsf{CNOT} \equiv \mathsf{CX} \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

The CNOT is equivalent to CX—if Control in $|1\rangle$ then apply a Pauli-X to the Target. The CNOT is similar to a classical XOR, except that there are two outputs. This makes the gate reversible. Writing out the operation of the CNOT on the basis states we have

$$|00\rangle \to |00\rangle \ , \ |01\rangle \to |01\rangle \ , \ |10\rangle \to |11\rangle \ , \ |11\rangle \to |10\rangle \ .$$

Entanglement is implicit in two-qubit gates. For example, the effect of CNOT on the product state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \ ,$$

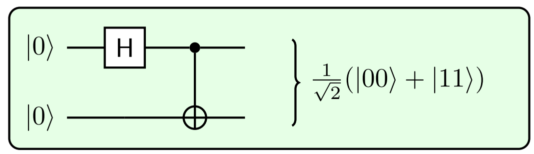
is to produce a Bell state,

$$\begin{aligned} \mathsf{CNOT}|\psi\rangle \quad &= \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \\ &= \quad \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \ . \end{aligned}$$

If we start with $|00\rangle$ we can produce a Bell state by first applying a Hadamard on the control and then a CNOT. Next, we show how to illustrate this using a circuit diagram.

[1] For a more complete list of quantum gates see the Wikipedia page quantum logic gate.

Quantum circuits Below is an example **quantum circuit**[2] that generates a Bell state starting from the product state $|00\rangle$.



To explore more complex circuits see e.g. IBM Quantum Experience and Quirk, Fig. 1.

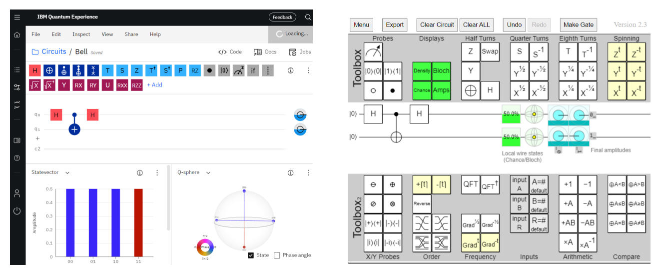


Figure 1: Example quantum circuits using the IBM Quantum Experience and Quirk.

CPHASE In practice, it is easier to realise a controlled-phase CPHASE than a CNOT and then build CNOT using CPHASE and Hadamard. For a phase shift of π on the $|11\rangle$ we realise a controlled-Z with matrix

$$\mathsf{CZ} \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

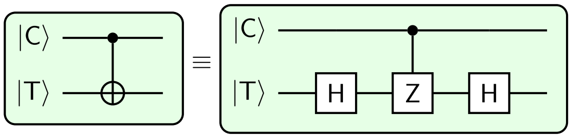
If we apply a Hadamard to the Target before and offer that CZ, see circuit below, then we obtain

$$\mathsf{H_T} \ \mathsf{CZ} \ \mathsf{H_T} = \mathsf{CNOT} \ ,$$

where $\mathsf{H_T} = \sigma_0 \otimes \mathsf{H}$.[3] CZ is depicted as a Z in a square or just a dot, see example circuits on p. 2.

[2] Quantum circuit diagrams are a variation on the **Penrose graphical notation** of matrix (or tensor) products.

[3] Check this by multiplying the matrices.



The phase version of the CNOT can be interpreted as a conditional Ramsey interferometer, where the Target is subject to a Ramsey sequence, e.g. two Hadamards, and if the Control is in state $|1\rangle$ then the qubit-qubit interaction introduces a π phase shift in the $|1\rangle$-arm of the interferometer.

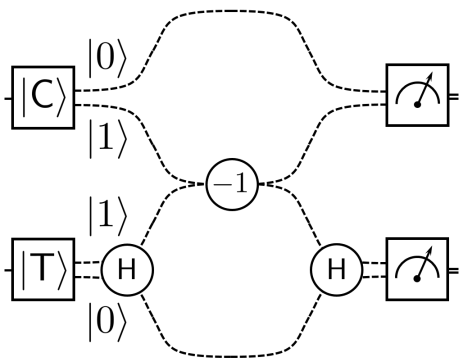


Figure 2: The CNOT as a conditional Ramsey interferometer.

$\sqrt{\text{SWAP}}$ A second example of a universal two-qubit gate is $\sqrt{\text{SWAP}}$ which is a part of a family of SWAP gates. The matrix for SWAP is

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

and

$$\sqrt{\text{SWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\pi/4} & \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\pi/4} & 0 \\ 0 & \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\pi/4} & \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\pi/4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

There is also a variation called iSWAP

$$\text{iSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

In practice, SWAP gates can be generated using spin-spin or dipole-dipole type interactions.[4] A spin-spin interaction has the form[5]

$$\mathcal{H} = \hbar V \hat{\boldsymbol{\sigma}}_{\mathrm{A}} \cdot \hat{\boldsymbol{\sigma}}_{\mathrm{B}} = \hbar V \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

where V is the interaction strength. This gives the unitary

$$\mathsf{U} = \mathrm{e}^{-\mathrm{i}Vt} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}2Vt}\cos 2Vt & -\mathrm{i}\mathrm{e}^{\mathrm{i}2Vt}\sin 2Vt & 0 \\ 0 & -\mathrm{i}\mathrm{e}^{\mathrm{i}2Vt}\sin 2Vt & \mathrm{e}^{-\mathrm{i}2Vt}\cos 2Vt & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ,$$

Chosing $Vt = \pi/4$ we get SWAP. Note that $\sqrt{\text{SWAP}}$ is also an entangling gate. For example,

$$\sqrt{\text{SWAP}}|01\rangle = \tfrac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}\pi/4}|01\rangle + \mathrm{e}^{-\mathrm{i}\pi/4}|10\rangle) ,$$

which is entangled. But if we apply $\sqrt{\text{SWAP}}$ twice to give SWAP we unentangle again,

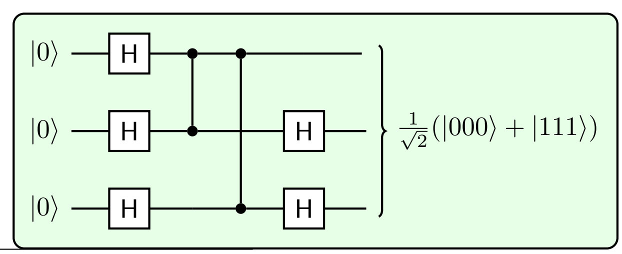
$$\text{SWAP}|01\rangle = \sqrt{\text{SWAP}}\sqrt{\text{SWAP}}|01\rangle = |10\rangle .$$

The system evolves back and forth between a product state and an entangled state.

Summary:

What do you need to be able to do?

1. Write down matrices for CNOT and $\sqrt{\text{SWAP}}$.
2. Understand the principle of CNOT in terms of a conditional Ramsey interferometer.
3. Evaluate the output of simple quantum circuits.
4. Work out the state at each layer of simple circuits like this one.



[4]We shall see interactions of this type when we consider highly-excited Rydberg atom pairs in QIC.17

[5]We can verify the matrix using

$$\hat{\sigma}_x \otimes \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} ,$$

$$\hat{\sigma}_y \otimes \hat{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} ,$$

$$\hat{\sigma}_z \otimes \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$