L4 Quantum information and computing (QIC) 2020-21

Lecture 6: Decoherence and the Optical Bloch equations

October 19, 2021

Aims of Lecture 6: To understand the effects of **decay** and **decoherence** using the Bloch vector picture.

Introduction: Decoherence If a qubit interacts with the external world it may decay or become entangled, which leads to **decoherence** (see QIC.10). A partially decohered qubit is described by a mixed state, and the extra degrees of freedom using the density matrix. In the Bloch sphere picture, pure states are described by points on the sphere (Bloch vector length $(u^2 + v^2 + w^2)^{1/2} = 1$) characterised by the two free parameters (θ, ϕ). Mixed states include points inside the sphere, characterised by three parameters, e.g. (u, v, w) where $(u^2 + v^2 + w^2)^{1/2} < 1$. In the latter case, the dynamics of the Bloch vector (i.e. the time evolution of the qubit) is described by the **optical Bloch equations**.

Optical Bloch equations First we derive the equations of motion for the components of the Bloch vector for a pure state (no decoherence). The time derivative of an expectation value of any operator can be derived the Schrödinger equation[1]

$$\frac{\partial}{\partial t}\langle\sigma\rangle = \frac{\mathrm{i}}{\hbar}\langle[\mathcal{H}_{\mathrm{int}}, \sigma]\rangle \ .$$

Using $\mathcal{H}_{\mathrm{int}} = (\hbar/2)(\Delta\sigma_z + \Omega\cos\phi_{\mathrm{L}}\sigma_x + \Omega\sin\phi_{\mathrm{L}}\sigma_y)$, for σ_x we have

$$\frac{\partial}{\partial t}\langle\sigma_x\rangle = \frac{\mathrm{i}}{2}\langle[\Delta\sigma_z + \Omega\cos\phi_{\mathrm{L}}\sigma_x + \Omega\sin\phi_{\mathrm{L}}\sigma_y, \sigma_x]\rangle \ .$$

Next, we use the commutation relations

$$[\sigma_x, \sigma_y] = 2\mathrm{i}\sigma_z \ , \ [\sigma_z, \sigma_x] = 2\mathrm{i}\sigma_y \ , \ [\sigma_y, \sigma_z] = 2\mathrm{i}\sigma_x \ ,$$

and $[\sigma_j, \sigma_j] = 0$ which gives

$$\begin{aligned}
\frac{\partial}{\partial t}\langle\sigma_x\rangle &= \frac{\mathrm{i}}{2}\left\{\Delta\langle[\sigma_z, \sigma_x]\rangle + \Omega\sin\phi_{\mathrm{L}}\langle[\sigma_y, \sigma_x]\rangle\right\} \ , \\
&= \frac{\mathrm{i}}{2}\left[\Delta\langle 2\mathrm{i}\sigma_y\rangle + \Omega\sin\phi_{\mathrm{L}}\langle -2\mathrm{i}\sigma_z\rangle\right] \ , \\
&= -\Delta\langle\sigma_y\rangle + \Omega\sin\phi_{\mathrm{L}}\langle\sigma_z\rangle \ .
\end{aligned}$$

Using $(u, v, w) = (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle)$ this becomes

$$\frac{\partial}{\partial t}u = -\Delta v + \Omega\sin\phi_{\mathrm{L}} w \ .$$

Repeating this derivation for the y and z components, e.g.

$$\begin{aligned}
\frac{\partial}{\partial t}\langle\sigma_z\rangle &= \frac{\mathrm{i}}{2}\left\{\Omega\cos\phi_{\mathrm{L}}\langle[\sigma_x, \sigma_z]\rangle + \Omega\sin\phi_{\mathrm{L}}\langle[\sigma_y, \sigma_z]\rangle\right\} \ , \\
&= \frac{\mathrm{i}}{2}\left[\Omega\cos\phi_{\mathrm{L}}\langle -2\mathrm{i}\sigma_y\rangle + \Omega\sin\phi_{\mathrm{L}}\langle 2\mathrm{i}\sigma_x\rangle\right] \ , \\
&= \Omega\cos\phi_{\mathrm{L}}\langle\sigma_y\rangle - \Omega\sin\phi_{\mathrm{L}}\langle\sigma_x\rangle \ ,
\end{aligned}$$

we obtain the coupled equations

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & -\Delta & \Omega\sin\phi_{\mathrm{L}} \\ \Delta & 0 & -\Omega\cos\phi_{\mathrm{L}} \\ -\Omega\sin\phi_{\mathrm{L}} & \Omega\cos\phi_{\mathrm{L}} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} \ .$$

Classically, we can think of this as a torque $(\Omega\cos\phi_{\mathrm{L}}, \Omega\sin\phi_{\mathrm{L}}, \Delta)$ acted on the Bloch vector (u, v, w).[2]

Decoherence may disrupt both the relative phase of the $|0\rangle$ and $|1\rangle$ components (encoded in u and v) and their amplitude (encoded in w). We assume that the population of the 'excited' state ($|1\rangle$) decays at a rate $\Gamma = 1/T_1$. This is known as the T_1 time. This decay plus other noise sources cause the relative phase (and hence u and v) to decay at a rate $1/T_2$. This is known as the T_2 time. For

[1] From

$$\frac{\partial}{\partial t}\langle\psi|\sigma|\psi\rangle = \frac{\partial\langle\psi|}{\partial t}\sigma|\psi\rangle + \langle\psi|\sigma\frac{\partial|\psi\rangle}{\partial t} \ ,$$

plus the Schrödinger equation and its complex conjugate (using the Hermitian property of the Hamiltonian),

$$\frac{\partial}{\partial t}|\psi\rangle = -\frac{\mathrm{i}}{\hbar}\mathcal{H}_{\mathrm{int}}|\psi\rangle \quad \text{and} \quad \frac{\partial}{\partial t}\langle\psi| = \frac{\mathrm{i}}{\hbar}\langle\psi|\mathcal{H}_{\mathrm{int}} \ ,$$

we obtain

$$\frac{\partial}{\partial t}\langle\psi|\sigma|\psi\rangle = \frac{\mathrm{i}}{\hbar}(\langle\psi|\mathcal{H}_{\mathrm{int}}\sigma|\psi\rangle - \langle\psi|\sigma\mathcal{H}_{\mathrm{int}}|\psi\rangle) \ .$$

[2] As noted earlier, historically atomic physics texts used a difference definition of u, v and w. w is sometimes defined with the opposite sign which flips the Bloch sphere such that $|1\rangle$ is at the North Pole. This also changes the sign of v such that Δ replaced by $-\Delta$, and changes the form of any damping terms.

$\phi_{\rm L} = 0$ the equations of motion become[3]

$$\begin{aligned} \dot{u} &= -\frac{u}{T_2} - \Delta v \ , \\ \dot{v} &= \Delta u - \frac{v}{T_2} - \Omega w \ , \\ \dot{w} &= \Omega v - \frac{1}{T_1}(w-1) \ . \end{aligned}$$

These equations are known as the **optical Bloch equations**. An analytic solution can be obtained for the case $T_1 \to \infty$, i.e. the excited state does not decay over the time scale of interest, and no driving, $\Omega = 0$. In this case $\dot{w} = 0$ and we only have two equations. Adding $\dot{u}$ and $\mathrm{i}\dot{v}$ gives,

$$\dot{u} + \mathrm{i}\dot{v} = -\left(\frac{1}{T_2} - \mathrm{i}\Delta\right)(u + \mathrm{i}v) \ .$$

We can integrate this to find that

$$u(t) + \mathrm{i}v(t) = \mathrm{e}^{-(1/T_2 - \mathrm{i}\Delta)t}\left[u(0) + \mathrm{i}v(0)\right] \ .$$

If we begin with the Bloch vector along the x axis, $(u, v, w) = (1, 0, 0)$, $v(0) = 0$ and then equating real and imaging parts we obtain $u(t) = \mathrm{e}^{-t/T_2}\cos\Delta t$ and $v(t) = \mathrm{e}^{-t/T_2}\sin\Delta t$, i.e. the Bloch vector spirals in towards the centre of the Bloch sphere, see Fig. 1(left). In the density matrix,

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{e}^{-t/T_2}\mathrm{e}^{-\mathrm{i}\Delta t} \\ \mathrm{e}^{-t/T_2}\mathrm{e}^{\mathrm{i}\Delta t} & 1 \end{pmatrix} \ ,$$

we see that it is the off-diagonal terms—the coherences—that decay due to decoherence. This type of behaviour can be observed in solid-state qubit systems where often $T_2 \gg T_1$. In the next lecture, we shall look at the Ramsey technique which gives us direct access to both the T_1 and T_2 times, i.e. the rate of decay of the coherences and the populations, respectively.

For isolated qubits where the only decay mechanism is radiative (i.e. spontaneous emission) we can write $1/T_1 = \Gamma$ and $1/T_2 = \Gamma/2$, where Γ is the spontaneous decay rate of the excited state ($|1\rangle$). In this case the optical Bloch equations become

$$\begin{aligned} \dot{u} &= -(\Gamma/2)u - \Delta v \ , \\ \dot{v} &= \Delta u - (\Gamma/2)v - \Omega w \ , \\ \dot{w} &= \Omega v - \Gamma(w-1) \ . \end{aligned}$$

The Bloch sphere dynamics for this case with $|\Delta| > \Gamma$ is shown in Fig. 1(right). Worth noting for later is that the

[3] The population decay, $\dot{\rho}_{bb} = -(1/T_1)\rho_{bb}$ and $\dot{\rho}_{aa} = (1/T_1)\rho_{bb}$, lead to a rate of change of the population difference, $\dot{w} = \dot{\rho}_{aa} - \dot{\rho}_{bb} = 2(1/T_1)\rho_{bb} = -(1/T_1)(w-1)$, using $\rho_{bb} = \frac{1}{2}(1-w)$.

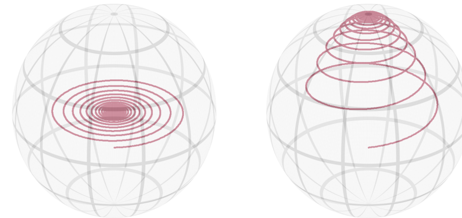

Figure 1: Bloch sphere dynamics illustrating decoherence (left) and dechorence plus decay (right): After creating a superposition using a $\pi/2$-pulse the coupling is turned off ($\Omega = 0$). Left: We show what happens if only the coherences decay, i.e. when $T_2 \gg T_1$. Right: In this case we set $T_2 = T_1/2$ which is characteristic of a system where state $|1\rangle$ decays at rate $\Gamma = 1/T_1$ and there are no other sources of decoherence.

steady-state solutions of the optical Bloch equations are particularly useful in the description of the initialisation of atomic qubits, e.g. the mathematical description of laser cooling of atoms and ions, and laser trapping of atoms. The steady-state solution is found by setting

$$\begin{aligned} \dot{u} &= -(\Gamma/2)u - \Delta v = 0 \ , \\ \dot{v} &= \Delta u - (\Gamma/2)v - \Omega w = 0 \ , \\ \dot{w} &= \Omega v - \Gamma(w-1) = 0 \ . \end{aligned}$$

we obtain the steady-state probability to be in state $|1\rangle$[4]

$$P_{|1\rangle} = \frac{1}{2}(1-w) = \frac{\Omega^2/4}{\Delta^2 + \Gamma^2/4 + \Omega^2/2} \ .$$

We shall use this result in Part II.

Summary: What do you need to be able to do?

1. Identify the terms in the optical Bloch equations.
2. Explain, the distinction between T_1 and T_2.
3. Sketch and describe the dynamics of the Bloch vector in the case of decay or decoherence.
4. Explain whether the steady-state predicted by the optical Bloch equations is Markovian or non-Markovian.

[4] From the first and third equation, we have

$$u = -\frac{2\Delta}{\Gamma}v \text{ and } v = \frac{\Gamma}{\Omega}(w-1) \ .$$

Substituting into the second equation gives

$$-\frac{2\Delta^2}{\Gamma}\frac{\Gamma}{\Omega}(w-1) - \frac{\Gamma^2}{2\Omega}(w-1) - \Omega w = 0 \ .$$

Rearranging we find that

$$w = \frac{4\Delta^2 + \Gamma^2}{2\Omega^2 + 4\Delta^2 + \Gamma^2} = \frac{1}{1+s} \ ,$$

where $s = \Omega^2/2/(\Delta^2 + \Gamma^2/4)$, is known as the saturation parameter.