L4 Quantum information and computing (QIC) 2020-21

Lecture 8: Two-qubits: Bell states and entanglement

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Aims of Lecture 8: To introduce a second qubit and the concept of entanglement.

Introduction: Two qubits: So far we have focused on the dynamics of single qubits. Adding a second qubit creates the possibility of **entanglement**—a concept that does not exist in classical physics. The Exponential scaling—Recall from Lecture 1 that for N qubit we have 2^N states but only $2N$ product state, so the extra $2^N - 2N$ states (and hence exponential scaling of the state space) arise from entanglement. Recall that a qubit computer with N qubits works with more states than N classical bits. Entangled means that we can separate the state vector of one qubit from the other. Mathematical, we cannot factorise the two-qubit state vector into a product of the state vector of qubit A and qubit B.

Bell states: The most important examples of two-qubit entangled states are the **Bell states**[1]:

$$
\begin{aligned}
|\Phi^{\pm}\rangle &= \tfrac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \ , \\
|\Psi^{\pm}\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \ .
\end{aligned}
$$

The Bell states are maximally entangled, we cannot factorise them into parts, at all! Looking at the Bell state vectors, we can see that $\mathsf{ENTANGLEMENT} = \mathsf{CORRELATION}$: For example, for the Bell state

$$
|\Phi^+\rangle_{\mathrm{AB}} = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \ ,
$$

if the first qubit, qubit A shown in red is detected in state $|0\rangle$, then the second, qubit B shown in blue will also be in found in state $|0\rangle$, or if A is detected in state $|1\rangle$ then B must be in $|1\rangle$. But quantum correlations offer more than classical correlations.[2] In the quantum world we do not just observe reality we create it—outcomes depends on what we choose to measure.[3]

Quantum entanglement is weird (you may be tempted to ask, 'but how can it like that?'). The Bell states, as the canonical example of entanglement, display some bizarre properties. For the state $|\Phi^+\rangle_{\mathrm{AB}}$, if we cover up terms associated with either A or B then it appears that the other qubit is in the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. But wait. Look what happens when we calculate the probability that either qubit A or B is in the state $|+\rangle$. To calculate the probability that A is in state $|+\rangle$ we construct an operator that projects A onto $|+\rangle$ and does nothing to B,[4]

$$
\begin{aligned}
\mathsf{P} &= |+\rangle\langle+| \otimes \hat{\sigma}_0 \ , = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} .
\end{aligned}
$$

The probability that we detect A in $|+\rangle$ regardless of B is given by the expectation value of this operator

$$
\begin{aligned}
\langle\Phi^+|\mathsf{P}|\Phi^+\rangle &= \tfrac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \\
&= \tfrac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \tfrac{1}{2} \ .
\end{aligned}
$$

So it turns out that qubit A only has a 50% probability to be detected in state $|+\rangle$. Similarly for B. So if the Bloch vector of A is not pointing in the direction $|+\rangle$,

[1] Named after **John Bell**.

[2] If someone gives us each one halve of a broken pencil and I have the blunt end then I can guess you have the sharp end.

[3] Bell used the example of Dr. Bertlmann's socks to illustrate the difference between classical and quantum correlations. Quantum correlations are exploited in **quantum cryptography**, allowing quantum-level secure communications, see Nielsen and Chang Sec. 12.6 in particular Box 12.7.

[4] For a single qubit, a projection operator finds the component of the Bloch vector along a particular direction. Note also for mixed states we have to use density matrices rather than state vectors. In this case, the expectation values are found using a trace,

$$
\langle\mathsf{P}\rangle = \mathrm{Tr}(\rho\mathsf{P}) \ .
$$

what direction does it point in? We can try to find out by using an operator that projects onto an arbitrary angle (θ, ϕ), i.e.

$$
\begin{aligned}
\mathsf{P}_{|\theta,\phi\rangle} &= |\theta,\phi\rangle\langle\theta,\phi| \otimes \sigma_0 \\
&= \tfrac{1}{2}\begin{pmatrix} 1+\cos\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta & 1-\cos\theta \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \tfrac{1}{2}\begin{pmatrix} 1+\cos\theta & 0 & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta & 0 \\ 0 & 1+\cos\theta & 0 & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta & 0 & 1-\cos\theta & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi}\sin\theta & 0 & 1-\cos\theta \end{pmatrix} .
\end{aligned}
$$

The expectation value is

$$
\langle\Phi^+|\mathsf{P}_{|\theta,\phi\rangle}|\Phi^+\rangle = \tfrac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1+\cos\theta \\ \mathrm{e}^{-\mathrm{i}\phi}\sin\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta \\ 1-\cos\theta \end{pmatrix} = \tfrac{1}{2} ,
$$

as before. But this result is strange because it says that the Bloch vector does not point in any particular direction. It is equally probable to detect it in any direction. Consequently, we can no longer draw a meaningful Bloch sphere picture for either A or B. Although A and B might be separate in real space we cannot separate their quantum states. This strange directionless aspect of the Bloch vector is related to rotational invariance. For example, the Bell state in the Z basis is equal to the Bell state in the X basis,

$$
|\Phi^+\rangle_{\mathrm{AB}} = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \tfrac{1}{\sqrt{2}}(|{++}\rangle + |{--}\rangle) .
$$

We shall look at this in more detail in Workshop 3.

Two-qubit operators: Any two-qubit operator can be written as a 4×4 matrix. As for states, we can have operators that factorize into products of operator on individual qubits, and 'entangling' operators that cannot be factorised.

First, we consider applying the same single-qubit operation to both qubits. Note that rotations (unitaries) and Hamiltonians behave differently. Unitaries multiple whereas Hamiltonians add. This follows because they are related via an exponential, $\mathsf{U} = \mathrm{e}^{-\mathrm{i}\mathcal{H}t/\hbar}$. The projection operator, considered in the previous section, is also multiplicative.

1. Two qubit rotations: Rotations performed independently on each qubit are given by the tensor product of the single-qubit rotation operators. For example, rotations R_{A} and R_{B} on the first and second qubits is described by the tensor product[5]

$$
\mathsf{R}_{\mathrm{AB}} = \mathsf{R}_{\mathrm{A}} \otimes \mathsf{R}_{\mathrm{B}} .
$$

2. For Hamiltonians $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{B}}$ acting on qubits A and B, the total Hamiltonian is

$$
\mathcal{H}_{\mathrm{AB}} = \mathcal{H}_{\mathrm{A}} \otimes \sigma_0 + \sigma_0 \otimes \mathcal{H}_{\mathrm{B}} .
$$

If a resonant field $(\Delta = 0)$ with Rabi frequency Ω and phase $\phi_{\mathrm{L}} = 0$ acts on both qubits, we have

$$
\begin{aligned}
\mathcal{H}_{\mathrm{AB}} &= \mathcal{H}_{\mathrm{int}} \otimes \hat{\sigma}_0 + \hat{\sigma}_0 \otimes \mathcal{H}_{\mathrm{int}} , \\
&= \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega & \Omega & 0 \\ \Omega & 0 & 0 & \Omega \\ \Omega & 0 & 0 & \Omega \\ 0 & \Omega & \Omega & 0 \end{pmatrix} .
\end{aligned}
$$

It follows that operators acting on three qubits corresponding to 8×8 matrices, and operators acting on N qubits require $2^N \times 2^N$ matrices.

3. Finally, we may also encounter entangling operators that cannot be factorised in terms that act independent on the two-qubit. The example we shall consider in Lecture 17 is the Rydberg blockade interaction Hamiltonian.

Summary: What do you need to be able to do?

1. Explain the principle of entanglement and identify entangled states.
2. Construct two-qubit operators.
3. Analyse the properties of Bell states.
4. Explain, why the Bloch sphere representation does not work for entangled states, and demonstrate examples such the directionless charactor of the Bloch vector and the irrotationality of Bell state.

[5] Recall that the tensor product of 2×2 matrices is given by

$$
\begin{aligned}
\mathsf{P}_1 \otimes \mathsf{P}_2 &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \otimes \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\
&= \begin{pmatrix} a_1a_2 & a_1b_2 & b_1a_2 & b_1b_2 \\ a_1c_2 & a_1d_2 & b_1c_2 & b_1d_2 \\ c_1a_2 & c_1b_2 & d_1a_2 & d_1b_2 \\ c_1c_2 & c_1d_2 & d_1c_2 & d_1d_2 \end{pmatrix} .
\end{aligned}
$$

i.e. we make 4 copies of the 2nd matrix with positions and prefactors given by the components of the first.