L4 Quantum information and computing (QIC) 2022-23

Lecture 2: DiVincenzo and Bloch

January 5, 2023

Background information: In the last lecture, we introduced the idea of a qubit, discussed 5 qubit candidates and gave 4 reasons why quantum computing is important.

Aims of Lecture 2: The central questions to be explored in this lecture are:

1. *How can we build a quantum computer?*
2. *How to build a mathematical framework that allows us to visualise qubit dynamics?*

How to build a quantum computer? How do we decide which of our 5 qubit candidates (atoms, ions, photons, semiconductors or superconductors) is best suited to building a quantum computer? We shall characterise what is needed using the **DiVincenzo Criteria (DVC)**[1]. The DVC provide a set of steps we need to climb to build a QC. In **Part II** the DVC will provide the framework for how to build a QC using atoms and lasers.

The DiVincenzo criteria The DiVincenzo criteria are often used to frame discussions about the advantages and disadvantages of different quantum computing platforms. The five criteria[2] are (re-arranged in this order, first we need some qubits, then we need to make a gate, then read-out, finally, we want to do this with low error rate and be able to scale to many qubits):

1. Initialization (**state preparation**) - typically means the ability to prepare identical qubits (cooling) and address each qubit independently (localisation).
2. A universal set of quantum gates.
3. Measurement (**read out**).
4. Low **decoherence** (errors, see e.g. Nielsen and Chang, Chap. 8).
5. **Scalability** - the ability to scale up to say 100 or 1000 or more qubits.

As initialisation is platform (qubit) specific, we shall begin by considering gates.

Quantum gates A combination of **single- and two-qubit gates** is sufficient to build **a universal quantum computer**, that can in principle perform any computation. Consequently, the course shall mainly focus on how to implement single- and two-qubit gates, however, we might include examples involving 3-qubits.[3] We begin with single-qubit gates and then consider the decoherence before discussing two-qubit gates. But to understand the operation of a single-qubit gate we need to develop the mathematics of qubits, and it will be convenient to introduce a way of visualising single-qubit operations.

Mathematical description of the qubit: As discussed in Lecture 1, a 'qubit' is described using a **state vector** of the form

$$|\psi\rangle = a|0\rangle + b|1\rangle \ ,$$

where a and b are complex coefficients that may be time dependent.[4] The state vector may also be written as a column vector

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \ . \tag{1}$$

The probabilities to detect the qubit in state $|0\rangle$ or $|1\rangle$ are given by the Born rule $|a|^2$ and $|b|^2$, respectively. If $|\psi\rangle$ is a

[1] DiVincenzo, David P. (2000). *The Physical Implementation of Quantum Computation*. Fortschr. der Physik. **48**, 771,arXiv:quant-ph/0002077. See also **Wikipedia** and Nielsen and Chang, Sec. 7.2.

[2] For quantum communication, **two additional criteria**:

1. High-fidelity qubit-to-photonic interconnects (**quantum interfaces**).
2. Ability to connect distance locations (e.g. **quantum repeaters**).

[3] Important 3-qubit gates include the **Toffoli** and **Fredkin**.

[4] In practice, the qubit also has some physical characteristic such a position and physical size which we have not included in the state vector but may play a role in a real computer.

complete description of the state then $|a|^2+|b|^2=1$. This is known as a **pure state**. If the qubit interacts with the environment leading to a loss of 'coherence' (decoherence) it is possible to have a state known as statistical mixture, where $\langle\psi|\psi\rangle = |a|^2+|b|^2 < 1$. We shall talk more about decoherence later. The state vector is specific to a particular basis. For a single qubit, $|0\rangle$ and $|1\rangle$ are referred to as the **computational basis**.[5]

We can use the Schrödinger equation to find out how the qubit state might evolve in time. For example, if states $|0\rangle$ and $|1\rangle$ have energies E_0 and E_1, respectively, the qubit has a resonant frequency, $\omega_0 = |E_1 - E_0|/\hbar$, and the solution to the time-dependent Schrödinger equation, assuming there are no perturbing fields is[6]

$$\begin{aligned}|\psi\rangle &= a\mathrm{e}^{-\mathrm{i}E_0t/\hbar}|0\rangle + b\mathrm{e}^{-\mathrm{i}E_1t/\hbar}|1\rangle\ ,\\ &= \mathrm{e}^{-\mathrm{i}E_0t/\hbar}\left[a|0\rangle + b\mathrm{e}^{-\mathrm{i}\omega_0t}|1\rangle\right]\ .\end{aligned}$$

The exponential term outside the square bracket is called a **global phase**. For a single qubit this disappears when we take the modulus squared of the state vector, however, it may still be important when we consider multiple qubits. The exponential term inside is called the **relative phase**. The concept of global versus relative phase is similar to the case of optics.

The Bloch sphere (see Nielsen and Chang, Sec. 1.2) If we ignore the global phase for now, and write the relative phase as ϕ,[7] then the state vector is

$$|\psi\rangle = \begin{pmatrix} a \\ \mathrm{e}^{\mathrm{i}\phi}b \end{pmatrix}\ ,$$

Due the normalisation condition $\langle\psi|\psi\rangle = |a|^2+|b|^2 = 1$, a and b are related and we can express in terms of a single parameter $a=\cos(\theta/2)$, $b=\sin(\theta/2)$, and the state becomes

$$|\psi\rangle = \begin{bmatrix} \cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2) \end{bmatrix}\ , \tag{2}$$

If we take θ and ϕ as spherical polar coordinates (θ and ϕ are known as the polar and azimuthal angles, respectively) then the state is represented by a point (θ,ϕ) on a sphere known as the **Bloch sphere**, see Fig. 1.[8] The line from the origin to (θ,ϕ) is known as the **Bloch vector**. All possible **pure states** lie on the surface of the sphere.[9]

Positions on the Bloch sphere

1. $\theta=0$ (North pole) corresponds to $|\psi\rangle=|0\rangle$ ('spin-up').

2. $\theta=\pi$ (South pole) corresponds to $|\psi\rangle=|1\rangle$ ('spin-down').

3. $\theta=\pi/2$ (equator) corresponds to the superposition $|\psi\rangle=\frac{1}{\sqrt{2}}(|0\rangle+\mathrm{e}^{\mathrm{i}\phi}|1\rangle)$. $\phi=0$ corresponds to the vector along the x axis

Note that the global phase is not specified by the Bloch sphere.[10] So multiplying the state vector by $\mathrm{e}^{\mathrm{i}\alpha}$ does not change the Bloch vector. For this reason, for multiple qubits other representations such as the **q-sphere** has been introduced.[11]

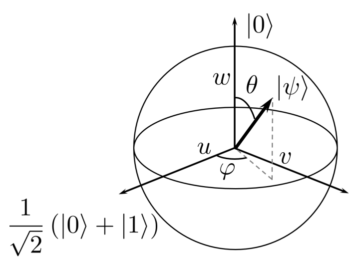

Figure 1: The qubit state $|\psi\rangle$ can be represented by a point on the Bloch sphere with spherical polar coordinate (θ,ϕ), or cartesian coordinates (u,v,w). The line from origin to the point (u,v,w), i.e. $\boldsymbol{b}=(u,v,w)$ is known as the Bloch vector.

Summary:

What do you need to be able to do?

1. List the 5 diVincenzo criteria.

2. Write the state vector in polar coordinates.

3. Sketch the Bloch sphere.

4. Identify different positions on the Bloch sphere with particular state vectors.

[5] Note that the choice of basis is not unique. We could also choose, e.g. $|+\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ and $|-\rangle=\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$.

[6] Both states individually satisfy the Schrödinger equation as follows: $\mathrm{i}\hbar\partial_t|0\rangle = E_0|0\rangle$ and $\mathrm{i}\hbar\partial_t|1\rangle = E_1|1\rangle$.

[7] For the example given above $\phi=-\mathrm{i}\omega_0t$.

[8] The Bloch sphere is analogous to the **Poincaré sphere** in optics. In the Poincaré sphere, the North and South Poles are left- and right-circularly polarized light and the equator is linearly polarized light.

[9] A statistical mixture can be represented by a point inside the sphere. See Nielsen and Chang, Sec. 8.3 for a discussion on how the Bloch sphere contracts due to interactions with the environment.

[10] Note that the Bloch sphere predates quantum computing. **Felix Bloch** won the Nobel Prize in 1952 for nuclear magnetic resonance, i.e. the dynamics of spin-1/2 which is a 2-level system, and hence similar maths to qubits.

[11] See e.g. towards the end of this blog post by JG Garcia.