L4 Quantum information and computing (QIC)

Lecture 5: Rabi oscillations, $\pi/2$, π and 2π pulses, and the Hadamard gate

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Aims of Lecture 5: In this lecture, we shall look at specific cases of the Rabi solution used as single-qubit gates.

Recap: The Rabi solution: In the last lecture, we derived the rotation matrix (the **Rabi solution**) describing the interaction between a qubit with an oscillatory EM field ,

$$\mathsf{R}_{\hat{\boldsymbol{n}}}^{\Theta} = \left[\begin{array}{cc} \cos\frac{\Theta}{2} - \mathrm{i}\frac{\Delta}{\Omega_{\rm eff}}\sin\frac{\Theta}{2} & -\mathrm{i}\frac{\Omega}{\Omega_{\rm eff}}\mathrm{e}^{-\mathrm{i}\phi_{\rm L}}\sin\frac{\Theta}{2} \\ -\mathrm{i}\frac{\Omega}{\Omega_{\rm eff}}\mathrm{e}^{\mathrm{i}\phi_{\rm L}}\sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} + \mathrm{i}\frac{\Delta}{\Omega_{\rm eff}}\sin\frac{\Theta}{2} \end{array} \right] ,$$

where $\Theta = \Omega_{\rm eff}t = (\Omega^2 + \Delta^2)^{1/2}t$. The time evolution of the state vector is given by

$$|\psi(t)\rangle = \mathsf{R}_{\hat{\boldsymbol{n}}}^{\Theta}|\psi(0)\rangle ,$$

where $|\psi(0)\rangle$ is the state vector at $t = 0$. Next, we shall look at specific cases that are particularly applicable to single-qubit gates. There are 4 free parameters, the detuning Δ, the Rabi frequency Ω, the phase $\phi_{\rm L}$ and the time t. However, as the rotation angle is equal to $\Omega_{\rm eff}t$, fixing either one, restricts the other which reduces the degrees of freedom to three.

Driving on resonance $\Delta = 0$: First, we shall look at what happens to the state vector when we apply a resonant field, $\Delta = 0$. The interaction Hamiltonian reduces to

$$\mathcal{H}_{\rm int} = \frac{\hbar}{2}\left[\Omega(\cos\phi_{\rm L}\sigma_x + \sin\phi_{\rm L}\sigma_y)\right] .$$

As there is no σ_z term this will give us a rotation about an axis in the xy (or equatorial) plane, described by the unit vector, $\boldsymbol{n}' = (\cos\phi_{\rm L}, \sin\phi_{\rm L}, 0)$, where $\phi_{\rm L}$ is the field phase. After fixing Δ there are only two remaining parameters, the desired rotation, $\Theta = \Omega t$ and the phase $\phi_{\rm L}$ so we write the rotation matrix as

$$\mathsf{R}_{\phi_{\rm L}}^{\Omega t} = \left[\begin{array}{cc} \cos(\Omega t/2) & -\mathrm{i}\mathrm{e}^{-\mathrm{i}\phi_{\rm L}}\sin(\Omega t/2) \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}\phi_{\rm L}}\sin(\Omega t/2) & \cos(\Omega t/2) \end{array} \right] .$$

Rabi oscillations For a qubit initially in state $|0\rangle$

$$|\psi(0)\rangle = \left(\begin{array}{c} 1 \\ 0 \end{array} \right) ,$$

an interaction of duration t, the Rabi solution gives

$$|\psi(t)\rangle = \left[\begin{array}{c} \cos(\Omega t/2) \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}\phi_{\rm L}}\sin(\Omega t/2) \end{array} \right] .$$

The populations in states $|0\rangle$ and $|1\rangle$, are given by

$$|a(t)|^2 = \cos^2(\Omega t/2) \quad \text{and} \quad |b(t)|^2 = \sin^2(\Omega t/2) .$$

The population oscillates between the two states at the Rabi frequency, Ω. These oscillations are known as **Rabi oscillations**. Remember that the Rabi frequency is proportional to the field amplitude, i.e. the square root of the field intensity (or power).

In the Bloch sphere picture, on resonance Rabi oscillations[1] appear as circuits from pole to pole. The field phase determine the plane of the circuit. With $\phi_{\rm L} = 0$, we obtain a rotation around the x-axis,[2]

$$\mathsf{R}_{x}^{\Omega t} = \left[\begin{array}{cc} \cos(\Omega t/2) & -\mathrm{i}\sin(\Omega t/2) \\ -\mathrm{i}\sin(\Omega t/2) & \cos(\Omega t/2) \end{array} \right] .$$

For $\phi_{\rm L} = \pi/2$, we rotate around the y-axis,

$$\mathsf{R}_{y}^{\Omega t} = \left[\begin{array}{cc} \cos(\Omega t/2) & -\sin(\Omega t/2) \\ \sin(\Omega t/2) & \cos(\Omega t/2) \end{array} \right] ,$$

and for $\phi_{\rm L} = \pi$, we rotation around the $-x$-axis,

$$\mathsf{R}_{-x}^{\Omega t} = \left[\begin{array}{cc} \cos(\Omega t/2) & \mathrm{i}\sin(\Omega t/2) \\ \mathrm{i}\sin(\Omega t/2) & \cos(\Omega t/2) \end{array} \right] ,$$

$\pi/2$, π and 2π pulses

The interactions times given by $\Omega t_{\pi/2} = \pi/2$, $\Omega t_{\pi} = \pi$, and $\Omega t_{2\pi} = 2\pi$ are known $\pi/2$, π and 2π pulses, respectively.

$\pi/2$ pulse: A $\pi/2$ pulse rotates the Bloch vector by $90°$, e.g. from a pole to the equator or from the equator to a pole, i.e. it converts a basis state $|0\rangle$ or $|1\rangle$ to an equal superposition of $|0\rangle$ and $|1\rangle$ or vice versa. We shall consider

[1] Rabi oscillations also occur for non-zero detuning but it this case do not go pole to pole.

[2] Rotations follow a right-hand rule: thump along axis of rotation; fingers direction of rotation.

two examples: First, consider a $\pi/2$-rotation about the $+y$-axis, i.e. we put $\Omega t = \pi/2$ and $\phi_{\rm L} = \pi/2$. This gives the rotation matrix,

$$\mathsf{R}_y^{\pi/2} = \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right) .$$

For an initial state $|0\rangle$, after the rotation we have

$$|\psi(t_{\pi/2})\rangle = \frac{1}{\sqrt{2}}\left(\begin{array}{c} 1 \\ 1 \end{array}\right) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) ,$$

corresponding to a Bloch vector pointing in the $+x$ direction with $u = \langle\hat{\sigma_x}\rangle = 1$, $v = \langle\hat{\sigma_y}\rangle = 0$ and $w = \langle\hat{\sigma_z}\rangle = 0$. Alternatively, applying a $\pi/2$ pulse with phase $\phi_{\rm L} = 0$ to state $|1\rangle$, we find

$$\begin{aligned} |\psi(t_{\pi/2})\rangle &= \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{array}\right)\left(\begin{array}{c} 0 \\ 1 \end{array}\right) = \frac{1}{\sqrt{2}}\left(\begin{array}{c} -\mathrm{i} \\ 1 \end{array}\right) \\ &= \frac{1}{\sqrt{2}}(-\mathrm{i}|0\rangle + |1\rangle) . \end{aligned}$$

It is not so obvious that this points in the $+y$ direction. To check we can either take out a global phase (as in QIC.4) or take the expectation of σ_y,

$$\langle\sigma_y\rangle = \frac{1}{\sqrt{2}}(\mathrm{i}\ \ 1)\left(\begin{array}{cc} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{array}\right)\frac{1}{\sqrt{2}}\left(\begin{array}{c} -\mathrm{i} \\ 1 \end{array}\right) = +1 .$$

The direction of the Bloch vector becomes more apparent if we take out a global phase factor and write it in the form,

$$|1\rangle \quad \rightarrow \quad -\mathrm{i}\frac{1}{\sqrt{2}}(|0\rangle + \mathrm{i}|1\rangle) .$$

Importantly, it is only the relative phase between the $|0\rangle$ and $|1\rangle$ components that determines the position on the Bloch sphere!

Hadamard gate: Example with off-resonant driving

Applying a $\pi/2$ pulse converts a basis state, $|0\rangle$ or $|1\rangle$ into an equal superposition of $|0\rangle$ and $|1\rangle$. Apply a second $\pi/2$ pulse continues the rotation in the same direction and we end up with a $|1\rangle$ or a $|0\rangle$ (a qubit flip). Two identical back-to-back $\pi/2$ pulses are equivalent to a π pulse, i.e. $\mathsf{R}_{\phi_{\rm L}}^{\pi} = \mathsf{R}_{\phi_{\rm L}}^{\pi/2}\mathsf{R}_{\phi_{\rm L}}^{\pi/2}$. It is also useful to have a pulse that take us into a superposition and then back to the same state. This is known as a **Hadamard gate** or **Hadamard transform**,

$$\mathsf{H} = \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right) .$$

The application of two Hadamard transforms leave the state unchanged, $\mathsf{H}^2 = \sigma_0$, the identity matrix. In practice, there are different ways to implement a Hadamard. If we put $\Delta = \Omega$ in the Rabi solution,

$$\mathsf{R} = \left[\begin{array}{cc} \cos\frac{\Omega}{\sqrt{2}}t - \frac{\mathrm{i}}{\sqrt{2}}\sin\frac{\Omega}{\sqrt{2}}t & -\frac{\mathrm{i}}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\phi_{\rm L}}\sin\frac{\Omega}{\sqrt{2}}t \\ -\frac{\mathrm{i}}{\sqrt{2}}\mathrm{e}^{+\mathrm{i}\phi_{\rm L}}\sin\frac{\Omega}{\sqrt{2}}t & \cos\frac{\Omega}{\sqrt{2}}t + \frac{\mathrm{i}}{\sqrt{2}}\sin\frac{\Omega}{\sqrt{2}}t \end{array}\right] ,$$

and set $(\Omega/\sqrt{2})t = \pi/2$ and $\phi_{\rm L} = 0$, we obtain

$$\mathsf{R} = -\mathrm{i}\ \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right) = -\mathrm{i}\mathsf{H} ,$$

which is a Hadamard with an additional global phase. This global phase can be cancelled by applying another field which shifts both states $|0\rangle$ and $|1\rangle$ by the same energy, E, for a time, $t = \pi\hbar/2E$. A sequence of two Hadamard operations is illustrated using the Bloch sphere in Fig. 1. Note how for $\Delta = \Omega$ the rotation axis is at 45°, as expected given that the rotation axis is given by

$$\hat{\boldsymbol{n}} = (1/\Theta)[\Omega\cos\phi_{\rm L}, \Omega\sin\phi_{\rm L}, \Delta] = (1/\sqrt{2})[1, 0, 1] .$$

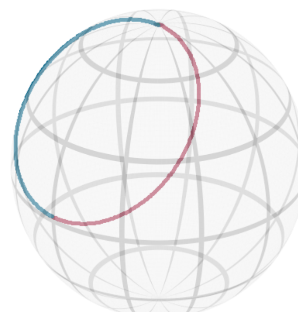

Figure 1: A sequence of two Hadamard gates in the Bloch sphere.

Summary:

What do you need to be able to do?

1. Explain the concept of Rabi oscillations and sketch the populations versus time with correct labelling of the axes.
2. Starting from the Rabi solution derive matrices for $\pi/2$, π, 2π pulses and sketch them on the Bloch sphere.
3. Explain the significance of the phase $\phi_{\rm L}$ in the Rabi solution.
4. Explain how to realise a Hadamard gate.
5. Sketch an Hadamard transform on the Bloch sphere.
6. Suggest five reasons why for real physical qubits the Rabi oscillations are less than perfect.