L4 Quantum information and computing (QIC) 2020-21

Lecture 10: Quantum circuit model of decoherence

November 9, 2021

Aims of Lecture 10: To use quantum circuit models to understand how decoherence is related to an interaction (and entanglement) between a quantum system and the external world (or environment).

Introduction:

In quantum computer we need to avoid decoherence as it leads to errors (DVC4). Decoherence arises because, in practice, it is difficult to isolate our quantum system from the rest of the World.[1] For physical qubits the dominant source of decoherence depend on the specifics of the system. Typically, it is related to random dephasing due to say thermal noise.

Decoherence: circuit model

To model decoherence, we make the simplest possible approximation for the environment—that it is a single qubit. We shall call our system qubit $|\mathsf{S}\rangle$, our environment or External World qubit $|\mathsf{E}\rangle$, and use a two-qubit basis $\{|\mathsf{S}\rangle, |\mathsf{E}\rangle\}$. To measure the coherence of our System qubit we perform a Ramsey sequence consisting of two Hadamard gates. In between the two Hadamards we apply a phase shift,

$$\mathsf{R}_\phi \quad = \quad \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi} \end{pmatrix} = \mathrm{e}^{\mathrm{i}\phi/2}\mathsf{R}_z^{\phi/2} \ .$$

By varying the phase ϕ we obtain Ramsey fringes, and the fringe visibility will tell us about the coherence of the qubit, similar to optical interferometry.

During the Ramsey sequence the System interacts with the External World. We shall assume that the interaction behave like a partial CNOT. If the System is in state $|1\rangle$, then perform a partial NOT on the External World state vector, i.e.

$$\mathsf{M}|10\rangle \quad = \quad \alpha|10\rangle + \beta|11\rangle \ ,$$

where $\alpha^2 + \beta^2 = 1$ and we have used the label M to distinguish the process from usual CNOT. In Fig. 1 we use a measurement symbol to represent M as the interaction but note that for $\beta = 1$ it is a CNOT. The parameter $\beta = \sqrt{1-\alpha^2}$ controls the strength of the interaction between the System and the External World: $\beta = 0$ no interaction, $\beta = 1$ complete 'bit-flip'. The bit-flip might be a spontaneous emission event. For example the $|1\rangle$-arm of the System interferometer decays via spontaneous emission and deposits the emitted photon into the External World. The appearance of the photon in the External World is represented by the change of its state vector from $|0\rangle$ to $|1\rangle$.

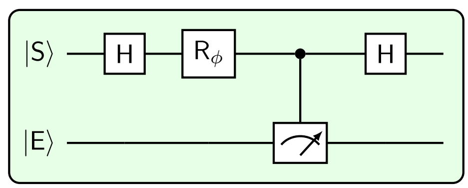

Figure 1: Quantum circuit model of the interaction of a quantum system $|\mathsf{S}\rangle$ with the external world $|\mathsf{E}\rangle$.

In general, the interaction is entangling. For example, if the System is in the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle$ then

$$\mathsf{M}\left[\tfrac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\right] = \tfrac{1}{\sqrt{2}}(|00\rangle + \alpha|10\rangle + \beta|11\rangle) \ ,$$

which unless $\beta = 0$, no interaction, is an entangled state. Entanglement between a quantum system and the environment is the underlying mechanism of decoherence.

Including the phase shift in Fig. 1, the state vector after the interaction with the environment is

$$|\psi\rangle \quad = \quad \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ \alpha\mathrm{e}^{\mathrm{i}\phi} \\ \beta\mathrm{e}^{\mathrm{i}\phi} \end{pmatrix} .$$

[1] Typically, both decoherence due to interactions and dephasing (due to say laser noise) get lumped together in the T_2 term. Sometimes the term dephasing is reserved for phase errors that can be reversed, where decoherence refers to an irreversible loss of information to the environment.

Finally, we apply the second Hadamard to obtain

$$|\psi\rangle \;=\; \tfrac{1}{2}\begin{pmatrix} 1+\alpha \mathrm{e}^{\mathrm{i}\phi} \\ \beta \mathrm{e}^{\mathrm{i}\phi} \\ 1-\alpha \mathrm{e}^{\mathrm{i}\phi} \\ -\beta \mathrm{e}^{\mathrm{i}\phi} \end{pmatrix} .$$

The probability to observe $|0\rangle$ at the output of the interferometer is given by

$$\tfrac{1}{4}\left[(1+\alpha \mathrm{e}^{\mathrm{i}\phi})(1+\alpha \mathrm{e}^{-\mathrm{i}\phi})+\beta^2\right] = \tfrac{1}{2}(1+\alpha\cos\phi) .$$

This is the same result as we obtained in a conventional Ramsey interferometer, except that the fringe visibility is reduced by factor $\alpha \leq 1$. If there is no coupling to the external world, $\alpha = 1$, we observe perfect fringes. However, if the interaction induces a complete bit-flip, $\alpha = 0$, then the Ramsey fringes disappear! The loss of fringe visibility as a function of the coupling α is an example of **decoherence**.

This scenario is a classic example of a **'which-way' experiment** and **complementarity**. If, in principle, one can detect the path ($|0\rangle$ or $|1\rangle$)—the particle-like property—then the interference—the wave-like property—disappears.[2] For $\beta = 0$, the final state of the E tells us whether the System was in the $|0\rangle$ or $|1\rangle$ arm of the interferometer, and so we cannot observe the any wave-like interference fringes. In **Wheeler's delayed-choice experiment**[3], the decision about whether to measure particle or wave is made after the particle is split in two paths. This tells us that whether think of a photon as a particle or wave is determined by the measurement.

Quantum eraser: The idea of a **quantum eraser** is that, may be, we can perform a measurement on the environment which erases the which-path information and at least partially recovers the interference fringes? In the **delayed-choice quantum eraser** we perform this eraser operation after the interference has taken place.[4]

[2] See The Feynman Lectures on Physics, vol. III Chaps. 1 and 3.

[3] This has been confirmed experimentally. Jacques *et al*, Science **315**, 966 (2007).

[4] This raises interesting questions about the timing of events. R. Omnés in his book *The interpretation of quantum mechanics* tells a story about aliens who in the year 1000 created an entangled pair of particles. They keep one part of the pair in a sealed box and release the other part on Earth. Due to quantum interference the Earth particle does not lead to the early death of an ancestor of Napoleon. One thousand years later, the aliens come back to Earth. They announce what they did and say that now they are going to perform an erasure measurement on their half of the entangled pair. Could this erasure measurement change 1000 years of human history? An eminent French professor of quantum physics, himself a descendant of Napoleon, is called before the World's media to explain. Will he cease to exist? Does the history of Western Civilisation need to be rewritten?

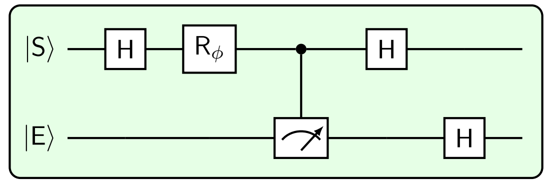

Figure 2: Quantum erasure circuit model. We try to erase the information encoded in the External world by applying Hadamard. What is the effect on the System qubit?

As the 'which-way' information is encoded in whether the External world is in state $|0\rangle$ or $|1\rangle$, we can try to erase it by projecting $|\mathsf{E}\rangle$ into a superposition of $|0\rangle$ and $|1\rangle$, using a Hadamard for example, see Fig. 2.

The probability that we detect the System in $|0\rangle$ **conditional** on projection the External world onto $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ is calculated by finding the expectation value of the projector operator[5]

$$\begin{aligned} \mathsf{P}_{|0+\rangle} &= |0\rangle\langle 0|\otimes|+\rangle\langle +| = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\otimes\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \end{aligned}$$

Using $\langle \mathsf{P}_{|0+\rangle}\rangle = \mathsf{Tr}[\rho \mathsf{P}_{|0+\rangle}]$, we find that the probability to that the System qubit in state $|0\rangle$ conditional on the enviroment being in $|+\rangle$ is

$$\langle \mathsf{P}_{|0+\rangle}\rangle = \tfrac{1}{4}[1+(\alpha+\beta)\cos\phi+\alpha\beta] .$$

By adding the 'eraser' condition, we can now observe fringes again even for the complete 'bit-flip' $\alpha = 0$ (and $\beta = 1$), but only if we correlate our measurements on S with measurements on E. Adding the Hadamard (or any single qubit rotation) cannot disentangle S and E. For a real world qubit there is also the complication that additional interactions will quickly lead to entanglement spreading more widely.

Summary: What do you need to be able to do?

1. Construct a quantum circuit to model the interaction between a quantum system and the external world.
2. Calculate the state vector after each layer of the circuit.
3. Interpret the results in term of which-path information and complementarity.
4. Use the quantum circuit model to analyse the delayed-choice quantum eraser.

[5] Note that the aliens cannot require that their particle is in $|+\rangle$.

Appendix I: More on the quantum eraser The quantum eraser is a good example of how quantum circuits can provide deep insight into the more murky aspect of quantum mechanics.[6] But we must be careful to avoid the 'danger to our soul' that Michael Atiyah talks about in this quote[7]:

> Algebra is the offer made by the devil to the mathematician. The devil says: *I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine* ...the danger to our soul is there, because when you pass over into algabraic calculation, essentially you stp thinking: you stop thinking geometrically, you stop thinking about the meaning.

First, we do the math, and then we think. Do not forget to think! Figure 3 is the simplest quantum circuit that describes the essence of the delayed-choice quantum erasure. If we can draw the circuit we can calculate everything. For an input state $|00\rangle$, the state are the first three steps is

$$|\Psi\rangle_3 = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}\phi/2}|00\rangle + \mathrm{e}^{\mathrm{i}\phi/2}|11\rangle\right) .$$

The final step is either: (i) Without eraser $\mathsf{H}\otimes\sigma_0$, or (ii) With eraser $\mathsf{H}\otimes\mathsf{H}$.

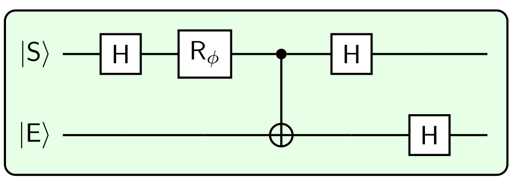

Figure 3: Quantum erasure circuit model. Note we have shown the Hadamard on E delayed but the outcome is the same whether it is before or simultaneous with the Hadamard on S.

(i) Without eraser. Using

$$\mathsf{H}\otimes\sigma_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

$$\begin{aligned}\mathsf{H}\otimes\sigma_0|\Psi\rangle_3 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} \\ 0 \\ 0 \\ \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} \\ \mathrm{e}^{\mathrm{i}\phi/2} \\ \mathrm{e}^{-\mathrm{i}\phi/2} \\ \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} .\end{aligned}$$

[6] See also this artcile Deflating Delayed Choice Quantum Erasure.
[7] Atiyah, Michael F. (2004), Collected works. Vol. 6, OUP.

The probability of measuring any of the four possible outcomes $|00\rangle$, $|01\rangle$, $|00\rangle$, $|01\rangle$ is independent of ϕ—it is NOT possible to observe interference.

(ii) With eraser. Now we look at how adding the Hadamard on E changes the outcome and what is observable.

Using

$$\mathsf{H}\otimes\mathsf{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

$$\begin{aligned}\mathsf{H}\otimes\mathsf{H}|\Psi\rangle_3 &= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} \\ 0 \\ 0 \\ \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\phi/2 \\ -\mathrm{i}\sin\phi/2 \\ -\mathrm{i}\sin\phi/2 \\ \cos\phi/2 \end{pmatrix} .\end{aligned}$$

Now the probability of measuring any one of the four possible outcomes $|00\rangle$, $|01\rangle$, $|00\rangle$, $|01\rangle$ is dependent on ϕ, and hence it is possible, in principle, to observe interference! But, if we only measure S then the interference pattern is still hidden. For example, if we ask what is the probability to measure the system in $|0\rangle$ (or $|1\rangle$) independent of E, e.g. using the projection operator $\hat{P} = |0\rangle\langle 0| \otimes \sigma_0$ we get

$$\begin{aligned}\mathrm{Tr}[\hat{P}\rho] &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} \cos^{\phi}/2 & \ldots & \ldots & \ldots \\ \ldots & \sin^{\phi}/2 & \ldots & \ldots \\ \ldots & \ldots & \sin^{\phi}/2 & \ldots \\ \ldots & \ldots & \ldots & \cos^{\phi}/2 \end{pmatrix} \\ &= \frac{1}{2} .\end{aligned}$$

No interference! In order to see the interference pattern, we have to perform a measurement on E aswell, and then post-select data where E is detected in $|0\rangle$ (or $|1\rangle$). This is because S and E are still entangled and the information we want is hidden in the correlations. We cannot unentangle S and E by performing single-qubit operations on either. Disentangling is hard—the only way is to the reverse the origin two-qubit operation that created entanglement in the first place.

Delayed-choice? What about the delayed-choice question? The fact that we can choose to perform the eraser operation after the interference has taken place. In our circuit model this simply means that we put the Hadamard on E later in time than the Hadamard on S. It is easy to show that this does not make any difference, because the ordering of single-photon rotations on different qubits does not change the output state, namely

$$\mathsf{H}\otimes\mathsf{H} = (\mathsf{H}\otimes\sigma_0)(\sigma_0\otimes\mathsf{H}) = (\sigma_0\otimes\mathsf{H})(\mathsf{H}\otimes\sigma_0) ,$$

i.e., the effect of two Hadamard simultaneously is the same as first one then the other, or the other way around.

Appendix II: Review of Part I Lectures 1–10 Eight key concepts and equations (Lecture numbers are specified using a superscript):

1. DiVincenzo 1: Initialisation requires **qubits**:[(1)]

$$|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix} .$$

$$|\psi\rangle \quad = \quad \mathrm{e}^{-\mathrm{i}E_0 t/\hbar}\left[a|0\rangle + b\mathrm{e}^{-\mathrm{i}\omega_0 t/\hbar}|1\rangle\right] .$$

2. Density matrix[(3)]

$$\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1+w & u-\mathrm{i}v \\ u+\mathrm{i}v & 1-w \end{pmatrix} ,$$

3. DiVincenzo 2: Bloch sphere[(2)] and single-qubit rotations[(4,5)]

$$|\psi\rangle \quad = \quad \begin{bmatrix} \cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2) \end{bmatrix} .$$

$$\mathsf{R} = \mathrm{e}^{-\mathrm{i}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}(\Theta/2)} = \sigma_0\cos\left(\frac{\Theta}{2}\right) - \mathrm{i}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\sin\left(\frac{\Theta}{2}\right) ,$$

4. Light-matter interaction[(4)]

$$\begin{aligned} \mathcal{H}_{\mathrm{int}} &= \frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega\mathrm{e}^{-\mathrm{i}\phi_{\mathrm{L}}} \\ \Omega\mathrm{e}^{\mathrm{i}\phi_{\mathrm{L}}} & -\Delta \end{pmatrix} , \\ &= \frac{\hbar}{2}\left[\Delta\sigma_z + \Omega(\cos\phi_{\mathrm{L}}\sigma_x + \sin\phi_{\mathrm{L}}\sigma_y)\right] . \end{aligned}$$

5. The Rabi solution[(4)]

$$\mathsf{R} \quad = \quad \begin{bmatrix} \cos\frac{\Theta}{2} - \mathrm{i}\frac{\Delta}{\Theta}\sin\frac{\Theta}{2} & -\mathrm{i}\frac{\Omega}{\Theta}\mathrm{e}^{\mathrm{i}\phi_{\mathrm{L}}}\sin\frac{\Theta}{2} \\ -\mathrm{i}\frac{\Omega}{\Theta}\mathrm{e}^{-\mathrm{i}\phi_{\mathrm{L}}}\sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} + \mathrm{i}\frac{\Delta}{\Theta}\sin\frac{\Theta}{2} \end{bmatrix} .$$

   Single-qubit rotations: $\pi/2$, π, 2π pulses. Hadamard.

6. DiVincenzo 4: Decoherence (decay): Steady-state solution of the optical Bloch equations.[(6)]

$$P_{|1\rangle} \quad = \quad \frac{1}{2}(1-w) = \frac{\Omega^2/4}{\Delta^2 + \Gamma^2/4 + \Omega^2/2} .$$

7. Ramsey interferometry[(7)]

$$\mathsf{HR}_\phi\mathsf{H} \quad = \quad \begin{pmatrix} \cos\phi/2 & -\mathrm{i}\sin\phi/2 \\ -\mathrm{i}\sin\phi/2 & \cos\phi/2 \end{pmatrix} .$$

8. Two-qubits[(8)]: gates[(9)] CNOT (conditional Ramsey interferometer)

$$\mathsf{CNOT} \equiv \mathsf{CX} \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} ,$$

   and quantum circuits:[(9,10)]