

L4 Quantum information and computing (QIC)

Lecture 9: Two-qubits gates and quantum circuits

January 6, 2023

Aims of Lecture 9: To introduce two-qubit gates (mainly CNOT, also SWAP) and simple quantum circuits.

Introduction: The two universal two-qubit gates are CNOT and $\sqrt{\text{SWAP}}$.¹ We can express any two-qubit gates as a 4×4 matrix.

CNOT The CNOT gate performs a NOT operation on a Target qubit conditional on another qubit—the Control being in state $|1\rangle$. In matrix form

$$\text{CNOT} \equiv \text{CX} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The CNOT is equivalent to CX—if Control in $|1\rangle$ then apply a Pauli-X to the Target. The CNOT is similar to a classical XOR, except that there are two outputs. This makes the gate reversible. Writing out the operation of the CNOT on the basis states we have

$$|00\rangle \rightarrow |00\rangle, |01\rangle \rightarrow |01\rangle, |10\rangle \rightarrow |11\rangle, |11\rangle \rightarrow |10\rangle.$$

Entanglement is implicit in two-qubit gates. For example, the effect of CNOT on the product state,

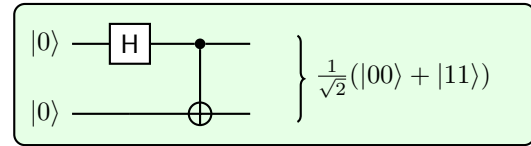
$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle,$$

is to produce a Bell state,

$$\begin{aligned} \text{CNOT}|\psi\rangle &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \\ &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \end{aligned}$$

If we start with $|00\rangle$ we can produce a Bell state by first applying a Hadamard on the control and then a CNOT. Next, we show how to illustrate this using a circuit diagram.

Quantum circuits Below is an example **quantum circuit**² that generates a Bell state starting from the product state $|00\rangle$.



To explore more complex circuits see e.g. [IBM Quantum Experience](#) and [Quirk](#), Fig. 1.

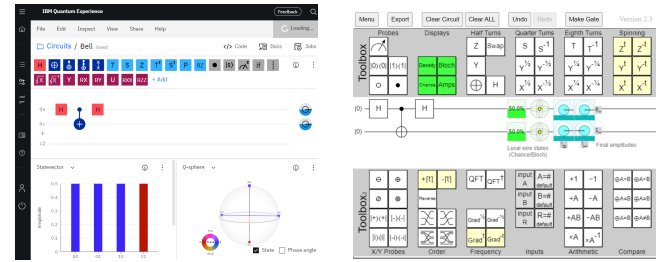


Figure 1: Example quantum circuits using the [IBM Quantum Experience](#) and [Quirk](#).

CPHASE In practice, it is easier to realise a controlled-phase CPHASE than a CNOT and then build CNOT using CPHASE and Hadamard. For a phase shift of π on the $|11\rangle$ we realise a controlled-Z with matrix

$$\text{CZ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

If we apply a Hadamard to the Target before and after that CZ, see circuit below, then we obtain

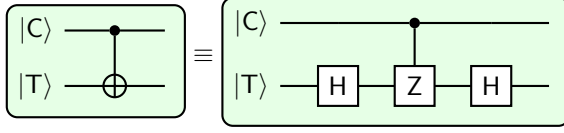
$$\text{H}_T \text{CZ} \text{H}_T = \text{CNOT},$$

where $\text{H}_T = \sigma_0 \otimes \text{H}$.³ CZ is depicted as a Z in a square or just a dot, see example circuits on p. 2.

²Quantum circuit diagrams are a variation on the [Penrose graphical notation](#) of matrix (or tensor) products.

³Check this by multiplying the matrices.

¹For a more complete list of quantum gates see the Wikipedia page [quantum logic gate](#).



The phase version of the CNOT can be interpreted as a conditional Ramsey interferometer, where the Target is subject to a Ramsey sequence, e.g. two Hadamards, and if the Control is in state $|1\rangle$ then the qubit-qubit interaction introduces a π phase shift in the $|1\rangle$ -arm of the interferometer.

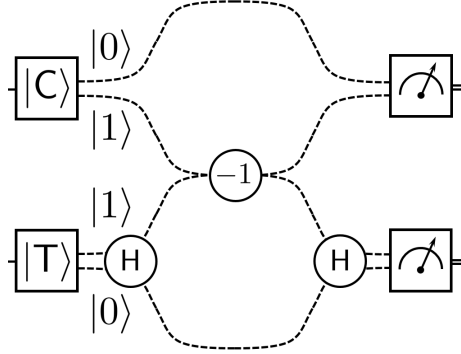


Figure 2: The CNOT as a conditional Ramsey interferometer.

$\sqrt{\text{SWAP}}$ A second example of a universal two-qubit gate is $\sqrt{\text{SWAP}}$ which is a part of a family of SWAP gates. The matrix for SWAP is

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

and

$$\sqrt{\text{SWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}e^{i\pi/4} & \frac{1}{\sqrt{2}}e^{-i\pi/4} & 0 \\ 0 & \frac{1}{\sqrt{2}}e^{-i\pi/4} & \frac{1}{\sqrt{2}}e^{i\pi/4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

There is also a variation called iSWAP

$$\text{iSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In practice, SWAP gates can be generated using spin-spin or dipole-dipole type interactions.⁴ A spin-spin

⁴We shall see interactions of this type when we consider highly-excited Rydberg atom pairs in QIC.17

interaction has the form⁵

$$\mathcal{H} = \hbar V \hat{\sigma}_A \cdot \hat{\sigma}_B = \hbar V \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

where V is the interaction strength. This gives the unitary

$$U = e^{-iVt} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-i2Vt} \cos 2Vt & -ie^{i2Vt} \sin 2Vt & 0 \\ 0 & -ie^{i2Vt} \sin 2Vt & e^{-i2Vt} \cos 2Vt & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Choosing $Vt = \pi/4$ we get SWAP. Note that $\sqrt{\text{SWAP}}$ is also an entangling gate. For example,

$$\sqrt{\text{SWAP}}|01\rangle = \frac{1}{\sqrt{2}}(e^{i\pi/4}|01\rangle + e^{-i\pi/4}|10\rangle),$$

which is entangled. But if we apply $\sqrt{\text{SWAP}}$ twice to give SWAP we unentangle again,

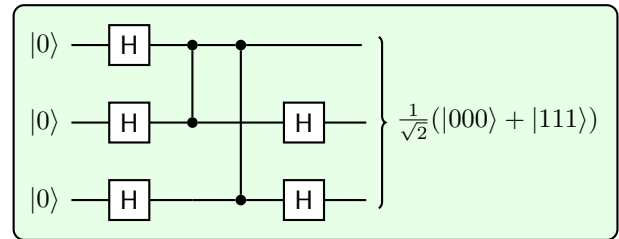
$$\text{SWAP}|01\rangle = \sqrt{\text{SWAP}}\sqrt{\text{SWAP}}|01\rangle = |10\rangle.$$

The system evolves back and forth between a product state and an entangled state.

Summary:

What do you need to be able to do?

1. Write down matrices for CNOT and $\sqrt{\text{SWAP}}$.
2. Understand the principle of CNOT in terms of a conditional Ramsey interferometer.
3. Evaluate the output of simple quantum circuits.
4. Work out the state at each layer of simple circuits like this one.



⁵We can verify the matrix using

$$\begin{aligned} \hat{\sigma}_x \otimes \hat{\sigma}_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ \hat{\sigma}_y \otimes \hat{\sigma}_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \hat{\sigma}_z \otimes \hat{\sigma}_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$