L4 Quantum information and computing (QIC) 2020-21

Lecture 7: Quantum interference and measurement

October 23, 2021

Aims of Lecture 7: To understand how we can use quantum interference to obtain information about the relative phases in the state vector. Later we shall see how quantum interference underpins quantum gate using the example of the CNOT gate, QIC.9.

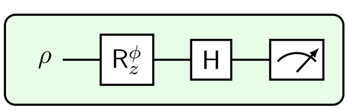
Introduction: Projective measurement In a projective measurement, on a qubit with state vector $|\psi\rangle = a|0\rangle + be^{i\phi}|1\rangle$ with a and b real, we measure either $|\langle 0|\psi\rangle|^2 = a^2$ or $|\langle 1|\psi\rangle|^2 = b^2$. This provides information about the probabilities to be in states $|0\rangle$ or $|1\rangle$, but we learn nothing about their relative phase ϕ. However, if we perform a Hadamard gate before the measurement then

$$\mathsf{H}|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ be^{i\phi} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} a + be^{i\phi} \\ a - be^{i\phi} \end{pmatrix} .$$

and now a projective measurement gives

$$\begin{aligned} |\langle 0|\mathsf{H}|\psi\rangle|^2 &= \left|(1,0)\,\frac{1}{\sqrt{2}}\begin{pmatrix} a + be^{i\phi} \\ a - be^{i\phi} \end{pmatrix}\right|^2 \\ &= \left|\frac{1}{\sqrt{2}}\left(a + be^{i\phi}\right)\right|^2 = \frac{1}{2}\left(a^2 + b^2 + 2ab\cos\phi\right) . \end{aligned}$$

If we vary the relative phase, ϕ,



then we obtain cosine with an offset, $\frac{1}{2}(a^2 + b^2)$, and an amplitude ab, which correspond to the sum of diagonal terms and the modulus of the off-diagonal terms in the density matrix respectively:

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} a^2 & abe^{-i\phi} \\ abe^{i\phi} & b^2 \end{pmatrix} .$$

Hence the Hadamard gate (or similarly a $\pi/2$ pulse) is particularly useful in extracting more information about the state of the qubit. Building on this idea, next we consider a sequence of two Hadamards(or $\pi/2$ pulses). The first one prepares a superposition and the second read-out the relative phase. This is known as a Ramsey interferometry.

Ramsey interferometry[1] The Ramsey interferometer consists of two $\pi/2$-rotations (or Hadamards) interspersed by a rotation about the z axis, see Fig. 1.

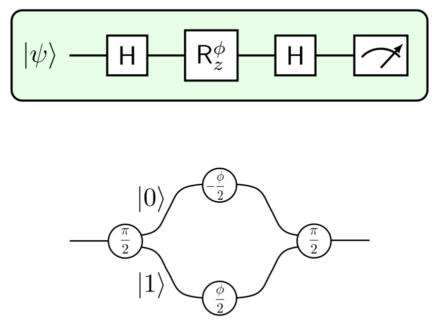


Figure 1: Schematic illustration of a Ramsey interferometer. A sequence of two $\pi/2$ pulses or two Hadamard gates is applied to the qubit. The output is sensitive to the relative phase difference between the two paths at the time of the second pulse.

If we choose a $\pi/2$-rotation about the $-y$-axis (to avoid too many factors of i), i.e. $\Omega t = \pi/2$ and $\phi_L = \pi/2$ in the Rabi solution,[2] then the rotation matrix is

$$\mathsf{R}_{-y}^{\pi/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .$$

[1] Norman Ramsey (Nobel Prize, 1989), inspired by the use of inverse apodization to achieve a higher spatial resolution in imaging, introduced an analogous idea to improve temporal resolution of atomic clocks. His idea was to replace a single long excitation pulse (e.g. a π pulse with duration T) with two short $\pi/2$ pulses separated by a time T. The first pulse prepares a superposition of $|0\rangle$ and $|1\rangle$ which then evolves for a time T. The second pulse rotates the state in a way that depends on the relative phase difference between the qubit and field. The Ramsey technique is analogous to an optical—Mach-Zehnder—interferometer: The first pulse 'splits' the state into two 'paths' in Hilbert space and the second pulse recombines the two paths in a way that is sensitive to their relative phase.

[2] Note that we are effectively assuming that duration of $\pi/2$ pulse is negligible, i.e. $t_\pi/2 \ll T$ and we can treat the $\mathsf{R}_{-y}^{\pi/2}$ rotations at instantaneous .

The rotation around the z-axis is given by

$$\mathsf{R}_z^\phi = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} .$$

This may arise due to a mismatch, $\Delta = \omega - \omega_0$, between the angular frequency of the field and the qubit resonance.[3] In this case, $\phi = \Delta T$, where T is the free evolution time. Note that we assumed that the field was on resonance in deriving the $\pi/2$ rotation. This is a reasonable approximation if $t_{\pi/2} \ll T$ as then the rotation about z during the $\pi/2$ pulse is small during the pulse. If the second $\pi/2$-pulse is identical to the first then complete Ramsey sequence is

$$\begin{aligned} & \mathsf{R}_{-y}^{\pi/2}\mathsf{R}_z^\phi\mathsf{R}_{-y}^{\pi/2} \\ & = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} , \\ & = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} & \mathrm{e}^{-\mathrm{i}\phi/2} \\ -\mathrm{e}^{\mathrm{i}\phi/2} & \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} , \\ & = \tfrac{1}{2}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} - \mathrm{e}^{\mathrm{i}\phi/2} & \mathrm{e}^{-\mathrm{i}\phi/2} + \mathrm{e}^{\mathrm{i}\phi/2} \\ -\mathrm{e}^{-\mathrm{i}\phi/2} - \mathrm{e}^{\mathrm{i}\phi/2} & -\mathrm{e}^{-\mathrm{i}\phi/2} + \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} , \\ & = \begin{pmatrix} -\mathrm{i}\sin\phi/2 & \cos\phi/2 \\ -\cos\phi/2 & \mathrm{i}\sin\phi/2 \end{pmatrix} . \end{aligned}$$

If we begin in state $|0\rangle$, then the probability to be in states $|0\rangle$ and $|1\rangle$ at the output are

$$\begin{aligned} |a|^2 &= \sin^2\frac{\phi}{2} = \frac{1}{2}(1-\cos\phi) \ , \text{ and} \\ |b|^2 &= \cos^2\frac{\phi}{2} = \frac{1}{2}(1+\cos\phi) \ . \end{aligned}$$

The output of the Ramsey interferometer oscillates between states $|0\rangle$ and $|1\rangle$ as a function of the relative phase between the qubit state and the phase of the EM field. These oscillations are known as **Ramsey fringes**. If the phase is an integer multiple of 2π then the second pulse completes the excitation as illustrated in Fig. 2. In this case, the interference between the two paths in Hilbert space is constructive and we complete the excitation to state $|1\rangle$. If phase difference between the two path is an odd multiple of π then the interference is destructive and we end up back in the initial state $|0\rangle$.

If we use Hadamards instead of $\pi/2$ pulses, then the Ramsey interferometer is described by

$$\mathsf{H}\mathsf{R}_z^\phi\mathsf{H} = \begin{pmatrix} \cos\phi/2 & -\mathrm{i}\sin\phi/2 \\ -\mathrm{i}\sin\phi/2 & \cos\phi/2 \end{pmatrix} . \qquad (1)$$

[3] Think of the qubit and the field as two clocks. For non-zero detuning they run at different rates and get out of phase.

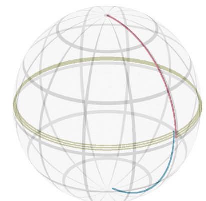


Figure 2: A Ramsey sequence in the Bloch sphere. The first and second $\pi/2$ pulses (red and blue) are assumed to be on resonance. The simulation is based on solving the optical Bloch equaitions (see Lecture 6). We have included a small amount of decay which causes the Bloch vector to spiral back towards to the North Pole during the free evolution time.

The evolution on the Bloch sphere for a Hadamard with $\Delta = \Omega$ (see Lecture 5) is illustrated in Fig. 3. In QIC.9, we shall see that quantum interference and Ramsey interferometry are a key ingrediens in how we realise particular two-qubit gates.

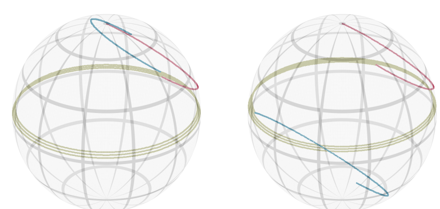


Figure 3: Ramsey sequence using the Hadamard gates for two different values of the free evolution time. Left: We complete close to an integer number of rotations during the free evolution time and end up back close to the start (state $|0\rangle$). Right: We complete close an extra π and end up back close to the South pole (state $|1\rangle$).

Question: Another option in the Ramsey sequence is to deliberately adjust the phase of the second pulse. For example, we could have

$$\mathsf{R}_{-y}^{\frac{\pi}{2}}\mathsf{R}_z^\phi\mathsf{R}_y^{\frac{\pi}{2}} .$$

Derive the matrix for this case. What is the probability to end up in $|0\rangle$ is we start in $|0\rangle$ for $\phi = 0$. How does this compare to the Hadamard case? See Workshop 2.

Summary: What do you need to be able to do?

1. Explain the principle of the Ramsey interferometer.
2. Derive the probabilities to be in the $|0\rangle$ or $|1\rangle$ at the output for different pulses sequences ($\pi/2$ pulses with different phase and Hadamards).

3. Sketch the evolution on the Bloch sphere.