Continuous Time Optimal Control of a Double Integrator

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Problem:

$$\min_{u(\cdot),t_f} \int_0^{t_f} 1 \; dt \tag{1}$$

$$\text{s.t.} \quad \forall t \in [0,t_f]: \quad \dot{x}_1(t) = x_2(t) \tag{2}$$

$$\forall t \in [0,t_f]: \quad \dot{x}_2(t) = u(t) \tag{3}$$

$$\forall t \in [0,t_f]: \quad -u_{\lim} \le u(t) \le u_{\lim} \tag{4}$$

$$x_1(0) = x_{10} \;, \quad x_1(t_f) = x_{1f} \tag{5}$$

$$x_2(0) = x_{20} \;, \quad x_2(t_f) = x_{2f} \tag{6}$$

The Hamiltonian for this problem is

$$H(x(t),u(t),\lambda(t)) = 1 + \lambda_1(t)x_2(t) + \lambda_2(t)u(t) \tag{7}$$

and the optimal controller (using Pontryagin's minimum principle) is given by

$$u^\star(t) = \arg\min_{u\in U} \left[H(x^\star(t),u,\lambda^\star(t))\right] = -\mathrm{sign}(\lambda_2^\star(t)) \; u_{\lim} \;. \tag{8}$$

The optimality conditions of the costates λ_1 and λ_2 are

$$\dot{\lambda}_1^\star(t) = -\frac{\partial H}{\partial x_1} = 0 \;, \qquad \dot{\lambda}_2^\star(t) = -\frac{\partial H}{\partial x_2} = -\lambda_1^\star(t) \tag{9}$$

which lead to the costate equations

$$\lambda_1^\star(t) = \lambda_{10} \;, \qquad \lambda_2^\star(t) = \lambda_{20} - \lambda_{10} t \;. \tag{10}$$

Equations (8) and (10) imply that the optimal solution is a bang-bang controller with at most one switch since $\lambda_2^\star(t)$ is a linear function that can change its sign at most once. This leads to the following candidate set of optimal control sequences

$$U^\star = \Big\{\{u_{\lim}\}, \{-u_{\lim}\}, \{-u_{\lim}, u_{\lim}\}, \{u_{\lim}, -u_{\lim}\}\Big\} \;. \tag{11}$$

If the optimal controller is one of the single element sequences in $U^\star$, then the corresponding command is applied over the full duration $[0,t_f]$. In the other case where the sequence has two element, a switching time t_s needs to be computed that defines the timepoint when to switch from the first command to the second command. In both cases, a constant control signal $u^\star \in \{-u_{\lim}, u_{\lim}\}$ is applied over a certain time period. The state trajectories of the double integrator for such a controller are

$$x_1^\star(t) = x_1^\star(0) + x_2^\star(0)t + \frac{1}{2}u^\star t^2 = x_{10} + x_{20}t + \frac{1}{2}u^\star t^2 \tag{12}$$

$$x_2^\star(t) = x_2^\star(0) + u^\star t = x_{20} + u^\star t \;. \tag{13}$$

Solving Equation (13) for t gives

$$t = \frac{x_2^\star(t) - x_{20}}{u^\star} \tag{14}$$

and inserting this term into Equation (12) leads to

$$x_1^\star(t) = x_{10} + \frac{x_{20}x_2^\star(t) - x_{20}^2}{u^\star} + \frac{1}{2}\left(\frac{x_2^\star(t)^2 - 2x_{20}x_2^\star(t) + x_{20}^2}{u^\star}\right) \tag{15}$$

$$= x_{10} - \frac{x_{20}^2}{2u^\star} + \frac{x_2^\star(t)^2}{2u^\star} \;. \tag{16}$$

By using the boundary conditions $x_1(t_f) = x_{1f}$ and $x_2(t_f) = x_{2f}$, we get the equation

$$x_{1f} = x_{10} - \frac{x_{20}^2}{2u^\star} + \frac{x_{2f}^2}{2u^\star} \;, \tag{17}$$

which is used in the following to select the optimal control sequence from $U^\star$. We first consider the easier case where the final states $x_{1f} = 0$ and $x_{2f} = 0$. Afterwards, a general solution for arbitrary x_{1f} and x_{2f} is provided.

Solution for $x_{1f} = 0$ and $x_{2f} = 0$

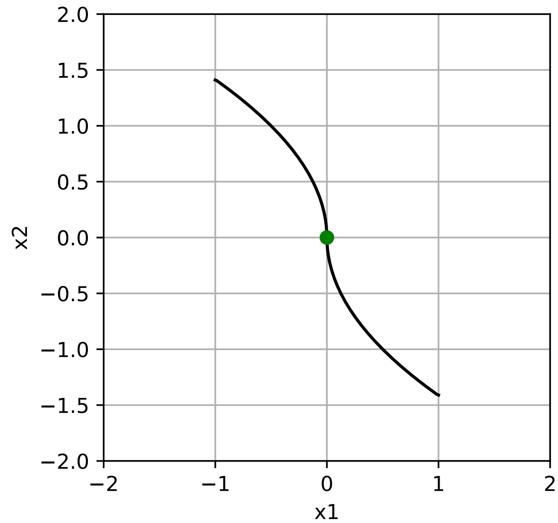

Figure 1: Switching curve $s(t) = 0$

Inserting $x_{1f} = x_{2f} = 0$ into Equation (17) and replacing the initial state with the current state gives

$$x_1(t) = \frac{x_2(t)^2}{2u^\star} \ . \tag{18}$$

This equation describes for each $u^\star$ a parabola in the state space that goes through the target $(0, 0)$. If the system is on one of those parabola at $x_1 > 0 \cap x_2 < 0$ or $x_1 < 0 \cap x_2 > 0$, then the optimal control command is

$$u^\star(t) = -\text{sign}(x_2(t))\ u_{\text{lim}} \tag{19}$$

which controls the system with the maximum/minimum command on the parabola to the target state. If the system is initially not on the parabola, a two element sequence in $U^\star$ is chosen where the first command moves the system onto the parabola and the second command moves the system to the target on the parabola. The type of sequence is chosen based on the switching criteria (derived by inserting Equation (19) into Equation (18)):

$$s(t) = x_1(t) + \text{sign}(x_2(t))\ \frac{x_2(t)^2}{2u_{\text{lim}}} \ . \tag{20}$$

The switching curve $s(t) = 0$ is visualized in Figure 1.
Optimal Controller:

$$u^\star(t) = \begin{cases} -u_{\text{lim}} & \text{if } s(t) > 0 \\ u_{\text{lim}} & \text{if } s(t) < 0 \\ -\text{sign}(x_2)u_{\text{lim}} & \text{if } s(t) = 0 \end{cases} \tag{21}$$

Optimal final time:

$$t_f^\star = \begin{cases} (x_{20} + 2\sqrt{u_{\text{lim}}x_{10} + 0.5x_{20}^2})/u_{\text{lim}} & \text{if } s(0) > 0 \\ (-x_{20} + 2\sqrt{-u_{\text{lim}}x_{10} + 0.5x_{20}^2})/u_{\text{lim}} & \text{if } s(0) < 0 \\ |x_{20}|/u_{\text{lim}} & \text{if } s(0) = 0 \end{cases} \tag{22}$$

Optimal switching time:

$$t_s^\star = \begin{cases} (x_{20} + \sqrt{u_{\text{lim}}x_{10} + 0.5x_{20}^2})/u_{\text{lim}} & \text{if } s(0) > 0 \\ (-x_{20} + \sqrt{-u_{\text{lim}}x_{10} + 0.5x_{20}^2})/u_{\text{lim}} & \text{if } s(0) < 0 \\ 0 & \text{if } s(0) = 0 \end{cases} \tag{23}$$

Optimal state trajectory:

$$
\begin{aligned}
x_1^\star(t) =& [t \le t_s] \cdot \Big(x_{10} + x_{20}t + \frac{1}{2}t^2 u^\star(t)\Big) + \\
& [t > t_s] \cdot \Big(x_{1s} + x_{2s}(t - t_s) + \frac{1}{2}(t - t_s)^2 u^\star(t)\Big) \qquad (24) \\
x_2^\star(t) =& [t \le t_s] \cdot \Big(x_{20} + t u^\star(t)\Big) + \\
& [t > t_s] \cdot \Big(x_{2s} + (t - t_s) u^\star(t)\Big) \qquad (25)
\end{aligned}
$$

Sample trajectories are visualized in Figure 2.

Solution for arbitrary x_{1f} and x_{2f}

In the general case where $x_{1f} \neq 0$ and $x_{2f} \neq 0$, the switching criteria is given by inserting the optimal policy

$$u^\star(t) = -\mathrm{sign}(x_2(t) - x_{2f}) u_{\mathrm{lim}} \qquad (26)$$

into Equation (17), which gives

$$s(t) = x_1(t) - x_{1f} + \mathrm{sign}(x_2(t) - x_{2f})\ \frac{x_2(t)^2 - x_{2f}^2}{2u_{\mathrm{lim}}^\star}\ . \qquad (27)$$

Optimal controller:

$$u^\star(t) = \begin{cases} -u_{\mathrm{lim}} & \text{if } s(t) > 0 \\ u_{\mathrm{lim}} & \text{if } s(t) < 0 \\ -\mathrm{sign}(x_2(t) - x_{2f}) u_{\mathrm{lim}} & \text{if } s(t) = 0 \end{cases} \qquad (28)$$

Optimal final time:

$$t_f^\star = \begin{cases} (x_{20} + x_{2f} + 2\sqrt{u_{\mathrm{lim}} x_{10} + 0.5x_{20}^2 + 0.5x_{2f}^2 - u_{\mathrm{lim}} x_{1f}})/u_{\mathrm{lim}} & \text{if } s(0) > 0 \\ (-x_{20} - x_{2f} + 2\sqrt{-u_{\mathrm{lim}} x_{10} + 0.5x_{20}^2 + 0.5x_{2f}^2 + u_{\mathrm{lim}} x_{1f}})/u_{\mathrm{lim}} & \text{if } s(0) < 0 \\ (|x_{20}| + |x_{2f}|)/u_{\mathrm{lim}} & \text{if } s(0) = 0 \end{cases} \qquad (29)$$

Optimal switching time:

$$t_s^\star = \begin{cases} (x_{20} + \sqrt{u_{\mathrm{lim}} x_{10} + 0.5x_{20}^2 + 0.5x_{2f}^2 - u_{\mathrm{lim}} x_{1f}})/u_{\mathrm{lim}} & \text{if } s(0) > 0 \\ (-x_{20} + \sqrt{-u_{\mathrm{lim}} x_{10} + 0.5x_{20}^2 + 0.5x_{2f}^2 + u_{\mathrm{lim}} x_{1f}})/u_{\mathrm{lim}} & \text{if } s(0) < 0 \\ 0 & \text{if } s(0) = 0 \end{cases} \qquad (30)$$

The state equations are equivalent to the ones given above. There always exists a solution to this problem for any initial and final conditions. Sample trajectories are visualized in Figure 3.

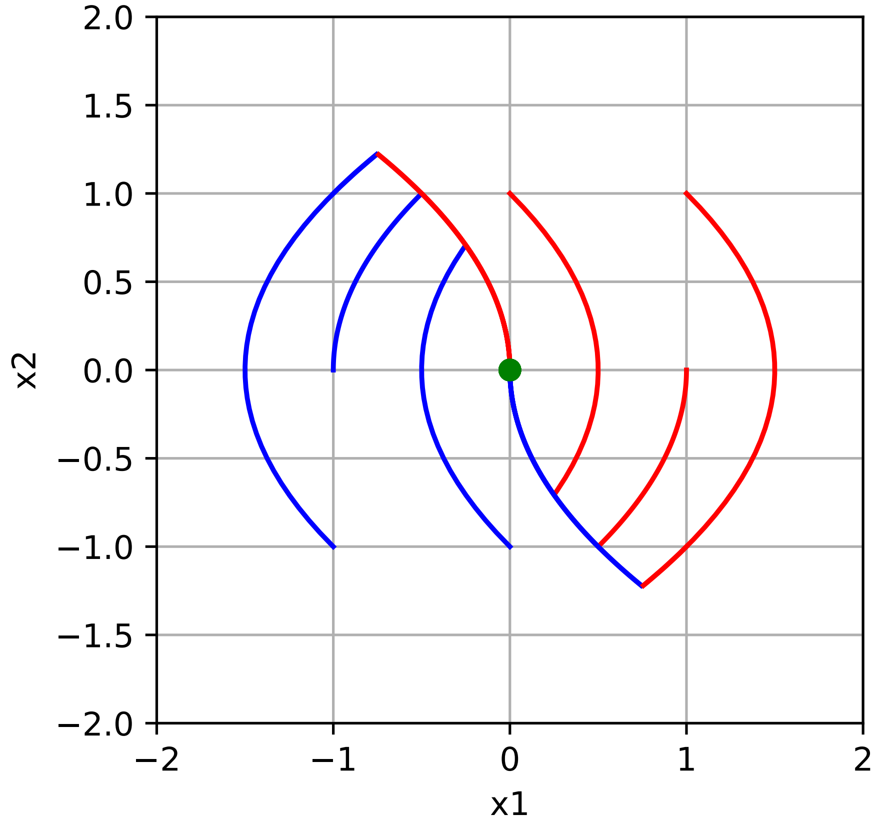

Figure 2: $x_{1f} = x_{2f} = 0$.

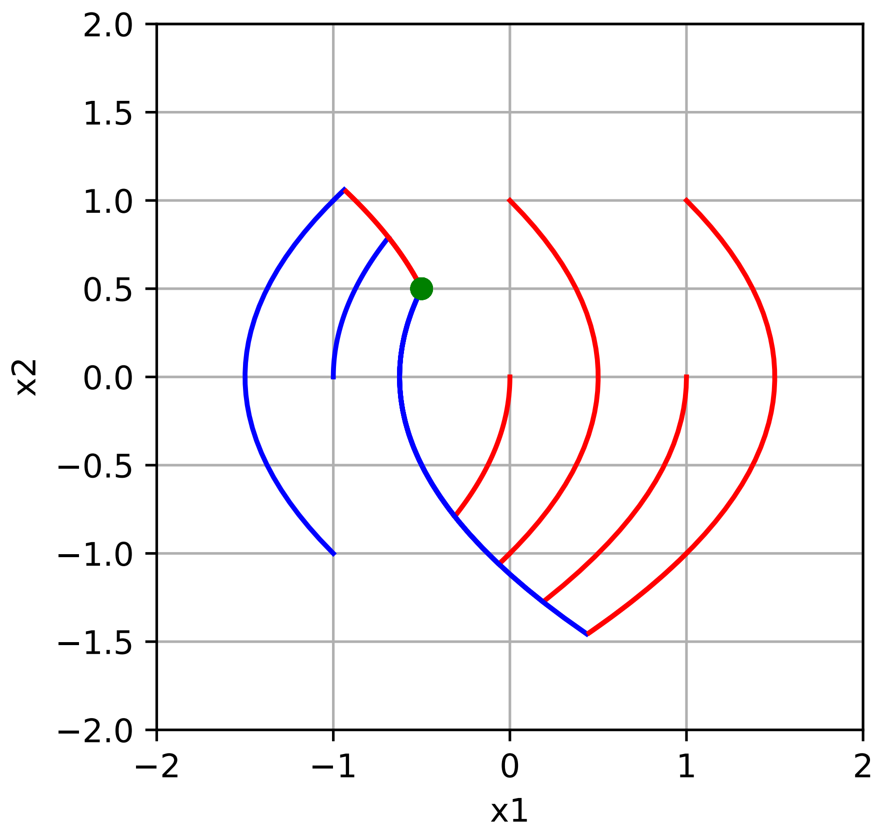

Figure 3: $x_{1f} = -0.5$ and $x_{2f} = 0.5$.