

# CME302 class notes

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2021 Fall quarter

## 1 Linear algebra review

### 1.1 Vector products

There are two vector products that are helpful for a lot of linear algebra. The **inner product**, also known as the dot product, results in a scalar. It is the inner sum of two vectors of the same length:  $x^T y = \sum x_i * y_i$

- $x^T y = \|x\|_2 \|y\|_2 \cos \theta$
- $x^T y = 0 \Leftrightarrow x \perp y$

And the **outer product** results in a matrix. It is the outer sum of the two vectors, which can be of different lengths.

### 1.2 Norms

Measures of the length of vectors and matrices. All norms satisfy

- Only zero vector has zero norm:  $\|x\|_x = 0 \Leftrightarrow x = 0$
- $\|\alpha x\|_x = |\alpha| \|x\|_x$
- $\|x + y\|_x \leq \|x\|_x + \|y\|_x$  (Triangle inequality)
- $\|x - y\|_x \geq \|x\|_x - \|y\|_x$  (another Triangle inequality)

#### 1.2.1 Vector norms

In machine learning applications, the selection of the norm can give you solutions with different properties. Many theorems and properties work out nicely when we choose the 2-norm. Types of **vector norms**,  $x \in \mathbb{R}^n$

- $\|x\|_1 = \sum_{i=1}^n |x_i|$
- $\|x\|_2 = \sqrt{\sum_{i=1}^n (x_i)^2}$
- $\|x\|_\infty = \max_{i \in \{1, \dots, n\}} |x_i|$
- $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$

**Cauchy-Schwartz Inequality:**  $|x^T y| \leq \|x\|_2 \|y\|_2$  (note equality when  $x^T y = 0$ )

**Holder's Inequality:**  $|x^T y| \leq \|x\|_p \|y\|_q$ , for  $p, q$ , s.t.  $\frac{1}{p} + \frac{1}{q} = 1$

#### 1.2.2 Matrix norms

Types of **matrix norms**,  $A \in \mathbb{R}^{n \times m}$

- $\|A\|_\infty = \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_i \|a_i^T\|_1$
- $\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p=1} \|Ax\|_p$
- $\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2} = \sqrt{\text{tr}(AA^T)} = \sqrt{\text{tr}(A^T A)} = \sqrt{\sum_{k=1}^{\min(m,n)} \sigma_k^2}$

**Submultiplicative inverse:**  $\|AB\|_p \leq \|A\|_p \|B\|_p$ . Note: this is not always true for Frobenius norms.

**Induced 2-norm:**  $\|Ay\|_p \leq \|A\|_p \|y\|_p$

**Orthogonally invariant:** Orthogonal matrices do not change the norms of vectors or matrices:

- $\|Qx\|_x = \|x\|_p$
- $\|QA\|_x = \|A\|_x, x \in \{p, F\}$

**Other norm properties**

- $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty$
- $\|A\|_2 \leq \sqrt{m} \|A\|_\infty$
- $\|A\|_\infty \leq \sqrt{n} \|A\|_2$

### 1.3 Matrix properties

Matrices represent the following linear operations on a vector

- Scaling
- 1D reflection
- 2D reflection (about a plane in N-dim space)
- Dimension reduction or increase ( $A : x \in \mathbb{R}^m \rightarrow y = Ax \in \mathbb{R}^n$ )

#### 1.3.1 Determinant

One way to think about the determinant of a matrix is that it represents how the volume of a hypercube is transformed by the matrix. And a few properties of the determinant

- For square matrix,  $\det(\alpha A) = \alpha^n \det(A)$
- For square matrices,  $\det(AB) = \det(A)\det(B)$
- $\det(A) = \det(A^T)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$
- For square matrix,  $A$  singular  $\Leftrightarrow \det(A) = 0 \Leftrightarrow$  columns of  $A$  are not linearly independent

#### 1.3.2 Trace

The trace of a matrix  $A \in \mathbb{R}^{m \times n}$ ,  $\text{tr}(A)$ , is equal to the sum of the entries in its diagonal

$$\text{tr}(A) = \sum_{i=1}^n a_{ii}$$

And a few properties of the trace

- $\text{tr}(A) = \text{tr}(A^T)$
- $\text{tr}(A + \alpha B) = \text{tr}(A) + \alpha \text{tr}(B)$
- Trace is invariant under cyclic permutations, that is  $\text{tr}(ABCD) = \text{tr}(BCDA) = \text{tr}(CDAB) = \text{tr}(DABC)$
- For two vectors,  $u, v \in \mathbb{R}, \text{tr}(uv^T) = v^T u$

#### 1.3.3 Inverses and transposes

The inverse of the transpose is the transpose of the inverse:

- $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$
- $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$

### 1.3.4 Sherman-Morrison-Woodbury formula

for  $A \in \mathbb{R}^{n \times n}$ ,  $U, V \in \mathbb{R}^{n \times k}$

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U(I + V^T A^{-1}U)^{-1}V^T A^{-1}$$

The significance of this formula is that you can compute the inverse of the sum of two matrices using the inverse of a known matrix,  $A$ , and the inverse of a much smaller matrix (assuming  $k < n$ ) in  $(I + V^T A^{-1}U)$

**Proof:** begin with the inverse of the *LHS* multiplied by the *RHS*:  $(A + UV^T)(A^{-1} - A^{-1}U(I + V^T A^{-1}U)^{-1}V^T A^{-1})$ . Next perform matrix multiplication. The end result will be  $I$ , implying that the *RHS* is an inverse of  $(A + UV^T)$

## 1.4 Matrix multiplication

Show:  $AB = a_1 b_1^T + a_2 b_2^T + \dots + a_n b_n^T$ ,  $A, B \in \mathbb{R}$

$$\text{Let } A = \begin{bmatrix} | & | & \dots & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix}, B = \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix}$$

$$a_1 b_1^T = \begin{bmatrix} a_{11} b_{11} & \dots & a_{1n} b_{1n} \\ \vdots & & \vdots \\ a_{n1} b_{11} & \dots & a_{nn} b_{1n} \end{bmatrix} \Rightarrow \sum_{i=1}^n a_i b_i^T = \begin{bmatrix} \sum a_{1i} b_{i1} & \dots & \sum a_{1i} b_{in} \\ \vdots & & \vdots \\ \sum a_{ni} b_{i1} & \dots & \sum a_{ni} b_{in} \end{bmatrix} \Rightarrow AB$$

## 1.5 Orthogonal matrices

An orthogonal matrix,  $Q$  is a matrix whose columns are orthonormal. That is

- $q_i^T q_j = 1$  for  $i = j$
- $q_i^T q_j = 0$  for  $i \neq j$

Equivalently,  $Q^T Q = I$ . For square matrices,  $Q^T Q = Q Q^T = I$

## 1.6 Projections, reflections, and rotations

### 1.6.1 Projections

A projection,  $v$ , of vector  $x$  onto vector  $y$  can be written in the form

$$v = \frac{y^T x}{y^T y} y$$

Which can be interpreted as the portion of  $x$  in the direction of  $y$  ( $y^T x$ ), times the direction of  $y$ , divided by the length of  $y$  twice ( $y^T y = \|y\|_2^2$ ), since  $y$  appears in the dot product and in the vector. Observe, the denominator would be 1 if  $y$  were a unit vector

**Projection matrices** are square matrices,  $P$ , s.t.,  $P^2 = P$ .

### 1.6.2 Reflection

- $P$  is a reflection matrix  $\Leftrightarrow P^2 = I$
- $P$  can be written in the form  $P = I - \beta v v^T$ , with  $\beta = \frac{2}{v^T v}$ , and  $v$  the vector orthogonal to the line/plane of reflection
- It can be shown that  $Px = x \Leftrightarrow v^T x = 0$ . These  $x$  are called the "fixed points" of  $P$

## 1.7 Symmetric Positive Definite (SPD) Matrices

For  $A$ , SPD

- $A = A^T$
- $x^T A x > 0 \forall x \neq 0$
- $a_{ii} > 0$
- $\lambda(A) \geq 0$
- for  $B$  nonsingular,  $B^T A B$  is also SPD

When proving properties of SPDs, use the following tricks

- Multiply by  $e_i$  since  $e_i \neq 0$
- Use matrix transpose property,  $x^T A^T = (Ax)^T$  to rearrange formulas

### 1.7.1 $B^T A B$ is also SPD

If  $A$  SPD  $\Rightarrow B^T A B$  SPD for  $B$  nonsingular:

$$x^T B^T A B x = (Bx)^T A (Bx) > 0, (\text{since } B \text{ nonsingular} \Rightarrow Bx \neq 0)$$

## 1.8 Eigenvalues

Observe by definition

$$\begin{aligned} Ax &= \lambda x \\ Ax - \lambda x &= 0 \\ (A - \lambda I)x &= 0 \end{aligned}$$

To find lambda, we solve for the system of equations to satisfy  $(A - \lambda I)x = 0$

The **algebraic multiplicity** of an eigenvalue,  $\lambda_i$ , is the number of times that the value  $\lambda_i$  appears as an eigenvalue of  $A$  e.g., for characteristic equation  $p(x) = (x-2)^3(x-1)^2$ ,  $\lambda = 2$  has algebraic multiplicity of 3, and  $\lambda = 1$  has algebraic multiplicity of 2

The **geometric multiplicity** of an eigenvalue,  $\lambda_i$ , is the dimension of the space spanned by the eigenvectors of  $\lambda_i$   
Other eigenvalue properties

- $\lambda(A) = \lambda(A^T)$

### 1.8.1 Determinants and trace

$$\begin{aligned} \det(A) &= \prod_{i=1}^n \lambda_i, \text{ Intuition: the lower triangular 0s cancel everything, but the values in the diagonal} \\ \text{tr}(A) &= \sum_{i=1}^n \lambda_i \end{aligned}$$

### 1.8.2 Triangular matrices

For  $T$  triangular, the eigenvalues appear on the diagonal:  $t_{ii} = \lambda_i, \forall i \in \{1, \dots, n\}$

**Corollary:**  $T$  nonsingular  $\Leftrightarrow$  all  $t_{ii} \neq 0$

### 1.8.3 Gershgorin disc theorem

Gershgorin disc,  $\mathbb{D}_i$ , defined

$$\mathbb{D}_i = \{z \in \mathbb{C} \mid |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|\}$$

All eigenvalues of  $A$ ,  $\lambda(A) \in \mathbb{C}$  are located in one of its Gershgorin discs

**Proof:**

$$\begin{aligned} Ax &= \lambda x \\ (A - \lambda I)x &= 0 \end{aligned}$$

$$\sum_{j \neq i} a_{ij}x_j + (a_{ii} - \lambda)x_i = 0, \forall i \in \{1, \dots, n\}$$

Choose  $i$  s.t.  $|x_i| = \max_i |x_i|$

$$|(a_{ii} - \lambda)| = \left| \sum_{j \neq i} \frac{a_{ij}x_j}{x_i} \right| \leq \sum_{j \neq i} \left| \frac{a_{ij}x_j}{x_i} \right|, \text{ by triangle inequality}$$

$$|(\lambda - a_{ii})| \leq \sum_{j \neq i} |a_{ij}|, \text{ since } \left| \frac{x_j}{x_i} \right| \leq 1$$

## 2 Matrix Decompositions

### 2.1 Schur Decomposition

For any  $A \in \mathbb{C}^{n \times n}$

$$A = QTQ^H, \text{ where } Q \text{ unitary } (Q^H Q = I), Q \in \mathbb{C}^{n \times n}, T \text{ upper triangular}$$

When  $A \in \mathbb{R}^{n \times n}$ , the Real Schur Decomposition becomes

$$A = QTQ^T, \text{ where } Q \text{ orthogonal } (Q^T Q = I), Q \in \mathbb{R}^{n \times n}, T \text{ upper triangular}$$

Note: If  $T$  is relaxed from strict upper triangular to block upper triangular (blocks of  $2 \times 2$  or  $1 \times 1$  on the diagonal), then  $Q$  can be selected to be in  $\mathbb{R}^{n \times n}$ .

### 2.2 Eigenvalue Decomposition

For  $A$  diagonalizable ( $A \in \mathbb{R}^{n \times n}$  with  $n$  linearly independent eigenvectors), it can be decomposed as

$$A = X\Lambda X^{-1}, \text{ where } \Lambda \text{ a diagonal matrix of the eigenvalues of } A$$

For  $A$  real symmetric,  $A$  can be decomposed as

$$A = Q\Lambda Q^T, Q \text{ orthogonal}$$

For  $A$  unitarily diagonalizable ( $\Leftrightarrow$  normal:  $A^H A = A A^H$ ),  $A$  can be decomposed as below. When  $A$  complex Hermitian ( $A = A^H$ ),  $\Lambda \in \mathbb{R}$

$$A = Q\Lambda Q^H, Q \text{ unitary}$$

### 2.3 Singular Value Decomposition

**Definition:** For any  $A \in \mathbb{C}^{m \times n}$  there exist two unitary matrices,  $U \in \mathbb{C}^{m \times m}$  and  $V \in \mathbb{C}^{n \times n}$ , and a diagonal matrix  $\Sigma \in \mathbb{R}^{m \times n}$  such that

$$A = U\Sigma V^H$$

$$A = U\Sigma V^T \text{ when } A \in \mathbb{R}^{m \times n}, \text{ with } U, V, \Sigma \in \mathbb{R}$$

The singular values,  $\sigma_i$  of  $\Sigma$  are always  $\geq 0$ . And by convention, they're ordered in decreasing order, so  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$

**Motivation:** Consider the action of a matrix,  $A$  on a sphere.  $A$  maps the sphere to a hyperellipsoid,  $E$

- The lengths of the semi-axes of  $E$  are denoted  $\sigma_1, \dots, \sigma_n$  called **singular values** of  $A$

- The directions of the semi axes are denoted by unit vectors,  $u_1, \dots, u_n$  called **left singular vectors** of  $A$
- For each  $u_i$  there is some unit vector  $v_i$  so that  $Av_i = \sigma_i u_i$ . The vectors  $v_1, \dots, v_n$  are called the **right singular vectors**

**Derivation:** Observe  $A^T A$  symmetric:  $(A^T A)^T = A^T A$

$A^T A$  symmetric  $\Rightarrow \exists Q$  orthogonal and  $\Lambda$  diagonal matrix of  $\lambda_i$  s.t.,

$$A^T A = Q \Lambda Q^T$$

$$Q^T A^T A Q = Q^T Q \Lambda Q^T Q$$

$(AQ)^T (AQ) = \Lambda$ , note  $AQ$  is orthogonal, but not scaled to 1. Instead, each row is scaled to the eigenvalue in that row:  $\lambda_i = \|Aq_i\|_2^2$

When  $A$  is full rank,

$$\begin{aligned} A &= AQQ^T \\ &= (AQ)Q^T \end{aligned}$$

$$= AQD^{-1}DQ^T, \text{ where } D = \begin{bmatrix} \sqrt{\lambda_1} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \sqrt{\lambda_n} \end{bmatrix} \text{ and } D^{-1} = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \frac{1}{\sqrt{\lambda_n}} \end{bmatrix}$$

$$A = U\Sigma V^T, \text{ where } U = AQD^{-1}, \Sigma = D, V^T = Q^T$$

When  $A$  is full rank, this does not hold since  $\lambda_i = 0$  for some  $i$  so we cannot construct  $U$  with  $D^{-1}$

$$\text{Start with } AQ = \begin{bmatrix} | & & | & | & & | \\ r_1 & \dots & r_r & 0 & \dots & 0 \\ | & & | & | & & | \end{bmatrix}$$

$$A = AQD^{-1}DQ^T, \text{ where } D = \begin{bmatrix} \sqrt{\lambda_1} & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \sqrt{\lambda_r} \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \quad (\text{observe this matrix has inverse, } D^{-1})$$

$A = U\Sigma V^T$ , where

$$U = [\text{left } r \text{ columns of } AQ] \times [\text{upper-left diagonal block of } D^{-1} \in \mathbb{R}^{r \times r}],$$

$$\Sigma = [\text{upper-left diagonal block of } D \in \mathbb{R}^{r \times r}]$$

$$V^T = [\text{left block of } Q, \text{ or upper block of } Q^T]$$

And a few properties and remarks of  $A \in \mathbb{R}^{n \times m}$  SVD

- $\|A\|_2 = \sigma_1$
- $\|A^{-1}\|_2 = \frac{1}{\sigma_n}$  when  $A$  nonsingular
- $\|A\|_F = \sqrt{\sum_i^{\min\{n,m\}} \sigma_i^2}$
- When  $A$  symmetric,  $\sigma_i = |\lambda_i|$
- The eigenvalues of  $A^T A$  and  $AA^T$  are the squares of the singular values of  $A$ ,  $\sigma_1^2, \dots, \sigma_n^2$
- By construction,  $V$  contains the eigenvectors of  $A^T A$  and  $U$  contains the eigenvectors of  $AA^T$ , so  $A^T A v_i = \sigma_i^2 v_i$  and  $AA^T u_i = \sigma_i^2 u_i$
- When  $A$  orthogonal,  $\sigma_1 = \dots = \sigma_n = 1$
- **Condition number**,  $\kappa(A) = \|A\|_2, \|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_n}$

### 3 Error analysis

#### 3.1 Floating point arithmetic

**Motivation:** Values stored in a computer are floating type, which means the digits of the number are stored, and then separately the location of the decimal place is stored. This means that when adding or subtracting numbers with limited overlap in where the digits lie can lead to truncation and accuracy errors.

General floating point number equation:

$$\pm \left( \sum_{i=1}^{t-1} d_i \beta^{-i} \right) \beta^e$$

Where

- $\beta$  is the base (in floating point computation,  $\beta = 2$ )
- $d_0 \geq 1$ , and  $d_i \leq \beta - 1$ .
- $e$  is called the **exponent**, this is the location of the decimal place.
- $t - 1$  in the summand is called the **precision** and indicates the number of digits (in base  $\beta$ ) that can be stored with the number.
- Lastly, the part of the equation in the parenthesis is referred to as the **significand** or **mantissa**

#### 3.2 Unit roundoff

The **unit roundoff** for a floating-point number is

$$u = \frac{1}{2} \times \beta^{-(t-1)} \text{ (distance between the smallest digits stored in a floating-point number)}$$

For double precision floating point numbers (64 bits)  $u \approx 10^{-16}$

The **floating point truncation operator**,  $fl(a)$ , takes as input  $a$  and returns the nearest floating point,  $fl(a)$ . Observe

$$fl(a + b) = a + b + \epsilon(a + b), \quad |\epsilon| \leq u, \text{ the unit roundoff}$$

To **prove** this inequality,

- start by writing  $fl(x)$  and  $x$  using floating point equations
- show that the difference between these numbers is bounded by the smallest bit that can be represented by  $fl(x)$ .
- The  $\frac{1}{2}$  enters the equation as a bound on the selection of the last digit of  $fl(x)$  to approximate  $x$ : it cannot be greater than  $\frac{1}{2}$  of the digit away, or a closer digit could have been chosen.

#### 3.3 Forward/Backward error analysis

**Forward error analysis** looks to create bounds between the computed quantity  $\tilde{f}(A, b)$  and true value  $f(A, b)$ . The forward error is  $\left\| \tilde{f}(x) - f(x) \right\|_p$ . i.e., What is the error in the solution computed with our algorithm? But this is difficult to compute.

**Backward error analysis** is an easier error estimation to solve and tries to find the error in  $A$  that lets us observe the  $\tilde{x}$  observed. More specifically, find  $\tilde{E}$  such that  $(A + \tilde{E})\tilde{x} = b$ . i.e., what is the problem that our algorithm actually solved? The backward error can also be referred to as  $\|E\|_p$ , and is regarded as *backward stable* if  $\|E\|_p \in O(u)$

The relative sensitivity of a problem is often called the **conditioning** of the problem

- Sensitivity:  $\frac{\left\| \tilde{f}(x) - f(x) \right\|_p}{\left\| \tilde{x} - x \right\|_p}$
- Relative sensitivity:  $\frac{\left\| \tilde{f}(x) - f(x) \right\|_p \|x\|_p}{\left\| \tilde{x} - x \right\|_p \|f(x)\|_p}$

## 4 LU Factorization

The **motivation** for the LU factorization is that it is relatively straightforward computationally to solve linear equations when the matrix is of a triangular form. So, if we can decompose a matrix,  $A$ , into a product of a lower triangular matrix,  $L$ , and an upper triangular matrix,  $U$ , then we can solve an easier problem.

Once we have  $A = LU$ , to solve  $Ax = b$ , we can start by solving  $Lz = b$ , and then  $Ux = z$ .  $x$ , here, is the solution!

### 4.1 Basic algorithm

In the basic LU factorization algorithm, we construct matrices  $L$  and  $U$  by iteratively subtracting the outer products of vectors that sequentially "zero-out" the rows and columns of  $A$ . We know  $LU = l_1 u_1^T + \dots + l_n u_n^T$ , and when  $l_1, u_1^T$  are from lower/upper respectively,  $LU - l_1 u_1^T$  yields a matrix with zeros in the first row and column. We use this principle for the basic algorithm

- Construct  $u_1^T$  equal to the first row of  $A, a_1^T$
- Construct  $l_1$  equal to each of the elements in the first column of  $A, a_1$ , divided by  $a_{11}$ , the "pivot"
- Calculate  $A' \leftarrow A - l_1 u_1^T$ . In practice (and somewhat confusingly),  $A'$  is now referred to as  $A$
- Repeat the algorithm with the updated  $A$ , and the next row/column. Observe each  $l_i, u_i^T$  constructed are the rows/columns of the lower and upper triangular matrices of  $L, U$  respectively.

#### 4.1.1 Gauss transforms

Another way to think about the basic LU factorization algorithm is with Gauss transforms. **Gauss transformation matrices** are linear transformations that zero out all entries below a certain entry. The columns of a Gauss transformation look like the values of  $l_i$ , where nonzero entries are divided by a pivot entry.

To compute  $A = LU$ , consider  $L^{-1}A = U$ , with  $L^{-1}$  that "zeros-out" the columns of  $A$  to get  $U$ . Call  $L^{-1}, G$ . As with the iterative algorithm above, we can multiply  $A$  by iterative  $G_i$ 's to get  $U$ :

$$\begin{aligned} G_n G_{n-1} \dots G_2 G_1 A &= U \\ A &= G_1^{-1} \dots G_n^{-1} U \end{aligned}$$

### 4.2 Pivoting

#### 4.2.1 When pivoting is needed

Notice that this algorithm relies on the pivots,  $a_{kk}$ , being nonzero. It turns out this will occur if none of the  $k \times k$  blocks of  $A, A[1:k, 1:k]$ , have a determinant of 0. **Proof by induction:**

Case  $k=1$ :

$$\begin{aligned} A_1 &= L_1 U_1 \\ \det(A_1) &= \det(L_1 U_1) \\ \det(A_1) &= \det(L_1) \det(U_1), \text{ by property of determinants} \\ \det(A_1) &= \det(U_1), \text{ since determinant of a triangular matrix is a product of the diagonals and the diagonal of } L_1 \text{ are } 1\text{'s} \\ \det(A_1) &= a_{11} = u_{11} \rightarrow \text{so when determinant is not zero, we have a nonzero pivot} \end{aligned}$$

Case  $k=n$ : assumed to be true

Case  $k=n+1$ :

$$\begin{aligned} A_1 &= L_1 U_1 \\ \det(A_{k+1}) &= \det(L_{k+1} U_{k+1}) \\ \det(A_{k+1}) &= \det(L_{k+1}) \det(U_{k+1}) \\ \det(A_{k+1}) &= \det(U_{k+1}) \\ \det(A_{k+1}) &= u_{11} * u_{22} * \dots * u_{kk} \\ &\text{but we know } u_{ii} \neq 0 \text{ for } i \leq k \text{ from induction step, so when determinant is not zero,} \\ &\text{we have pivot, } a_{k+1,k+1} \text{ nonzero} \end{aligned}$$

What's more, if the entries of  $L$  are large (which occurs when entries in  $A$  are really small and land on the pivot locations), then because of roundoff errors in a computer, this algorithm can generate errors. The **key** is to not have small values in the



diagonal! Consider  $A \in \mathbb{R}^{2 \times 2}$  below. The issue arises when we need to calculate  $\epsilon^{-1} + (\pi - \epsilon^{-1})$ . With finite precision and  $\epsilon$  small, this value is very different from  $\pi$ .

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & \pi \end{bmatrix}, L = \begin{bmatrix} 1 & 0 \\ \epsilon^{-1} & 1 \end{bmatrix}, U = \begin{bmatrix} \epsilon & 1 \\ 0 & \pi - \epsilon^{-1} \end{bmatrix},$$

#### 4.2.2 Pivoting algorithms

Pivoting algorithms pivot the iterative version of  $A$  in each iteration to avoid the numerical issues identified above. There are several pivoting algorithms

- **Partial/Row pivoting** performs row swaps at each step in the LU factorization so that the largest entry in a column appears in the pivot location. And we solve  $PA = LU$ , with  $P$  being a matrix storing the successive row swaps of  $A$
- **Full pivoting** performs row and column swaps at each step in the LU factorization so that at each step, the largest remaining entry appears in the next pivot location. Here we solve  $PAQ^T = LU$ , with  $P$  swapping rows of  $A$ , and  $Q^T$  swapping columns.

Full pivoting is **rank-revealing** since once the rank of the matrix  $r$  iterations have been performed, the remaining block will contain only zeros and the algorithm can stop early (plus we learned something about the rank of  $A$ !)

- **Rook pivoting** performs row and column swaps at each step in the LU algorithm, but instead of swapping the pivot for the largest remaining entry, it swaps the next pivot for the first entry encountered that is maximum in its row and column. This pivoting approach is also rank-revealing and computationally less expensive!

### 4.3 Cholesky factorization

The Cholesky factorization is an LU factorization for Symmetric Positive Definite (SPD) matrices, where SPD matrix,  $A = GG^T$ , with  $G$  lower triangular.

**Intuition:** An SPD matrix,  $A$ , can be written of the form

$$A = \begin{bmatrix} a & C^T \\ C & B \end{bmatrix} \text{ where } a \text{ is } 1 \times 1, C \text{ is } n-1 \times 1, \text{ and } B \text{ is } (n-1) \times (n-1)$$

After the first step of the LU factorization, we have the following matrix product,  $A = L_1 U_1$

$$\begin{bmatrix} a & C^T \\ C & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ C/a & I \end{bmatrix} \begin{bmatrix} a & C^T \\ 0 & B - (1/a)CC^T \end{bmatrix}$$

Notice since  $A$  is symmetric,  $B$  is also symmetric, so  $B - (1/a)CC^T$  must be symmetric by construction. We are also guaranteed to have the pivot,  $a$  in entry  $(1,1)$  of  $A$ , to be strictly greater than zero since  $A$  is SPD:  $a = e_1^T A e_1 > 0$ . Next, we can further decompose the second matrix to

$$A = \begin{bmatrix} 1 & 0 \\ C/a & I \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & B - (1/a)CC^T \end{bmatrix} \begin{bmatrix} 1 & C^T/a \\ 0 & I \end{bmatrix}$$

Using the fact that  $A \text{ SPD} \Rightarrow B^T A B \text{ SPD}$  for  $B$  nonsingular, observe that matrix  $\begin{bmatrix} 1 & 0 \\ C/a & I \end{bmatrix}$  is nonsingular so therefore the

matrix  $\begin{bmatrix} a & 0 \\ 0 & B - (1/a)CC^T \end{bmatrix}$  must be SPD. Which also means the submatrix  $B - (1/a)CC^T$  is SPD

We can use induction to prove that the Cholesky factorization exists.

Continuing with this factorization, we get an equation of the form  $A = LDL^T$  for  $D$ , diagonal, and  $L$ , lower triangular. It's common to rewrite  $A = LDL^T$  in the form  $A = GG^T$ , where  $G = LD^{\frac{1}{2}}$

### 4.3.1 Cholesky factorization is unique

By contradiction, suppose  $A = GG^T = MM^T$  for  $G \neq M$ . We know  $G, M$  nonsingular (consider the determinants of the equation above) so

$$\begin{aligned}
GG^T &= MM^T \\
I &= G^{-1}MM^TG^{-T} \\
I &= (G^{-1}M)(G^{-1}M)^T, \text{ since } (A^{-1})^T = (A^T)^{-1} \\
(G^{-1}M)^{-T} &= (G^{-1}M) \\
\Rightarrow G^{-1}M &\text{ diagonal since } G^{-1}M \text{ lower triangular and } (G^{-1}M)^{-T} \text{ upper triangular} \\
\Rightarrow G^{-1}M &= D \Rightarrow M = GD \\
I &= (G^{-1}GD)(G^{-1}GD)^T \\
I &= DD^T = D^2 \Rightarrow \text{so the entries of } D \text{ are on the order of } 1
\end{aligned}$$

## 4.4 Schur complement

A useful way to think about the LU factorization is with the **Schur complement** matrix structure. First observe  $A$  can be written in the following form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

If we run the LU factorization algorithm for  $k$  steps, the resulting  $A' = A$  is equal to

$$A = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \begin{bmatrix} I & A_{21}A_{11}^{-1} \\ 0 & I \end{bmatrix}$$

The bottom-right block of  $A' = A$ ,  $A'_{22} = A_{22}$  is equal to  $A_{22} - A_{21}A_{11}^{-1}A_{12}$  from the original matrix. This is called the **Schur complement** of  $A$

### 4.4.1 Schur complement derivation

At any step in the LU factorization,  $A$  can be written in the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

From this equality, we can create a system of equations and derive

$$\begin{aligned}
U_{12} &= L_{11}^{-1}A_{12} \\
L_{11}^{-1} &= L_{21}A_{21}A_{11}^{-1} \\
A_{22} - L_{21}U_{12} &= L_{22}U_{22} \\
A_{22} - A_{21}A_{11}^{-1}A_{12} &= L_{22}U_{22}
\end{aligned}$$

Notice that the Schur complement equals the product of  $L_{22}U_{22}$ . The next step in the derivation is to show that  $A'_{22}$  in the LU factorization is equal to  $A_{22} - L_{21}U_{12}$  since at each step we're subtracting  $l_i U_i^T$ , which can be stored as the nonzero rows/columns of  $L_{21}U_{12}$ . So

$$\begin{aligned}
A'_{22} &= A_{22} - L_{21}U_{12} \\
&= (L_{21}U_{12} + L_{22}U_{22}) - L_{21}U_{12} \\
&= L_{22}U_{22} \\
&= A_{22} - A_{21}A_{11}^{-1}A_{12}
\end{aligned}$$

## 5 QR factorization

The QR factorization decomposes a matrix,  $A \in \mathbb{R}^{m \times n}, m \geq n$  into an orthogonal (orthonormal) matrix,  $Q$  and an upper triangular matrix,  $R$ . When  $A \in \mathbb{C}^{m \times n}$ ,  $Q$  is unitary. Recall

- for  $Q \in \mathbb{R}$ , orthogonal,  $Q^T Q = I$

- for  $Q \in \mathbb{C}$ , unitary,  $Q^H Q = I$
- $\|Qx\|_2 = \|x\|_2$

If  $A$  is skinny (i.e.,  $n \ll m$ ),  $QR$  can take two different forms.  $Q \in \mathbb{R}^{m \times m}$  can be square and  $R \in \mathbb{R}^{m \times n}$  can be skinny. Or  $Q \in \mathbb{R}^{m \times n}$  can be skinny and  $R \in \mathbb{R}^{n \times n}$  can be square.

## 5.1 The QR factorization is unique

**Proof** that the QR factorization is unique for full rank matrix,  $A$ :

$$\begin{aligned} A &= QR \\ Q^T A &= R \\ R^T Q^T A &= R^T R \\ (QR)^T A &= R^T R \\ A^T A &= R^T R \end{aligned}$$

We now have a matrix,  $A^T A$  that can be written of the form  $R^T R$ , which is the structure of the Cholesky factorization. Suffice to show that  $A^T A$  is Symmetric and Positive Definite (SPD) to prove the uniqueness of  $R$ . Since  $A$  is full rank, it follows that  $Q$  is also unique (since  $AR^{-1} = Q$ ).  $A^T A$  SPD:

$$\text{Symmetric: } (A^T A)^T = A^T A$$

Positive definite: for  $x \neq 0$ ,

$$x^T A^T A x = (Ax)^T (Ax) = (QRx)^T (QRx) = x^T R^T Q^T Q R x = (Rx)^T (Rx)$$

$$Rx \text{ is of the form } Rx = \begin{bmatrix} r_{11}x_1 \\ r_{12}x_1 + r_{22}x_2 \\ \vdots \\ \sum_{i=1}^n r_{in}x_i \end{bmatrix}, \text{ so } (Rx)^T (Rx) = \sum_{i=1}^n \left( \sum_{j \leq i} r_{ij}x_j \right)^2$$

$$\text{So, } (Rx)^T (Rx) > 0 \text{ for } x \neq 0$$

## 5.2 Householder reflection

The Householder reflection is an algorithm to construct matrices  $Q$  and  $R$  for the  $QR$  factorization of  $A$ . The Householder reflection relies on the principles of reflection matrices, which (turns out) are easier to construct than rotation matrices.

### 5.2.1 Householder reflection algorithm

The high-level approach of the algorithm is to construct  $Q^T$  for each column in  $A$  that projects it onto a corresponding column of an upper right triangular matrix,  $R$ . It's easiest to see this in the case of column  $a_1$ :

- Want  $Q_1^T$  such that  $Q_1^T a_1 = r_1$ , where  $r_1 = \pm \|a_1\|_2 e_1$  (since  $Q^T$  is orthogonal)
- Put differently, want  $Q_1^T$  that reflects  $a_1$  onto  $e_1$

**The key** to the iterative part of the algorithm is to construct  $Q_i^T, i > 1$  with an identity matrix in the upper-left  $i - 1 \times i - 1$  quadrant, and a smaller  $Q_i^{*T}$  in the lower right  $n - i \times n - i$  quadrant, filling the remaining sections of the matrix with 0's.

### 5.2.2 Constructing the Householder reflection permutation

The **Householder reflection** maps  $a \rightarrow \|a\|_2 e_1$  with

$$P = I - \beta v v^T, \text{ where } v = a - \|a\|_2 e_1, \text{ and } \beta = 2/v^T v$$

- Notice that multiplying  $Px$  is the same as taking the vector  $x$  and subtracting  $\frac{2vv^T}{v^T v}x$  from it, where  $\frac{2vv^T}{v^T v}x$  is twice the projection of  $x$  onto  $v$ . When you draw this out, you'll see this vector subtraction is the same thing as a reflection about the plane defined by  $v$
- In our case where we want to reflect  $a$  onto  $\|a\|_2 e_1$ ,  $a + \|a\|_2 e_1$  is the line of reflection. Perpendicular to this line is  $a - \|a\|_2 e_1$ , the vector that defines the line of reflection

- In some cases where the other entries in  $a$  are much smaller than  $a_1$ , it may be advantageous to choose to project onto  $-\|a\|_2 e_1$  instead of  $-\|a\|_2 e_1$  to avoid roundoff errors from subtracting small numbers. In this case, we choose  $v = a + \|a\|_2 e_1$ .

**Aside:** The fixed points of a reflection,  $P$ , are the points that remain unchanged when multiplied by the reflection,  $Px = x$ . Geometrically, these are the points that are *orthogonal* to the vector  $v$  defining the reflection (i.e.,  $v^T x = 0$ )

### 5.3 Givens transformation

The Givens transformation is a much more precise algorithm for creating an upper triangular matrix,  $R$ , through an orthogonal transformation,  $Q^T$ , of  $A$ ,  $Q^T A = R$ . While the householder reflection is useful for operating on dense matrices, if we are presented with a sparse matrix, the sequential reflections of the House transformation will create more work for us.

For example, consider a matrix,  $A$ , that has a dense upper triangular portion, and a diagonal row of nonzero entries just below the diagonal. In this case, the **Givens transformation** will allow us to zero out these limited rows with less complexity!

#### 5.3.1 Givens transformation algorithm

A **Givens rotation** rotates  $u = (u_1, u_2)^T$  to  $\|u\|_2 e_1$ . The matrix that does this,  $G^T$ , is defined by

$$G^T = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}, c = \frac{u_1}{\|u\|_2}, s = -\frac{u_2}{\|u\|_2}$$

A full matrix,  $P_i$ , can be constructed to only contain this targeted transformation. Sequentially, the  $P_i$ 's can multiply  $A$  to arrive at  $R$

### 5.4 Gram-Schmidt transformation

The Householder and Givens transformations produce square  $Q \in \mathbb{R}^{n \times n}$  matrices. However, if  $A \in \mathbb{R}^{m \times n}$  is tall and thin, with  $m \gg n$ , then we want a method to create a tall and thin  $Q \in \mathbb{R}^{m \times n}$ . The Gram Schmidt transformation does this.

Similar to the  $LU$  factorization, the **Gram-Schmidt Transformation** starts with the property that  $A = QR$  can be written as a sum of the outer products of the columns of  $Q$  and rows of  $R$ :  $A = QR = q_1 r_1^T + \dots + q_m r_m^T$ .

The algorithm proceeds as follows

$$\begin{aligned} r_{11} &= \|a_1\|_2, \text{ since } \|a_1\|_2 = \|q_1 r_{11}\|_2 \text{ and } q_i \text{ orthogonal} \\ q_1 &= \frac{1}{r_{11}} a_1, \text{ since } a_1 = q_1 r_{11} \text{ by construction of } QR \\ r_{1j} &= q_1^T a_j, \text{ (repeat for all } j) \text{ since} \\ &(a_j = q_1 r_{1j} + \dots + q_j r_{jj}) \\ (q_1^T a_j &= q_1^T q_1 r_{1j} + \dots + q_1^T q_j r_{jj}) \\ (q_1^T a_j &= r_{1j}), \text{ since } q_i \text{ orthonormal} \\ A' &= A - q_1 r_1^T \end{aligned}$$

Repeat for  $A'$ , the construction of  $r_{11}, q_1, r_{1j}$

### 5.5 QR factorization to solve least-squares problems

When  $A$  is really tall and thin, it is very unlikely that we get a solution to  $Ax = b$ . Instead, we choose to solve the least-squares problem, where we seek to find  $\argmin_x \|Ax - b\|_2$ .

#### 5.5.1 Method of normal equations

Assuming  $A$  full rank. Geometrically, the point,  $x$  which solves  $\argmin_x \|Ax - b\|_2$  is one where  $b - Ax$  is orthogonal to the range of  $A$ . To solve for this,  $x$ :

$$\begin{aligned} \text{Want: } (b - Ax) &\perp \{z | z = Ay\} \\ (b - Ax) &\perp \text{range}(A) \\ (b - Ax) &\perp a_i, \forall i \in A \\ a_1^T (b - Ax) &= 0, \forall i \in A \\ A^T (b - Ax) &= 0 \\ x &= (A^T A)^{-1} A^T b \end{aligned}$$

We can use Cholesky method for fast and accurate solve of this method since  $A^T A$  is SPD. This method can run into issues when  $A$  is poorly conditioned. Notice, condition number of  $A^T A, \kappa(A^T A) = \kappa(A)^2$ , so if  $A$  is poorly conditioned, this method can get inaccurate.

### 5.5.2 QR method for least squares

Assuming  $A$  full rank. The QR method for least squares attempts to address the issue of poor conditioning and may also lead to faster computation. We construct the QR method for least squares with one of the normal equation equalities:

$$\begin{aligned} A^T(Ax - b) &= 0 \\ R^T Q^T(Ax - b) &= 0 \\ Q^T(Ax - b) &= 0, \text{ since we assume } A, R \text{ full rank (multiply both sides by } R^{-T}) \\ Q^T Q R x - Q^T b &= 0 \\ R x &= Q^T b \\ x &= R^{-1} Q^T b \end{aligned}$$

### 5.5.3 SVD for rank-deficient $A$

It may be that  $A$  is not full rank. When this is the case, we can get infinite solutions (a line of points that satisfy  $\operatorname{argmin}_x \|Ax - b\|_2$ ). To choose  $x$ , we add the additional criteria  $\min_x \|x\|_2$  to our original objective function of  $\operatorname{argmin}_x \|Ax - b\|_2$ .

We can use the "thin" version of the Singular Value Decomposition to solve this! By thin we mean, for  $A \in \mathbb{R}^{m \times n}$  with rank,  $r$ , construct  $U \in \mathbb{R}^{m \times r}$ ,  $\Sigma \in \mathbb{R}^{r \times r}$  (notice this  $\Sigma$  has an inverse),  $V^T \in \mathbb{R}^{r \times n}$ . And calculate  $x$  as

$$\begin{aligned} (Ax - b) &\perp \operatorname{range}(A) \\ (Ax - b) &\perp \operatorname{range}(U), \text{ since } R(A) = R(U) \text{ for } A = U\Sigma V^T \\ U^T(Ax - b) &= 0 \\ U^T(U\Sigma V^T x - b) &= 0 \\ \Sigma V^T x &= U^T b \\ x &= V\Sigma^{-1}U^T b \text{ (the "thin" SVD here provides a nonsingular } \Sigma \in \mathbb{R}^{r \times r}, \text{ so we can take the inverse)} \end{aligned}$$

Observe  $\min_x \|x\|_2$  the  $x \perp N(A)$ , the shortest vector between  $N(A)$  and the vector/plane of solutions to  $\operatorname{argmin}_x \|Ax - b\|_2$ . This value it turns out must be in  $R(V)$  since  $R(V) = N(A)^\perp$ .

## 6 Iterative methods to find eigenvalues

### 6.1 Power iteration

Given  $\lambda_1 > \lambda_2 \geq \dots \geq \lambda_n \in \lambda(A)$ , the **Power iteration** is a process for finding  $\lambda_1$ . The basic idea is to repeatedly multiply matrix  $A$  times a vector (normalizing each time) and eventually the first eigenvalue and eigenvector will emerge. This process assumes  $A$  is diagonalizable, meaning it can be written of the form  $A = X\Lambda X^{-1}$ .

Reminder: for  $A$  diagonalizable  $\implies A^k = X\Lambda^k X^{-1}$

$$\begin{aligned} A^k &= \sum_{i=1}^n \lambda_i^k x_i y_i^T \text{ where } Y = X^{-1} \\ A^k &\approx \lambda_1^k x_1 y_1^T \text{ since } \lambda_1 > \lambda_2 \\ A^k q &\approx \lambda_1^k x_1 y_1^T q = \lambda_1^k (y_1^T q) x_1, \text{ since } y_1^T q \text{ is a scalar. Observe } A^k q \sim \|x_1\| \end{aligned}$$

This theory is implemented in practice with the following formula

1.  $q_0$ , vector chosen at random
2.  $z_k = Aq_k = A^k q_0$ , evaluating for convergence if  $z_k \sim \|q_k\| \rightarrow z_k^T x_k = \sqrt{\|z\|_2 \|x\|_2}$
3.  $q_{k+1} = \frac{z_k}{\|z_k\|_2} = \frac{A^k q_0}{\|A^k q_0\|_2} \approx \left(\frac{\lambda_1}{|\lambda_1|}\right)^k x_1$

Since  $A^k q_0 = A q_k \approx \lambda_1 x_1$ , where  $\|x_1\|_2 = 1$  (WLOG) and  $q_k \sim \|x_1\|$ , we can solve for  $\lambda$ :

$$A q_k \approx \lambda_1 x_1 \implies A x_1 \approx \lambda_1 x_1 \Rightarrow x_1^H A x_1 \approx \lambda_1$$

Convergence:  $O((|\frac{\lambda_1}{\lambda_2}|)^K)$ , bounds the error since  $\lambda_2$  is the second largest term.

## 6.2 Inverse iteration

This process finds the eigenvector (and corresponding eigenvalue) of  $A$  that is closest to the value  $\mu$ . The basic idea is to multiply matrix  $(A - \mu I)^{-1}$  iteratively by a random vector,  $z$ , normalizing each time. Eventually, you will get the eigenvector for the eigenvalue closest to  $\mu$ .

Observe  $(A - \mu I)^{-1}$  has the same eigenvectors of  $A$ :

$$\begin{aligned} (A - \mu I)^{-1} x &= \lambda x \\ x &= (A - \mu I)x = \lambda A x - \lambda \mu x \\ \lambda A x &= x + \lambda \mu x \\ A x &= \frac{(1 + \lambda \mu)}{\lambda} x \end{aligned}$$

Performing the power iteration on  $(A - \mu I)^{-1}$ , the largest eigenvalue to emerge will be of the form  $\frac{1}{\lambda_i - \mu}$ , and we get

$$(A - \mu I)^{-1k} q_0 = (A - \mu I)^{-1} q_k \approx \lambda_i x_i, \text{ where } \|x_i\|_2 = 1 \text{ (WLOG) and } q_k \parallel x_i$$

Since  $x_i$  is also an eigenvalue of  $A$ , we can solve  $x_i^H A x_i = \lambda_i$  for the  $\lambda_i$  closest in magnitude to  $\mu$ . **Convergence:**  $O((|\frac{\lambda_i - \mu}{\lambda_j - \mu}|)^k)$ , where  $\lambda_j$  is the next closest eigenvalue to  $\mu$

## 6.3 Orthogonal iteration

This process finds all eigenvalues and eigenvectors of  $A$  in a single iterative process. We arrive at the final result with three building blocks. First, we show how we can compute each  $q_i$  to capture  $\lambda_i$ . Next, we justify a matrix decomposition (based on Schur) that allows us to reveal  $Q$ . And lastly, we combine these blocks to show how this can be done at the same time for all of  $A$ .

First, consider how orthogonal columns could reveal subsequent eigenvalues. Assume we have used the power iteration to compute  $q_1$ . To get  $q_2$

$$\begin{aligned} A^k &= \lambda_1 x_1 y_1^T + \lambda_2 x_2 y_2^T + \dots \\ P A^k &= \lambda_1 P x_1 y_1^T + \lambda_2 P x_2 y_2^T + \dots, \text{ where } P = I - x_1 x_1^T \\ P A^k &= 0 + \lambda_2 P x_2 y_2^T + \dots, \text{ since } P x_1 = I x_1 - x_1 x_1^T x_1 = x_1 - x_1 = 0 \\ P A &\text{ can now be used to apply the power iteration to reveal } x_2 \text{ and } \lambda_2 \end{aligned}$$

The general process is:

- Start with  $\lambda_1, q_1$  from power iteration
- Build  $P_2$ , projector orthogonal to  $q_1$ , use power iteration to reveal  $(\lambda_2, q_2)$
- Build  $P_3$ , projector orthogonal to  $\{q_1, q_2\}$ , use power iteration to reveal  $(\lambda_3, q_3)$
- Build  $P_i$ , projector orthogonal to  $\{q_1, \dots, q_{i-1}\}$ , use power iteration to reveal  $(\lambda_i, q_i)$

Now, consider the QR decomposition of  $X$ :

$$\begin{aligned} A &= X \Lambda X^{-1} \\ A &= Q R \Lambda R^{-1} Q^H \\ A &= Q T Q^H, \text{ where upper triangular } T = R \Lambda R^{-1} \end{aligned}$$

Observe, the last line is the Schur Decomposition, so

- The eigenvalues of  $A$  are on the diagonal of  $T$

- By construction, each column of  $Q$  is projecting the corresponding column of  $X$  onto a vector orthogonal the preceding ones
- The span of the columns of  $Q$ ,  $\text{span}\{q_1, \dots, q_n\}$  will be equal to the span of the columns of  $X$

The process for the **orthogonal iteration** is:

1.  $AQ_k \rightarrow Z$ , where  $k$  is the iteration and  $Q_0 = I$
2.  $Z \rightarrow Q_{k+1}R_{k+1}$ , the QR factorization of  $Z$
3. Repeat  $AQ_{k+1} \rightarrow Z$  and eventually  $Q_k \rightarrow Q$  revealing  $A = QR\Lambda R^{-1}Q^H$

Note in each iteration we are calculating  $Q_{k+1}^H A Q_k = R_{k+1}$  (This  $R_k$  is commonly referred to as  $T_k$ )

### 6.3.1 Reveal eigenvectors of $A$ from $T$

Motivation:  $A = X\Lambda X^{-1}$  can be hard to calculate.

$$A = X\Lambda X^{-1} = QR\Lambda R^{-1}Q^H = QTQ^H, \text{ where } T = R\Lambda R^{-1}$$

$$A = QY\Lambda Y^{-1}Q^H, \text{ where } T = Y\Lambda Y^{-1} \text{ is easier to compute}$$

Focusing on  $T = Y\Lambda Y^{-1}$ , choose some  $\lambda_i$  (we could get from power or QR iteration).

$$Tx = \lambda_i x$$

$$(T - \lambda_i I)x = 0$$

$$(T - \lambda_i I)x = \begin{bmatrix} T_{11} - \lambda_i I & T_{12} & T_{13} \\ 0 & 0 & T_{23} \\ 0 & 0 & T_{33} - \lambda_i I \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ where one diagonal element is 0}$$

And solve with back substitution:

$$X_3 = 0 : (T_{33} - \lambda_i I)X_3 = 0$$

$$X_2 \text{ is a free parameter } \in \mathbb{R} : 0X_2 + T_{33}X_3 = 0 \implies 0X_2 = 0$$

$$X_1 = -(T_{11} - \lambda_i I)^{-1}T_{12}X_2 : (T_{11} - \lambda_i I)X_1 + T_{12}X_2 + T_{13}X_3 = 0$$

So, if  $T$  upper triangular with  $\lambda_i$  on diagonal of  $T$ , you can figure out all the columns of  $Y$  for  $T = Y\Lambda Y^{-1}$ . Note,  $(T_{11} - \lambda_i I)$  nonsingular as long as the algebraic multiplicity of  $\lambda_i$  is 1.

### 6.3.2 Rate of convergence in orthogonal (and QR) iteration

Convergence of a given eigenvalue is dictated the the eigenvalues in the adjacent rows. For row  $i$ , the rate of convergence is

$$O\left(\left|\frac{\lambda_{i+1}}{\lambda_i}\right|^k\right)$$

The span of those  $i$  columns of  $Q_k$ ,  $\text{span}\{q_1, \dots, q_i\} \longrightarrow$  the span of those columns of  $X$ ,  $\text{span}\{x_1, \dots, x_i\}$ . **Note:** difficulties arise when  $\left|\frac{\lambda_{i+1}}{\lambda_i}\right|$  is close to 1.

## 6.4 QR iteration

The QR iteration builds directly on the framework of the orthogonal iteration. In orthogonal iteration, we compute  $T_{k+1}$  with the eigenvalues of  $A$  appearing on the diagonal of  $T_{k+1}$

$$Q_{k+1}^H A Q_k = T_{k+1} \text{ with } A Q_k = Z = Q_{k+1} T_{k+1}$$

In the QR iteration, we ask if we can go from  $T_k$  to  $T_{k+1}$  directly. Observe

$$A = Q_k T_k Q_k^H \implies T_k = Q_k^H A Q_k$$

$$A = Q_{k+1} R_{k+1} \text{ and } T_k = U_{k+1} R_{k+1} \implies U_{k+1}^H T_k = Q_{k+1}^H A$$

$$T_{k+1} = Q_{k+1}^H A Q_{k+1} = Q_{k+1}^H Q_k T_k Q_k^H Q_{k+1} = U_{k+1}^H T_k U_{k+1}$$

$$\implies T_k = U_{k+1} R_{k+1}$$

$$\implies T_{k+1} = R_{k+1} U_{k+1}$$

So we have an algorithm for  $T_k \rightarrow T_{k+1}$ , this process is the **QR iteration**:

1.  $T_k \rightarrow Q_{k+1}R_{k+1}$ , the QR factorization of  $T_k$
2.  $R_{k+1}U_{k+1} \rightarrow T_{k+1}$
3. Repeat with  $T_{k+1}$

**Proof** for why the  $R_{k+1}$  is the same in both QR factorization of  $A = Q_{k+1}R_{k+1}$  and  $T_k = U_{k+1}R_{k+1}$

#### 6.4.1 QR iteration on upper Hessenberg

Each QR iteration step of a dense matrix is  $O(n^3)$ . To reduce flops (floating point operations) in QR iteration, we can first convert  $A$  to upper Hessenberg ( $H = Q^H A Q$ ), and proceed with QR iteration on  $H$  using Givens rotations with complexity  $O(n^2)$ :

Choose  $Q_1^T = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{P}_1 \end{bmatrix}$  to perform a Householder rotation onto the first two entries of  $a_1 \in A$

$$\text{Observe } Q_1^T A Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{P}_1 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & \tilde{P}_1^T \end{bmatrix} = \begin{bmatrix} x & x & \cdots \\ x & x & \cdots \\ 0 & x & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \text{ where } a_{11} \text{ is never changed, and}$$

the rest of  $a_1$  is only operated on by  $\tilde{P}_1^T$  and the rest of  $a_1^T$  is only operated on by  $\tilde{P}_1$

Continuing on,  $Q_n^T \dots Q_2^T Q_1^T A Q_1 Q_2 \dots Q_n = H = Q^H A Q$  where  $Q_k^T = \begin{bmatrix} I_k & 0 \\ 0 & \tilde{P}_k \end{bmatrix}$

#### 6.4.2 QR iteration with shift

Recall, convergence is dictated by  $|(\lambda_{i+1}/\lambda_i)|^k$ . When  $\lambda_{i+1}$  is close to  $\lambda_i$ , **QR iteration with shift** helps accelerate convergence. First observe for  $\lambda_i \in \lambda(A) \rightarrow (\lambda_i - \mu) \in \lambda(A - \mu I)$ . In this algorithm, at each step we shift  $T_k$  by  $\mu I$ . For  $\mu$  close to  $\lambda_{i+1}$  close to  $\lambda_i$ , the resulting convergence,  $|[(\lambda_{i+1} - \mu)/(\lambda_i - \mu)]|^k$  will be faster.

In general, the **QR iteration with shift** process works by shifting the last eigenvalue (smallest in absolute value), updating the shift in each iteration. It works as follows:

1.  $\mu_k = T_k[n, n]$
2.  $(T_k - \mu_k I) \rightarrow U_{k+1}R_{k+1}$ , QR factorization of the shifted  $T_k$
3.  $R_k U_k + \mu_k I \rightarrow T_{k+1}$
4. Repeat with  $T_{k+1}$

Observe, this shift preserves the original QR iteration

$$\begin{aligned} (T_k - \mu I) &= U_{k+1}R_{k+1} \implies U_{k+1}^H T_k - \mu_k U_{k+1}^H = R_{k+1} \\ T_{k+1} &= R_{k+1}U_{k+1} + \mu_k I \implies T_{k+1} = (U_{k+1}^H T_k - \mu_k U_{k+1}^H)U_{k+1} + \mu_k I \\ T_{k+1} &= U_{k+1}^H T_k U_{k+1} - \mu_k I + \mu_k I = U_{k+1}^H T_k U_{k+1} \end{aligned}$$

And lastly, another useful speed-up property is called **deflation**, which is a process that allows us to break up the current QR iteration process into two smaller/easier problems.

- If any sub-diagonal element of an upper Hessenberg matrix,  $H$ , is 0, it can be written as  $H = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}$  with  $H_{11}$  and  $H_{22}$  upper Hessenberg and  $\lambda(H) = \lambda(H_{11}) \cup \lambda(H_{22})$
- Therefore, if when updating  $T_k = R_k U_k + \mu_k I$ , any sub-diagonal element of  $T_k = 0$ , then  $T_k$  can be written in this form and the QR iteration can be performed on  $(T_k)_{11}$  and  $(T_k)_{22}$  separately (simpler problems)

**Proof** of  $\lambda(H) = \lambda(H_{11}) \cup \lambda(H_{22})$



### 6.4.3 QR iteration on symmetric matrices

Upper Hessenberg symmetric matrices are Tri-diagonal matrices

- Unsymmetric case complexity:
  - Transform to upper Hessenberg:  $O(n^3)$
  - QR iteration step:  $O(n^2)$
  - Overall QR iteration:  $O(pn^3)$ , where  $p$  is the number of iterations per eval (assume quadratic convergence)
- Symmetric case complexity:
  - Transform to upper Hessenberg:  $O(n^3)$
  - QR iteration step:  $O(n)$
  - Overall QR iteration:  $O(pn^2)$ , where  $p$  is the number of iterations per eval (assume cubic convergence)

## 6.5 Bulge chasing

## 6.6 Fracis shifts

# 7 Finding eigenvalues of sparse matrices

When matrix,  $A$ , is dense, we're typically limited to  $10^4 - 10^5$  rows because of computing restrictions. We need different methods for dealing with sparse matrices. Define a **sparse matrix** as a matrix with the number of nonzero entries on the order of  $O(1)$  (i.e., does not scale with matrix size).

The main operation for reducing complexity is matrix-vector multiplication,  $Ax = \sum_j a_{ij}x_j$ , where you can skip all  $a_{ij}$  when  $a_{ij} = 0$ .

## 7.1 Arnoldi process

A similar process to *Gram-Schmidt process for QR factorization*. The **Arnoldi process** is used to reveal the first  $k$  eigenvalues of a sparse matrix.

Like Gram-Schmidt, where we use  $Q$  to make  $R$  where each subsequent column of  $A$  is made orthogonal to all preceding columns, with Arnoldi process we start with equation  $Q^H A Q = H \implies A Q = Q H$  and use  $Q$  to make  $H$  where each subsequent column of  $AQ$  orthogonal to all preceding columns.

The process follows:

1. Begin with random  $q_1 \in Q$ , such that  $\|q_1\|_2 = 1$

Iterate through each of the first  $k$  columns of  $Q$  with

2.  $Aq_j = \sum_{k=1}^{j+1} h_{kj} q_k$ , observing  $q_k^T Aq_j = h_{kj}$  the projection of  $Aq_j$  onto  $q_k$
3.  $Aq_j = \sum_{k=1}^j h_{kj} q_k + h_{j+1,j} q_{j+1}$
4.  $r = Aq_j - \sum_{k=1}^j h_{kj} q_k = h_{j+1,j} q_{j+1}$ , where only  $r$  is unknown
5.  $\|q_{j+1}\|_2 = 1 \implies h_{j+1,j} = \|r\|_2$  and  $q_{j+1} = \frac{r}{h_{j+1,j}}$

The output of this process is  $k$  columns of  $Q$  and the upper  $k \times k$  block of upper Hessenberg matrix,  $H$ , which can be used in the QR iteration to reveal  $k$  eigenvalues close to  $\lambda(A)$ . **Proof:**

$$\begin{aligned}
 AQ &= QH \implies AQ_k = Q_k H_k + h_{k+1,k} q_{k+1} e_k^T, \text{ where } Q_k = Q[:, 1 : k], H_k = [1 : k, 1 : k] \\
 AQ_k &= Q_k X_k \Lambda_k X_k^{-1} + h_{k+1,k} q_{k+1} e_k^T, \text{ where } H_k = X_k \Lambda_k X_k^{-1} \text{ through QR iteration} \\
 A(Q_k X_k) &= (Q_k X_k) \Lambda_k + h_{k+1,k} q_{k+1} x_k^T, \text{ where } x_k^T \text{ is the } k^{th} \text{ column of } X
 \end{aligned}$$

And we get an equation where

- $\Lambda_k$  contains  $k$  eigenvalues close to  $\lambda_i \in \lambda(A)$
- $(Q_k X_k)$  serve as eigenvectors for those eigenvalues
- $h_{k+1,k} q_{k+1} x_k^T$  represents something like an error term

## 7.2 Krylov spaces

A **Krylov subspace** is defined as a space of sparse Matrix-vector products:

$$K(A, q, k) = \text{span}\{q_1, Aq_1, A^2q_1, \dots, A^k q_1\}$$

In general, Krylov spaces can be used approximate linear algebra problems of  $A \in \mathbb{R}^{n \times n}$  in a  $K(A, q_1, k) \in \mathbb{R}^{k \times k}$  space instead.

### 7.2.1 QR factorization of Krylov subspace contains $Q_k$ from Arnoldi

**Proof:** We show for  $K_k = Q_k R_k$ , that  $R_k$  is upper triangular.

$$\begin{aligned} \text{Want } Q^T K_k &= R_k \text{ upper triangular for } K_k = \begin{bmatrix} | & | & \dots & | \\ q_1 & Aq_1 & \dots & A^k q_1 \\ | & | & & | \end{bmatrix} \\ Q^T k_j &= Q^T A^{j-1} q_1 = Q^T Q H^{j-1} Q^T q_1, \text{ since } A^k = Q^T H^k Q \\ &= H^{j-1} Q^T q_1 = H^{j-1} e_1, \text{ since } Q \text{ orthogonal} \\ \Rightarrow r_j \in R_k &= h_1 \in H^{j-1}, \text{ which has top } j \text{ rows nonzero} \end{aligned}$$

The last statement can be checked by iteratively checking the first column of  $H^i$ . Furthermore, this result indicates that  $Q_k$  forms a basis for  $K(A, q_1, k)$ .

### 7.2.2 Arnoldi process generates a minimal polynomial

For vector  $x \in K(A, q_1, k)$ :

$$\begin{aligned} x &= Q_k y = \sum_i y_i q_i, \text{ since } Q_k \text{ forms a basis of Krylov space} \\ &= \sum_i z_i A^{i-1} q_1 = \left( \sum_i z_i A^{i-1} \right) q_1 \\ &= p_{k-1}(A) q_1, \text{ a polynomial of degree } k-1 \end{aligned}$$

## 7.3 Lanczos process

The **Lanczos process** is a parallel process to the Arnoldi process, but for symmetric matrices. Reminder: A symmetric upper Hessenberg matrix,  $T$  is tri-diagonal. Note this tri-diagonal matrix is of the form

$$T = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ 0 & \ddots & \ddots & \ddots & \\ & & & & \ddots \end{bmatrix}$$

The process follows

1.  $\alpha_k = q_k^T A q_k \implies \alpha_k q_k = A q_k$
2.  $r_k = A q_k - \beta_{k-1} q_{k-1} - \alpha_k q_k \implies r_k = \beta_{k-1} q_{k-1}$ ,  $r_k$  becomes the orthogonal part of  $A q_k$
3.  $\beta_k$  PLACEHOLDER, NEED TO REVIEW TEXTBOOK

The orthogonalization in step 2 is reduced from  $O(k)$  in Arnoldi to  $O(1)$  in Lanczos because of the symmetry of  $A$ . Professor Darve notes the "magic" in this.

### 7.3.1 Process for revealing the max eigenvalue of $A$

$$\lambda(T_k) \approx \lambda(A)$$

$$\lambda_1 \in \lambda(T_k) = \max_{x \neq 0} \frac{y^T Q_k^T A Q_k y}{\|y\|_2^2}, \text{ by property that } \lambda_1 \in \lambda(A) = \max_{x \neq 0} \frac{x^T A x}{\|x\|_2^2}$$

$$\implies \text{ want max } x \text{ of the form } Q_k y$$

$$\implies \text{ want max } x \text{ in Krylov space, a subspace of } \mathbb{R}^k$$

$$\implies \lambda_1 \in \lambda(T_k) \leq \lambda_1 \in \lambda(A), \text{ since it is the max in a smaller space}$$

$$\lambda_1 \in \lambda(T_k) = \max_{x \neq 0} \frac{q_1^T p(A) A p(A) q_1}{q_1^T p(A)^2 q_1}, \text{ and see textbook for step from here to next step}$$

$$\implies \lambda_1 \in \lambda(T_k) \leq \lambda_1 - (\lambda_1 - \lambda_n) \left( \frac{\tan(\theta)}{T_{k-1}^{Cheb}(1 + 2p_1)} \right), \text{ where } p_1 = \frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_n}$$

Observe that the *RHS* approaches  $\lambda_1$  when  $\lambda_1$  is well separated from the other eigenvalues.