STATS219: Stochastic Processes

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1 Measure theory

1.1 Definitions of measure theory

1.1.1 Powerset (2^Ω)

Set of all possible subsets of Ω

1.1.2 σ-algebra ($\mathcal{F}$)

(a) $\Omega \in \mathcal{F}$
(b) $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$
(c) $A_1, \ldots, A_\infty \in \mathcal{F} \Rightarrow \cup_i^\infty A_i \in \mathcal{F}$

1.1.3 Probability space ($(\Omega, \mathcal{F}, P)$)

$P : \mathcal{F} \longrightarrow [0, 1]$ such that

(a) $0 \leq P(A) \leq 1 \; \forall A \in \mathcal{F}$
(b) $P(\Omega) = 1$
(c) $P(A) = \cup_i^\infty P(A_i)$ whenever $A = \cup_i^\infty A_i$ and $A_n \cap A_m = \emptyset$ for $n \neq m$

2 Random variables

2.1 Definitions of random variables

2.1.1 Random varible (X, "$\mathcal{F}$-measurable function")

$X : \Omega \longrightarrow \mathbb{R}$ such that

$$\{\omega : X(\omega) \leq \alpha\} \in \mathcal{F}, \; \forall \alpha \in \mathbb{R}$$

2.1.2 Indicator function ($\mathbb{I}_\mathbb{A}$)

A rv $\forall A \in \mathcal{F}$ such that

$$I_A(\omega) = \begin{cases} 1 & if \; \omega \in A \\ 0 & else \end{cases}$$

2.1.3 Borel function

$g : \mathbb{R} \longrightarrow \mathbb{R} \Longrightarrow g$ is an rv on $(\mathbb{R}, \mathcal{B})$

2.1.4 σ-algebra generator ($\sigma(\{A_\alpha\})$)

$$\begin{aligned} \sigma(\{A_\alpha\}) &= \cap \{\mathcal{G} : \mathcal{G} \subseteq 2^\Omega \; a\sigma - field, \; A_\alpha \in \mathcal{G}, \; \forall \alpha \in \Gamma, \text{ a countable or uncountable index}\} \\ \sigma(X) &= \mathcal{F}_x = \sigma(\{\omega : X(\omega) \leq \alpha\} \forall \alpha) \end{aligned}$$

2.1.5 Expectation of an rv ($E[X]$)

$$E[X] = \lim_{n \to \infty} \left[\sum_{k=0}^{\infty} x_{k,n} * P(\{\omega : X(\omega) \in I_{k,n}\}) \right] \text{ for } x_{k,n} = k2^{-n}, I_{k,n} = (x_{k,n}, x_{k+1,n}]$$

2.1.6 Independence

$$\text{events } A, B \text{ independent} \Longrightarrow P(A \cap B) = P(A)P(B) \text{ for } A, B \in \mathcal{F}$$
$$\sigma\text{-fields } \mathcal{H}, \mathcal{G} \subseteq \mathcal{F} \text{ independent} \Longrightarrow P(G \cap H) = P(G)P(H), \forall G \in \mathcal{G}, \forall H \in \mathcal{H}$$

2.1.7 Uncorrelated

$E(XY) = E(X)E(Y)$, for $X, Y \in L^2$

2.1.8 L^q spaces ($L^q(\Omega, \mathcal{F}, P)$)

The collection of all rv X on $(\Omega, \mathcal{F})$ where $E(|X|^q) < \infty$

2.1.9 Law of an rv ($\mathcal{P}_x$)

Probability measure on $(\mathbb{R}, \mathcal{B})$ such that $\mathcal{P}_x(B) = P(\{\omega : X(\omega) \in B\})$ for all $B \in \mathcal{B}$

2.1.10 Distribution function of an rv (F_X)

$F_X(\alpha) = P(\{\omega : X(\omega) \leq \alpha\}) = \mathcal{P}_X((-\infty, \alpha]) \ \forall \alpha \in \mathbb{R}$

2.1.11 Convergence and equality almost surely ($a.s.$)

$$X \overset{a.s.}{=} Y \Longleftrightarrow P(\{\omega : X(\omega) \neq Y(\omega)\}) = 0$$
$$X \overset{a.s.}{\leq} 0 \Longleftrightarrow P(\{\omega : X(\omega) > 0\}) = 0$$
$$X_n \overset{a.s.}{\to} X \Longleftrightarrow X_n(\omega) \to X(\omega) \text{ as } n \to \infty \ \forall \omega \in A \in \mathcal{F} \text{ with } P(A) = 1$$

2.1.12 Convergence in probability (p)

$$X_n \xrightarrow{p} X \Longleftrightarrow P(\{\omega : |X_n(\omega) - X(\omega)| > \epsilon\}) \longrightarrow 0 \text{ as } n \to \infty \ \forall \epsilon \in \mathbb{R}$$

2.1.13 Convergence in L^q (or in q-mean)

$$X_n \xrightarrow{q.m} X \Longleftrightarrow \|X_n - X\|_q = [E(|X_n - X|^q)]^{\frac{1}{q}} \longrightarrow 0 \text{ as } n \to 0, \text{ for } X_n, X \in L^q$$

2.1.14 Convergence in law (or weak convergence or in distribution)

$$X_n \xrightarrow{d} X \Longleftrightarrow F_{X_n}(\alpha) \longrightarrow F_X(\alpha) \text{ as } n \to \infty \ \forall \alpha \text{ continuity points of } F_X$$

3 Conditional expectation

3.1 Definitions of conditional expectation

3.1.1 In discrete space

$$f(y) := E(X|Y = y) = \frac{E(X * I\{Y = y\})}{P(Y = y)} \forall y \text{ requiring } P(Y = y) > 0$$

3.1.2 In L^2 space (Hilbert space)

$$Z = E(X|Y) \text{ unique random variable satisfying :}$$
$$Z \in H_Y = L^2(\Omega, \sigma(Y), P)$$
$$E[(X - Z)V] = 0 \ \forall V \in H_Y$$
$$\min_Z \{E[(X - Z)^2] : Z \in H_Y\} \text{ (interchangeable with line above it b/c of } L^2 \text{ properties)}$$

3.1.3 In L^1 space

$$Z = E(X|Y) \text{ such that :}$$
$$E[(X-Z)I\{Y=y\}] = 0, \forall y \in \mathbb{R}$$
$$E[(X-Z)I\{Y \in B\}] = 0, \forall B \in \mathcal{B}$$

3.2 Properties of conditional expectation

For $X, Y \in L^1$, σ-fields $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$

- $X \geq 0 \Longrightarrow E(X|\mathcal{G}) \overset{a.s.}{\geq} 0$
- If $\mathcal{G}, \sigma(X)$ independent $\Longrightarrow E(X|\mathcal{G}) = E(X)$
- If X is $\mathcal{G}$-measurable $\Longrightarrow E(X|\mathcal{G}) = X$
- $E[\alpha X + \beta Y|\mathcal{G}] = \alpha E(X|\mathcal{G}) + \beta E(Y|\mathcal{G})\ \forall \alpha, \beta \in \mathbb{R}$
- $E[E(X|\mathcal{G})|\mathcal{H}] = E(X|\mathcal{H})$
- $E[E(X|\mathcal{G})] = E(X)$
- If Y is $\mathcal{G}$-measurable and $X, XY \in L^1 \Longrightarrow E[XY|\mathcal{G}] = YE[X|\mathcal{G}]$

For $Var(Y|\mathcal{G}) := E(Y^2|\mathcal{G}) + E(Y|\mathcal{G})^2$, $Y \in L^2$

- If $Y \in L^2(\Omega, \mathcal{G}, P) \Longrightarrow Var(Y|\mathcal{G}) = 0$
- $Var(Y) = E[Var(Y|\mathcal{G})] + Var[E(Y|\mathcal{G})]$
- If $E(Y|\mathcal{G}) = X$ and $E(X^2) = E(Y^2) < \infty \Longrightarrow X \overset{a.s.}{=} Y$
- If $\mathcal{H} \subseteq \mathcal{G}$ and $X \in L^2(\Omega, \mathcal{F}, P) \Longrightarrow E[(X - E(X|\mathcal{G}))^2] \leq E[(X - E(X|\mathcal{H}))^2]$