STATS370: Final Project

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2022 Fall Quarter

1 Introduction

1.1 Description

- After a sample (or a weighted sample) has been generated by any method, you are expected to explore the posterior distribution of the parameters based on the sample. For example, compute quantiles, plot histograms, obtain means, variances and correlations.

Priors

$$\begin{aligned}
\theta &= (\sigma^2, \tau, \mu_1, \mu_2, \gamma_1, \gamma_2) \\
p(\sigma^2) &\propto \frac{1}{\sigma^2} \\
p(\tau) &\sim Unif[0,1] \\
p(\mu) &= p(\mu_1, \mu_2) \propto 1 \text{ (improper uniform)} \\
p(\gamma) &= p(\gamma_1, \gamma_2) \propto 1 \text{ (improper uniform)}
\end{aligned}$$

Gene expression distributions

$$\begin{aligned}
(y_i|g_i = 1) &\sim N(\mu, \sigma^2 I) \\
(y_i|g_i = 2) &\sim N(\gamma, \sigma^2 I) \\
(y_i|g_i = 3) &\sim N(\frac{1}{2}(\mu + \gamma), \sigma^2 I) \\
(y_i|g_i = 4) &\sim N(\tau\mu + (1-\tau)\gamma, \sigma^2 I)
\end{aligned}$$

Likelihood

$$
\begin{aligned}
L(Y|\theta) &= \prod_{i=1}^{n} p(y_i|\theta), \text{ for } Y = (y_i, \ldots, y_n) \\
&= \prod_{i \in g_1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(y_i - \mu)^T (y_i - \mu)] \times \prod_{i \in g_2} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(y_i - \gamma)^T (y_i - \gamma)] \\
&\quad \times \prod_{i \in g_3} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(y_i - \frac{1}{2}(\mu + \gamma))^T (y_i - \frac{1}{2}(\mu + \gamma))] \\
&\quad \times \prod_{i \in g_4} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(y_i - (\tau\mu + (1-\tau)\gamma))^T (y_i - (\tau\mu + (1-\tau)\gamma))] \\
&= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp[-\frac{1}{2\sigma^2}(\sum_{i \in g_1} (y_i - \mu)^T (y_i - \mu) + \sum_{i \in g_2} (y_i - \gamma)^T (y_i - \gamma) \\
&\quad + \sum_{i \in g_3} (y_i - \frac{1}{2}(\mu + \gamma))^T (y_i - \frac{1}{2}(\mu + \gamma)) + \sum_{i \in g_4} (y_i - (\tau\mu + (1-\tau)\gamma))^T (y_i - (\tau\mu + (1-\tau)\gamma)))]
\end{aligned}
$$

Posterior

$$p(\theta|Y) \propto p(\theta) L(Y|\theta)$$

2 Metropolis Hasting

3 Gibbs sampling

Posterior conditioned on each parameter

$$p(\theta[i]|Y, \theta[-i]) = \frac{p(\theta[i], \theta[-i]|Y)}{p(\theta[-i]|Y)} = f(\theta[i]) \propto p(\theta|Y) \text{ with fixed } \theta[-i], Y$$

Posterior conditional probability of τ

$$
\begin{aligned}
p(\tau|Y,\theta[-\tau]) &\propto p(\theta)L(Y|\theta) \\
&\propto p(\tau)\prod_{i\in g_4}\frac{1}{\sqrt{2\pi\sigma^2}}\exp[-\frac{1}{2\sigma^2}(y_i-(\tau\mu+(1-\tau)\gamma))^T(y_i-(\tau\mu+(1-\tau)\gamma))] \\
&\propto \exp[-\frac{1}{2\sigma^2}\sum_{i\in g_4}(y_i-(\tau\mu+(1-\tau)\gamma))^T(y_i-(\tau\mu+(1-\tau)\gamma))], \text{ for } \tau\in[0,1] \\
&\propto \exp[-\frac{1}{2\sigma^2}\sum_{i\in g_4}(y_i^Ty_i-2y_i^T(\tau\mu+(1-\tau)\gamma)+(\tau\mu+(1-\tau)\gamma)^T(\tau\mu+(1-\tau)\gamma))], \text{ for } \tau\in[0,1] \\
&\propto \exp[-\frac{1}{2\sigma^2}\sum_{i\in g_4}\tau^2(\mu^T\mu-2\mu^T\gamma+\gamma^T\gamma)-2\tau(y_i^T\mu-y_i^T\gamma-\mu^T\gamma+\gamma^T\gamma)], \text{ for } \tau\in[0,1] \\
&\propto \exp[-\frac{1}{2\sigma^2}(n_4\tau^2(\mu-\gamma)^T(\mu-\gamma)-2\tau(\mu-\gamma)^T[\sum_{i\in g_4}(y_i-\gamma), \text{ for } \tau\in[0,1] \\
&\propto \exp\left[-\frac{1}{2}*\frac{n_4(\mu-\gamma)^T(\mu-\gamma)}{\sigma^2}\left(\tau-\frac{(\mu-\gamma)^T(\sum_{i\in g_4}(y_i-\gamma))}{n_4(\mu-\gamma)^T(\mu-\gamma)}\right)^2\right], \text{ for } \tau\in[0,1] \\
&\sim Norm\left(\mu=\frac{(\mu-\gamma)^T(\sum_{i\in g_4}(y_i-\gamma))}{n_4(\mu-\gamma)^T(\mu-\gamma)},\sigma^2=\frac{\sigma^2}{n_4(\mu-\gamma)^T(\mu-\gamma)}\right), \text{ truncated to } [0,1]
\end{aligned}
$$

Posterior conditional probability of σ^2

$$
\begin{aligned}
p(\sigma^2|Y,\theta[-\sigma^2]) \propto & p(\theta)L(Y|\theta)\propto p(\sigma^2)\prod_{i=1}^n p(y_i|\theta) \\
\propto & \frac{1}{\sigma^2}*\left(\frac{1}{\sigma}\right)^n\exp\left[-\frac{1}{2\sigma^2}M\right], \text{ where } M= \\
& \sum_{i\in g_1}(y_i-\mu)^T(y_i-\mu)+\sum_{i\in g_2}(y_i-\gamma)^T(y_i-\gamma)+\sum_{i\in g_3}(y_i-\frac{1}{2}(\mu+\gamma))^T(y_i-\frac{1}{2}(\mu+\gamma)) \\
& +\sum_{i\in g_4}(y_i-(\tau\mu+(1-\tau)\gamma))^T(y_i-(\tau\mu+(1-\tau)\gamma)) \\
\propto & (\sigma^2)^{-\frac{n}{2}-1}\exp\left[-\frac{M}{2}\frac{1}{\sigma^2}\right] \\
\sim & InvGamma\left(\alpha=\frac{n}{2},\beta=\frac{M}{2}\right)
\end{aligned}
$$

Posterior conditional probability of μ

$$
\begin{aligned}
p(\mu|Y,\theta[-\mu]) \propto & p(\theta)L(Y|\theta) \\
\propto & \prod_{i\in g_1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(y_i-\mu)^T(y_i-\mu)] \times \prod_{i\in g_3} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(y_i-\frac{1}{2}(\mu+\gamma))^T(y_i-\frac{1}{2}(\mu+\gamma))] \\
& \times \prod_{i\in g_4} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(y_i-(\tau\mu+(1-\tau)\gamma))^T(y_i-(\tau\mu+(1-\tau)\gamma))] \\
\propto & \exp[-\frac{1}{2\sigma^2}(n_1\mu^T\mu - 2\mu^T\left(\sum_{i\in g_1} y_i\right) - \mu^T\left(\sum_{i\in g_3} y_i\right) + \frac{n_3}{4}\mu^T\mu + \frac{n_3}{2}\mu^T\gamma \\
& - 2\tau\mu^T\left(\sum_{i\in g_4} y_i\right) + \tau^2 n_4\mu^T\mu + 2\tau(1-\tau)n_4\mu^T\gamma)] \\
\propto & \exp\left[-\frac{1}{2\sigma^2}(n_1+\frac{n_3}{4}+n_4\tau^2)\left(\mu^T\mu - 2\mu^T\frac{\sum_{i\in g_1} y_i + \frac{1}{2}\sum_{i\in g_3} y_i + \tau\sum_{i\in g_4} y_i - (\frac{n_3}{4}+n_4\tau(1-\tau))\gamma}{n_1+\frac{n_3}{4}+n_4\tau^2}\right)\right] \\
\propto & \exp\left[-\frac{1}{2\phi^2}(\mu-\psi)^T(\mu-\psi)\right], \text{ where} \\
& \psi = \frac{\sum_{i\in g_1} y_i + \frac{1}{2}\sum_{i\in g_3} y_i + \tau\sum_{i\in g_4} y_i - (\frac{n_3}{4}+n_4\tau(1-\tau))\gamma}{n_1+\frac{n_3}{4}+n_4\tau^2}, \ \phi^2 = \frac{\sigma^2}{n_1+\frac{n_3}{4}+n_4\tau^2} \\
\sim & N(\mu=\psi, \Sigma=\phi^2 I)
\end{aligned}
$$

Posterior conditional probability of γ

By symmetry with posterior conditional probability of μ,

$$
\begin{aligned}
p(\gamma|Y,\theta[-\gamma]) \sim & N(\mu=\psi', \Sigma=\phi'^2 I), \text{ where} \\
& \psi' = \frac{\sum_{i\in g_2} y_i + \frac{1}{2}\sum_{i\in g_3} y_i + (1-\tau)\sum_{i\in g_4} y_i - (\frac{n_3}{4}+n_4\tau(1-\tau))\mu}{n_2+\frac{n_3}{4}+n_4(1-\tau)^2}, \ \phi'^2 = \frac{\sigma^2}{n_2+\frac{n_3}{4}+n_4(1-\tau)^2}
\end{aligned}
$$

4 Hamiltonian Monte Carlo

$$
\begin{aligned}
H(q,p) =& U(q) + K(p), \text{ Hamiltonian function} \\
& \text{with } U(\cdot), \text{ potential energy, } K(\cdot), \text{ kinetic energy, } q \in \mathbb{R}^k, \text{ position, } p \in \mathbb{R}^k \text{ momentum} \\
& \text{let } U(\theta) := -\log f(\theta), \text{ where } p(\theta|Y) = \frac{1}{C}f(\theta), \text{ and } p \sim N(0, \mathbf{M})
\end{aligned}
$$

5 Importance sampling