Data Structures and Algorithms II

Assignment 4

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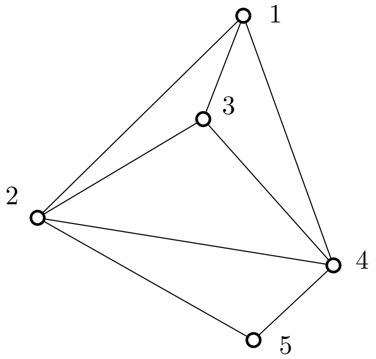
1 Task Description

Triangulation 3-coloring
Your are given a triangulation of a point set. Your task is to design an efficient algorithm that constructs a valid 3-coloring of the points of the triangulation or determines that such a 3-coloring does not exist. A 3-coloring of the points is valid if any two points that are connected with an edge have different colors. The n points of the triangulation are labeled with the integers $\{1, \ldots, n\}$. The triangulation is given by a list of edges with additional triangle points (see Figure 1 for an example):

- Every edge is given by the labels of its two end points (first the smaller point label, then the larger one).
- For every edge, the labels of the point(s) with which the edge forms a triangle (a bounded triangular face) in the triangulation is given (two labels for interior edges and one label for edges on the boundary of the convex hull).



edges	**triangle points**
$1-2$	3
$2-4$	$3, 5$
$3-4$	$1, 2$
$1-3$	$2, 4$
$4-5$	2
$2-5$	4
$2-3$	$1, 4$
$1-4$	3

Figure 1: Example of a triangulation and a list of its edges with triangle points.

Explain and describe your algorithm in detail, analyze its runtime and memory requirements, and give reasons for the correctness of your solution.

2 Description of algorithm

Note: It is assumed from the example, that the edges are always given with the lower vertex number in the first place e.g. 1-5 instead of 5-1. If this is not a given, the edge representation could be changed to conform to the above constraint in linear time, thus not increasing the asymptotic runtime.

General: The algorithm takes a list of edges with additional triangle points `EDGES` (as described above) as input and outputs a 3-coloring of the graph `COLORS` if it is possible. If no valid 3-coloring of the given graph exists, the algorithm exits and indicates that no such coloring is possible (`COLORS` is not returned in this case).

The algorithm starts with the **Setup** step, then the points on the first edge and it's triangle points are colored in **Start**, after which **Loop** takes care of the remaining points. Afterwards the validity of the coloring is checked.

1. **Setup:**
   First get number of points. The Maximum is given, by the maximum (number) in triangle points. Use this number to create an array `COLORS` with this size for the colors of the vertices and set each entry to NULL (= no color). Store edges (keys[hash]) and triangle points (+optional flags) (values) in a hashmap `EDGE_DIC` for fast access. Setup counter `counter` which tracks how many vertices need to be colored in. Setup empty queue `NEXTEDGES` for neighboring edges.

2. **Start:**
   An arbitrary start edge, for example the first in `EDGES` is chosen as `cur_edge`. The two endpoints p_1 and p_2 of `cur_edge` are colored with c_1 and c_2 respectively. Then the triangle points are colored[1] in. Finally the `counter` is decreased by the number of vertices which were colored in and `cur_edge` is deleted from `EDGE_DIC`.

3. **Loop:** The Loop is executed while `counter` is not 0.
   All neighbors of `cur_edge` which were not visited before (meaning they were never `cur_edge`) are added to the queue `NEXTEDGES`. Neighbors to the `cur_edge` are edges which form a triangle with the `cur_edge` if the two endpoints of the so given path are connected.[2] Now `cur_edge` is set to the last edge in `NEXTEDGES` (according to the FIFO-principle). The colors c_1 and c_2 of p_1 and p_2 of `cur_edge` are checked. If they are the same, algorithm terminates because no valid 3-coloring of the given graph exists. Otherwise the algorithm tries to color the triangle points of `cur_edge` (at least one [outer edge] and at most 2 triangle points for inner edges). If the triangle point is already colored with c_3, it is check if the color is neither c_1 nor c_2, if the color would match either c_1 or c_2 the algorithm terminates as in the aforementioned case c_1 equals c_2. If the the triangle point is properly colored nothing happens and if applicable the second triangle point is checked. In the case that the triangle point isn't colored already, it is colored with c_3 such that c_3 is different than c_2 and c_1. Finally the `counter` is decreased by the number of vertices which were colored in and `cur_edge` is deleted from `EDGE_DIC`. → Loop

4. **Check validity of coloring of remaining edges:**
   For the graph in Figure 1 **without Point 5** (without edges 2-5 and 4-5) the algorithm could produce a impossible coloring and end.[3] To avoid these edge cases one could check all edges and their triangle points for matching colors as described in 'Loop' above. However color-checking only the remaining edge-triangle-point-pairs

[1] Depending on the edge type either one (outer edge e.g. 1-2 in Figure 1) or a maximum of two triangle points (inner edge e.g. 1-3 in Figure 1) are colored in.

[2] This definition of neighbors guarantees that the so formed triangle already has two colored points, thus making the coloring of the third trivial.
Example: The edge 1-2 in Figure 1 has 2 neighbors (1-3 and 2-3) according to the above definition, whereas an inner edge has 4: e.g. 3-4 has 1-3, 1-4, 2-3 and 2-4 as neighbors

[3] The `counter` would reach 0 if the start edge would be 1-3 (The loop would be skipped and thus no checks would be done)

in `EDGE_DIC` is sufficient, because these are the edges which were never `cur_edge`. Asymptotically this reduction in remaining edge-triangle-point-pair color checks doesn't matter as we'll see below.

3 Space complexity / memory requirements

Note: As derived from Euler's formula in the lecture for a connected, simple, planar graph the following holds true for $v \geq 3$:

$$e \leq 3v - 6 \tag{1}$$

Where e is the number of edges, and v the number of vertices in the graph. For this reason asymptotically the number of edges as well as the number of vertices are the same ($\mathcal{O}(e) = \mathcal{O}(v)$). For this reason the analysis in the current section, as well as section 4 will use the size of the input, more precisely the number of edges n in `EDGES`.

The memory requirements of the algorithm is given as follows:

- `EDGES` [input] stores all n edges of the graph which results in $\mathcal{O}(n)$ space complexity.
- `COLORS` [output] stores asymptotically (as shown above) $\mathcal{O}(n)$ colors of the vertices which results in $\mathcal{O}(n)$ space complexity
- `EDGE_DIC` stores all the edges from the [input] `EDGES` as key/hash-value pair in a hashmap which requires $\mathcal{O}(n)$ space.
- `NEXTEDGES` stores at most $n - 1$ edges[4] which results in $\mathcal{O}(n)$ space complexity.
- Local variables: `counter` and `cur_edge` have $\mathcal{O}(1)$ space complexity.

All together the data structures require $\mathcal{O}(n)$ space.

4 Runtime complexity

The runtime complexity analysis is based on the number of edges n in the input `EDGES`. As shown above in Equation 1 n, being the number of edges, is asymptotically the same as v the number of vertices.
The Analysis is split into the same four parts as in section 2:

1. **Setup:** The number of the points is found by a linear scan over all n elements in `EDGES`, thus resulting in $\mathcal{O}(n)$ time complexity. Then `COLORS` and `EDGE_DIC` are allocated and filled with NULL and the edges with corresponding triangle points respectively, also resulting in $\mathcal{O}(n)$ time complexity each. The instantiation of `NEXTEDGES`, `counter` and `cur_edge` happens asymptotically in constant time.
   Time complexity of this section: $\mathcal{O}(n)$

[4] $n - 1$ is for example true in the trivial case with $n = 3$

2. **Start:** An edge is chosen from `EDGES` and assigned to `cur_edge` which takes constant time. Then the vertices of this edge as well as the triangle points (at most 2) are colored. This means that the corresponding colors in the `COLORS` array are set (at most 4) which takes $\mathcal{O}(4)$ time at most. Finally `counter` is decreased by the number of colored points (3 or 4) and `cur_edge` is deleted from `EDGE_DIC` which both take constant time.
   Time complexity of this section: $\mathcal{O}(1)$

3. **Loop:** The Loop is executed at most n-times because every point is at least a 'triangle point' to one edge. Per loop pass non or at most one triangle point gets colored. First, the neighboring edges to `cur_edge` are added to `NEXTEDGES`. In each pass at most 4 edges are added (adding elements into a queue with size n which takes at most n elements can always be achieved in constant time); It's important that edges are only added once to `NEXTEDGES`. This can be accomplished by setting a flag in the corresponding hashmap value in `EDGE_DIC` (takes constant time). Furthermore edges are only added if they were not visited before, this can be checked in constant time through the `EDGE_DIC`. Next `cur_edge` is set as the last edge from `NEXTEDGES` (this edge gets deleted in the queue [pop-operation]) which also happens in $\mathcal{O}(1)$ time.
   Now all the color checks of the points in the edge and the triangle point are performed. First the colors of the edge points are checked, then 2 checks (if triangle point color is the same as one of the edge points and if triangle point is already colored) are performed for each of at most 2 triangle points. This yields at most 5 checks which are all performed in constant time as well. Should colorable points exist, they are colored in constant time, as described above. Last but not least `counter` is decreased by the number of colored points (0 or 1) and `cur_edge` is deleted from `EDGE_DIC` which takes constant time. The overall time complexity of the loop is determined by the time complexity of each pass multiplied by the number of passes (n). Every step inside the loop happens in constant time, resulting in the following overall complexity:
   Time complexity of this section: $\mathcal{O}(n)$
   Note: If no valid 3-coloring of the graph exists the algorithm simply stops and skips the next step, thus not impacting the asymptotic runtime.

4. **Check validity of coloring of remaining edges:** As just described checking the validity of the coloring of an edge and it's triangle point(s) is accomplished in constant time. Time complexity wise checking all n edges would take $\mathcal{O}(n)$ time. Checking fewer, as stated in section 2, will indeed in decrease the runtime but will not have an effect on the asymptotic bound of $\mathcal{O}(n)$.
   Time complexity of this section: $\mathcal{O}(n)$

The overall runtime of the algorithm is now given by the sum of the aforementioned 4 parts which are run successively.
All together the time complexity of the algorithm is $\mathcal{O}(n)$.

5 Correctness of the algorithm

The proof of correctness is left to the tutor!