Homework Assignment 1

Data Structures and Algorithms I, WT 2021

Due: 22. 10. 2021

1. (2 Points) We use o-notation to denote an upper bound that is not asymptotically tight. We formally define $o(g(n))$ ("little-oh of g of n") as the set

$$o(g(n)) = \{\, f(n) \mid \text{for any positive constant } c > 0 \text{, there exists a constant } n_0 > 0 \text{ such that } 0 \leq f(n) < cg(n) \text{ for all } n \geq n_0 \,\}.$$

Similarly, we use ω-notation to denote a lower bound that is not asymptotically tight. We formally define $\omega(g(n))$ ("little-omega of g of n") as the set

$$\omega(g(n)) = \{\, f(n) \mid \text{for any positive constant } c > 0 \text{, there exists a constant } n_0 > 0 \text{ such that } 0 \leq cg(n) < f(n) \text{ for all } n \geq n_0 \,\}.$$

Using the definitions from above, prove that $o(g(n)) \cap \omega(g(n))$ is the empty set.

Solution: By definition, $o(g(n))$ is the set of functions $f(n)$ such that $0 \leq f(n) < c_1 g(n)$ for any positive constant $c_1 > 0$ and all $n \geq n_0$. And, $\omega(g(n))$ is the set of functions $f(n)$ such that $0 \leq c_2 g(n) < f(n)$ for any positive constant $c_2 > 0$ and all $n \geq n_0$. Hence, $o(g(n)) \cap \omega(g(n))$ is the set of functions $f(n)$ such that,

$$0 \leq c_2 g(n) < f(n) < c_1 g(n) \tag{1}$$

The above inequality cannot be true asymptotically as n becomes very large, $f(n)$ cannot be simultaneously greater than $c_2 g(n)$ and less than $c_1 g(n)$ for any constants $c_1, c_2 > 0$. Hence, no such $f(n)$ exists, that is, the intersection is indeed the empty set.

2. (6 Points) We use the notation $f^{(i)}(n)$ to denote the function $f(n)$ iteratively applied i times to an initial value of n. Formally, let $f(n)$ be a function over the reals. For non-negative integers i, we recursively define

$$f^{(i)}(n) = \begin{cases} n, & \text{if } i = 0, \\ f(f^{(i-1)}(n)), & \text{if } i > 0. \end{cases}$$

Let $\lg^* n = \min\{i \geq 0 \mid \lg^{(i)} n \leq 1\}$. Note that $\lg^{(i)} n$ is defined only if $\lg^{(i-1)} n > 0$. Which is asymptotically larger: $\lg(\lg^* n)$ or $\lg^*(\lg n)$? **Hint**: Note that $\lg^*(2^n) = 1 + \lg^*(n)$.

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\lg(\lg^*(n))}{\lg^*(\lg(n))} &= \lim_{n\to\infty} \frac{\lg(\lg^*(2^n))}{\lg^*(\lg(2^n))} && (1) \\
&= \lim_{n\to\infty} \frac{\lg(1 + \lg^*(n))}{\lg^*(n)} && (2) \\
&= \lim_{n\to\infty} \frac{\lg(1+n)}{n} && (3) \\
&= \lim_{n\to\infty} \frac{1}{1+n} && (4) \\
&= 0. && (5)
\end{aligned}$$

Hence, we have that $\lg^*(\lg(n))$ grows more quickly.

3. (6 Points) Using the basic definition of Θ-notation, prove $\forall a, b \in \mathbb{R}$ with $b > 0$, that n^b is an asymptotically tight bound for $(n+a)^b$.

Solution: To show that $(n+a)^b = \Theta(n^b)$ we have to find the constants $c_1, c_2, n_0 > 0$, such that

$$0 \leq c_1 n^b \leq (n+a)^b \leq c_2 n^b \ \forall n \geq n_0. \tag{1}$$

We have

$$\begin{aligned} n + a &\leq n + |a| \qquad (2) \\ &\leq 2n \text{ if } |a| \leq n \qquad (3) \end{aligned}$$

and

$$\begin{aligned} n + a &\geq n - |a| \qquad (4) \\ &\geq 1/2n \text{ if } |a| \leq 1/2n. \qquad (5) \end{aligned}$$

Since $b > 0$ the following holds as well:

$$\begin{aligned} 0 \leq (1/2n)^b \leq (n+a)^b \leq (2n)^b \qquad (6) \\ 0 \leq (1/2)^b n^b \leq (n+a)^b \leq 2^b n^b. \qquad (7) \end{aligned}$$

Hence, $c_1 = (1/2)^b, c_2 = 2^b$ and $n_0 = 2|a|$.