Introduction: Counting states which might have a special property (like same energy, number of particles, etc.) is one of the key abilities in statistical physics which are the foundation to express partition sum and derive thermodynamical quantities from microscopic properties. As warm up we look at two counting problems which we will come across in some future exercise classes.

In the third problem we will discuss the concept of equilibration and description via probability distributions.

1 Counting problem: Valid sequence of brackets

Consider bracket sequences $a(k, j)$ of k open and j closed brackets.
E.g. $a(3, 2) =$ `()((`, `)(()(`, `((())`, ...

- How many different bracket sequences $|a(k, j)|$ are there?

We call a bracket sequence **valid** if – when reading from left to right – there are always more (or equal) open brackets than closed brackets.

A sequence like "`(()()))(`" is not valid. A sequence like "`(()())(`" is valid.

We call such a sequence **closed** if the sequence is valid and has the same number of open and closed brackets.

- Find out the number of valid sequences with $n + m$ open brackets and $n - m$ closed brackets[1].

  Hint: Use a similar trick (reflection method) as in Bertrand's ballot theorem.

- Find out the number C_n[2] of closed sequences with n open and n closed brackets.

2 Counting problem: Telephone numbers

Given are ordinary phone numbers with N digits. Imagine you have a very old disc telephone. When dialing a number it costs you a different amount of time t_d) depending (not linearly) on the digit $d \in \{0, \ldots, 9\}$ you are dialing (e.g. dialing a 9 takes the longest time).

- **Binary warmup:** How many binary telephone numbers ($d \in \{0, 1\}$) have N digits and need a total time of T when dialing them on the binary disc phone? The total time is given by:

$$T = \sum_{j=1}^{N} t_{d_j}, \qquad \begin{cases} d_j \in \{0, 1\} & \text{binary case} \\ d_j \in \{0, \ldots, 9\} & \text{decimal case} \end{cases} \tag{1}$$

- How many telephone numbers $Z(N, T)$ have N digits and need a total time of T when dialing them on the decimal disc phone.
  Hint: For both the questions, think how to characterize those numbers which have the same total dial time T.

[1] These are the so-called Lobb numbers $L_{m,n}$.
[2] These are the so-called Catalan numbers are present in many combinatorial counting problems.

3 The dog-flea problem according to Ehrenfest

The two dogs Arko and Bello meet along the Mur promenade. Arko has N fleas, Bello has none before the meeting. The N (even natural number, e.g. $N = 50$) fleas represent distinguishable, classical particles! (s. chapter 6.1 in the lecture notes). Each second ($\Delta t = 1\text{s}$) one of the N fleas is selected randomly (uniformly flat probability distribution) and hops onto the other dog.

The aim of this task is to describe and analyze this process by statistical means.

- What will happen after very long time (at equilibrium)? How does the probability distribution in equilibrium look like? What is the probability to find 20 of 50 fleas on Bello?
- Start in an equilibrium state. How long does it take on average to gain the initial state (all fleas on Arko)?
- At which time the probability to have equal fleas on both dogs becomes bigger than zero?

Exercises modified from the original work licensed below.