5. GREEDY STRATEGY

⇒ with the SVD we're able to chose a reduces space V_N s.t. the average error $\Phi(y_N)$ is minimum

The POD produce a space V_N in which we can minimize the function

$$\Phi(y_N) := \frac{1}{M}\sum_{m=1}^{M} \inf_{v \in V_N} \|w_m - v\|_V^2 \quad \Rightarrow \quad \min \Phi(y_N) := \frac{1}{M}\sum_{m=1}^{M} \inf_{v \in V_N} \|w_m - v\|_V^2$$

This is a way to produce a space,

Hence, we chose the V_N space st the error on average is the smallest one → we can find other way

AIM_ we want to find this relation : $\|u_\delta(\mu) - u_N(\mu)\|_V < \varepsilon_{toe}$ → we have to produce a space in this way **GREEDY**

↳ To do this we need the generation of $M \gg N$ snapshots and the computation of SVD decomposition (expensive), even if it is off-line

↓ for this reason the greedy strategy is implemented ⇒ This strategy requires an **a posteriori** error estimator $\eta = \eta(\mu, V_N)$

5.1 Idea of greedy strategy.

We start from $P_H \subseteq P$, we pick one value of the parameter, compute the HF solution and if it is good, this enters in the set

↓

Starting from $V_N = \{\}$

We select $\mu_1 \in P_H$ and we compute $u_\delta(\mu_1)$ ⟶ we add $u_\delta(\mu_1)$ to V_N but to do that we have to modify it a little bit

Indeed, we compute $w_1 := \dfrac{u_\delta(\mu_1)}{\|u_\delta(\mu_1)\|_V}$, so we have just to normalize it before to put in the basis

↳ computed one each iteration ⇒ cheap!

Now we have $V_N := \text{span}\{w_1\}$ and $N = 1$

- Is the space enough? $\max_{\mu \in P} \|u_\delta(\mu) - u_N(\mu)\|_V < \varepsilon_{toe}$ so all the error has to be under a tolerance

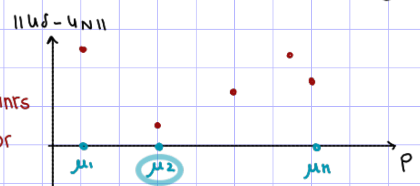
  - Yes we have finished
  - No we need an iterative algorithm to enlarge the space : $\mu_{i+1} = \arg\max_{\mu \in P_H} \|u_\delta(\mu) - u_N(\mu)\|_V$

So I take the parameter that gives me the largest error

$i = 0 \quad V_N = \{w_1 = u(\mu_1)/\|u(\mu_1)\|_V\}$

$i = 1 \quad V_N = \{w_1, w_2\}$

all the other points has biggest error



Starting from this parameter : here there is the smallest error

I compute the high fidelity solution $u_\delta(\mu_{i+1})$

Hence, I normalize $w_{i+1} := \dfrac{u_\delta(\mu_{i+1})}{\|u_\delta(\mu_{i+1})\|_V} \perp V_N$

NB Has to be orthogonal to all the other function inside the space V_N to do that we exploit **GRAM SCHMIDT**

So that $V_{N+1} := \text{span}\{V_N, w_{i+1}\}$

We check again the tolerance

BUT, we have to know all the value of the HF solution? No, we want to compute that in a cheaply way

So we will use a quantity that controls the error η, so that when this quantity $\eta < \varepsilon$ we know also that all our value are under the tolerance ⇒ we will use an a posteriori estimator that controls the error

$$\|u_\delta(\mu) - u_N(\mu)\|_V \le \eta(\mu, V_N) < \varepsilon_{toe}$$

⇒ $\eta(\mu)$ shall be computed for many values of $\mu \in P$, then we can find $\eta(\mu)$ cheaply

↳ is the **reduced basis approximation** of $u_\delta(\mu)$ in the space V_N

Why greedy is better than POD?

1. Controls the maximum error and not the average error (as POD)
2. Compute at most N solution
3. It is cheapier

5.2 A posteriori error computation

We recall the VP$_N(\mu)$ and we want to find $u_N(\mu) \in V_N$ ⇒ $a(u_N(\mu), v_N; \mu) = F(v_N; \mu) \quad \forall v_N \in V_N \subseteq V_\delta,\ \forall \mu \in P$

We introduce the **weak residual** : $R_\delta(u_N(\mu); \mu) \in V_\delta'$ → Stay in the dual of V_δ

$$\boxed{\langle R_\delta(u_N;\mu), v_\delta \rangle_{V_\delta' V_\delta}} := F(v_\delta;\mu) - \underbrace{a(u_N(\mu), v_\delta;\mu)}_{= r_\delta(u_N(\mu), v_\delta;\mu)} \quad \forall v_\delta \in V_\delta$$

from the definition of $\|\cdot\|_{V_\delta'}$, we have

This is a sort of bilinear function

$$\|R_\delta(u_N(\mu);\mu)\|_{V_\delta'} := \sup_{v_\delta \in V_\delta} \frac{r_\delta(u_N(\mu), v_\delta;\mu)}{\|v_\delta\|_{V_\delta}} \quad \forall v_\delta \in V_\delta$$

This is the duality pair

Property - It is true that: $\frac{1}{\gamma_\delta(\mu)} \|R_\delta(u_N(\mu);\mu)\|_{V_\delta'} \le \|u_\delta(\mu) - u_N(\mu)\|_V \le \frac{1}{\alpha_\delta(\mu)} \|R_\delta(u_N(\mu);\mu)\|_{V_\delta'}$

high-fidelity solution

reduce solution

coercivity constant

We are in the discrete formulation of the problem

Proof - From coercivity we have
$$\alpha_\delta(\mu)\|u_\delta(\mu)-u_N(\mu)\|_V^2 \le a(u_\delta(\mu)-u_N(\mu);u_\delta(\mu)-u_N(\mu);\mu)$$
$$= F(u_\delta(\mu)-u_N(\mu);\mu) - a(u_N(\mu), u_\delta(\mu)-u_N(\mu);\mu)$$
$$= \langle R_\delta(u_N(\mu);\mu), u_\delta(\mu)-u_N(\mu)\rangle_{V_\delta' V_\delta} \le \|R_\delta(u_N(\mu);\mu)\|_{V_\delta'}\|u_\delta(\mu)-u_N(\mu)\|_V$$

The other direction follow from the continuity hypothesis $\forall v_\delta \in V_\delta$
$$r_\delta(u_N(\mu), v_\delta;\mu) := a(u_\delta(\mu), v_\delta;\mu) - a(u_N(\mu), v_\delta;\mu) = a(u_\delta(\mu)-u_N(\mu), v_\delta;\mu) \le \gamma_\delta(\mu)\|u_\delta(\mu)-u_N(\mu)\|_V\|v_\delta\|_V$$
Using the definition of $\|R_\delta(u_N(\mu),\mu)\|_{V'}$.

It is good if it can be computer faster (cheaper!)

Now, using the upper bound of the property, we can derive an **a-posteriori error estimator**. Let's make it computable using the Riesz' representation theorem

Riesz' Representation Theorem - $\exists\, \hat{r}_\delta(u_N(\mu);\mu) \in V_\delta$ s.t. $\langle \hat{r}_\delta(u_N(\mu);\mu), v_\delta\rangle_V = \langle R_\delta(u_N(\mu),\mu), v_\delta\rangle_{V_\delta' V_\delta}$ $\forall v_\delta \in V_\delta$
and $\|\hat{r}_\delta(u_N(\mu);\mu)\|_V = \|R_\delta(u_N(\mu),\mu)\|_{V_\delta'}$

in the original space V_δ

Duality pair

The dual space V_δ'

NB
1. The computation of $\hat{r}_\delta(\mu) := \hat{r}_\delta(u_N(\mu);\mu)$ requires to solve an high-fidelity problem (expensive)
   However, in the affine case the computation can be efficient
   but we can perfore offline-online phase
2. The error a-posteriori estimation $\|u_\delta(\mu)-u_N(\mu)\|_V \le \frac{1}{\alpha(\mu)}\|\hat{r}_\delta(\mu)\|_V := \eta_V(\mu)$ requires also the computation of $\alpha(\mu)$ → we need to know for all the parameter
   This can be accomplished with the lower bound estimation $\alpha_{LB}(\mu) \le \alpha(\mu)$ cheap to be computed

To use in algorithm with relative tolerance

There is the possibility to compute also an **a-priori relative error estimator**, observing that
$$\|u_N(\mu)\|_V = \|u_\delta(\mu) + u_N(\mu) - u_\delta(\mu)\|_V \le \|u_\delta(\mu)\|_V + \|u_N(\mu) - u_\delta(\mu)\|_V \le \|u_\delta(\mu)\|_V + \eta_V(\mu)$$
If we define $\eta_{rel,V}(\mu) := \frac{2\,\eta_V(\mu)}{\|u_N(\mu)\|_V}$, we obtain

→ Norm of the reduce solution

$$\|u_N(\mu)\|_V \le \|u_\delta(\mu)\|_V + \frac{1}{2}\|u_N(\mu)\|\,\underbrace{\eta_{rel,V}(\mu)}_{\le 1} \le \|u_\delta(\mu)\|_V + \frac{1}{2}\|u_N(\mu)\|_V \quad \Rightarrow \quad \boxed{\|u_N(\mu)\|_V \le 2\|u_\delta(\mu)\|_V}$$

$$\frac{\|u_\delta(\mu)-u_N(\mu)\|_V}{\|u_\delta(\mu)\|_V} \le \frac{\eta_V(\mu)}{\frac{1}{2}\|u_N(\mu)\|_V}$$

Property

1. $\boxed{\|u_\delta(\mu)-u_N(\mu)\|_V \le \eta_V(\mu) := \frac{\|\hat{r}_\delta(\mu)\|_V}{\alpha(\mu)}}$
2. $\boxed{\frac{\|u_\delta(\mu)-u_N(\mu)\|_V}{\|u_\delta(\mu)\|_V} \le \eta_{rel,V}(\mu) := \frac{1}{2}\frac{\eta_V(\mu)}{\|u_N(\mu)\|_V}}$
3. $\mathrm{eff}_V(\mu) := \frac{\eta_V(\mu)}{\|u_\delta(\mu)-u_N(\mu)\|_V}$

   $\mathrm{eff}_{rel,V}(\mu) := \frac{\eta_{rel,V}(\mu)\|u_\delta(\mu)\|_V}{\|u_\delta(\mu)-u_N(\mu)\|_V}$

Algorithm 1 Greedy Algorithm for Reduced Basis Construction in POD

Initialization:
Choose a set of training parameters (or a continuous parameter domain) $\mathcal{P}_H$.
Select an initial parameter $\mu_1 \in \mathcal{P}_H$.
Compute the high-fidelity solution $u(\mu_1)$.
Define the initial reduced basis $V_1 = \mathrm{span}\{u(\mu_1)\}$ (or its normalization w_1).
Set $N = 1$.
Define an error tolerance ϵ_{tol}.
while $\max_{\mu\in\mathcal{P}_H} ||u(\mu) - u_N(\mu)||_{E_{tol}} > \epsilon_{tol}$ **do**
    Find the parameter $\mu_{max} \in \mathcal{P}_H$ that maximizes the estimated error: $\mu_{max} = \arg\max_{\mu\in\mathcal{P}_H} ||u(\mu) - u_N(\mu)||_{E_{tol}}$ where $u_N(\mu)$ is the approximated solution in the space V_N.
    Compute the high-fidelity solution for this parameter: $u(\mu_{max})$.
    Orthogonalize $u(\mu_{max})$ with respect to the functions already in V_N using Gram-Schmidt to obtain a new basis function w_{N+1}.
    Extend the reduced basis: $V_{N+1} = \mathrm{span}\{V_N, w_{N+1}\}$.
    Increment the counter: $N = N + 1$.
end while
Result:
The final reduced basis $V_N = \mathrm{span}\{w_1, w_2, ..., w_N\}$.