15. NOT AFFINE PROBLEM

Affinity property: consider $g: \bar{\Omega} \times P \longrightarrow \mathbb{R}$, $(u, \mu) \longmapsto g(x, \mu)$ that can be split in this formulation $g(x,\mu) = \sum_{q=1}^{Q} \vartheta_q(\mu) g_q(x)$

be able to do offline vs online phase

BUT if we do not have this hypothesis?
1. NN
2. Interpolation with Empirical Interpolation Method

15.1 EIM: empirical interpolation method

We want to exploit a NOT-AFFINE approximation $\boxed{g(x,\mu) = \sum_{q=1}^{Q} \vartheta_q(\mu) g_q(x)}$

To do that, we have to define the interpolation operator $\mathcal{I}_Q^x : \mathcal{G} \longrightarrow X_Q^{EIM}$ where

$\mathcal{G} = \{ g(\cdot, \mu) : \mu \in P \} \subset C^0(\bar{\Omega})$

X_Q^{EIM} of dimension Q generated by $\{ h_1(x), \dots, h_Q(x) \}$

such that, the operator can be written as

$$\mathcal{I}_Q^x : \mathcal{G} \longrightarrow X_Q^{EIM}$$
$$g(x,\mu) \longmapsto (\mathcal{I}_Q^x g)(x,\mu) := \sum_{q=1}^{Q} \vartheta_q(\mu) h_q(x) \quad \forall \mu \in P$$

$\longrightarrow$ IDEA: via interpolation, find an approximation of g that is affine so that we can do offline/online

$\{\vartheta_q(\mu)\}_{q=1,\dots,Q}$ are computed $(\mathcal{I}_Q^x g)(t^q, \mu) = g(t^q, \mu) \quad \forall \mu \in P, \ \forall q \in \{1, \dots, Q\}$

This interpolation points are called magic points := points in which the rule of interpolation holds

We do that with a greedy strategy

AIM: minimize the error of the interpolator operator

Initialization

1. Select some parameter $\mu^1_{EIM} := \underset{\mu \in P}{\operatorname{argmax}} \| g(\cdot, \mu) \|_{L^\infty(\Omega)}$ — can be also the forcing term
2. Create a set to collect the parameter $S_1 := \{ \mu^1_{EIM} \}$
3. Define the function $\xi_1(x) := g(x, \mu^1_{EIM})$
4. Define the magic point $t^1 := \underset{x \in \bar{\Omega}}{\operatorname{argmax}} | \xi_1(x) |$
5. Define a set to collect the magic point $T_1 = \{ t^1 \}$
6. Define the basis function $h_1(x) := \dfrac{\xi_1(x)}{\xi_1(t^1)} \Rightarrow X_1^{EIM} = \operatorname{span}\{h_1\}$

Now, this is an iterative method, hence we have to define the q-th step

Iterative orthogonalization steps

7. We have $S_q = \{ \mu^1_{EIM}, \dots, \mu^q_{EIM} \}$, $\{ \xi_1(x), \dots, \xi_q(x) \}$, $T_q = \{ t^1, \dots, t^q \}$, $X_q^{EIM} = \{ h_1(x), \dots, h_q(x) \}$
8. Compute new parameter $\mu^{q+1}_{EIM} := \underset{\mu \in P}{\operatorname{argmax}} \| g(\cdot, \mu) - (\mathcal{I}_q^x)(\cdot, \mu) \|_{L^\infty(\Omega)}$
9. Enlarge $S \longrightarrow \xi_{q+1}(x) := g(x, \mu^{q+1}_{EIM})$, $S_{q+1} := S_q \cup \{ \mu^{q+1}_{EIM} \}$
10. Update the magic point $\longrightarrow t^{q+1} := \underset{x \in \bar{\Omega}}{\operatorname{argmax}} \left| \xi_{q+1}(x) - (\mathcal{I}_q^x \xi_{q+1})(x) \right|$ the parameter is fixed
11. Update the basis $h_{q+1}(x) := \dfrac{\xi_{q+1}(x) - (\mathcal{I}_q^x \xi_{q+1})(x)}{\xi_{q+1}(t^{q+1}) - (\mathcal{I}_q^x \xi_{q+1})(t^{q+1})}$

12. Than $X^{EIM}_{q+1} := X^{EIM}_{q} \cup \text{span}\{h_{q+1}\}$

13. Hence, the stopping criteria is $\max_{\mu \in P} \|g(x,\mu) - (\mathcal{I}^{x}_{q}\, g)(x,\mu)\|_{L^\infty(\Omega)} < \varepsilon_{tol}$

How fast this approximation converges to the real function?

This property holds

$$\boxed{\sup_{\mu \in P} \|g(\cdot,\mu) - (\mathcal{I}^{x}_{Q}\, g)(\cdot,\mu)\|_{L^\infty(\Omega)} < c\, e^{-\alpha Q}}$$

Is this the best method?

If the non-linearity is very complex, this method is NOT a good method via computation error of approximation

Indeed every time we do

- $\arg\max_{\mu \in P}$ we have to discretize the space P with $P^{EIM}_{train} \subset P$
- $\arg\max_{x \in \Omega}$ with $\bar{\Omega}_h \subseteq \Omega$