16 — OPTIMAL CONTROL PROBLEMS

In this setting the PDE is: a state system needs to be OPTIMIZED with some "external help", i.e. other variables called CONTROLS

The goal of OCP is minimize an objective functional to reach beneficial configuration
related to known data

INPUT ⟶ FUNCTIONAL TO MINIMIZE ⟶ OPTIMAL STATE
data of the PDE

control, no more a snapshot

Then, our problem is: given a control $u \in \mathcal{U}$ with $\mathcal{U}$ a Hilbert space, the state variable $y(u) \in \mathcal{Y}$, with $\mathcal{Y}$ an Hilbert space, verifies the strong form of the controlled PDE

1 $\boxed{G(y(u), u; \mu) = 0}$ in $\mathcal{Y}'$ with $\mu :=$ physic or geometric parameter

The main goal is to minimize over $\mathcal{Y} \times \mathcal{U}$

$J : \mathcal{Y} \times \mathcal{U} \times P \longrightarrow \mathbb{R}$ defined as

$\boxed{J(y(u), u; \mu) = \frac{1}{2} \| y(u) - y_d \|^2_{\mathcal{Y}_d} + \frac{\alpha}{2} \| u \|^2_{\mathcal{U}}}$ such that the strong form holds ⟶ The function is quadratic to garantee the uniqueness

where
- $y_d \in \mathcal{Y}_d \supset \mathcal{Y}$ is the observation (is bigger than $\mathcal{Y}$ because you required less property)
- $\alpha > 0$ is the penalization parameter ⟶ $\alpha :=$ small big control; $\alpha :=$ big small control

In an abstract way, an OCP can be recast as

$$\boxed{\begin{aligned} &\text{find } u^* := \underset{u \in \mathcal{U}}{\operatorname{argmin}}\ J(y(u), u; \mu) \\ &\qquad \text{s.t. } G(y(u), u; \mu) = 0 \end{aligned}}$$

$u^* :=$ the optimal control
$y^*(u^*) :=$ the optimal state
fixing a parameter, if you change μ you have to recompute everything

From now on, we work with linear PDEs:

$\boxed{G(y(u), u; \mu) = K(\mu) y - C(\mu) u - f(\mu) = 0}$ where
- $C(\mu) : \mathcal{U} \longrightarrow \mathcal{Y}'$, continuous
- $K(\mu) : \mathcal{Y} \longrightarrow \mathcal{Y}'$, continuous, positive definite — coercive problem
- $f(\mu) \in \mathcal{Y}'$

The related bilinear form are
- $a : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$
- $c : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R}$
- $f : \mathcal{Y} \rightarrow \mathbb{R}$

and the weak formulation of 1 is

2 $\boxed{a(y, v; \mu) - c(u, v; \mu) - f(v; \mu) = 0 \quad \forall v \in \mathcal{Y}}$

But also the functional can be written in "weak form", rewriting the norm into integral

$\boxed{J(y, u; \mu) = \frac{1}{2} m(y - y_d, y - y_d; \mu) + \frac{\alpha}{2} n(u, u; \mu)}$ where
- $m : \mathcal{Y}_d \times \mathcal{Y}_d \rightarrow \mathbb{R}$ is $\mathcal{Y}_d$-norm product
- $n : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}$ is $\mathcal{U}$-inner product

Example. $\mathcal{Y}_d = L^2(\Omega_o)$ $\Omega_o \subseteq \Omega$, $\mathcal{U} = L^2(\Omega_u)$ $\Omega_u \subseteq \Omega$

Then J in strong form is $J(y, u; \mu) = \boxed{\frac{1}{2} \| y - y_d \|^2_{L^2(\Omega_o)}} + \frac{\alpha}{2} \| u \|^2_{L^2(\Omega_u)}$

$\downarrow$
$(y - y_d, y - y_d)_{L^2(\Omega_o)}$
$\downarrow$
$\int_{\Omega_o} (y - y_d)(y - y_d)\, dx$

What happen if I add another term? Consider $J(y, u; \mu) = \frac{1}{2} m(y - y_d, y - y_d; \mu) + \frac{\alpha}{2} n(u, u; \mu) - \frac{1}{2} m(y_d, y_d)$, but the solution remain the same

To solve I use The Lagrangian Approach

Let us define $p \in Y$ (same regularity of y) and define the lagrangian functional:

(3)
$$\boxed{\begin{aligned}\mathcal{L}(y,u,p;\mu) : Y \times U \times Y \times P &\longrightarrow \mathbb{R} \\ (y,u,p;\mu) &\longmapsto J(y,u;\mu) + \underbrace{a(y,p) - c(u,p) - f(p)}\end{aligned}}$$

testing the weak formulation on p

$\downarrow$ in strong formulation

$\langle \underbrace{K(\mu)y - C(\mu)u - F(\mu)}_{\in Y'}, p \rangle_Y$

where p is the adjoint variable and the role is the Lagrangian multiplier

We can derive the necessary conditions differentiating the Lagrangian and imponing the quantities to zero.

Explicity,

$$\begin{aligned}\mathcal{L}(y,u,p;\mu) &= \tfrac{1}{2}m(y-y^d, y-y^d) + \tfrac{\alpha}{2}n(u,u) + a(y,p;\mu) - c(u,p;\mu) - f(p;\mu) - \tfrac{1}{2}m(y^d,y^d) \\ &= \tfrac{1}{2}m(y,y) - m(y^d,y) + \tfrac{1}{2}m(y^d,y^d) + \tfrac{\alpha}{2}n(u,u) + a(y,p;\mu) - c(u,p;\mu) - f(p;\mu) - \tfrac{1}{2}m(y^d,y^d) \\ &= \boxed{\tfrac{1}{2}m(y,y)}\ \boxed{-m(y^d,y)} + \tfrac{\alpha}{2}n(u,u)\ \boxed{+a(y,p;\mu)} - c(u,p;\mu) - f(p;\mu)\end{aligned}$$

The conditions are related to the derivatives:

(1) $D_y \mathcal{L}(y,u,p;\mu)[q] = \boxed{\tfrac{1}{2}m(q,y) + \tfrac{1}{2}m(y,q)}\ \boxed{-m(y^d,q)}\ \boxed{+a(q,p;\mu)}$ $\longrightarrow$ all the other terms does not depend on y

direction of differentiation

This is the way to variational differentiation $y \cdot y = 1 \cdot y + y \cdot 1$ Identity, but we compute on q so that is the reason why a put q in $m(q,y)$ or $m(y,q)$

$= \underbrace{m(y,q)}_{L^2 \text{ product between } y \,\&\, q} - m(y^d,q) + a(q,p;\mu) \rightarrow a(q,p;\mu)$ the test function is on the left, this is not good, we have to switch

If the operator is not symmetric, we have to built the adjoint operator of a to put q on the right

Hence

$D_y \mathcal{L}(y,u,p;\mu)[q] = m(y-y^d, q) + a^*(p,q;\mu)$ with a^* the adjoint operator, i.e. $a^*(p,q) = a(q,p)$

<u>Example</u>. Linear operators $\langle Aq, p\rangle = \langle q, A^T p\rangle \longleftrightarrow a(q,p) = a^*(p,q)$

i.e. $A :=$ grad$\cdot$grad $\longrightarrow \int_\Omega \nabla q \nabla p \, dx = \int_\Omega \nabla p \nabla q \, dx \qquad a(p,q) = a(q,p)$ symmetric

$A :=$ convection $\longrightarrow \int_\Omega \frac{\partial q}{\partial x} p \, dx = \underbrace{\cancel{+\int_\Omega \frac{d}{dx}(qp)}}_{=0 \text{ for boundary condition}} - \int_\Omega \frac{\partial p}{\partial x} q \, dx = a^*(q,p) \neq a(p,q)$

(2) $D_u \mathcal{L}(y,u,p;\mu)[v] = \frac{\alpha}{2}n(v,u) + \frac{\alpha}{2}n(u,v) - c(v,p) = \alpha n(u,v) - c(v,p) = \alpha n(u,v) - c^*(p,v) = 0$

(3) $D_p \mathcal{L}(y,u,p;\mu)[z] = a(y,z;\mu) - c(u,z,\mu) - f(z) = 0$

we obtain the constrain equal to zero, hence we obtain the week formulation

The minimum has reached at the stationary point of the Lagrangian, imposing

$$\begin{cases} D_y \mathcal{L}(y,u,p;\mu)[q] = 0 & \forall q \in Y & \text{adjoint equation} \\ D_u \mathcal{L}(y,u,p;\mu)[v] = 0 & \forall v \in U & \text{optimality equation} \\ D_p \mathcal{L}(y,u,p;\mu)[z] = 0 & \forall z \in Y & \text{state equation} \end{cases}$$

Namely

$$\begin{cases} m(y,q) + a^*(p,q;\mu) = m(y^d,q) & \forall q \in Y \\ \alpha n(u,v) - c^*(p,v;\mu) = 0 & \forall v \in U \\ a(y,z;\mu) - c(u,z;\mu) = f(z,\mu) & \forall z \in Y \end{cases}$$

Hence, <u>matrix wise</u> (FOM)
$$\begin{cases} M_\delta(\mu) y + K_\delta^T(\mu) p = M_\delta(\mu) y_d \\ \alpha N_\delta(\mu) u - C_\delta^T(\mu) p = 0 \\ K_\delta(\mu) y - C_\delta(\mu) u = f_\delta(\mu) \end{cases}$$
finite element matrix

Algebrically, in compact form

$$\begin{array}{c}\text{observation vector}\end{array}$$

$$\begin{bmatrix} \boxed{\begin{matrix} M_\delta(\mu) & 0 \\ 0 & \alpha N_\delta(\mu) \end{matrix}} & \boxed{\begin{matrix} K_\delta^T(\mu) \\ -C_\delta^T(\mu) \end{matrix}} \\ \boxed{\begin{matrix} K_\delta(\mu) & -C_\delta(\mu) \end{matrix}} & 0 \end{bmatrix} \begin{bmatrix} y \\ u \\ p \end{bmatrix} = \begin{bmatrix} \boxed{M_\delta(\mu) y_d} \\ 0 \\ f_\delta \end{bmatrix}$$

Saddle-point structure. Consider the problem that is in this structure, we can write

$$\begin{bmatrix} \boxed{A(\mu)} & \boxed{B^T(\mu)} \\ \boxed{B(\mu)} & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} g_\delta \\ f_\delta \end{bmatrix} \quad \text{where } x = \begin{bmatrix} y \\ u \end{bmatrix}, \quad g_\delta = \begin{bmatrix} M_\delta(\mu) y_d \\ 0 \end{bmatrix}$$

- $A(\mu)$ is always well-poseness
- For $B(\mu)$, the well-poseness is related to the inf-sup stability and it is provable if $p \in Y$

↓

ROM level. Collect three sets of snapshots and build three different basis matrices $\mathbb{B}_y, \mathbb{B}_u, \mathbb{B}_p$
- y_N spamed by $\mathbb{B}_y$
- u_N spamed by $\mathbb{B}_u$
- p_N spamed by $\mathbb{B}_p \neq \mathbb{B}_y$ → How can I recover the inf-sup aggregated spaces?

We define $\mathbb{B}_{yp} = [\mathbb{B}_y \mid \mathbb{B}_p]$ the aggregated space and consider y_N, p_N spamed by $\mathbb{B}_{yp}$

↓

In order to garantee well-poseness, we need $3N$ basis wrt N basis without control!!

Once we build the basis, assuming affinity, we can pre-compute

$M_N = \mathbb{B}_{yp}^T M_\delta \mathbb{B}_{yp}$

$N_N = \mathbb{B}_u^T N_\delta \mathbb{B}_u$

$K_N = \mathbb{B}_{yp}^T K_\delta \mathbb{B}_{yp}$

$C_N = \mathbb{B}_{yp}^T C_\delta \mathbb{B}_u$

and solve

$$\begin{pmatrix} M_N & 0 & K_N^T \\ 0 & \alpha N_N & C_N^T \\ K_N & C_N & 0 \end{pmatrix} \begin{pmatrix} y_N \\ u_N \\ p_N \end{pmatrix} = \begin{pmatrix} M_N y_d^N \\ 0 \\ f_N \end{pmatrix} \quad \text{where} \quad y_d^N = \mathbb{B}_{yp}^T y_d, \quad f_N = \mathbb{B}_{yp}^T f_\delta$$

10.1 A very simple practice example

Consider $\Omega = [0,1] \times [0,1]$, $y_d \in L^2(\Omega_0)$, $\Omega_0 \subseteq \Omega$, $\mu \in P \subset \mathbb{R}$, $\alpha > 0$

Given a $\mu \in P$, find $(y,u) \in H_0^1(\Omega) \times L^2(\Omega)$ s.t. they verify

$$\min_{(y,u)} \boxed{\frac{1}{2} \| y - y_d \|^2_{L^2(\Omega_0)}} + \boxed{\frac{\alpha}{2} \| u \|^2_{L^2(\Omega)}}$$

$$\text{s.t.} \quad \begin{cases} -\mu \Delta y = u + f & \text{in } \Omega \\ y = 0 & \text{in } \partial\Omega \end{cases}$$

Let us define $p \in H_0^1(\Omega)$ and the Lagrangian

$$\mathcal{L}(y,u,p;\mu) = \boxed{\frac{1}{2} \int_{\Omega_0} (y - y_d)^2 \, dx} + \boxed{\frac{\alpha}{2} \int_\Omega u^2 dx} + \underbrace{\boxed{\mu \int_\Omega \nabla y \nabla p \, dx}}_{a(y,p)} \underbrace{\boxed{- \int_\Omega u p \, dx}}_{-c(u,p)} - \int_\Omega f p \, dx$$

Now we have to differenciate

$$D_y \mathcal{L}(y,u,p;\mu)[q] = \int_{\Omega_0} (y - y_d) q \, dx + \underbrace{\mu \int_\Omega \nabla q \nabla p \, dx}_{\text{self adjoint}}$$

$$D_u \mathcal{L}(y,u,p;\mu)[v] = \alpha \int_\Omega u v \, dx \underbrace{- \int_\Omega p v \, dx}_{\text{self adjoint}}$$

$$D_p \mathcal{L}(y,u,p;\mu)[z] = \mu \int_\Omega \nabla y \nabla z \, dx - \int_\Omega u z \, dx - \int_\Omega f z \, dx \longrightarrow \text{This is the variational formulation}$$

⚠ If you have an advection term $a(y,z;\mu) = \int_\Omega \beta \nabla y \, z \, dx = \int_\Omega \operatorname{div}(\beta y z) \, dx - \int \beta \nabla z y \, dx = \underbrace{\cancel{\int_{\partial\Omega} \beta y z n \, ds}}_{=0 \text{ b.c.}} - \int_\Omega \beta \cdot \nabla z y \, dx$

$= a^*(z,y;\mu)$