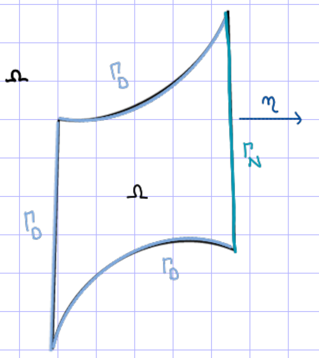
12 - STOKES EQUATIONS in HDR → we're in a linear setting

Consider the domain Ω and we want to study (find)
- The fluid velocity (vectorial variable) $u = (u_1, u_2)$ on Ω
- The pressure



Make some assumption
- H1. The fluid is incompressible → volume costant in time, it holds $\nabla \cdot u = 0$
- H2. Do NOT consider the time: stationary system

Let us define the system

$$(1)\quad \boxed{\begin{cases} -\gamma\,\Delta u + \nabla p = f & \text{in } \Omega \\ \nabla\cdot u = 0 & \text{in } \Omega \\ u = g & \text{in } \Gamma_D \subseteq \partial\Omega \neq 0 \\ -\gamma(\nabla u)^T \eta + p\eta = \Psi & \text{in } \Gamma_N := \partial\Omega \setminus \Gamma_D \end{cases}}$$

$\gamma :=$ viscosity, that for WE it is a constant

→ strong boundary condition: we need to have a unique solution (Dirichlet)

→ Neuman boundary condition

If this boundary exist, we fix some condition on the pressure
So if this boundary does NOT exist, we solve the system and find the solution of p up to a constant

i.e. If $\Gamma_N = \phi$, for example you can state
- $p = 0$ on a point of Ω
- $\int_\Omega p = 0$ on Ω

to solve the uniqueness of the solution

Definition of the term

The definition of $\nabla u \in \mathbb{R}^{2,2}$ → $(\nabla u)_{ji} = \dfrac{\partial u_i}{\partial x_j} = (J u)_{ij}$ because it's the transpose of the jacobian

The divergence is $\nabla \cdot u = \sum_{i=1}^{2} \partial x_i u_i = \partial_x u_1 + \partial_y u_2$

The last definitio is $\boxed{\Delta u := \underbrace{\nabla(\nabla\cdot u)}_{0} - \nabla\times(\nabla\times u) = -\nabla\times(\nabla\times u)}$, where the vectorial product is defined as

$$\nabla\times u := \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ u_1 & u_2 & 0 \end{vmatrix} = i(-\partial_z u_2) - j(-\partial_z u_1) + k(\partial_x u_2 - \partial_y u_1) = (0, 0, \partial_x u_2 - \partial_y u_1)$$

Hence, doing the computation we have

$$\nabla\times(\nabla\times u) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \partial_x u_2 - \partial_y u_1 \end{vmatrix} = i(\partial_y\partial_x u_2 - \partial_y\partial_y u_1) - j(\partial_x\partial_x u_2 - \partial_x\partial_y u_1) + k\,0$$

Then $\nabla\cdot u = 0 \iff \partial_x u_1 + \partial_y u_2 = 0 \iff \partial_x u_1 \overset{*}{=} -\partial_y u_2$

$$\longrightarrow (-\partial_y^2 u_1 + \underbrace{\partial_y\partial_x u_2}_{\text{we remove } u_2}, -\partial_x^2 u_2 + \underbrace{\partial_x\partial_y u_1}_{\text{we remove } u_1}, 0) \overset{*}{=} (-\partial_y^2 u_1 - \partial_x^2 u_1, -\partial_x^2 u_2 - \partial_y^2 u_1, 0) = -(\Delta u_1, \Delta u_2)$$

Hence, returning to $\Delta u := \nabla(\nabla\cdot u) - \nabla\times(\nabla\times u) = (\Delta u_1, \Delta u_2)$ → $\Delta u = (\Delta u_1, \Delta u_2)$ → we can split and compute the Laplacian of the 2 variable separately

Can we go to the weak form? Yes, rewrite the system

$$(2)\quad \boxed{\begin{cases} -\int_\Omega \gamma\,\Delta u\cdot v + \int_\Omega \nabla p\cdot v = \int_\Omega f\cdot v & \text{in } \Omega \quad \forall v \in ? \\ -\int (\nabla\cdot u) q = 0 & \text{in } \Omega \quad \forall q \in ? \end{cases}}$$

in which space? we now do NOT know, it is a test function for velocity

another space that we do NOT know, it is a test function for the pressure

Now, we want to define the Green's identities

(1) $\int_{\partial\Omega} \Psi\,\nabla\phi\cdot\eta = \int_\Omega \Psi\,\Delta\phi + \int_\Omega \nabla\Psi\cdot\nabla\phi$ → in Stokes equation: $\nabla\phi \equiv u$, $\Psi = p$

(2) for vectorial variables

$$\int_{\partial\Omega} v\cdot[(\nabla u)^T\eta] = \int_\Omega v\cdot\Delta u + \int_\Omega \nabla u : \nabla v \quad \text{where} \quad A:B := \sum_{ij=1} A_{ij}B_{ij}$$

double dot product — for square matrix, and obtain a scalar

Applying to the system, we obtain

$$\begin{cases} \int_\Omega \gamma\,\nabla u:\nabla v - \int_\Omega p\,\nabla\cdot v - \int_{\partial\Omega}\gamma v\cdot((\nabla u)^T\eta) + \int_{\partial\Omega} p\,v\cdot\eta = \int_\Omega f\cdot v & \forall v \\ -\int_\Omega (\nabla\cdot u)q = 0 & \forall q \end{cases}$$

BUT we are in a weak formulation, hence we want that $v|_{\Gamma_D} = 0 = (0,0)^T$ so we can consider that

$$-\int_{\partial\Omega} v\cdot((\nabla u)^T n) + \int_{\partial\Omega} p\, v\cdot n \longrightarrow \begin{cases} 0 \text{ in } \Gamma_D \\ \neq 0 \text{ in } \Gamma_N \end{cases}$$

Rewrite the system considering this observations $\longrightarrow$ **Weak form stokes equations**

$$\begin{cases} \int_\Omega \nu \nabla u : \nabla v - \int_\Omega \boxed{p}\, \nabla\cdot v = \int_\Omega f\cdot v - \int_{\Gamma_N} \psi\cdot v & \forall v \in H^1_0(\Omega) \\ -\int_\Omega (\nabla\cdot u)\boxed{q} = 0 & \forall q \in L^2(\Omega) \end{cases}$$

velocity

pressure

This terms are "the same"

Let's now deal with the space :
- preassure space $p, q \in L^2(\Omega)$, and also the divergence has to belong to the same space
  - we have not to derive it, no gradient, very weak hypothesis
- forcing term $f \in [L^2(\Omega)]^2$
  - because they are array
- velocity space $u \in [H^1_g(\Omega)]^2 := \{ v \in [H^1(\Omega)]^2 : v|_{\Gamma_D} = g \}$
  $v \in [H^1_0(\Omega)]^2 := \{ v \in [H^1(\Omega)]^2 : v|_{\Gamma_D} = 0 \}$

Remark -
R1. If $\Gamma_N = \emptyset$, we have to set conditions to have a unique solution $\longrightarrow p \in L^2_0(\Omega) := \{ v \in L^2(\Omega) : \int_\Omega v = 0 \}$
R2. Implementing this problem, we define $u = u_0 + G$, where $u_0 \in [H^1_0(\Omega)]^2$, $G|_{\Gamma_D} = g$ is the «rilevamento» of the Dirichler condition

Make all these informations together...

$$\begin{cases} \underbrace{\boxed{\int_\Omega \nu \nabla u_0 : \nabla v}}_{a(\cdot,\cdot)} - \underbrace{\boxed{\int_\Omega p \nabla\cdot v}}_{b(\cdot,\cdot)} = \underbrace{\boxed{\int_\Omega f\cdot v - \int_{\Gamma_N} \psi\cdot v - \int_\Omega \nu\nabla G : \nabla v}}_{F(\cdot,\cdot)} & \forall v \in [H^1_0(\Omega)]^2 \equiv V \\ \underbrace{\boxed{-\int_\Omega (\nabla\cdot u)\, q}}_{b(\cdot,\cdot)} = 0 & \forall q \in L^2(\Omega) \equiv Q \end{cases}$$

Introducing the bilinear form, to write everything in a contract way, we have $\begin{cases} a(\cdot,\cdot) - b(\cdot,\cdot) = F(\cdot,\cdot) & \forall v \in V \\ -b(\cdot,\cdot) = 0 & \forall q \in Q \end{cases}$ [VP]

variational problem

where $a(\cdot,\cdot): V\times V \to \mathbb{R}$, $a(u,v) = \int_\Omega \nu \nabla u : \nabla v$, $b(\cdot,\cdot): V\times Q \to \mathbb{R}$, $b(v,q) := -\int_\Omega q\, \nabla\cdot v$
$F(\cdot,\cdot): V \to \mathbb{R}$, $F(v) = \int_\Omega f\cdot v - \int_{\Gamma_N} \psi\cdot v - \int_\Omega \nabla G : \nabla v$

Hence we obtain the saddle point problem

$$\begin{cases} \text{Find } u\in V,\ p \in Q \text{ s.t.} \\ a(u,v) + b(v,p) = {}_{V'}\langle F, v\rangle_V & \forall v \in V \\ b(u,q) = 0 & \forall q \in Q \end{cases} \quad \text{VSP}$$

Theorem - Brezzi's Theorem

Hypothesis -
H1. $a(\cdot,\cdot)$ is continuous in $V \longrightarrow \exists\, \gamma_a > 0 : |a(u,v)| \le \gamma_a \|u\|_V \|v\|_V \quad \forall u, v \in V$
H2. $a(\cdot,\cdot)$ is cohercive in $V_0 \longrightarrow V_0 := \{ v \in V : b(v,q) = 0 \quad \forall q \in Q \} \subset V$
$\exists\, \alpha_a > 0 : a(u,u) \ge \alpha_a \|u\|^2_V \quad \forall u \in V_0 \iff \sup_{w\in V_0} \frac{a(w,v)}{\|w\|_V} \ge \alpha_a \|v\|_V \quad \forall v \in V_0$ (equivalence)
H3. $b(\cdot,\cdot)$ is continuous on $V\times Q \longrightarrow \exists\, \gamma_b > 0 : b(v,q) \le \gamma_b \|v\|_V \|q\|_Q \quad \forall v\in V\ \forall q \in Q$
H4. $b(\cdot,\cdot)$ is inf-sup stable in $V \times Q \longrightarrow \exists\, \beta_b > 0 : \inf_{q\in Q} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \ge \beta_b > 0 \iff \sup_{v\in V} \frac{b(v,q)}{\|v\|_V} \ge \beta_b \|q\|_Q \quad \forall q \in Q$ (equivalence)
extension of cohercivity for bilinear form in 2 dimension

Thesis - Then $\exists!\ (u,p) \in V\times Q$ solution of [VP]
Unique continuous solution

1. $\boxed{\|u\|_V \le \frac{1}{\alpha_a} \|F\|_{V'}}$

2. $\boxed{\|p\|_Q \le \frac{1}{\beta_b}\left(1 + \frac{\gamma_a}{\alpha_a}\right) \|F\|_{V'}}$ (•)

What happen in the Galerkin approximation? See lecture 11 for more details

1.2.1 Supermizer operator

Is the operator define such that $T: Q \longrightarrow V$, where $\forall q \in Q \quad Tq \in V: \ (Tq, w)_V = b(w,q) \quad \forall w \in V$ that is an implicite definition of the supermizer operator T

↓ This is usefull only in the discrete space, not in the continuity one

Property - $\forall \ni Tq \in \arg\sup_{w \in V} \frac{b(w,q)}{\|w\|_V}$ when you apply the operator to the pression you obtain the velocity

1.2.2 What happen when we want to study the stokes equation in a discrete space?

We discretize the domain V_δ, Q_δ with the same tassellation δ to study the discrete space

↓

What happen to the Hp of the Brezzi Theorem?

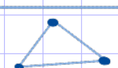
Consider $\boxed{\beta_0 := \inf_{q \in Q} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} > 0}$, if I restrict Q

↳ We have to study the discrete inf-sup stability

- The pressure resulting from the inf sup belongs to Q_δ: ✓
- The pressure is NOT in Q_δ: ✓ it is inf
- The velocity is in V_δ: ✓
- The velocity is NOT in V_δ: this is really a big problem! ✗
  The value β_δ could be 0 ⟶ HUGE PROBLEM (•)

Hence, NOT ALL THE DISCRETIZATIONS ARE GOOD, then the usual space used is the **Taylor-Hood FEM spaces**

$\begin{cases} v \in FEM \quad K=2 \\ q \in FEM \quad K=1 \end{cases}$



basis function $\varphi_i: \begin{cases} 1 \text{ in the vertex } i \\ 0 \text{ in the other points} \end{cases}$

basis also in the middle of the edge

This is a good and stable tassellation, moreover the error of the velocity goes to zero faster wrt the one on the q

1.2.3 Reduced Model Order for Stokes equation

: we want to reduce V_{δ}, Q_δ in the classic way, hence we can take the snapshots $\{u_N = u_\delta(\mu_N)\}$ $\{p_N = p_\delta(\mu_N)\}$

The same observation as before for the inf sup value, How can we effort? Add v to the space so that we have nor problem with the value of inf sup ⟶ We have to build the reduced space in order that the inf-sup stability holds, indeed in high fidelity $\{u_\delta, p_\delta\}$ the inf-sup stability is given by the choice of Q_δ, V_δ

↓ Add other velocity to the ones find with POD until the new basis $\{\psi_i\}_N \{\varphi_i\}_N$ satisfies the condition of inf sup

To obtain the stability we enlarge the space with T-operator

To do that we exploit the supermizer operator. Let's see the discrete version

$T: Q_\delta \longrightarrow V_\delta$ s.t. $\forall q_\delta \in Q_\delta \quad Tq_\delta \in V_\delta$

$\forall w_\delta \in V_\delta \quad \boxed{\langle Tq_\delta, w_\delta \rangle_V = b(w_\delta, q_\delta)}$ ⟶ This now is a linear system

But, it holds the same property ⟶ $Tq_\delta \in \arg\sup_{w_\delta \in V_\delta} \frac{b(w_\delta, q_\delta)}{\|w_\delta\|_V}$ ⟶ This is good because in that way it garantees the existance of the infsup

Corollary - The constants $\beta_\delta := \inf_{q_\delta \in Q_\delta} \sup_{w_\delta \in V_\delta} \frac{b(w_\delta, q_\delta)}{\|w_\delta\|_V \|q_\delta\|_Q} := \inf_{q_\delta \in Q_\delta} \frac{\|Tq_\delta\|_V}{\|q_\delta\|_Q}$ ⟶ This holds thanks to the properties
Then $\beta_N \geq \beta_\delta \geq \beta_0 > 0$ ✓

So that we can build the reduced spaces

Fixing $Q_N \subset Q_\delta$, $\boxed{Q_N := \text{span}\{p_\delta^i\}_{i=1}^m}$ find, i.e. with POD, the right choice for the space of the velocity is

$\boxed{V_N := \text{span}\{w_\delta^i\}_{i=1}^m \cup \text{span}\{Tp_\delta^i\}_{i=1}^m}$ ⟶ this is infsup stable & $\beta_N \geq \beta_\delta \geq \beta_0 > 0$ ✓

NB This approach can be done also with other model of PDE to deal with the same problems

12.4 Remark on Stokes High-Fidelity implementation

Lab 10

We have this specific problem, that is the high-fidelity problem

$$\begin{cases} a(u_\delta, v_\delta) - b(v_\delta, p_\delta) = F(v_\delta) & \forall v_\delta \in V_\delta \\ -b(u_\delta, q_\delta) = 0 & \forall q_\delta \in Q_\delta \end{cases} \qquad a(u_\delta, v_\delta) := \int_\Omega \nu \nabla u_\delta : \nabla v_\delta, \quad b(u_\delta, q_\delta) := \int_\Omega q_\delta \nabla \cdot u_\delta \qquad F(v_\delta) = \int_\Omega f \cdot v_\delta$$

In FEM, we fix the basis function
- Q_δ: $\{\varepsilon_k\}_{k=1}^{N_\delta^p}$ standard FEM basis $\mathbb{P}_1(\Omega)$
- V_δ: $\{\varphi_j\}_{j=1}^{N_\delta^u}$ $\longrightarrow$ vectorial basis $\varphi_j = \{\varphi_j^1, \varphi_j^2\} \in \{[\varphi_i, 0], [0, \varphi_i]\}$

How to discretize with the FEM?

↳ We need to define a smart basis

standard FEM basis $\mathbb{P}_2(\Omega)$ where $\{\varphi_i\}_{i=1}^{N_\delta}$ is the FEM standard basis $N_\delta = \frac{N_\delta^u}{2}$

$$\begin{Bmatrix} [\varphi_1, 0] \\ [\varphi_2, 0] \\ \vdots \\ [0, \varphi_{N-1}] \\ [0, \varphi_N] \end{Bmatrix} = \varphi_j$$

Consider now (1) $\boxed{A_{ij} = a(\varphi_j, \varphi_i) = \int_\Omega \nu \nabla \varphi_i : \nabla \varphi_j}$ $\Rightarrow$ if $k \in (0, .., N_\delta)$ $\varphi_k = (\varphi_j, 0)$

we can split the computation thanks to the basis that we choose

$i \in (0, .., N_\delta)$, $j \in (0, .., N_\delta)$
$$A^1_{ij} = \int_\Omega \nu \nabla \begin{pmatrix} \varphi_i \\ 0 \end{pmatrix} : \nabla \begin{pmatrix} \varphi_j \\ 0 \end{pmatrix} = \int_\Omega \nu \begin{bmatrix} \partial_x \varphi_i & 0 \\ \partial_y \varphi_i & 0 \end{bmatrix} : \begin{bmatrix} \partial_x \varphi_j & 0 \\ \partial_y \varphi_j & 0 \end{bmatrix} = \int_\Omega \nu (\partial_x \varphi_i \partial_x \varphi_j + \partial_y \varphi_i \partial_y \varphi_j)$$
$$= \int_\Omega \nu \nabla \varphi_i \cdot \nabla \varphi_j$$ gradient φ_j dot gradient φ_j

$i \in (0, .., N_\delta)$, $j \in (N_\delta+1, .., 2N_\delta)$ $\quad A_{ij} = \int_\Omega \nu \begin{bmatrix} \partial_x \varphi_i & 0 \\ \partial_y \varphi_i & 0 \end{bmatrix} : \begin{bmatrix} 0 & \partial_x \varphi_j \\ 0 & \partial_y \varphi_j \end{bmatrix} = 0$ by definition dot product

$i \in (N_\delta+1, .., 2N_\delta)$ $j \in (0, .., N_\delta)$ $\quad A_{ij} = 0$

$i \in (N_\delta+1, .., 2N_\delta)$ $j \in (N_\delta+1, .., 2N_\delta)$ $\quad A^2_{ij} = \int_\Omega \nu \nabla \varphi_i : \nabla \varphi_j$

(2) $$\boxed{B_{kj} = b(\varphi_j, \varepsilon_k) = \int_\Omega \varepsilon_k \nabla \cdot \varphi_j} = \left(\int_\Omega \varepsilon_k \partial_x \varphi_j^1\right)_{j \in (1, .., N_\delta)} \left(\int_\Omega \varepsilon_k \partial_y \varphi_j^2\right)_{j \in (N_\delta+1, .., 2N_\delta)}$$
$$= \underbrace{\left(\int_\Omega \varepsilon_k [1,0] \cdot \nabla \varphi_j^1\right)_{j \in (1, .., N_\delta)}}_{(B^1_{FEM})_{kj}} \underbrace{\left(\int_\Omega \varepsilon_k [0,1] \cdot \nabla \varphi_j^2\right)_{j \in (N_\delta+1, .., 2N_\delta)}}_{(B^2_{FEM})_{kj}}$$

(3) $$\boxed{q_i = F(\varphi_i) = \int_\Omega f \cdot \varphi_i} = \underbrace{\left(\int_\Omega f_1 \varphi_i^1\right)_{i \in (1, .., N_\delta)}}_{q^1_{FEM}} \underbrace{\left(\int_\Omega f_2 \varphi_i^2\right)_{i \in (N_\delta, .., 2N_\delta)}}_{q^2_{FEM}}$$

Hence, the Stokes linear system becomes

$$\underbrace{\begin{array}{c c} & \begin{array}{ccc} \multicolumn{2}{c}{N_\delta^u} & N_\delta^p \end{array} \\ \begin{array}{c} N_\delta^u \ (N_\delta) \\ N_\delta^u \ (N_\delta) \\ N_\delta^p \end{array} & \begin{pmatrix} A^1 & 0 & {B^1_{FEM}}^T \\ 0 & A^2 & {B^2_{FEM}}^T \\ B^1_{FEM} & B^2_{FEM} & 0 \end{pmatrix} \\ & \begin{array}{cc} N_\delta & N_\delta \end{array} \end{array}}_{:= J} \begin{pmatrix} u_1 \\ u_2 \\ p \end{pmatrix} \begin{array}{c} N_\delta \\ N_\delta \\ N_\delta^p \end{array} = \begin{pmatrix} q^1_{FEM} \\ q^2_{FEM} \\ 0 \end{pmatrix} \begin{array}{c} N_\delta \\ N_\delta \\ N_\delta^p \end{array}$$

$\downarrow$ This is the solution that we want to find