13 - NON LINEAR PROBLEMS

The non-linearity is in the bilinear form $a(u,v) = F(v)$, hence $a(u,v)$ is non linear in u, i.e. $a(u,v) = \max(u,v)$
$a(u,v) = u \cdot |u| \cdot v$

↓ We have to linearize the problem, i.e. using Newton Method

13.1 Newton Method : linear system resolver

It's an iterative method in which we assume $f(u)=0$ and the iterative step is $u_{n+1} = u_n + \delta_u$ ← what we have to compute

Exploits the Taylor expansion centered in u_n, we can compute, using $\delta_u = u_{n+1} - u_n$

$$f(u_{n+1}) = f(u_n) + J_f(\delta_u)\big|_{u_n} \delta_u + O(\delta_u^2)$$

Jacobian first order — difference — negligible second order term

This term has to go to zero

→ IDEA: solve a non-linear PDE by iteratively solving a sequence of linear PDE. At each step, you linearize the non-linear PDE around your current best guess for the solution & solving the resulting linear PDE to find update to go closer to the solution

Hence ⟶ while $\delta_u \approx 0$ do

$$J_f(\delta_u)\big|_{u_n} \delta_u = -f(u_n)$$

⟶ $\delta_u \approx 0$ or $f(u_{n+1}) \approx 0$ are two equivalent stopping criteria

Definition of the Jacobian: $\boxed{J_f[v]\big|_{x_0} := \lim_{h\to 0} \frac{f(x_0 + hv) - f(x_0)}{h}}$ — direction v — if we can compute this, we can solve the system

Gateaux differentiable: if exists the limit of the incremental ratio

In the laboratory 12 we'll study the burger equation $\boxed{-\Delta u + u \nabla\cdot u = g}$, where $\nabla\cdot u := \partial_x u + \partial_y u$ where u is scalar

Sort of with scalar variable — definition of this strange divergence

↓

The first thing to do is to go to the variational form to write the correct form of f

$$F: \underbrace{\int_\Omega \nabla u \cdot \nabla v}_{f_1} + \underbrace{\int_\Omega u \nabla\cdot u\, v}_{f_2} - \underbrace{\int_\Omega g v}_{f_3} = 0$$

Now we can split f in f_i so that we can compute the Jacobian

- Both f_1 & f_3 are linear function, so let's try to compute the Jacobian

Diffusion $\quad J_{f_1}(\delta_u)\big|_{u_N} = \lim_{h\to 0} \frac{f_1(u_N + h\delta_u) - f_1(u_N)}{h} = \lim_{h\to 0} \frac{f_1(u_N) + h f_1(\delta_u) - f_1(u_N)}{h} = f_1(\delta_u) = \int_\Omega \nabla u \cdot \nabla v$ ⟶ This holds for all linear function

(f_1 is linear)

$J_{f_3}(\delta_u)\big|_{u_N} = 0$ because it doesn't appear u in the function

- For f_2, that is not linear, we have to compute the Jacobian in a different way

$$J_{f_2}(\delta_u)\big|_{u_N} = \lim_{h\to 0} \frac{f_2(u_N + h\delta_u) - f_2(u_N)}{h} = \lim_{h\to 0} \frac{\int_\Omega (u_N + h\delta_u)\nabla(u_N + h\delta_u) v - \int_\Omega (u_N)\nabla(u_N) v}{h} \overset{\downarrow}{=}$$

The integral is linear, we can split

$$= \lim_{h\to 0} \frac{1}{h}\Big(\underbrace{\int_\Omega u_N \nabla u_N v - \int_\Omega u_N \nabla\cdot u_N}_{=0}\Big) + \frac{h^2(\ldots)}{h} + \underbrace{\int_\Omega u_N \nabla\cdot(\delta_u) v}_{\text{advection term}} + \underbrace{\int_\Omega \delta_u \nabla\cdot u_N v}_{\text{reaction term}}$$

neglectable

advection term ↓ $\int_\Omega (u_N [1,1]) \nabla\cdot(\delta_u) v$ — reaction term ↓ $\int_\Omega (\nabla\cdot u_N)\delta_u v$

Newton Schema $\boxed{J_f[\delta_u]\big|_{u_N} \cdot \delta_u = -f(u_N)}$

$$\longrightarrow \Big(\int_\Omega \nabla\delta_u \cdot \nabla v + \int_\Omega (u_N[1,1]) \cdot \nabla\delta_u v + \int_\Omega (\nabla\cdot u_N)\delta_u v\Big) = -\int_\Omega \nabla u_N \nabla v - \int_\Omega u_N \nabla\cdot u_N v + \int_\Omega g v \qquad \forall v \in V = H_0^1(\Omega)$$

With respect to Lab 12.

$$\int_\Omega h(u_N)\, \ell(\delta_u)\, m(v) \simeq \sum_{q=1}^{Q} h(u_N, x_q)\, \ell(\delta_u, x_q)\, m(v, x_q)\, w_q$$ ⟶ quadrature formulae

burger-non-linear.c => $h(u_N) = \nabla\cdot(u_N) = \partial_x u_N\big|_{x_q} + \partial_y u_N\big|_{y_p}$

Algorithm 3 Newton's Method for Nonlinear PDEs

Input: Nonlinear PDE operator R, Tolerance ϵ, Maximum iterations m_{max}, Initial guess u_0.
Output: Approximate solution u.
Initialize $m = 0$.
Choose an initial guess u_0.
for $m = 0, 1, 2, \dots, m_{max} - 1$ **do**
    Evaluate the residual $R(u_m)$. *#Often in variational form:* $F(u_m)(v)$
    Evaluate the linearized operator $R'(u_m)$. *#Often in variational form: find* $J_F[\cdot](\cdot)$
    Solve the linear equation $R'(u_m)[\delta u_m] = -R(u_m)$ for the correction δu_m. *#Often in variational form:* $J_F[\delta u_m](v) = -F(u_m)(v)$
    Update the solution: $u_{m+1} = u_m + \delta u_m$.
    Check for convergence.
    if $||R(u_{m+1})|| < \epsilon$ or $||\delta u_m|| < \epsilon$ **then**
        Set $u = u_{m+1}$.
        break
    end if
end for
if $m == m_{max} - 1$ and convergence not reached **then**
    Warning: Maximum number of iterations reached without convergence.
end if
Set $u = u_{m+1}$.
Return: u.