INSTITUT DES SCIENCES ACTUARIALES ET FINANCIERES

MASTER 1 INTERNSHIP

On the range of Admissible Term Structure

Author:
Mohammed ETTABIB

Supervisor:
Areski COUSIN

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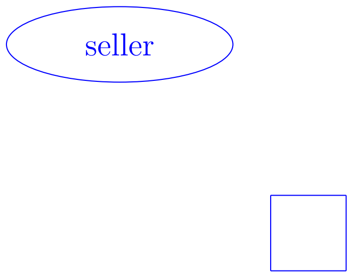
Contents

1 Credit default swap (CDS)

In the credit derivative market, the credit default swaps takes a big place. A CDS contains one only underlying, it allowed the market investors to manage dynamically the underlying default risk.

A CDS looks like an interest rate swap, because there is an exchanging floating payment and fixed payment between the buyer and the seller. A CDS carry the advantage of an insurance product witch give to his owner a protection over the underlying default risk .

The CDS contract involve 3 entities : the *buyer*, the *seller* and the *reference entity*. The seller of a CDS will garanty, a recovery $1-R$ of the nominal, in a period of time T called the *maturity*, if the reference entity fails.



Instead, the buyer of the protection pays a fixed amount s called the *spread*, at a regular and prefixed dates ($t_1, t_2, \ldots, T$)[1], until the default date τ if the default occurs before the maturity T. Else, he will pays the previous amount until T.

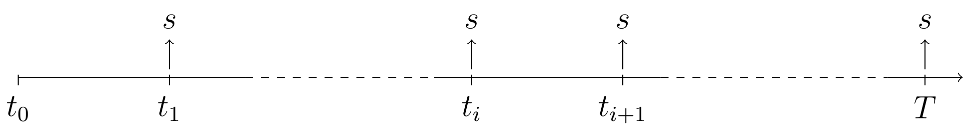


Figure 1: In the case of no default

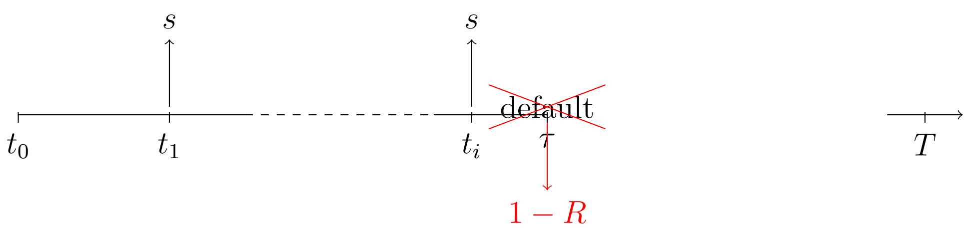


Figure 2: In the case of occuring default ($\tau < T$)

[1]In the market, the spreads are issued evry 3 months ($t_{k+1} - t_k$=3 months)

Example : In the market place, the spreads are presented for several maturities, as we can see in the following example. The legs then are emitted every 3 months (i.e $\delta t = \frac{1}{4}$) :

Maturity (year)	1	2	3	4	5	6	7
CDS spread (bp)	28,125	35,33	42,96	51,815	57,15	58,23	58,795

Figure 3: *A CDS spread cotation from AIG on 21/04/2010*

The floating part payed by the protection seller depends on the default condition of the underlying before the maturity. In the case of default, the seller will refund to the buyer a part R of the nominal, depending on the recovery rate R of the underlying. In the case of no defect, the seller will pay nothing.

The recovery rate will remain unknown until the maturity date. Not easy to estimate, he varies depending on the company.

The CDS's price or the spread is determined at the initial date (t_0) by equalizing the expected value of the two previous cash flows.

Let's specify first some notations :

τ The underlying default date.

R His recovery rate wich is a predictable process of $[0,1]$

T_0 =0 The CDS signature date

T The maturity of the CDS

t_i The payment dates of the buyer where $\delta t = t_i - t_{i-1}$ are equal $\forall i \in 1, \ldots, n$

$\beta(t)$ An index in which $t \in [t_{\beta(t)-1}, t_{\beta(t)}]$.

r The short rate and $P^D(t,T) = \exp(-\int_t^T r_s ds)$

For the seller, the future cash flow that he will **receive** at $t < T \wedge \tau$:

$$s\left\{(T_{\beta(t)} - t)P^D(t, T_{\beta(t)})\mathbb{1}_{\tau > T_{\beta(t)}} + \sum_{i=\beta(t)+1}^{n} \delta_k P^D(t, t_i)\mathbb{1}_{\tau > t_i} + (\tau - T_{\beta(\tau)-1}P^D(t,\tau)\mathbb{1}_{\tau \le T})\right\}$$

this formula can be approximated by the continuous flow

$$\int_t^T sP^D(t,u)du$$

Instead the seller will pay

$$\mathbb{1}_{\tau\leq T}P^D(t,\tau)(1-R)$$

We have then at $t = t_0$ the following result :

$$S\mathbb{E}_{\mathbb{P}}\left[\mathbb{1}_{\tau\leq T}(1-R)P^D(t_0,\tau)\right] = \mathbb{E}_{\mathbb{P}}\left[\int_{t_0}^{T}\mathbb{1}_{\tau>u}P^D(t,u)du\right] \tag{1}$$

where $\mathbb{P}$ is a free-risk probability.

Let's introduce the following probability , called *survival probability* :

$$\mathbb{Q}(t) = P(\tau > t)$$

NB : the corresponding path of $\mathbb{Q}(t)$ is called *survival curve* or *credit curve.*

then equation 1 can be written as following (under the survival probability):

$$S\mathbb{E}_{\mathbb{Q}}\left[\int_{t_0}^{T}P^D(t,u)du\right] = \mathbb{E}_{\mathbb{Q}}\left[(1-R)P^D(t_0,\tau)\right] \tag{2}$$

A more practical expression of the last equation 4 can be obtained by approximating the integrals involved :

$$\boxed{\mathrm{S}\sum_{k=1}^{n}\delta_k P^D(t_0,t_k)Q(t_0,t_k) = -(1-R)\int_{t_0}^{T}P^D(t_0,t)dQ(t_0,t)(3)}$$

Several financial calculations involves knowing the credit curve. For this reason, we will need a *yield-curve construction method* since the number of market isn't sufficient for a good curve approximation.

2 Admissible Term Structures

The previous problematic is also present for some other derivative product such as :

- Corporate bond yield curve
- Discounting curve based on OIS
- Forward curve based on fixed-vs-Ibor-floating IRS

All this term-structure construction consist in finding a function $T \longrightarrow P(t_0, T)$ given a small number market quotes $S_1, ..., S_n$.

Indeed, We have to rely on interpolation/calibration schemes to construct the curve for the missing maturities.
This will lead us to define what will be understood by a good yield curve construction.

2.1 Notation

First Let's de fine some notations in order to be concise :

n The number of maturities

$\mathbf{t_0}$ The cotation date

S $= (S_1, S_2, \ldots, S_n)$ The set of market quotes at t_0

T $= (T_1, \ldots, T_n)$ the corresponding set of increasing maturities

t $= (t_1, \ldots, t_{p_n})$ payment time grid

The payement time grid are arranged as : $\forall i \in 1 \ldots n \ T_i = t_{p_i}$

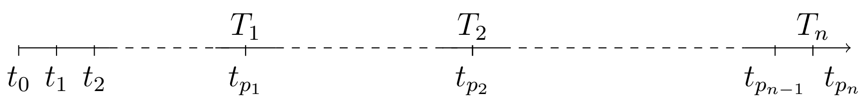


Figure 4: Time grid

2.2 Market fit condition

We assume first that this quotes can be presented in a linear form, which is the case for the previous instruments presented :

Assumption 2.1 (*Linear representation of present values [AC14]*)

Present values of products used in the curve construction process can be expressed as linear combination of some elementary quantities. Depending on the context, the latter can be either zero-coupon prices, discount factors, Libor or Euribor forward rates or CDS-implied survival probabilities.

Credit curve based on CDS : From equation (3), by integration by parts we can get:

$$S_i \sum_{k=1}^{p_i} \delta_k P^D(t_0,t_k)Q(t_0,t_k) - (1-R)P^D(t_0,T)Q(t_0,T)$$
$$+(1-R)\int_{t_0}^{T_i} f^D(t_0,t)P^D(t_0,t)Q(t_0,t)dt \quad = 1-R,\ i=1,\dots,n.$$

where $f^D(t0,u)$ is the instantaneous forward (discount) rate associated with maturity date u and the . Therefore CDS can be written as a linear equation of *survival probability* $(Q(t_{p_i}))$.

This condition can be written more generally and more algebraically in this form : The Term structure function $T \longrightarrow P(t_0,T)$ is built from market quotes of standard product.

Let $P = (P^D(t_0,t_1),\dots,P^D(t_0,t_m))$ be the vector of the values of the curve $T \longrightarrow P(t_0,T)$ at the payment dates $t_1,\dots,t_{p_n}$:

The market fit condition can be restated as a rectangular system of linear equations :

$$A \cdot P = B$$

where :

P $=(P(t_0,t_1),\dots,P(t_0,t_{p_n}))'$ A vector of curve $T \longrightarrow P(t_0,T)$ at the payement time grid

A is a $n \times p_n$ matrix

B is a $n \times 1$ matrix with positive coefficients

A and B only depend on current market quotes S, on standard maturities T, on payment dates t and on products characteristics.

A lot of curves can satisfies the market, in order to get an admissible curve $T \longrightarrow P(t_0,T)$ have to fit some others conditions.

2.3 Arbitrage-free conditions

Definition 2.1 (*arbitrage-free condition*)

A credit curve is said to be arbitrage-free if the curve corresponds to a well-defined default distribution function. In other words P had to verify the following conditions :

- $P(t_0, t_0) = 1$
- $T \longmapsto P(t_0, T)$ *is non increasing function (i.e* $\nexists x, y, P(t_0, x) < P(t_0, y) \& x > y$*)*

NB : If a credit curve didn't satisfy the arbitrage free condition, then there is some arbitrage opportunities in the market.

Therefore we have the following inequalities, called *Arbitrage-free inequalities* :

$$
\begin{aligned}
P(t_0, T_1) \leq P(t_0, t_k) \leq 1 & \qquad \forall k \in 1, \dots, p_1 \\
P(t_0, T_i) \leq P(t_0, t_k) \leq P(t_0, T_{i-1}) & \qquad \forall k \in [p_{i-1} + 1, p_i - 1]
\end{aligned}
$$

Admissible curve require also to be smooth :

Definition 2.2 (*Smoothness condition*)

A curve is said to be smooth if it is differentiable and his derivative is continuous.

Remark 2.1

- *For example, a credit curve (CDS) satisfies the smoothness condition if the associated default density function exists and is continuous.*
- *Also an IR curve (OIS) satisfies the smoothness condition if the associated instantaneous forward rates exist for all maturities and are continuous.*

In summary :

Definition 2.3 (*Admissible Curve [AC14]*)

Given a set of observed market quotes. A curve is said to be admissible if it satisfies the following three constraints :

- *The selected market quotes are perfectly reproduced by the curve.*
- *The curve is arbitrage-free in the sense of Definition 2.1.*
- *The curve satisfies the smoothness condition presented in Definition 2.2.*

2.4 The geometrical side of admissible curves

Let denote by $\mathcal{C}$ the set of admissible Curves.

Proposition 2.1

the set of admissible curves $\mathcal{C}$ is convex

Proof: Let $P_1, P_2 \in \mathcal{C}$ then by the Assumption 2.1, $\exists A \in \mathcal{M}_{n,p_n}(\mathbb{R})$, $\exists B \in \mathcal{M}_{n,1}(\mathbb{R})$[2] that verifies :

$$AP_1 = B$$
$$AP_2 = B$$

Let $\alpha \in [0,1]$ then : $(\alpha P_1 + (1-\alpha)P_2)\,A = B$ so $P = \alpha P_1 + (1-\alpha)P_2$ verifies the Assumption 2.1. P is smooth(resp continuous) as far as P_1, P_2 are smooth(resp continuous). So $P \in \mathcal{C}$. ■

All the admissible curves are bounded : $\forall f \in \mathcal{C},\ 0 \leq f \leq 1$. One brave question is to say if there is min (resp max) for the set $\mathcal{C}$? . This will naturally lead us to study the *compactness* of the set $\mathcal{C}$ since it's already convex.

Let's remind a theorem in compactness of function spaces :

Theorem 2.2 (*Ascoli-Arzela*)

Let (E,d) a compact metric space, and (F,δ) a full metric space.
A subset A of $\mathcal{C}(E,F)$ is relatively compact if and only if :

1. *A is equicontinuous:*

$$\forall x \in E, \forall \epsilon > 0, \exists \eta > 0 / \forall f \in A, \forall y \in E, (d(x,y) \leq \eta) \Longrightarrow (\delta(f(x), f(y)) < \epsilon)$$

2. *$\forall x \in E$, the set $A(x) = f(x), f \in A$ is relatively compact.*

[2]Until we are working at one product at a time (CDS, OIS , ...) the matrices A, B are uniques for all admissible curves

3 Credit curve for CDS over AIG Data

It's quite obvious that admissible curves are bounded. A useful idea treated in the article of A.cousin [AC14], consist in finding an absolute bounds on the intervals $[T_i, T_{i+1}]$. This idea is treated also for other term-structure.

The market fit conditions include n equations for n unknowns $Q(t_{p_i})_{i\in[1,n]}$:

$$S_i \sum_{k=1}^{p_i} \delta_k P^D(t_0,t_k)Q(t_0,t_k) - (1-R)P^D(t_0,T)Q(t_0,T) \\ +(1-R)\sum_{k=0}^{p_i} f^D(t_0,t_k)P^D(t_0,t_k)Q(t_0,t_k)\delta t_k \quad = 1-R,\ i = 1,\dots,n.$$

Each equation (i) depends on i variables $Q(T_1),\dots,Q(T_i)$. So we can proceed by a recursion to find the bounds for each $Q(T_i)$.

By using the arbitrage-free 2.3 inequalities over equation (4) we have the following proposition :

Proposition 3.1

Assume that, at time t_0, quoted fair spreads $S_1,\dots,S_n$ are reliable for standard CDS maturities $T_1 < \cdots < T_n$. For any $i = 1,\dots,n$ the survival probability $Q(t_0,T_i)$ associated with a market-compatible and arbitrage-free credit curve is such that:

$$Q_{min}(t_0,T_i) \le Q(t_0,T_i) \le Q_{max}(t_0,T_i)$$

where :

$$Q_{max}(t_0,T_i) = \frac{1-R-\sum_{k=1}^{i-1}((1-R)M_k + S_iN_k)Q(t_0,T_k)}{P^D(t_0,T_{i-1})(1-R) + S_i(N_i + \delta_{p_i}P^D(t_0,T_i))}$$

$$Q_{min}(t_0,T_i) = \frac{1-R-\sum_{k=1}^{i}((1-R)M_k + S_iN_k)Q(t_0,T_{k-1})}{P^D(t_0,T_i)(1-R+S_i\delta_{p_i})}$$

with :

- $p_0 = 1,\ T_0 = t_0$ *and* $P^D(t_0,T_0) = Q(t_0,t_0) = 1$
- $\forall i \in 1,\dots,n,\ M_i = P^D(t_0,T_{i-1}) - P^D(t_0,T_i)$ *and* $N_i = \sum_{k=p_{i-1}}^{p_i-1} \delta_k P^D(t_0,t_k)$

This bounds are not calculable since they depends on $Q(T_i)$. Instead, we can remark that $Q_{max}(T_i)$ are a decreasing fonction of $Q(T_k)_{k\in[1,i-1]}$. So at must $Q_{min}(T_k)_{k\in[1,i-1]}$ is lower values that $Q(T_k)_{k\in[1,i-1]}$ can takes.

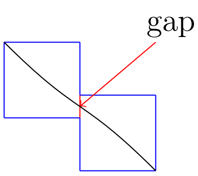
Proposition 3.2 ***([AC14])***

for each standard CDS maturity, model-free bounds for implied survival probabilities can be computed using the following recursive procedure. For $i = 1, \dots, n$

$$
\begin{aligned}
Q_{max}(T_i) &= \frac{1 - R - \sum_{k=1}^{i-1}((1-R)M_k + S_iN_k)Q_{min}(T_k)}{P^D(t_0, T_{i-1})(1-R) + S_i(N_i + \delta_{p_i}P^D(t_0, T_i))} \\
Q_{min}(T_i) &= \frac{1 - R - \sum_{k=1}^{i}((1-R)M_k + S_iN_k)Q_{max}(T_{k-1})}{P^D(t_0, T_i)(1 - R + S_i\delta_{p_i})}
\end{aligned}
$$

This last proposition can be used to identify a union of rectangles $\cup\mathcal{R}_i$ that are defined by the points : $\{(Q_{max}(T_i), T_i)\,;(Q_{min}(T_{i+1}), T_i)\}$. This rectangles are well defined because of the arbitrage free inequalities (2.3). They also have to be decreasing $Q_{max}(T_{i+1}) < Q_{max}(T_i)$ (resp $Q_{min}(T_{i+1}) < Q_{min}(T_i)$).

The effectiveness of borders can be measured by the length of the gap $G_i = Q_{max}(T_i) - Q_{min}(T_i)$:



3.1 Application on AIG Datas

In this section we will apply the previous results on quoted spreads data of the company AIG. The data provided is a series of CDS quoted spreads for the maturities $1, \ldots, 10$ between 16/08/2005 and 30/09/2010.

The first idea consist on studying statistically the behavior of spreads over 2005 - 2010.

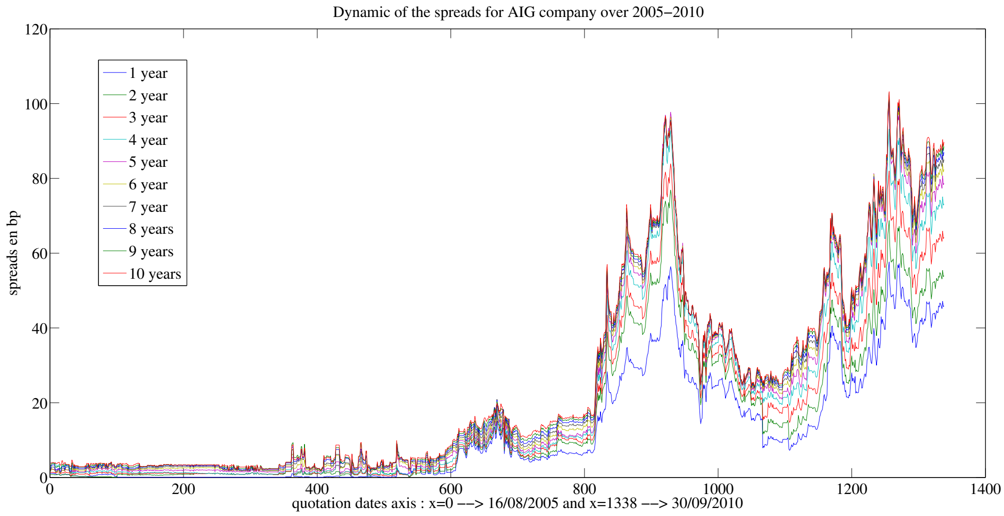


Figure 5: *Evolution of spreads of the company AIG*

We can naturally remark two phases in the spreads evolution: the first one corresponds to a stagnation of CDS market from 0 to 600 corresponding to 16/08/2005-03/12/2007 . The second one remains from the surrounding of 600 to the end of the period 30/09/2010. This period is exactly superimposed to the subprime crisis. We can remark that the company have reached a hight level of spreads. This policy is quite natural in a period of crisis, the company had had to increase her spreads in presence of a hight risk of bankruptcy.

In almost all quotations, spread is an increasing function of time. Obviously, the more years protection recover, higher is the risk that the protecter most care. But how much would the company be vulnerable to maturity ? The following graphic shows the spreads volatility over the available maturities :

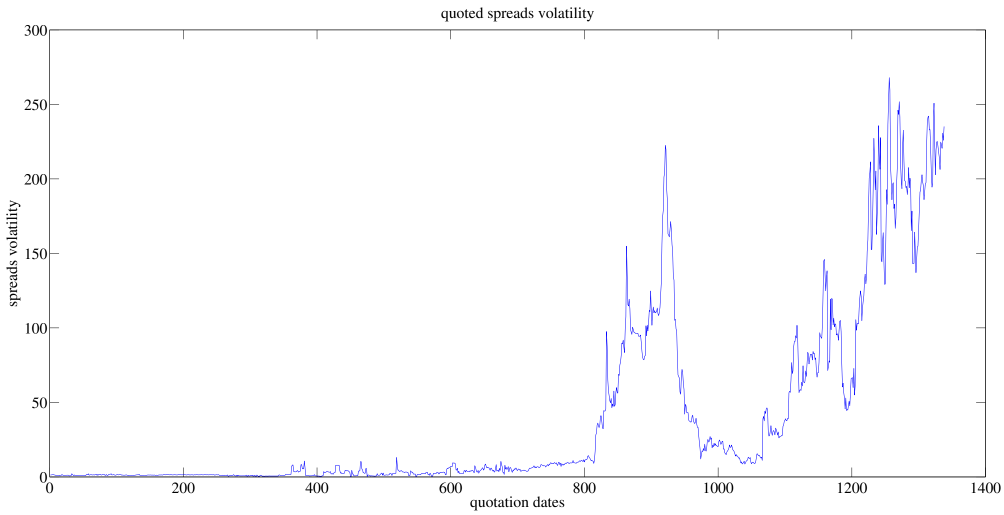


Figure 6: Quoted spreads volatility

The company in fact, in a crisis period, is too vulnerable to the maturity.

3.2 Dynamic of bounds

Under *arbitrage-free* conditions, proposition (3.2) gives bounds for credit curves at each quotation :

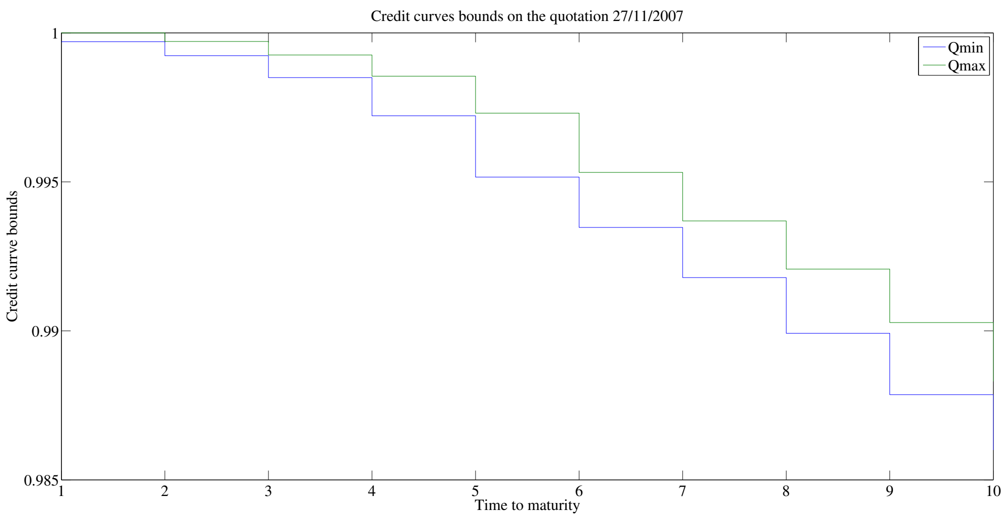


Figure 7: *Credit curve among 18/08/2005 - 28/09/2007*

We can use this proposition to identify either arbitrage free opportunities : if the arbitrage-free inequalities are not violated that means that there are arbitrage at the specified quotation. Indeed figure (8) shows an example of a quotation that present an arbitrage opportunity :

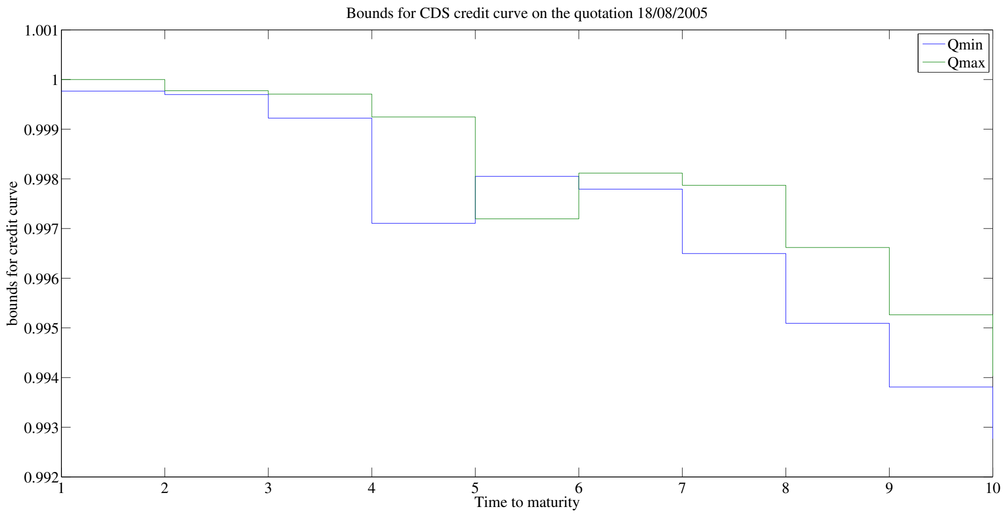


Figure 8: *Market fit condition non verified*

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References

[AC14] Ibrahima Niang Areski Cousin. On the range of admissible term structures. *Hal*, 2014.