MA3227 Numerical Analysis II

Lab Session 7

1 Trajectory of a cannonball (continued)

Recall from Tutorial 6 the ODE

$$\ddot{x} = -D\,\|\dot{x}\|_2\,\dot{x} - g\,e_2, \qquad x(0) = 0, \qquad \dot{x}(0) = v_0 \tag{1}$$

which describes the trajectory of a cannonball of mass $m = 1$ subject to drag with drag coefficient $D > 0$ and gravity with gravitational constant $g > 0$. Equivalently, the ODE (1) can be split into the two coupled ODEs

$$\dot{v} = -D\,\|v\|_2\,v - g\,e_2 \quad \text{with} \quad v(0) = v_0 \qquad \text{and} \qquad \dot{x} = v \quad \text{with} \quad x(0) = 0$$

which is more convenient for some of the following tasks.

1. We observe that the ODE $\dot{v} = f(v)$ has a fixed point $f(v_F) = 0$ for $v_F = -\sqrt{\frac{g}{D}}\,e_2$, which corresponds to the cannonball falling straight down with a velocity such that drag exactly balances the acceleration due to gravity. Verify that

$$\nabla f(v_F) = \begin{pmatrix} -\sqrt{gD} & 0 \\ 0 & -2\sqrt{gD} \end{pmatrix}.$$

2. Let us denote by $\lambda = -2\sqrt{gD}$ the more negative of the two eigenvalues of $\nabla f(v_F)$. We have seen in class that explicit Runge-Kutta methods applied to $\dot{v} = f(v)$ are stable only if the time step Δt is chosen such that $|R(\lambda\,\Delta t)| \le 1$, where

$$R(z) = \begin{cases} 1 + z & \text{for Euler's method, and} \\ 1 + z + \frac{z^2}{2} & \text{for the trapezoidal method.} \end{cases}$$

   For both methods, determine $\Delta t > 0$ such that $|R(\lambda\,\Delta t)| = 1$. Test your answer by replacing the placeholder `TODO` in `stability()` with the determined values of `dt` $= \Delta t$. If your answer is correct, then the distance $d = |v_2 - v_{F,2}|$ between v_2 and its fixed-point value $v_{F,2} = -\sqrt{g/D}$ is approximately constant rather than exponentially decaying.

   Hint. You will find that d decays slightly even for the correct `dt`. This is due to the nonlinearity of the ODE, which is not captured by our linearised analysis around the fixed point.

3. The time-step constraints derived in the previous two tasks can be avoided by switching to an implicit Runge-Kutta scheme, but doing so would require us to solve a system of nonlinear equations which we would like to avoid. Instead, let us consider a semi-implicit Euler method given by

$$\tilde{v}(t) = v(0) - D\,\|v(0)\|_2\,\tilde{v}(t)\,t - g\,e_2\,t, \qquad \tilde{x}(t) = x(0) + v(0)\,t.$$

   These are almost the equations of the explicit Runge-Kutta method except that we have a single factor of $\tilde{v}(t)$ appearing on the right-hand side of the equation for $\tilde{v}(t)$. The advantage of these equations is that we can write down an explicit formula for $\tilde{v}(t)$, namely

$$\tilde{v}(t) = \frac{1}{1 + D\,\|v(0)\|_2\,t}\,\big(v(0) - g\,e_2\,t\big).$$

Implement this scheme in the function `semi_implicit_euler_step()`. You can test your code using the provided function `convergence()`. If your code is correct, you will find that the error e_n decays as $e_n = O\big(n^{-1}\big)$ for this semi-implicit Euler method.

4. Uncomment the line for the semi-implicit Euler method in `stability()`. Note how even with a time step `dt` $= 10^3$, the distance $d = |v_2 + \sqrt{g/D}|$ still decays (albeit slowly) for the semi-implicit Euler method.

   The practical implications of this are as follows: Once the velocity $v(t)$ approaches its steady state v_F, we should be able to take arbitrarily large time steps `dt` since the two ODEs for position and velocity simplify to $\dot{v} = 0$, $\dot{x} = v$ and these equations can be solved *exactly* with a single step of any of the methods considered above. However, it is not possible to increase the time step `dt` beyond some `max_dt` $< \infty$ for the explicit methods due to the stability constraint. The semi-implicit method has no such constraint; hence for large enough time intervals $[0, T]$ the semi-implicit method can be arbitrarily much faster than the explicit methods.