MA3227 Numerical Analysis II

Assignment 2

Deadline: 1 April 2021, 12 noon
Total marks: 20

1 Explicit Simpson's method [10 marks]

Consider the Runge-Kutta method whose one-step equation is given by

$$\tilde y(t) = y(0) \; + \; f\big(\tilde y_1(0)\big)\,\tfrac{t}{6} \; + \; f\big(\tilde y_2(\tfrac{t}{2})\big)\,\tfrac{4t}{6} \; + \; f\big(\tilde y_3(t)\big)\,\tfrac{t}{6} \tag{1}$$

where

$$\tilde y_1(0) = y(0), \qquad \tilde y_2(\tfrac{t}{2}) = y(0) \; + \; f\big(\tilde y_1(0)\big)\,\tfrac{t}{2}, \qquad \tilde y_3(t) = y(0) \; + \; f\big(\tilde y_1(0)\big)\,t.$$

1. [2 marks] Write down the Butcher tableau for this method.
2. [2 marks] Show that this method is third-order consistent, i.e. show that
$$\tilde y(t) - y(t) = O(t^3).$$
3. [2 marks] Complete the function `simpson_step()` such that it implements a single step according to this scheme. Check that your code is correct using the provided function `convergence()`.
4. [2 mark] State one possible reason why this scheme is not listed under `https://en.wikipedia.org/wiki/List_of_Runge-Kutta_methods`.

   Hint. This question has quite a clear answer of the type "you know it when you see it". Revising the discussion regarding the tradeoffs involved when choosing Runge-Kutta quadrature rules may help you find this answer.
5. [2 marks] Determine the stability function $R(z)$ of this method.

2 Implicit Simpson's method [10 marks]

Consider the Runge-Kutta method whose one-step equation is given by

$$\tilde y(t) = y(0) + f\big(\tilde y_1(0)\big)\,\tfrac{t}{6} + f\big(\tilde y_2(\tfrac{t}{2})\big)\,\tfrac{4t}{6} + f\big(\tilde y(t)\big)\,\tfrac{t}{6} \tag{2}$$

where

$$\tilde y_1(0) = y(0), \qquad \tilde y_2(\tfrac{t}{2}) = y(0) + f\big(\tilde y_1(0)\big)\,\tfrac{t}{2}.$$

Note that equation (2) is obtained from equation (1) by replacing $\tilde y_3(t)$ with $\tilde y(t)$.

6. [2 marks] Write down the Butcher tableau for this scheme.
7. [4 marks] Complete the function `implicit_simpson_step()`.

   You may solve nonlinear equations using the function `find_zero(f,x0)` provided by the `Roots` package. This function returns a number `x` such that `f(x) = 0`, using `x0` as an initial guess for the root-finding algorithm. Use $y(0) + f\big(\tilde y_1(0)\big)\,t$ as initial guess for $\tilde y(t)$.

   Check that your code is correct using the provided function `convergence()`. You may assume without proof that the proposed method is second-order convergent.

8. [2 marks] Determine the stability function $R(z)$ of this scheme.
9. [2 marks] Does this scheme produce a numerical solution which converges to zero when applied to the ODE $\dot{y} = -y$ with a very large time step? Motivate your answer by referring to the stability function determined in Task 8.