Domain Theory 2024 Exercise Solutions

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Chapter 0

Preamble

These are my solutions to the exercises of TypeSig's Domain Theory lecture series. Find out more about it here: `https://typesig.comp-soc.com/pages/dt2024/`.

Chapter 1

First Steps to Scott Domains

1.1 Exercises

1. Consider the functions $\mathbb{B}_\perp \to \mathbb{B}_\perp$. Which ones are monotonic? There are a total of 27 such functions but only three significant classes.
2. Let $\mathbf{K}_K$ denote the chain of values $x_1, x_2, x_3, \ldots, x_K$ where $a \leq b$ implies $x_a \sqsubseteq x_b$. There is one monotonic function $\mathbf{K}_1 \to \mathbf{K}_1$, and there are three monotonic functions $\mathbf{K}_2 \to \mathbf{K}_2$.
   a) Write down the monotonic functions $\mathbf{K}_3 \to \mathbf{K}_3$.
   b) Write a simple recursive program to calculate the number of monotonic functions $\mathbf{K}_n \to \mathbf{K}_m$.
3. Give a semantics to the proc construct:
   a) $[\![\texttt{proc}\ c]\!]_{\mathcal{E}\sigma} =?$
   b) $[\![x]\!]_{\mathcal{E}\sigma} =?$

1.2 Solutions

I've only attempted one question, 2a.

It's simplest to see with a visual proof. When drawn like in the question, if a line crosses another (excluding lines ending in the same point, like those shown in the question), then the function is non-monotonic.

Let $x_1, x_2 \in \mathbf{K}_3$. As established, $x_1 \leq x_2$. If the lines drawn from x_1 and x_2 cross, then that means $f(x_1) > f(x_2)$, because if they were equal, then the lines would be ending in the same point, and if they were less than, they would not be crossing at all (they may be coming closer together, but they wouldn't cross). This means f is non-monotonic because $x_1 \leq x_2$ does not imply $f(x_1) \leq f(x_2)$ for this choice of f.

Thus, the monotonic functions are all those where the lines do not cross.

Chapter 2

Recursively Defined Programs

2.1 Exercises

1. Give an example of a poset A and a monotonic function $f\colon A \to A$ such that f *doesn't* a fixed point.

2. What is the lub operator on subsets $X \subseteq \mathbb{N}$ of the poset $(\mathbb{N}, \leq)$?

3. Show that if a function $f\colon A \to B$ on cpos A and B is monotonic and A is finite, then f is continuous.

   Hint: Finite directed sets contain their lub

4. Show that if a function $f\colon A \to B$ on cpos A and B is monotonic, then $\bigsqcup\{f(x) \mid x \in X\}$ exists.

   Hint: It suffices to show that $\{f(x) \mid x \in X\}$ is directed.

2.2 Solutions

1. Take the poset $(\mathbb{N}, \leq)$ and the function $f\colon \mathbb{N} \to \mathbb{N}$ defined to be $f(n) = n + 1$. Since f doesn't approach any particular value, and since we don't have ∞, $\mathbf{fix}(f)$ doesn't exist.

2. We require a binary operator $\sqcup$ such that it is idempotent, symmetric, and associative, and $x \leq y$ iff $x \sqcup y = y$. The function $\max\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined in the usual way satisfies these requirements. For all $n, m, k \in \mathbb{N}$ where $n \leq m \leq k$,

   i. it's idempotent, since $\max(n, n) = n$;
   ii. it's symmetric, since $\max(n, m) = m = \max(m, n)$;
   iii. it's associative, since

   $$\max(n, \max(m, k)) = \max(n, k) = k,$$

   and

   $$\max(\max(n, m), k) = \max(m, k) = k;$$

iv. and finally, $x \leq y$ iff $x \sqcup y = y$ by the definition of the max function.

3. By the definition of continuity, it only remains to prove that for all directed subsets $X \subseteq A$, it holds that $f(\bigsqcup X) = \bigsqcup\{f(x) \mid x \in X\}$.

   Recall that f being monotonic means that for all $a, b \in A$ such that $a \sqsubseteq b$, it holds that $f(a) \sqsubseteq f(b)$.

   Since any subset X of A will be finite, because A is finite, $\bigsqcup X$ will be a single $x_0 \in X$ such that, for all other $x \in X$, $x \sqsubseteq x_0$ (since X is a directed set). Because f is monotonic, $f(\bigsqcup X) = f(x_0) \sqsupseteq f(x)$ for all other $x \in X$.

   The image of f under X, represented by $\{f(x) \mid x \in X\}$, will have a single member which is "bigger" (via the $\sqsubseteq$ relation) than any other; this element is $f(x_0)$, because f is monotonic. Thus, $\bigsqcup\{f(x) \mid x \in X\} = f(x_0) = f(\bigsqcup X)$. $\square$

4. Consider some $a, b \in \{f(x) \mid x \in X\}$. Let a_X and b_X be the elements which f has mapped a and b from, so $f(a_X) = a$ and $f(b_X) = b$. There exists a $c_X \in X$ such that $a_X \sqsubseteq c_X$ and $b_X \sqsubseteq c_X$, since X is a directed set. Let $c \in \{f(x) \mid x \in X\}$ be defined as $f(c_X) = c$. Because f is monotonic, $f(a_X) = a \sqsubseteq c = f(c_X)$, and similarly $f(b_X) = b \sqsubseteq c = f(c_X)$. Thus, for any choice of $a, b \in \{f(x) \mid x \in X\}$, one can find a c such that $a \sqsubseteq c$ and $b \sqsubseteq c$, and therefore $\{f(x) \mid x \in X\}$ is directed. $\square$