



Georgios Kollidas/Fotolia

Sir Isaac Newton (1642–1727) was an English natural philosopher, a professor of mathematics at Cambridge University, and president of the Royal Society. His *Principia Mathematica* (1687), which deals with the laws and conditions of motion, is considered to be the greatest scientific work ever produced. The definitions of force, mass, and momentum and his three laws of motion crop up continually in dynamics. Quite fittingly, the unit of force named “newton” in SI units happens to be the approximate weight of an average apple, the falling object that inspired him to study the laws of gravity.

CHAPTER 2

Free Vibration of Single-Degree-of-Freedom Systems

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This chapter starts with a consideration of the free vibration of an undamped single-degree-of-freedom (spring-mass) system. Free vibration means that the mass is set into motion due to initial disturbance with no externally applied force other than the spring force, damper force, or gravitational force. To study the free-vibration response of the mass, we need to derive the governing equation, known as the *equation of motion*. The equation of motion of the undamped translational system is derived using four methods. The natural frequency of vibration of the system is defined and the solution of the equation of motion is presented using appropriate initial conditions. The solution is shown to represent harmonic motion. The equation of motion and the solution corresponding to free vibration of an undamped torsional system are presented. The response of first-order systems and the time constant are considered. Rayleigh's method, based on the principle of conservation of energy, is presented with illustrative examples.

Next, the derivation of the equation for the free vibration of a viscously damped single-degree-of-freedom system and its solution are considered. The concepts of critical damping constant, damping ratio, and frequency of damped vibration are introduced. The distinctions between underdamped, critically damped, and overdamped systems are explained. The energy dissipated in viscous damping and the concepts of specific damping and loss coefficient are considered. Viscously damped torsional systems are also considered analogous to viscously damped translational systems with applications. The graphical representation of characteristic roots and the corresponding solutions as well as the concept of parameter variations and root locus plots are considered. The equations of motion and their solutions of single-degree-of-freedom systems with Coulomb and hysteretic damping are presented. The concept of complex stiffness is also presented. The idea of stability and its importance is explained along with an example. The determination of the responses of single-degree-of-freedom systems using MATLAB is illustrated with examples.

Learning Objectives

After completing this chapter, you should be able to do the following:

- Derive the equation of motion of a single-degree-of-freedom system using a suitable technique such as Newton's second law of motion, D'Alembert's principle, the principle of virtual displacements, and the principle of conservation of energy.
- Linearize the nonlinear equation of motion.
- Solve a spring-mass-damper system for different types of free-vibration response depending on the amount of damping.
- Compute the natural frequency, damped frequency, logarithmic decrement, and time constant.
- Determine whether a given system is stable or not.
- Find the responses of systems with Coulomb and hysteretic damping.
- Find the free-vibration response using MATLAB.

2.1 Introduction

A system is said to undergo free vibration when it oscillates only under an initial disturbance with no external forces acting afterward. Some examples are the oscillations of the pendulum of a grandfather clock, the vertical oscillatory motion felt by a bicyclist after hitting a road bump, and the motion of a child on a swing after an initial push.

Figure 2.1(a) shows a spring-mass system that represents the simplest possible vibratory system. It is called a *single-degree-of-freedom system*, since one coordinate (x) is sufficient to specify the position of the mass at any time. There is no external force applied to the mass; hence the motion resulting from an initial disturbance will be free vibration. Since there is no element that causes dissipation of energy during the motion of the mass, the amplitude of motion remains constant with time; it is an *undamped* system. In actual practice, except in a vacuum, the amplitude of free vibration diminishes gradually over time, due to the resistance offered by the surrounding medium (such as air). Such vibrations are said to be *damped*. The study of the free vibration of undamped and damped single-degree-of-freedom systems is fundamental to the understanding of more advanced topics in vibrations.

Several mechanical and structural systems can be idealized as single-degree-of-freedom systems. In many practical systems, the mass is distributed, but for a simple analysis, it can be approximated by a single point mass. Similarly, the elasticity of the system, which may be distributed throughout the system, can also be idealized by a single spring. For the cam-follower system shown in Fig. 1.39, for example, the various masses were replaced by an equivalent mass (m_{eq}) in Example 1.12. The elements of the follower system (pushrod, rocker arm, valve, and valve spring) are all elastic but can be reduced to a single equivalent spring of stiffness k_{eq} . For a simple analysis, the cam-follower system can thus be idealized as a single-degree-of-freedom spring-mass system, as shown in Fig. 2.2.

Similarly, the structure shown in Fig. 2.3 can be considered a cantilever beam that is fixed at the ground. For the study of transverse vibration, the top mass can be considered a point mass and the supporting structure (beam) can be approximated as a spring to obtain the single-degree-of-freedom model shown in Fig. 2.4. The building frame shown

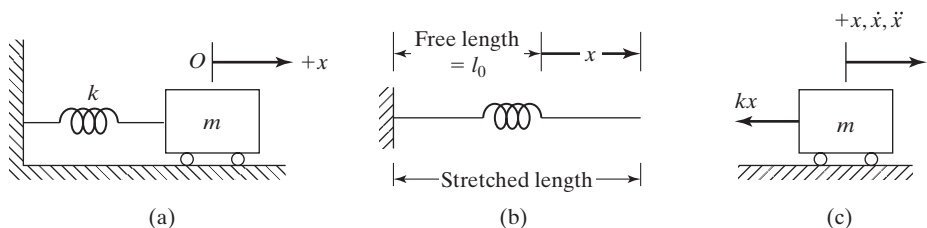


FIGURE 2.1 A spring-mass system in horizontal position.

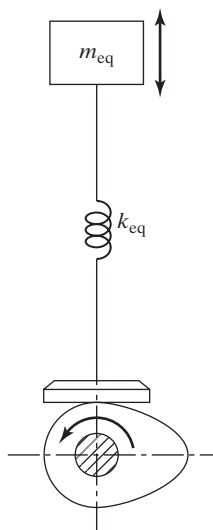


FIGURE 2.2 Equivalent spring-mass system for the cam-follower system of Fig. 1.39.



FIGURE 2.3 The space needle (structure). (RG/Fotolia)

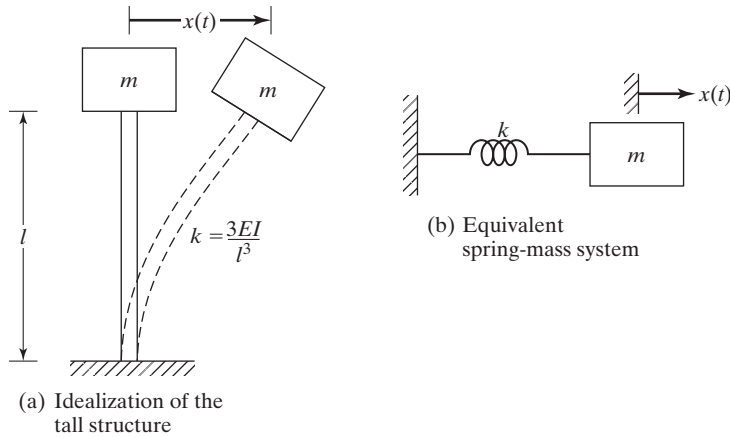


FIGURE 2.4 Modeling of tall structure as spring-mass system.

in Fig. 2.5(a) can also be idealized as a spring-mass system, as shown in Fig. 2.5(b). In this case, since the spring constant k is merely the ratio of force to deflection, it can be determined from the geometric and material properties of the columns. The mass of the idealized system is the same as that of the floor if we assume the mass of the columns to be negligible.

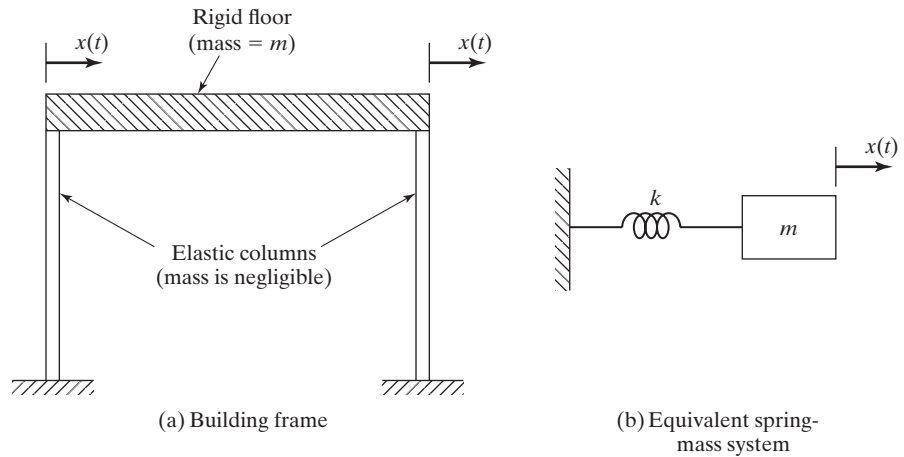


FIGURE 2.5 Idealization of a building frame.

2.2 Free Vibration of an Undamped Translational System

2.2.1 Equation of Motion Using Newton's Second Law of Motion

Using Newton's second law of motion, in this section we will consider the derivation of the equation of motion. The procedure we will use can be summarized as follows:

1. Select a suitable coordinate to describe the position of the mass or rigid body in the system. Use a linear coordinate to describe the linear motion of a point mass or the centroid of a rigid body, and an angular coordinate to describe the angular motion of a rigid body.
2. Determine the static equilibrium configuration of the system and measure the displacement of the mass or rigid body from its static equilibrium position.
3. Draw the free-body diagram of the mass or rigid body when a positive displacement and velocity are given to it. Indicate all the active and reactive forces acting on the mass or rigid body.
4. Apply Newton's second law of motion to the mass or rigid body shown by the free-body diagram. Newton's second law of motion can be stated as follows:

The rate of change of momentum of a mass is equal to the force acting on it.

Thus, if mass m is displaced a distance $\vec{x}(t)$ when acted upon by a resultant force $\vec{F}(t)$ in the same direction, Newton's second law of motion gives

$$\vec{F}(t) = \frac{d}{dt} \left(m \frac{d\vec{x}(t)}{dt} \right)$$

If mass m is constant, this equation reduces to

$$\vec{F}(t) = m \frac{d^2 \vec{x}(t)}{dt^2} = m \ddot{\vec{x}} \quad (2.1)$$

where

$$\ddot{\vec{x}} = \frac{d^2 \vec{x}(t)}{dt^2}$$

is the acceleration of the mass. Equation (2.1) can be stated in words as

Resultant force on the mass = mass \times acceleration

For a rigid body undergoing rotational motion, Newton's law gives

$$\vec{M}(t) = J \ddot{\theta} \quad (2.2)$$

where \vec{M} is the resultant moment acting on the body and $\vec{\theta}$ and $\ddot{\theta} = d^2\theta(t)/dt^2$ are the resulting angular displacement and angular acceleration, respectively. Equation (2.1), or (2.2), represents the equation of motion of the vibrating system.

The procedure is now applied to the undamped single-degree-of-freedom system shown in Fig. 2.1(a). Here the mass is supported on frictionless rollers and can have

translatory motion in the horizontal direction. When the mass is displaced a distance $+x$ from its static equilibrium position, the force in the spring is kx , and the free-body diagram of the mass can be represented as shown in Fig. 2.1(c). The application of Eq. (2.1) to mass m yields the equation of motion

$$F(t) = -kx = m\ddot{x}$$

or

$$m\ddot{x} + kx = 0 \quad (2.3)$$

2.2.2 Equation of Motion Using Other Methods

As stated in Section 1.6, the equations of motion of a vibrating system can be derived using several methods. The applications of D'Alembert's principle, the principle of virtual displacements, and the principle of conservation of energy are considered in this section.

D'Alembert's Principle. The equations of motion, Eqs. (2.1) and (2.2), can be rewritten as

$$\vec{F}(t) - m\ddot{\vec{x}} = 0 \quad (2.4a)$$

$$\vec{M}(t) - J\ddot{\theta} = 0 \quad (2.4b)$$

These equations can be considered equilibrium equations provided that $-m\ddot{\vec{x}}$ and $-J\ddot{\theta}$ are treated as a force and a moment, respectively. This fictitious force (or moment) is known as the inertia force (or inertia moment) and the artificial state of equilibrium implied by Eq. (2.4a) or (2.4b) is known as dynamic equilibrium. This principle, implied in Eq. (2.4a) or (2.4b), is called D'Alembert's principle. Applying it to the system shown in Fig. 2.1(c) yields the equation of motion:

$$-kx - m\ddot{x} = 0 \quad \text{or} \quad m\ddot{x} + kx = 0 \quad (2.3)$$

Principle of Virtual Displacements. The principle of virtual displacements states that "if a system that is in equilibrium under the action of a set of forces is subjected to a virtual displacement, then the total virtual work done by the forces will be zero." Here the virtual displacement is defined as an imaginary infinitesimal displacement given instantaneously. It must be a physically possible displacement that is compatible with the constraints of the system. The virtual work is defined as the work done by all the forces, including the inertia forces for a dynamic problem, due to a virtual displacement.

Consider a spring-mass system in a displaced position as shown in Fig. 2.6(a), where x denotes the displacement of the mass. Figure 2.6(b) shows the free-body diagram of the mass with the reactive and inertia forces indicated. When the mass is given a virtual displacement δx , as shown in Fig. 2.6(b), the virtual work done by each force can be computed as follows:

$$\text{Virtual work done by the spring force} = \delta W_s = -(kx)\delta x$$

$$\text{Virtual work done by the inertia force} = \delta W_i = -(m\ddot{x})\delta x$$

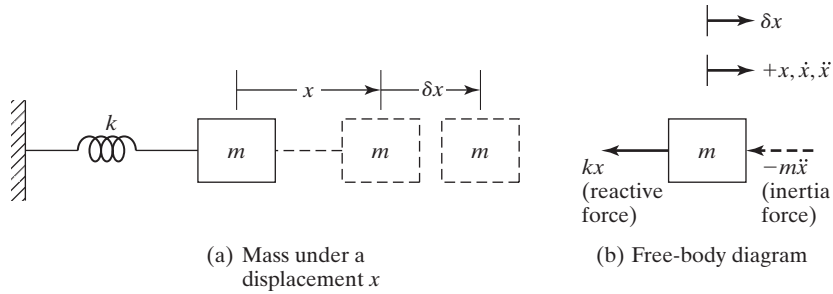


FIGURE 2.6 Mass under virtual displacement.

When the total virtual work done by all the forces is set equal to zero, we obtain

$$-m\ddot{x}\delta x - kx\delta x = 0 \quad (2.5)$$

Since the virtual displacement can have an arbitrary value, $\delta x \neq 0$, Eq. (2.5) gives the equation of motion of the spring-mass system as

$$m\ddot{x} + kx = 0 \quad (2.3)$$

Principle of Conservation of Energy. A system is said to be conservative if no energy is lost due to friction or energy-dissipating nonelastic members. If no work is done on a conservative system by external forces (other than gravity or other potential forces), then the total energy of the system remains constant. Since the energy of a vibrating system is partly potential and partly kinetic, the sum of these two energies remains constant. The kinetic energy T is stored in the mass by virtue of its velocity, and the potential energy U is stored in the spring by virtue of its elastic deformation. Thus the principle of conservation of energy can be expressed as

$$T + U = \text{constant}$$

or

$$\frac{d}{dt}(T + U) = 0 \quad (2.6)$$

The kinetic and potential energies are given by

$$T = \frac{1}{2}m\dot{x}^2 \quad (2.7)$$

and

$$U = \frac{1}{2}kx^2 \quad (2.8)$$

Substitution of Eqs. (2.7) and (2.8) into Eq. (2.6) yields the desired equation

$$m\ddot{x} + kx = 0 \quad (2.3)$$

2.2.3 Equation of Motion of a Spring-Mass System in Vertical Position

Consider the configuration of the spring-mass system shown in Fig. 2.7(a). The mass hangs at the lower end of a spring, which in turn is attached to a rigid support at its upper end. At rest, the mass will hang in a position called the *static equilibrium position*, in which the upward spring force exactly balances the downward gravitational force on the mass. In this position the length of the spring is $l_0 + \delta_{st}$, where δ_{st} is the static deflection—the elongation due to the weight W of the mass m . From Fig. 2.7(a), we find that, for static equilibrium,

$$W = mg = k\delta_{st} \quad (2.9)$$

where g is the acceleration due to gravity. Let the mass be deflected a distance $+x$ from its static equilibrium position; then the spring force is $-k(x + \delta_{st})$, as shown in Fig. 2.7(c). The application of Newton's second law of motion to mass m gives

$$m\ddot{x} = -k(x + \delta_{st}) + W$$

and since $k\delta_{st} = W$, we obtain

$$m\ddot{x} + kx = 0 \quad (2.10)$$

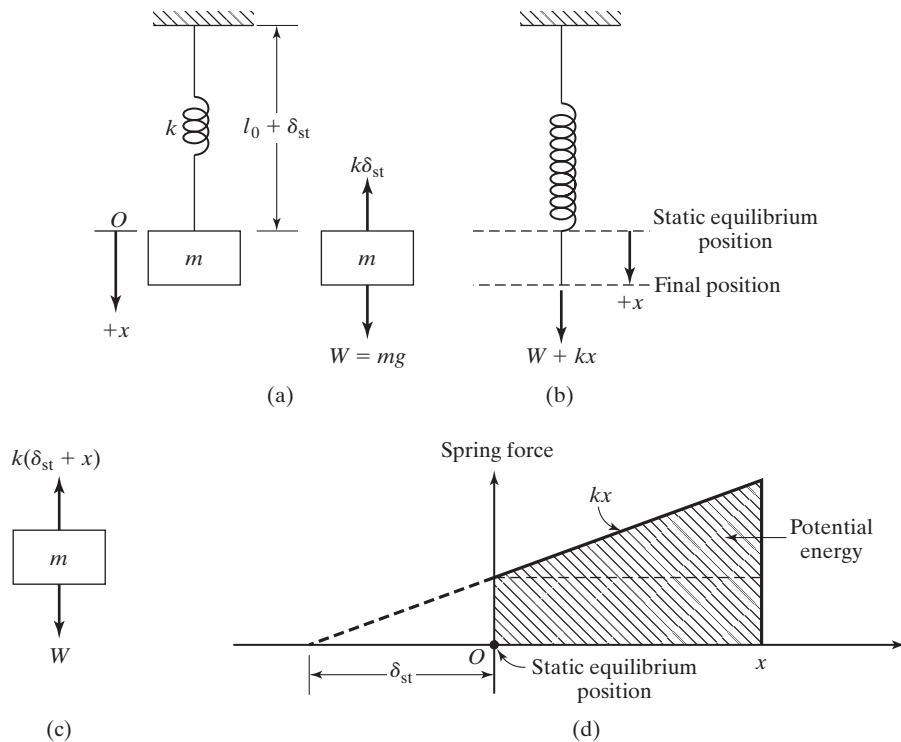


FIGURE 2.7 A spring-mass system in vertical position.

Notice that Eqs. (2.3) and (2.10) are identical. This indicates that when a mass moves in a vertical direction, we can ignore its weight, provided we measure x from its static equilibrium position.

Note: Equation (2.10), the equation of motion of the system shown in Fig. 2.7, can also be derived using D'Alembert's principle, the principle of virtual displacements, or the principle of conservation of energy. If we use the latter, for example, we note that the expression for the kinetic energy, T , remains the same as Eq. (2.7). However, the expression for the potential energy, U , is to be derived by considering the weight of the mass. For this we note that the spring force at static equilibrium position ($x = 0$) is mg . When the spring deflects by an amount x , its potential energy is given by (see Fig. 2.7(d)):

$$mgx + \frac{1}{2}kx^2$$

Furthermore, the potential energy of the system due to the change in elevation of the mass (note that $+x$ is downward) is $-mgx$. Thus the net potential energy of the system about the static equilibrium position is given by

$$\begin{aligned} U &= \text{potential energy of the spring} \\ &\quad + \text{change in potential energy due to change in elevation of the mass } m \\ &= mgx + \frac{1}{2}kx^2 - mgx = \frac{1}{2}kx^2 \end{aligned}$$

Since the expressions of T and U remain unchanged, the application of the principle of conservation of energy gives the same equation of motion, Eq. (2.3).

2.2.4 Solution

The solution of Eq. (2.3) can be found by assuming

$$x(t) = Ce^{st} \quad (2.11)$$

where C and s are constants to be determined. Substitution of Eq. (2.11) into Eq. (2.3) gives

$$C(ms^2 + k) = 0$$

Since C cannot be zero, we have

$$ms^2 + k = 0 \quad (2.12)$$

and hence

$$s = \pm \left(-\frac{k}{m} \right)^{1/2} = \pm i\omega_n \quad (2.13)$$

where $i = (-1)^{1/2}$ and

$$\omega_n = \left(\frac{k}{m} \right)^{1/2} \quad (2.14)$$

Equation (2.12) is called the *auxiliary* or the *characteristic* equation corresponding to the differential Eq. (2.3). The two values of s given by Eq. (2.13) are the roots of the characteristic equation, also known as the *eigenvalues* or the *characteristic values* of the problem. Since both values of s satisfy Eq. (2.12), the general solution of Eq. (2.3) can be expressed as

$$x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t} \quad (2.15)$$

where C_1 and C_2 are constants. By using the identities

$$e^{\pm i\alpha t} = \cos \alpha t \pm i \sin \alpha t$$

Eq. (2.15) can be rewritten as

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \quad (2.16)$$

where A_1 and A_2 are new constants. The constants C_1 and C_2 or A_1 and A_2 can be determined from the initial conditions of the system. Two conditions are to be specified to evaluate these constants uniquely. Note that the number of conditions to be specified is the same as the order of the governing differential equation. In the present case, if the values of displacement $x(t)$ and velocity $\dot{x}(t) = (dx/dt)(t)$ are specified as x_0 and \dot{x}_0 at $t = 0$, we have, from Eq. (2.16),

$$\begin{aligned} x(t=0) &= A_1 = x_0 \\ \dot{x}(t=0) &= \omega_n A_2 = \dot{x}_0 \end{aligned} \quad (2.17)$$

Hence $A_1 = x_0$ and $A_2 = \dot{x}_0/\omega_n$. Thus the solution of Eq. (2.3) subject to the initial conditions of Eq. (2.17) is given by

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \quad (2.18)$$

2.2.5 Harmonic Motion

Equations (2.15), (2.16), and (2.18) are harmonic functions of time. The motion is symmetric about the equilibrium position of the mass m . The velocity is a maximum and the acceleration is zero each time the mass passes through this position. At the extreme displacements, the velocity is zero and the acceleration is a maximum. Since this represents simple harmonic motion (see Section 1.10), the spring-mass system itself is called a *harmonic oscillator*. The quantity ω_n given by Eq. (2.14), represents the system's natural frequency of vibration.

Equation (2.16) can be expressed in a different form by introducing the notation

$$\begin{aligned} A_1 &= A \cos \phi \\ A_2 &= A \sin \phi \end{aligned} \quad (2.19)$$

where A and ϕ are the new constants, which can be expressed in terms of A_1 and A_2 as

$$\begin{aligned} A &= (A_1^2 + A_2^2)^{1/2} = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} = \text{amplitude} \\ \phi &= \tan^{-1} \left(\frac{A_2}{A_1} \right) = \tan^{-1} \left(\frac{\dot{x}_0}{x_0 \omega_n} \right) = \text{phase angle} \end{aligned} \quad (2.20)$$

Introducing Eq. (2.19) into Eq. (2.16), the solution can be written as

$$x(t) = A \cos (\omega_n t - \phi) \quad (2.21)$$

By using the relations

$$\begin{aligned} A_1 &= A_0 \sin \phi_0 \\ A_2 &= A_0 \cos \phi_0 \end{aligned} \quad (2.22)$$

Equation (2.16) can also be expressed as

$$x(t) = A_0 \sin (\omega_n t + \phi_0) \quad (2.23)$$

where

$$A_0 = A = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} \quad (2.24)$$

and

$$\phi_0 = \tan^{-1} \left(\frac{x_0 \omega_n}{\dot{x}_0} \right) \quad (2.25)$$

The nature of harmonic oscillation can be represented graphically as in Fig. 2.8(a). If \vec{A} denotes a vector of magnitude A , which makes an angle $\omega_n t - \phi$ with respect to the vertical (x) axis, then the solution, Eq. (2.21), can be seen to be the projection of the vector \vec{A} on the x -axis. The constants A_1 and A_2 of Eq. (2.16), given by Eq. (2.19), are merely the rectangular components of \vec{A} along two orthogonal axes making angles ϕ and $-(\frac{\pi}{2} - \phi)$ with respect to the vector \vec{A} . Since the angle $\omega_n t - \phi$ is a linear function of time, it increases linearly with time; the entire diagram thus rotates counterclockwise at an angular velocity ω_n . As the diagram (Fig. 2.8(a)) rotates, the projection of \vec{A} onto the x -axis varies harmonically so that the motion repeats itself every time the vector \vec{A} sweeps an angle of 2π . The projection of \vec{A} , namely $x(t)$, is shown plotted in Fig. 2.8(b) as a function of $\omega_n t$, and as a function of t in Fig. 2.8(c). The phase angle ϕ can also be interpreted as the angle between the origin and the first peak.

Note the following aspects of the spring-mass system:

1. If the spring-mass system is in a vertical position, as shown in Fig. 2.7(a), the circular natural frequency can be expressed as

$$\omega_n = \left(\frac{k}{m} \right)^{1/2} \quad (2.26)$$

The spring constant k can be expressed in terms of the mass m from Eq. (2.9) as

$$k = \frac{W}{\delta_{st}} = \frac{mg}{\delta_{st}} \quad (2.27)$$

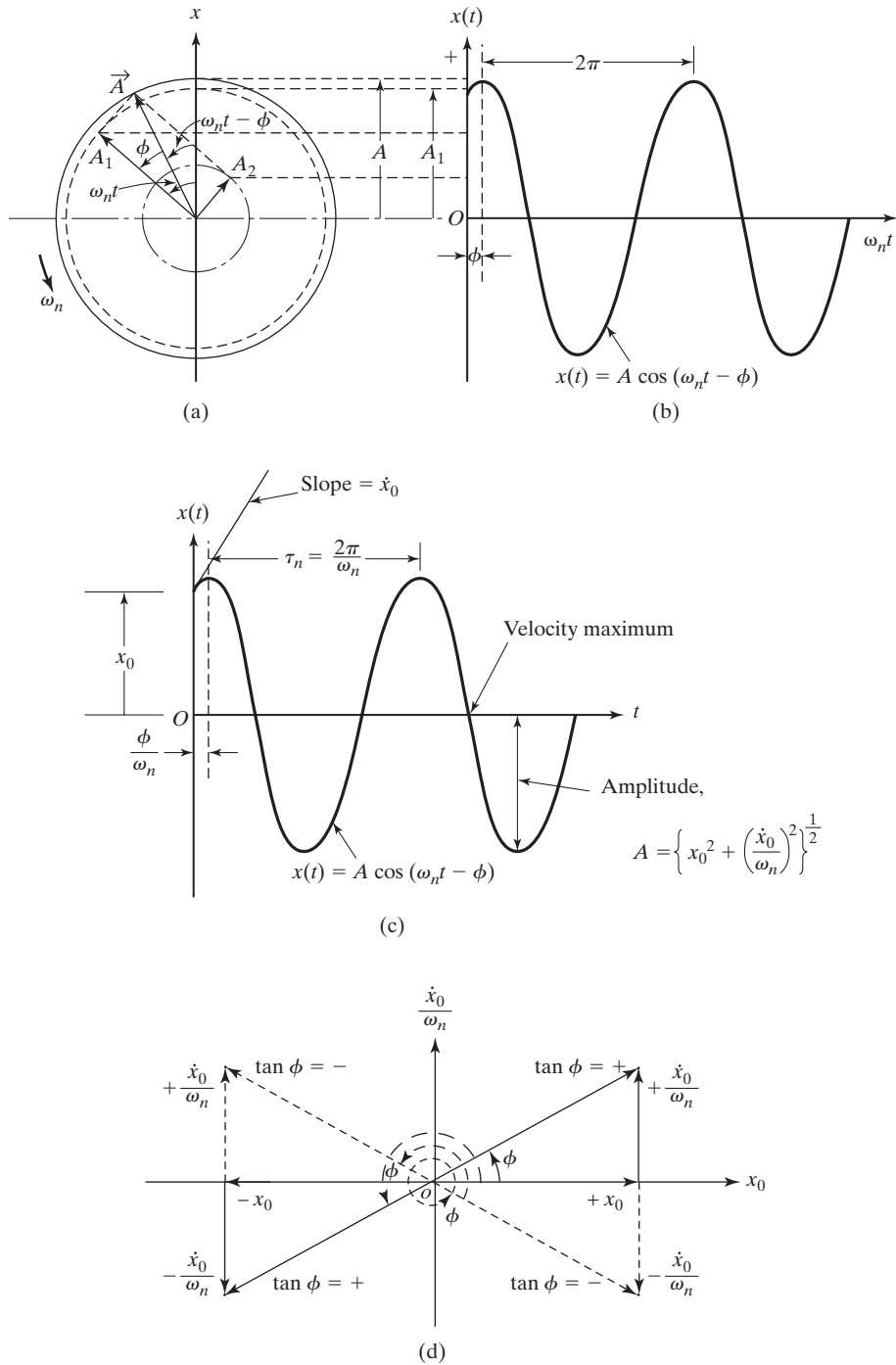


FIGURE 2.8 Graphical representation of the motion of a harmonic oscillator.

Substitution of Eq. (2.27) into Eq. (2.14) yields

$$\omega_n = \left(\frac{g}{\delta_{st}} \right)^{1/2} \quad (2.28)$$

Hence the natural frequency in cycles per second and the natural period in seconds are given by

$$f_n = \frac{1}{2\pi} \left(\frac{g}{\delta_{st}} \right)^{1/2} \quad (2.29)$$

$$\tau_n = \frac{1}{f_n} = 2\pi \left(\frac{\delta_{st}}{g} \right)^{1/2} \quad (2.30)$$

Thus, when the mass vibrates in a vertical direction, we can compute the natural frequency and the period of vibration by simply measuring the static deflection δ_{st} . We don't need to know the spring stiffness k and the mass m .

2. From Eq. (2.21), the velocity $\dot{x}(t)$ and the acceleration $\ddot{x}(t)$ of the mass m at time t can be obtained as

$$\begin{aligned} \dot{x}(t) &= \frac{dx}{dt}(t) = -\omega_n A \sin(\omega_n t - \phi) = \omega_n A \cos\left(\omega_n t - \phi + \frac{\pi}{2}\right) \\ \ddot{x}(t) &= \frac{d^2x}{dt^2}(t) = -\omega_n^2 A \cos(\omega_n t - \phi) = \omega_n^2 A \cos(\omega_n t - \phi + \pi) \end{aligned} \quad (2.31)$$

Equation (2.31) shows that the velocity leads the displacement by $\pi/2$ and the acceleration leads the displacement by π .

3. If the initial displacement (x_0) is zero, Eq. (2.21) becomes

$$x(t) = \frac{\dot{x}_0}{\omega_n} \cos\left(\omega_n t - \frac{\pi}{2}\right) = \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \quad (2.32)$$

If the initial velocity (\dot{x}_0) is zero, however, the solution becomes

$$x(t) = x_0 \cos \omega_n t \quad (2.33)$$

4. The value of the phase angle ϕ given by Eq. (2.20) [or ϕ_0 given by Eq. (2.23)] needs to be calculated with care. As indicated in Fig. 2.8 (d), $\tan \phi$ can be positive when both x_0 and $\frac{\dot{x}_0}{\omega_n}$ are either positive or negative. Thus, we need to use the first quadrant value of ϕ when both x_0 and $\frac{\dot{x}_0}{\omega_n}$ are positive and the third quadrant value of ϕ when both x_0 and $\frac{\dot{x}_0}{\omega_n}$ are negative. Similarly, since $\tan \phi$ can be negative when x_0 and $\frac{\dot{x}_0}{\omega_n}$ have opposite signs, we need to use the second quadrant value of ϕ when x_0 is negative and $\frac{\dot{x}_0}{\omega_n}$ is positive and the fourth quadrant value of ϕ when x_0 is positive and $\frac{\dot{x}_0}{\omega_n}$ is negative.

5. The response of a single-degree-of-freedom system can be represented in the displacement (x)-velocity plane, known as the state space or phase plane. For this, we consider the displacement given by Eq. (2.21) and the corresponding velocity:

$$x(t) = A \cos(\omega_n t - \phi)$$

or

$$\begin{aligned} \cos(\omega_n t - \phi) &= \frac{x}{A} \\ \dot{x}(t) &= -A\omega_n \sin(\omega_n t - \phi) \end{aligned} \quad (2.34)$$

or

$$\sin(\omega_n t - \phi) = -\frac{\dot{x}}{A\omega_n} = -\frac{y}{A} \quad (2.35)$$

where $y = \dot{x}/\omega_n$. By squaring and adding Eqs. (2.34) and (2.35), we obtain

$$\cos^2(\omega_n t - \phi) + \sin^2(\omega_n t - \phi) = 1$$

or

$$\frac{x^2}{A^2} + \frac{y^2}{A^2} = 1 \quad (2.36)$$

The graph of Eq. (2.36) in the (x, y) plane is a circle, as shown in Fig. 2.9(a), and it constitutes the phase-plane or state-space representation of the undamped system. The radius of the circle, A , is determined by the initial conditions of motion. Note that the graph of Eq. (2.36) in the (x, \dot{x}) plane will be an ellipse, as shown in Fig. 2.9(b).

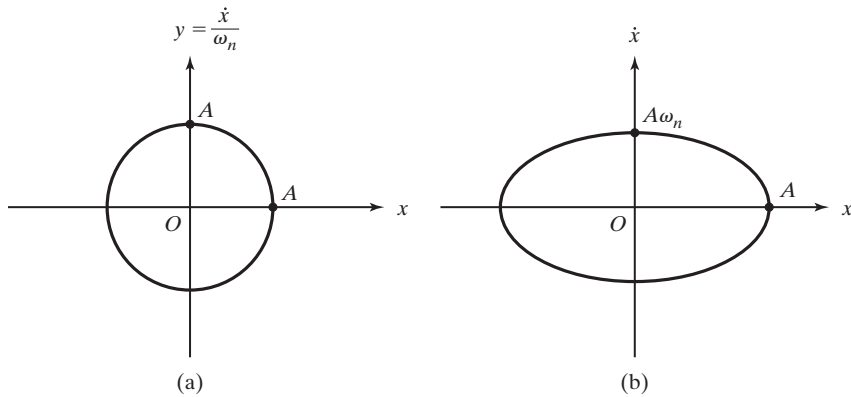


FIGURE 2.9 Phase-plane representation of an undamped system.

EXAMPLE 2.1**Response of a Spring-Mass System to Initial Conditions**

An undamped single-degree-of-freedom system has a mass of 1 kg and a stiffness of 2500 N/m. Find the magnitude and the phase of the response of the system when the initial displacement is -2 mm and initial velocity of 100 mm/s.

Solution: The natural frequency of the system is given by Eq. (2.14):

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{1}} = 50 \text{ rad/s}$$

The amplitude and phase of the response are given by Eqs. (2.20):

$$A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = \sqrt{(-2)^2 + \left(\frac{100}{50}\right)^2} = \sqrt{4 + 4} = 2.8284 \text{ mm}$$

$$\begin{aligned} \phi &= \tan^{-1}\left(\frac{\dot{x}_0}{x_0\omega_n}\right) = \tan^{-1}\left(\frac{100}{-2(50)}\right) = \tan^{-1}(-1) = -45.0^\circ \text{ or } -45.0^\circ + 180^\circ = 135.0^\circ \\ &= -0.7854 \text{ rad or } -0.7854 + \pi = 2.3562 \text{ rad} \end{aligned}$$

Since $\frac{\dot{x}_0}{\omega_n}$ is positive and x_0 is negative, ϕ should be in the second quadrant. Thus $\phi = 135.0^\circ$ or 2.3562 rad. Thus the response of the system can be expressed as (Eq. (2.21)):

$$x(t) = A \cos(\omega_n t - \phi) = 2.8284 \cos(50t - 2.3562) \text{ mm.}$$

■

EXAMPLE 2.2**Harmonic Response of a Water Tank**

The column of the water tank shown in Fig. 2.10(a) is 100 m high and is made of reinforced concrete with a tubular cross section of inner diameter 2.5 m and outer diameter 3 m. The tank has a mass of 275,000 kg when filled with water. By neglecting the mass of the column and assuming the Young's modulus of reinforced concrete as 30 GPa, determine the following:

- The natural frequency and the natural time period of transverse vibration of the water tank.
- The vibration response of the water tank due to an initial transverse displacement of 25 cm.
- The maximum values of the velocity and acceleration experienced by the water tank.

Solution: Assuming that the water tank is a point mass, the column has a uniform cross section, and the mass of the column is negligible, the system can be modeled as a cantilever beam with a concentrated load (weight) at the free end as shown in Fig. 2.10(b).

- The transverse deflection of the beam, δ , due to a load P is given by $\frac{Pl^3}{3EI}$, where l is the length, E is the Young's modulus, and I is the area moment of inertia of the beam's cross section. The stiffness of the beam (column of the tank) is given by

$$k = \frac{P}{\delta} = \frac{3EI}{l^3}$$

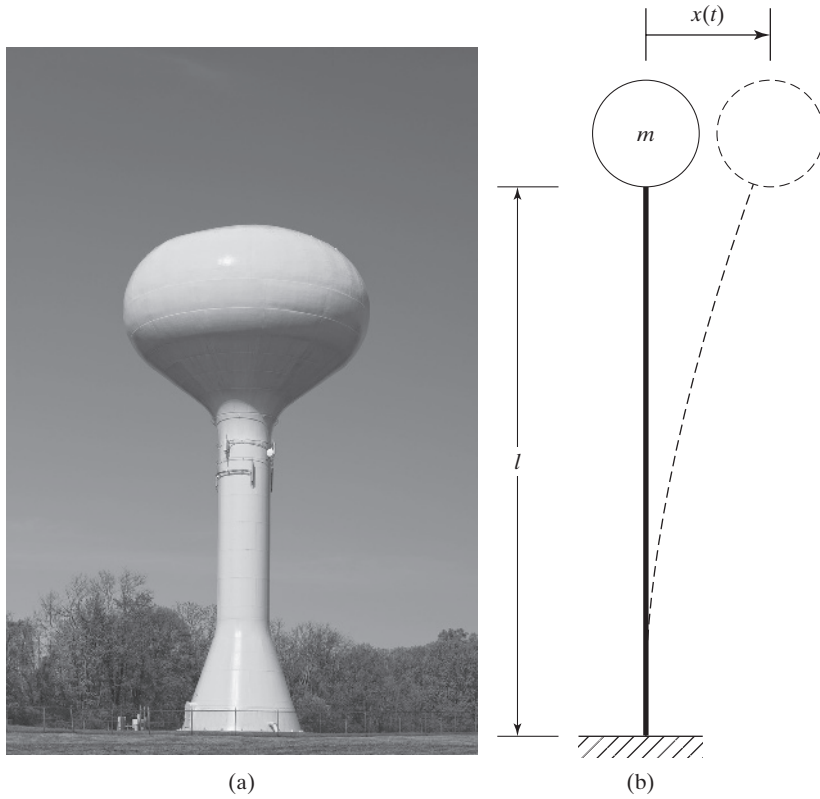


FIGURE 2.10 Elevated tank. (Andrea Izzotti/Fotolia.)

In the present case, $l = 100$ m, $E = 30 \times 10^9$ Pa,

$$I = \frac{\pi}{64}(d_0^4 - d_i^4) = \frac{\pi}{64}(3^4 - 2.5^4) = 2.059 \text{ m}^4$$

and hence

$$k = \frac{3(30 \times 10^9) \times (2.059)}{100^3} = 185,310 \text{ N/m}$$

The natural frequency of the water tank in the transverse direction is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{185,310}{275,000}} = 0.8209 \text{ rad/s}$$

The natural time period of transverse vibration of the tank is given by

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{0.8209} = 7.65 \text{ s}$$

- b. Using the initial displacement of $x_0 = 0.25$ m and the initial velocity of the water tank (\dot{x}_0) as zero, the harmonic response of the water tank can be expressed, using Eq. (2.23), as

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where the amplitude of transverse displacement (A_0) is given by

$$A_0 = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} = x_0 = 0.25 \text{ m}$$

and the phase angle (ϕ_0) by

$$\phi_0 = \tan^{-1} \left(\frac{x_0 \omega_n}{0} \right) = \frac{\pi}{2}$$

Thus

$$x(t) = 0.25 \sin \left(0.8209t + \frac{\pi}{2} \right) = 0.25 \cos(0.8209t) \text{ m} \quad (\text{E.1})$$

- c. The velocity of the water tank can be found by differentiating Eq. (E.1) as

$$\dot{x}(t) = 0.25(0.8209) \cos \left(0.8209t + \frac{\pi}{2} \right) \quad (\text{E.2})$$

and hence

$$\dot{x}_{\max} = A_0 \omega_n = 0.25(0.8209) = 0.2052 \text{ m/s}$$

The acceleration of the water tank can be determined by differentiating Eq. (E.2) as

$$\ddot{x}(t) = -0.25(0.8209)^2 \sin \left(0.8209t + \frac{\pi}{2} \right) \quad (\text{E.3})$$

and hence the maximum value of acceleration is given by

$$\ddot{x}_{\max} = A_0 (\omega_n)^2 = 0.25(0.8209)^2 = 0.1684 \text{ m/s}^2$$

■

EXAMPLE 2.3

Free-Vibration Response Due to Impact

A cantilever beam carries a mass M at the free end as shown in Fig. 2.11(a). A mass m falls from a height h onto the mass M and adheres to it without rebounding. Determine the resulting transverse vibration of the beam.

Solution: When the mass m falls through a height h , it will strike the mass M with a velocity of $v_m = \sqrt{2gh}$, where g is the acceleration due to gravity. Since the mass m adheres to M without

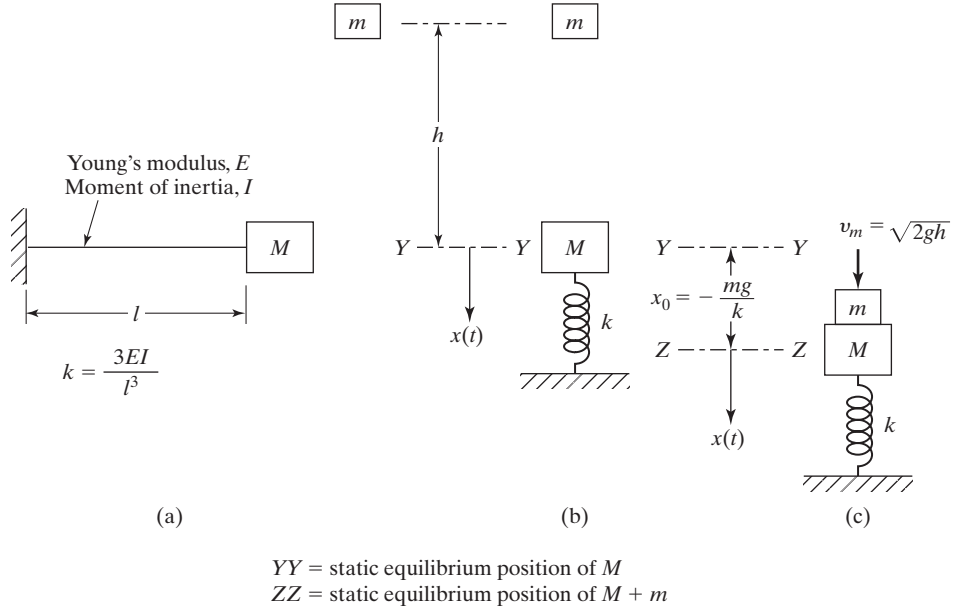


FIGURE 2.11 Response due to impact.

rebounding, the velocity of the combined mass $(M + m)$ immediately after the impact (\dot{x}_0) can be found using the principle of conservation of momentum:

$$mv_m = (M + m)\dot{x}_0$$

or

$$\dot{x}_0 = \left(\frac{m}{M + m} \right) v_m = \left(\frac{m}{M + m} \right) \sqrt{2gh} \quad (\text{E.1})$$

The static equilibrium position of the beam with the new mass $(M + m)$ is located at a distance of $\frac{mg}{k}$ below the static equilibrium position of the original mass (M) as shown in Fig. 2.11(c). Here k denotes the stiffness of the cantilever beam, given by

$$k = \frac{3EI}{l^3}$$

Since free vibration of the beam with the new mass $(M + m)$ occurs about its own static equilibrium position, the initial conditions of the problem can be stated as

$$x_0 = -\frac{mg}{k}, \quad \dot{x}_0 = \left(\frac{m}{M + m} \right) \sqrt{2gh} \quad (\text{E.2})$$

Thus the resulting free transverse vibration of the beam can be expressed as (see Eq. (2.21)):

$$x(t) = A \cos(\omega_n t - \phi)$$

where

$$A = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2}$$

$$\phi = \tan^{-1} \left(\frac{\dot{x}_0}{x_0 \omega_n} \right)$$

$$\omega_n = \sqrt{\frac{k}{M + m}} = \sqrt{\frac{3EI}{l^3(M + m)}}$$

with x_0 and \dot{x}_0 given by Eq. (E.2).

■

EXAMPLE 2.4

Young's Modulus from Natural Frequency Measurement

A simply supported beam of square cross section 5 mm \times 5 mm and length 1 m, carrying a mass of 2.3 kg at the middle, is found to have a natural frequency of transverse vibration of 30 rad/s. Determine the Young's modulus of elasticity of the beam.

Solution: By neglecting the self weight of the beam, the natural frequency of transverse vibration of the beam can be expressed as

$$\omega_n = \sqrt{\frac{k}{m}} \quad (\text{E.1})$$

where

$$k = \frac{192EI}{l^3} \quad (\text{E.2})$$

where E is the Young's modulus, l is the length, and I is the area moment of inertia of the beam:

$$I = \frac{1}{12} (5 \times 10^{-3})(5 \times 10^{-3})^3 = 0.5208 \times 10^{-10} \text{ m}^4$$

Since $m = 2.3 \text{ kg}$, $l = 1.0 \text{ m}$, and $\omega_n = 30.0 \text{ rad/s}$, Eqs. (E.1) and (E.2) yield

$$k = \frac{192EI}{l^3} = m\omega_n^2$$

or

$$E = \frac{m\omega_n^2 l^3}{192I} = \frac{2.3(30.0)^2(1.0)^3}{192(0.5208 \times 10^{-10})} = 207.0132 \times 10^9 \text{ N/m}^2$$

This indicates that the material of the beam is probably carbon steel.

■

EXAMPLE 2.5**Natural Frequency of Cockpit of a Firetruck**

The cockpit of a firetruck is located at the end of a telescoping boom, as shown in Fig. 2.12(a). The cockpit, along with the fireman, weighs 2000 N. Find the cockpit's natural frequency of vibration in the vertical direction.

Data: Young's modulus of the material: $E = 2.1 \times 10^{11} \text{ N/m}^2$; lengths: $l_1 = l_2 = l_3 = 3 \text{ m}$; cross-sectional areas: $A_1 = 20 \text{ cm}^2$, $A_2 = 10 \text{ cm}^2$, $A_3 = 5 \text{ cm}^2$.

Solution: To determine the system's natural frequency of vibration, we find the equivalent stiffness of the boom in the vertical direction and use a single-degree-of-freedom idealization. For this, we assume that the mass of the telescoping boom is negligible and the telescoping boom can deform only in the axial direction (with no bending). Since the force induced at any cross section $O_1 O_2$ is equal to the axial load applied at the end of the boom, as shown in Fig. 2.12(b), the axial stiffness of the boom (k_b) is given by

$$\frac{1}{k_b} = \frac{1}{k_{b_1}} + \frac{1}{k_{b_2}} + \frac{1}{k_{b_3}} \quad (\text{E.1})$$

where k_{b_i} denotes the axial stiffness of the i th segment of the boom:

$$k_{b_i} = \frac{A_i E_i}{l_i}, \quad i = 1, 2, 3 \quad (\text{E.2})$$

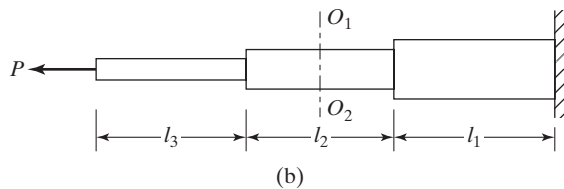
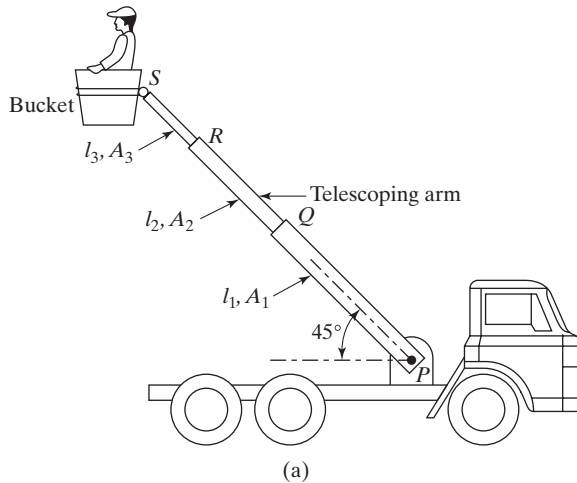


FIGURE 2.12 Telescoping boom of a fire truck.

From the known data ($l_1 = l_2 = l_3 = 3 \text{ m}$, $A_1 = 20 \text{ cm}^2$, $A_2 = 10 \text{ cm}^2$, $A_3 = 5 \text{ cm}^2$, $E_1 = E_2 = E_3 = 2.1 \times 10^{11} \text{ N/m}^2$),

$$k_{b_1} = \frac{(20 \times 10^{-4})(2.1 \times 10^{11})}{3} = 14 \times 10^7 \text{ N/m}$$

$$k_{b_2} = \frac{(10 \times 10^{-4})(2.1 \times 10^{11})}{3} = 7 \times 10^7 \text{ N/m}$$

$$k_{b_3} = \frac{(5 \times 10^{-4})(2.1 \times 10^{11})}{3} = 3.5 \times 10^7 \text{ N/m}$$

Thus Eq. (E.1) gives

$$\frac{1}{k_b} = \frac{1}{14 \times 10^7} + \frac{1}{7 \times 10^7} + \frac{1}{3.5 \times 10^7} = \frac{1}{2 \times 10^7}$$

or

$$k_b = 2 \times 10^7 \text{ N/m}$$

The stiffness of the telescoping boom in the vertical direction, k , can be determined as

$$k = k_b \cos^2 45^\circ = 10^7 \text{ N/m}$$

The natural frequency of vibration of the cockpit in the vertical direction is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{(10^7)(9.81)}{2000}} = 221.4723 \text{ rad/s}$$

■

EXAMPLE 2.6

Natural Frequency of Pulley System

Determine the natural frequency of the system shown in Fig. 2.13(a). Assume the pulleys to be frictionless and of negligible mass.

Solution: To determine the natural frequency, we find the equivalent stiffness of the system and solve it as a single-degree-of-freedom problem. Since the pulleys are frictionless and massless, the tension in the rope is constant and is equal to the weight W of the mass m . From the static equilibrium of the pulleys and the mass (see Fig. 2.13(b)), it can be seen that the upward force acting on pulley 1 is $2W$ and the downward force acting on pulley 2 is $2W$. The center of pulley 1 (point A) moves up by a distance $2W/k_1$, and the center of pulley 2 (point B) moves down by $2W/k_2$. Thus the total movement of the mass m (point O) is

$$2\left(\frac{2W}{k_1} + \frac{2W}{k_2}\right)$$

as the rope on either side of the pulley is free to move the mass downward. If k_{eq} denotes the equivalent spring constant of the system,

$$\frac{\text{Weight of the mass}}{\text{Equivalent spring constant}} = \text{Net displacement of the mass}$$

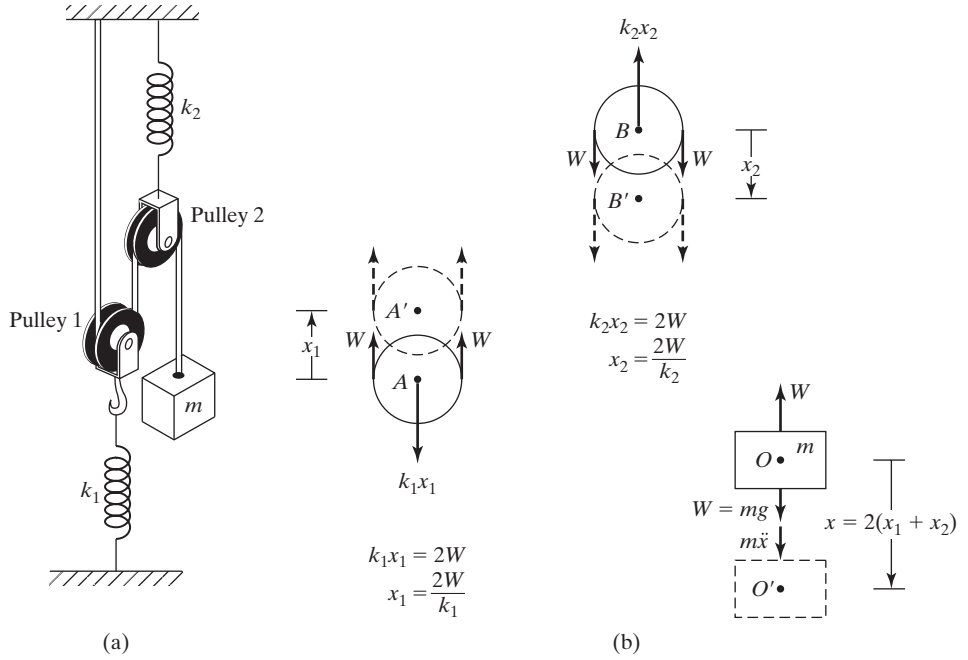


FIGURE 2.13 Pulley system.

$$\begin{aligned}\frac{W}{k_{\text{eq}}} &= 4W \left(\frac{1}{k_1} + \frac{1}{k_2} \right) = \frac{4W(k_1 + k_2)}{k_1 k_2} \\ k_{\text{eq}} &= \frac{k_1 k_2}{4(k_1 + k_2)}\end{aligned}\quad (\text{E.1})$$

By displacing mass m from the static equilibrium position by x , the equation of motion of the mass can be written as

$$m\ddot{x} + k_{\text{eq}}x = 0 \quad (\text{E.2})$$

and hence the natural frequency is given by

$$\omega_n = \left(\frac{k_{\text{eq}}}{m} \right)^{1/2} = \left[\frac{k_1 k_2}{4m(k_1 + k_2)} \right]^{1/2} \text{ rad/s} \quad (\text{E.3})$$

or

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{4\pi} \left[\frac{k_1 k_2}{m(k_1 + k_2)} \right]^{1/2} \text{ cycles/s} \quad (\text{E.4})$$

2.3 Free Vibration of an Undamped Torsional System

If a rigid body oscillates about a specific reference axis, the resulting motion is called *torsional vibration*. In this case, the displacement of the body is measured in terms of an angular coordinate. In a torsional vibration problem, the restoring moment may be due to the torsion of an elastic member or to the unbalanced moment of a force or couple.

Figure 2.14 shows a disc, which has a polar mass moment of inertia J_0 , mounted at one end of a solid circular shaft, the other end of which is fixed. Let the angular rotation of the disc about the axis of the shaft be θ ; θ also represents the shaft's angle of twist. From the theory of torsion of circular shafts [2.1], we have the relation

$$M_t = \frac{GI_0}{l}\theta \quad (2.37)$$

where M_t is the torque that produces the twist θ , G is the shear modulus, l is the length of the shaft, I_0 is the polar moment of inertia of the cross section of the shaft, given by

$$I_0 = \frac{\pi d^4}{32} \quad (2.38)$$

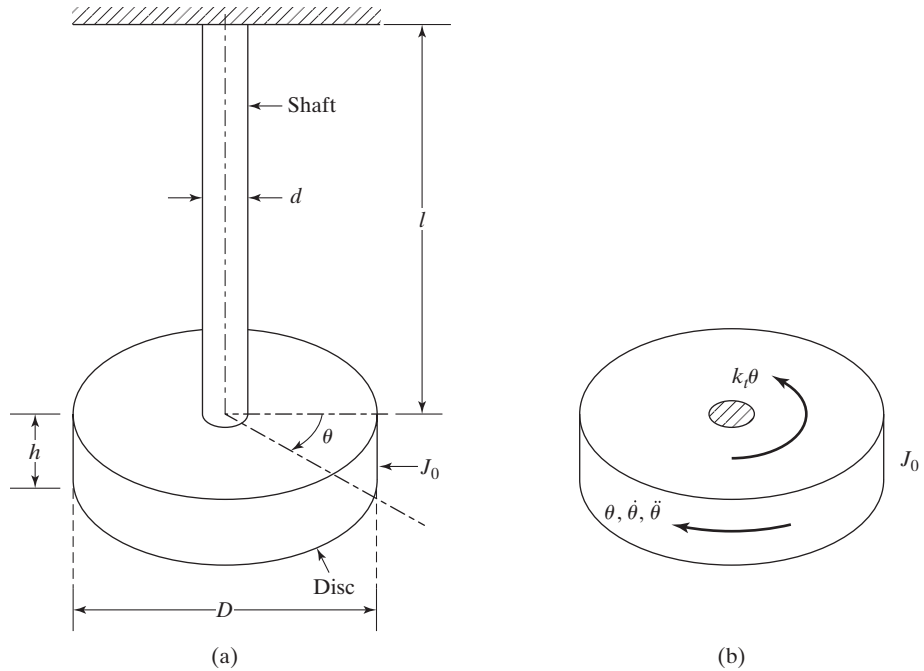


FIGURE 2.14 Torsional vibration of a disc.

and d is the diameter of the shaft. If the disc is displaced by θ from its equilibrium position, the shaft provides a restoring torque of magnitude M_t . Thus the shaft acts as a torsional spring with a torsional spring constant

$$k_t = \frac{M_t}{\theta} = \frac{GI_0}{l} = \frac{\pi G d^4}{32l} \quad (2.39)$$

2.3.1 Equation of Motion

The equation of the angular motion of the disc about its axis can be derived by using Newton's second law or any of the methods discussed in Section 2.2.2. By considering the free-body diagram of the disc (Fig. 2.14(b)), we can derive the equation of motion by applying Newton's second law of motion:

$$J_0 \ddot{\theta} + k_t \theta = 0 \quad (2.40)$$

which can be seen to be identical to Eq. (2.3) if the polar mass moment of inertia J_0 , the angular displacement θ , and the torsional spring constant k_t are replaced by the mass m , the displacement x , and the linear spring constant k , respectively. Thus the natural circular frequency of the torsional system is

$$\omega_n = \left(\frac{k_t}{J_0} \right)^{1/2} \quad (2.41)$$

and the period in seconds and frequency of vibration in cycles per second are

$$\tau_n = 2\pi \left(\frac{J_0}{k_t} \right)^{1/2} \quad (2.42)$$

$$f_n = \frac{1}{2\pi} \left(\frac{k_t}{J_0} \right)^{1/2} \quad (2.43)$$

Note the following aspects of this system:

1. If the cross section of the shaft supporting the disc is not circular, an appropriate torsional spring constant is to be used [2.4, 2.5].
2. The polar mass moment of inertia of a disc is given by

$$J_0 = \frac{\rho h \pi D^4}{32} = \frac{WD^2}{8g}$$

where ρ is the mass density, h is the thickness, D is the diameter, and W is the weight of the disc.

3. The torsional spring-inertia system shown in Fig. 2.14 is referred to as a *torsional pendulum*. One of the most important applications of a torsional pendulum is in a mechanical clock, where a ratchet and pawl convert the regular oscillation of a small torsional pendulum into the movements of the hands.

2.3.2 Solution

The general solution of Eq. (2.40) can be obtained, as in the case of Eq. (2.3):

$$\theta(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \quad (2.44)$$

where ω_n is given by Eq. (2.41) and A_1 and A_2 can be determined from the initial conditions. If

$$\theta(t = 0) = \theta_0 \quad \text{and} \quad \dot{\theta}(t = 0) = \frac{d\theta}{dt}(t = 0) = \dot{\theta}_0 \quad (2.45)$$

the constants A_1 and A_2 can be found:

$$\begin{aligned} A_1 &= \theta_0 \\ A_2 &= \dot{\theta}_0 / \omega_n \end{aligned} \quad (2.46)$$

Equation (2.44) can also be seen to represent a simple harmonic motion.

EXAMPLE 2.7

Natural Frequency of Compound Pendulum

Any rigid body pivoted at a point other than its center of mass will oscillate about the pivot point under its own gravitational force. Such a system is known as a compound pendulum (Fig. 2.15). Find the natural frequency of such a system.

Solution: Let O be the point of suspension and G be the center of mass of the compound pendulum, as shown in Fig. 2.15. Let the rigid body oscillate in the xy -plane so that the coordinate θ can be used to describe its motion. Let d denote the distance between O and G , and J_0 the mass moment of inertia

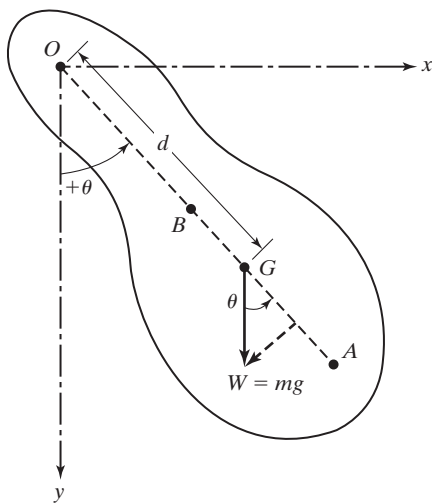


FIGURE 2.15 Compound pendulum.

of the body about the z -axis (perpendicular to both x and y). For a displacement θ , the restoring torque (due to the weight of the body W) is ($Wd \sin \theta$) and the equation of motion is

$$J_0 \ddot{\theta} + Wd \sin \theta = 0 \quad (\text{E.1})$$

Note that Eq. (E.1) is a second-order nonlinear ordinary differential equation. Although it is possible to find an exact solution of this equation (see Section 13.3), exact solutions cannot be found for most nonlinear differential equations. An approximate solution of Eq. (E.1) can be found by one of two methods. A numerical procedure can be used to integrate Eq. (E.1). Alternatively, Eq. (E.1) can be approximated by a linear equation whose exact solution can be determined readily. To use the latter approach, we assume small angular displacements so that θ is small and $\sin \theta \approx \theta$. Hence Eq. (E.1) can be approximated by the linear equation:

$$J_0 \ddot{\theta} + Wd\theta = 0 \quad (\text{E.2})$$

This gives the natural frequency of the compound pendulum:

$$\omega_n = \left(\frac{Wd}{J_0} \right)^{1/2} = \left(\frac{mgd}{J_0} \right)^{1/2} \quad (\text{E.3})$$

Comparing Eq. (E.3) with the natural frequency of a simple pendulum, $\omega_n = (g/l)^{1/2}$ (see Problem 2.84), we can find the length of the equivalent simple pendulum:

$$l = \frac{J_0}{md} \quad (\text{E.4})$$

If J_0 is replaced by mk_0^2 , where k_0 is the radius of gyration of the body about O , Eqs. (E.3) and (E.4) become

$$\omega_n = \left(\frac{gd}{k_0^2} \right)^{1/2} \quad (\text{E.5})$$

$$l = \left(\frac{k_0^2}{d} \right) \quad (\text{E.6})$$

If k_G denotes the radius of gyration of the body about G , we have

$$k_0^2 = k_G^2 + d^2 \quad (\text{E.7})$$

and Eq. (E.6) becomes

$$l = \left(\frac{k_G^2}{d} + d \right) \quad (\text{E.8})$$

If the line OG is extended to point A such that

$$GA = \frac{k_G^2}{d} \quad (\text{E.9})$$

Eq. (E.8) becomes

$$l = GA + d = OA \quad (\text{E.10})$$

Hence, from Eq. (E.5), ω_n is given by

$$\omega_n = \left\{ \frac{g}{(k_O^2/d)} \right\}^{1/2} = \left(\frac{g}{l} \right)^{1/2} = \left(\frac{g}{OA} \right)^{1/2} \quad (\text{E.11})$$

This equation shows that, no matter whether the body is pivoted from O or A , its natural frequency is the same. The point A is called the *center of percussion*.

■

Center of Percussion. The concepts of compound pendulum and center of percussion can be used in many practical applications:

1. A hammer can be shaped to have the center of percussion at the hammer head while the center of rotation is at the handle. In this case, the impact force at the hammer head will not cause any normal reaction at the handle (Fig. 2.16(a)).
2. In a baseball bat, if on one hand the ball is made to strike at the center of percussion while the center of rotation is at the hands, no reaction perpendicular to the bat will be experienced by the batter (Fig. 2.16(b)). On the other hand, if the ball strikes the bat near the free end or near the hands, the batter will experience pain in the hands as a result of the reaction perpendicular to the bat.
3. In Izod (impact) testing of materials, the specimen is suitably notched and held in a vise fixed to the base of the machine (see Fig. 2.16(c)). A pendulum is released from a standard height, and the free end of the specimen is struck by the pendulum as it passes through its lowest position. The deformation and bending of the pendulum can be reduced if the center of percussion is located near the striking edge. In this case, the pivot will be free of any impulsive reaction.
4. In a car (shown in Fig. 2.16(d)), if the front wheels strike a bump, the passengers will not feel any reaction if the center of percussion of the vehicle is located

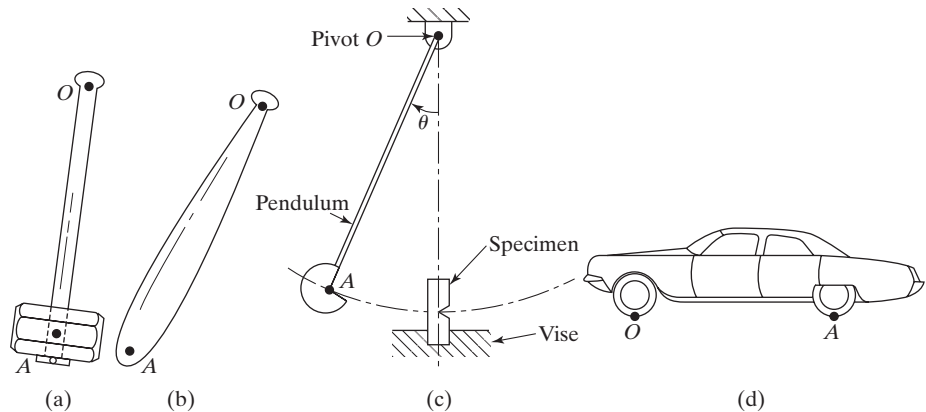


FIGURE 2.16 Applications of center of percussion.

near the rear axle. Similarly, if the rear wheels strike a bump at point A , no reaction will be felt at the front axle (point O) if the center of percussion is located near the front axle. It is desirable, therefore, to have the center of oscillation of the vehicle at one axle and the center of percussion at the other axle [2.2].

2.4 Response of First-Order Systems and Time Constant

Consider a turbine rotor mounted in bearings as shown in Fig. 2.17(a). The viscous fluid (lubricant) in the bearings offers viscous damping torque during the rotation of the turbine rotor. Assuming the mass moment of inertia of the rotor about the axis of rotation as J and

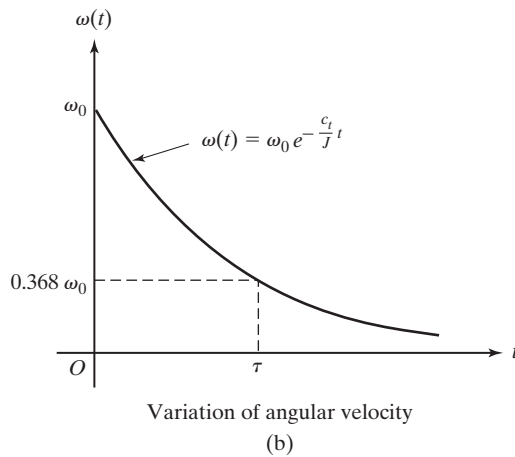
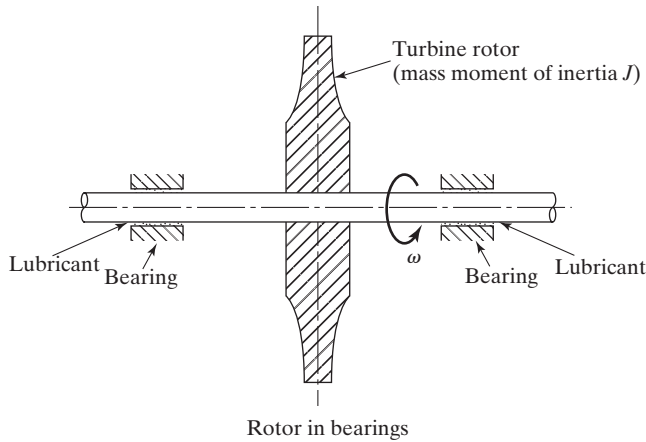


FIGURE 2.17 Rotor mounted in bearings and its angular velocity variation.

the rotational damping constant of the bearings as c_t , the application of Newton's second law of motion yields the equation of motion of the rotor as

$$J\dot{\omega} + c_t\omega = 0 \quad (2.47)$$

where ω is the angular velocity of the rotor, $\dot{\omega} = \frac{d\omega}{dt}$ is the time rate of change of the angular velocity, and the external torque applied to the system is assumed to be zero. We assume the initial angular velocity, $\omega(t = 0) = \omega_0$, as the input and the angular velocity of the rotor as the output of the system. Note that the angular velocity, instead of the angular displacement, is considered as the output in order to obtain the equation of motion as a first order differential equation.

The solution of the equation of motion of the rotor, Eq. (2.47), can be found by assuming the trial solution as

$$\omega(t) = Ae^{st} \quad (2.48)$$

where A and s are unknown constants. By using the initial condition, $\omega(t = 0) = \omega_0$, Eq. (2.48) can be written as

$$\omega(t) = \omega_0 e^{st} \quad (2.49)$$

By substituting Eq. (2.49) into Eq. (2.47), we obtain

$$\omega_0 e^{st}(Js + c_t) = 0 \quad (2.50)$$

Since $\omega_0 = 0$ leads to "no motion" of the rotor, we assume $\omega_0 \neq 0$ and Eq. (2.50) can be satisfied only if

$$Js + c_t = 0 \quad (2.51)$$

Equation (2.51) is known as the characteristic equation which yields $s = -\frac{c_t}{J}$. Thus the solution, Eq. (2.49), becomes

$$\omega(t) = \omega_0 e^{-\frac{c_t}{J}t} \quad (2.52)$$

The variation of the angular velocity, given by Eq. (2.52), with time is shown in Fig. 2.17(b). The curve starts at ω_0 , decays and approaches zero as t increases without limit. In dealing with exponentially decaying responses, such as the one given by Eq. (2.52), it is convenient to describe the response in terms of a quantity known as the *time constant* (τ). The time constant is defined as the value of time which makes the exponent in Eq. (2.52) equal to -1 . Because the exponent of Eq. (2.52) is known to be $-\frac{c_t}{J}t$, the time constant will be equal to

$$\tau = \frac{J}{c_t} \quad (2.53)$$

so that Eq. (2.52) gives, for $t = \tau$,

$$\omega(t) = \omega_0 e^{-\frac{c_t}{J}\tau} = \omega_0 e^{-1} = 0.368\omega_0 \quad (2.54)$$

Thus the response reduces to 0.368 times its initial value at a time equal to the time constant of the system.

2.5 Rayleigh's Energy Method

For a single-degree-of-freedom system, the equation of motion was derived using the energy method in Section 2.2.2. In this section, we shall use the energy method to find the natural frequencies of single-degree-of-freedom systems. The principle of conservation of energy, in the context of an undamped vibrating system, can be restated as

$$T_1 + U_1 = T_2 + U_2 \quad (2.55)$$

where the subscripts 1 and 2 denote two different instants of time. Specifically, we use the subscript 1 to denote the time when the mass is passing through its static equilibrium position and choose $U_1 = 0$ as reference for the potential energy. If we let the subscript 2 indicate the time corresponding to the maximum displacement of the mass, we have $T_2 = 0$. Thus Eq. (2.55) becomes

$$T_1 + 0 = 0 + U_2 \quad (2.56)$$

If the system is undergoing harmonic motion, then T_1 and U_2 denote the maximum values of T and U , respectively, and Eq. (2.56) becomes

$$T_{\max} = U_{\max} \quad (2.57)$$

The application of Eq. (2.57), which is also known as *Rayleigh's energy method*, gives the natural frequency of the system directly, as illustrated in the following examples.

EXAMPLE 2.8

Manometer for Diesel Engine

The exhaust from a single-cylinder four-stroke diesel engine is to be connected to a silencer, and the pressure therein is to be measured with a simple U-tube manometer (see Fig. 2.18). Calculate the minimum length of the manometer tube so that the natural frequency of oscillation of the mercury column will be 3.5 times slower than the frequency of the pressure fluctuations in the silencer at an engine speed of 600 rpm. The frequency of pressure fluctuation in the silencer is equal to

$$\frac{\text{Number of cylinders} \times \text{Speed of the engine}}{2}$$

Solution

1. *Natural frequency of oscillation of the liquid column:* Let the datum in Fig. 2.18 be taken as the equilibrium position of the liquid. If the displacement of the liquid column from the equilibrium position is denoted by x , the change in potential energy is given by

$$\begin{aligned} U &= \text{potential energy of raised liquid column} + \text{potential energy of depressed liquid column} \\ &= (\text{weight of mercury raised} \times \text{displacement of the C.G. of the segment}) + (\text{weight of} \\ &\quad \text{mercury depressed} \times \text{displacement of the C.G. of the segment}) \\ &= (Ax\gamma)\frac{x}{2} + (Ax\gamma)\frac{x}{2} = A\gamma x^2 \end{aligned} \quad (E.1)$$

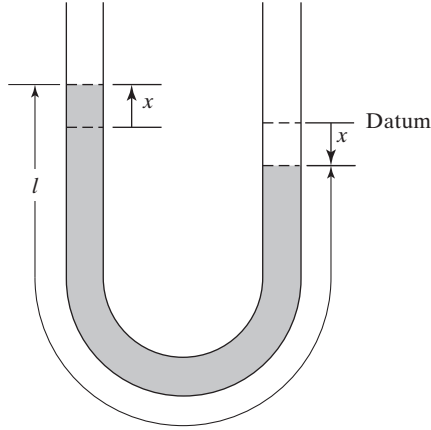


FIGURE 2.18 U-tube manometer.

where A is the cross-sectional area of the mercury column and γ is the specific weight of mercury. The change in kinetic energy is given by

$$\begin{aligned} T &= \frac{1}{2}(\text{mass of mercury})(\text{velocity})^2 \\ &= \frac{1}{2} \frac{Al\gamma}{g} \dot{x}^2 \end{aligned} \quad (\text{E.2})$$

where l is the length of the mercury column. By assuming harmonic motion, we can write

$$x(t) = X \cos \omega_n t \quad (\text{E.3})$$

where X is the maximum displacement and ω_n is the natural frequency. By substituting Eq. (E.3) into Eqs. (E.1) and (E.2), we obtain

$$U = U_{\max} \cos^2 \omega_n t \quad (\text{E.4})$$

$$T = T_{\max} \sin^2 \omega_n t \quad (\text{E.5})$$

where

$$U_{\max} = A\gamma X^2 \quad (\text{E.6})$$

and

$$T_{\max} = \frac{1A\gamma l\omega_n^2}{2g} X^2 \quad (\text{E.7})$$

By equating U_{\max} to T_{\max} , we obtain the natural frequency:

$$\omega_n = \left(\frac{2g}{l} \right)^{1/2} \quad (\text{E.8})$$

2. *Length of the mercury column:* The frequency of pressure fluctuations in the silencer

$$\begin{aligned}
 &= \frac{1 \times 600}{2} \\
 &= 300 \text{ rpm} \\
 &= \frac{300 \times 2\pi}{60} = 10\pi \text{ rad/s}
 \end{aligned} \tag{E.9}$$

Thus the frequency of oscillations of the liquid column in the manometer is $10\pi/3.5 = 9.0 \text{ rad/s}$. By using Eq. (E.8), we obtain

$$\left(\frac{2g}{l}\right)^{1/2} = 9.0 \tag{E.10}$$

or

$$l = \frac{2.0 \times 9.81}{(9.0)^2} = 0.243 \text{ m} \tag{E.11}$$

■

EXAMPLE 2.9

Effect of Mass on ω_n of a Spring

Determine the effect of the mass of the spring on the natural frequency of the spring-mass system shown in Fig. 2.19.

Solution: To find the effect of the mass of the spring on the natural frequency of the spring-mass system, we add the kinetic energy of the system to that of the attached mass and use the energy method to determine the natural frequency. Let l be the total length of the spring. If x denotes the displacement of the lower end of the spring (or mass m), the displacement at distance y from the support is given by $y(x/l)$. Similarly, if \dot{x} denotes the velocity of the mass m , the velocity of a spring

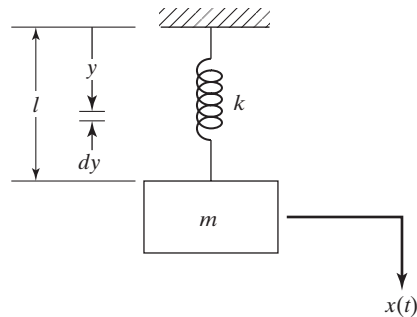


FIGURE 2.19 Equivalent mass of a spring.

element located at distance y from the support is given by $y(\dot{x}/l)$. The kinetic energy of the spring element of length dy is

$$dT_s = \frac{1}{2} \left(\frac{m_s}{l} dy \right) \left(\frac{y\dot{x}}{l} \right)^2 \quad (\text{E.1})$$

where m_s is the mass of the spring. The total kinetic energy of the system can be expressed as

$$\begin{aligned} T &= \text{kinetic energy of mass } (T_m) + \text{kinetic energy of spring } (T_s) \\ &= \frac{1}{2} m \dot{x}^2 + \int_{y=0}^l \frac{1}{2} \left(\frac{m_s}{l} dy \right) \left(\frac{y^2 \dot{x}^2}{l^2} \right) \\ &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \frac{m_s}{3} \dot{x}^2 \end{aligned} \quad (\text{E.2})$$

The total potential energy of the system is given by

$$U = \frac{1}{2} k x^2 \quad (\text{E.3})$$

By assuming a harmonic motion

$$x(t) = X \cos \omega_n t \quad (\text{E.4})$$

where X is the maximum displacement of the mass and ω_n is the natural frequency, the maximum kinetic and potential energies can be expressed as

$$T_{\max} = \frac{1}{2} \left(m + \frac{m_s}{3} \right) X^2 \omega_n^2 \quad (\text{E.5})$$

$$U_{\max} = \frac{1}{2} k X^2 \quad (\text{E.6})$$

By equating T_{\max} and U_{\max} , we obtain the expression for the natural frequency:

$$\omega_n = \left(\frac{k}{m + \frac{m_s}{3}} \right)^{1/2} \quad (\text{E.7})$$

Thus the effect of the mass of the spring can be accounted for by adding one-third of its mass to the main mass [2.3].

■

EXAMPLE 2.10

Effect of Mass of Column on Natural Frequency of Water Tank

Find the natural frequency of transverse vibration of the water tank considered in Example 2.2 and Fig. 2.10 by including the mass of the column.

Solution: To include the mass of the column, we find the equivalent mass of the column at the free end using the equivalence of kinetic energy and use a single-degree-of-freedom model to find the natural frequency of vibration. The column of the tank is considered as a cantilever beam fixed

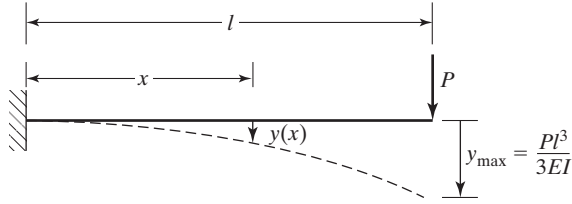


FIGURE 2.20 Equivalent mass of the column.

at one end (ground) and carrying a mass M (water tank) at the other end. The static deflection of a cantilever beam under a concentrated end load is given by (see Fig. 2.20):

$$\begin{aligned} y(x) &= \frac{Px^2}{6EI}(3l - x) = \frac{y_{\max}x^2}{2l^3}(3l - x) \\ &= \frac{y_{\max}}{2l^3}(3x^2l - x^3) \end{aligned} \quad (\text{E.1})$$

The maximum kinetic energy of the beam itself (T_{\max}) is given by

$$T_{\max} = \frac{1}{2} \int_0^l \frac{m}{l} \{\dot{y}(x)\}^2 dx \quad (\text{E.2})$$

where m is the total mass and (m/l) is the mass per unit length of the beam. Equation (E.1) can be used to express the velocity variation, $\dot{y}(x)$, as

$$\dot{y}(x) = \frac{\dot{y}_{\max}}{2l^3}(3x^2l - x^3) \quad (\text{E.3})$$

and hence Eq. (E.2) becomes

$$\begin{aligned} T_{\max} &= \frac{m}{2l} \left(\frac{\dot{y}_{\max}}{2l^3} \right)^2 \int_0^l (3x^2l - x^3)^2 dx \\ &= \frac{1}{2} \frac{m}{l} \frac{\dot{y}_{\max}^2}{4l^6} \left(\frac{33}{35} l^7 \right) = \frac{1}{2} \left(\frac{33}{140} m \right) \dot{y}_{\max}^2 \end{aligned} \quad (\text{E.4})$$

If m_{eq} denotes the equivalent mass of the cantilever (water tank) at the free end, its maximum kinetic energy can be expressed as

$$T_{\max} = \frac{1}{2} m_{\text{eq}} \dot{y}_{\max}^2 \quad (\text{E.5})$$

By equating Eqs. (E.4) and (E.5), we obtain

$$m_{\text{eq}} = \frac{33}{140} m \quad (\text{E.6})$$

Thus the total effective mass acting at the end of the cantilever beam is given by

$$M_{\text{eff}} = M + m_{\text{eq}} \quad (\text{E.7})$$

where M is the mass of the water tank. The natural frequency of transverse vibration of the water tank is given by

$$\omega_n = \sqrt{\frac{k}{M_{\text{eff}}}} = \sqrt{\frac{k}{M + \frac{33}{140}m}} \quad (\text{E.8})$$

■

2.6 Free Vibration with Viscous Damping

2.6.1 Equation of Motion

As stated in Section 1.9, the viscous damping force F is proportional to the velocity \dot{x} or v and can be expressed as

$$F = -c\dot{x} \quad (2.58)$$

where c is the damping constant or coefficient of viscous damping and the negative sign indicates that the damping force is opposite to the direction of velocity. A single-degree-of-freedom system with a viscous damper is shown in Fig. 2.21. If x is measured from the equilibrium position of the mass m , the application of Newton's law yields the equation of motion:

$$m\ddot{x} = -c\dot{x} - kx$$

or

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (2.59)$$

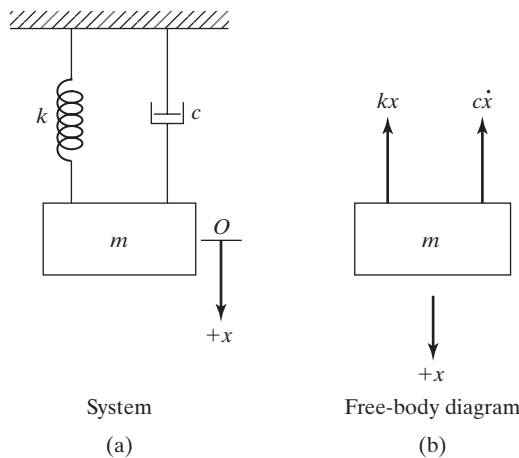


FIGURE 2.21 Single-degree-of-freedom system with viscous damper.

2.6.2 Solution

To solve Eq. (2.59), we assume a solution in the form

$$x(t) = Ce^{st} \quad (2.60)$$

where C and s are undetermined constants. Inserting this function into Eq. (2.59) leads to the characteristic equation

$$ms^2 + cs + k = 0 \quad (2.61)$$

the roots of which are

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \quad (2.62)$$

These roots give two solutions to Eq. (2.59):

$$x_1(t) = C_1 e^{s_1 t} \quad \text{and} \quad x_2(t) = C_2 e^{s_2 t} \quad (2.63)$$

Thus the general solution of Eq. (2.59) is given by a combination of the two solutions $x_1(t)$ and $x_2(t)$:

$$\begin{aligned} x(t) &= C_1 e^{s_1 t} + C_2 e^{s_2 t} \\ &= C_1 e^{\left\{-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} + C_2 e^{\left\{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} \end{aligned} \quad (2.64)$$

where C_1 and C_2 are arbitrary constants to be determined from the initial conditions of the system.

Critical Damping Constant and the Damping Ratio. The critical damping c_c is defined as the value of the damping constant c for which the radical in Eq. (2.62) becomes zero:

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

or

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n \quad (2.65)$$

For any damped system, the damping ratio ζ is defined as the ratio of the damping constant to the critical damping constant:

$$\zeta = c/c_c \quad (2.66)$$

Using Eqs. (2.66) and (2.65), we can write

$$\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \zeta \omega_n \quad (2.67)$$

and hence

$$s_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n \quad (2.68)$$

Thus the solution, Eq. (2.64), can be written as

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \quad (2.69)$$

The nature of the roots s_1 and s_2 and hence the behavior of the solution, Eq. (2.69), depends upon the magnitude of damping. It can be seen that the case $\zeta = 0$ leads to the undamped vibrations discussed in Section 2.2. Hence we assume that $\zeta \neq 0$ and consider the following three cases.

Case 1. *Underdamped system* ($\zeta < 1$ or $c < c_c$ or $c/2m < \sqrt{k/m}$). For this condition, $(\zeta^2 - 1)$ is negative and the roots s_1 and s_2 can be expressed as

$$\begin{aligned} s_1 &= (-\zeta + i\sqrt{1 - \zeta^2})\omega_n \\ s_2 &= (-\zeta - i\sqrt{1 - \zeta^2})\omega_n \end{aligned}$$

and the solution, Eq. (2.69), can be written in different forms:

$$x(t) = C_1 e^{(-\zeta + i\sqrt{1 - \zeta^2})\omega_n t} + C_2 e^{(-\zeta - i\sqrt{1 - \zeta^2})\omega_n t} \quad (2.70a)$$

$$= e^{-\zeta\omega_n t} \left\{ C_1 e^{i\sqrt{1 - \zeta^2}\omega_n t} + C_2 e^{-i\sqrt{1 - \zeta^2}\omega_n t} \right\} \quad (2.70b)$$

$$= e^{-\zeta\omega_n t} \left\{ (C_1 + C_2) \cos \sqrt{1 - \zeta^2}\omega_n t + i(C_1 - C_2) \sin \sqrt{1 - \zeta^2}\omega_n t \right\} \quad (2.70c)$$

$$= e^{-\zeta\omega_n t} \left\{ C'_1 \cos \sqrt{1 - \zeta^2}\omega_n t + C'_2 \sin \sqrt{1 - \zeta^2}\omega_n t \right\} \quad (2.70d)$$

$$= X_0 e^{-\zeta\omega_n t} \sin \left(\sqrt{1 - \zeta^2}\omega_n t + \phi_0 \right) \quad (2.70e)$$

$$= X e^{-\zeta\omega_n t} \cos \left(\sqrt{1 - \zeta^2}\omega_n t - \phi \right) \quad (2.70f)$$

where (C'_1, C'_2) , (X, ϕ) , and (X_0, ϕ_0) are arbitrary constants to be determined from the initial conditions.

To determine the constants C'_1 and C'_2 , for example, first Eq. (2.70d) is differentiated to find the velocity, $\dot{x}(t)$, as

$$\begin{aligned} \dot{x}(t) &= -\zeta\omega_n e^{-\zeta\omega_n t} \{ C'_1 \cos \sqrt{1 - \zeta^2}\omega_n t + C'_2 \sin \sqrt{1 - \zeta^2}\omega_n t \} \\ &\quad + e^{-\zeta\omega_n t} \{ -\sqrt{1 - \zeta^2}\omega_n C'_1 \sin \sqrt{1 - \zeta^2}\omega_n t \\ &\quad + \sqrt{1 - \zeta^2}\omega_n C'_2 \cos \sqrt{1 - \zeta^2}\omega_n t \} \end{aligned} \quad (2.71a)$$

By using the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, respectively, in Eqs. (2.70d) and (2.71a), we obtain

$$x_0 = C'_1, \quad \dot{x}_0 = -\zeta\omega_n C'_1 + \sqrt{1 - \zeta^2}\omega_n$$

or

$$C'_1 = x_0, \quad C'_2 = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1 - \zeta^2}\omega_n} \quad (2.71b)$$

Thus the solution given by Eq. (2.70d) can be expressed as

$$x(t) = e^{-\zeta\omega_n t} \left\{ x_0 \cos \sqrt{1 - \zeta^2}\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1 - \zeta^2}\omega_n} \sin \sqrt{1 - \zeta^2}\omega_n t \right\} \quad (2.72a)$$

To determine the constants X_0 and ϕ_0 , first we differentiate Eq. (2.70e) to find $\dot{x}(t)$

as

$$\begin{aligned} \dot{x}(t) = & -X_0\zeta\omega_n e^{-\zeta\omega_n t} \sin(\sqrt{1 - \zeta^2}\omega_n t + \phi_0) \\ & + X_0 e^{-\zeta\omega_n t} \sqrt{1 - \zeta^2}\omega_n \cos(\sqrt{1 - \zeta^2}\omega_n t + \phi_0) \end{aligned} \quad (2.72b)$$

By using the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, respectively, in Eqs. (2.70e) and (2.72b), we obtain

$$x_0 = X_0 \sin \phi_0, \quad \dot{x}_0 = -X_0\zeta\omega_n (\sin \phi_0 + X_0\omega_d \cos \phi_0)$$

or

$$\sin \phi_0 = \frac{x_0}{X_0}, \quad \cos \phi_0 = \frac{\dot{x}_0 + \zeta\omega_n x_0}{X_0\omega_d} \quad (2.72c)$$

so that

$$\tan \phi_0 = \frac{x_0\omega_d}{\dot{x}_0 + \zeta\omega_n x_0}$$

or

$$\phi_0 = \tan^{-1} \left(\frac{x_0\omega_n}{\dot{x}_0 + \zeta\omega_n x_0} \right) \quad (2.72d)$$

The relation $\sin^2\phi_0 + \cos^2\phi_0 = 1$ gives

$$\frac{x_0^2}{X_0^2} + \left(\frac{\dot{x} + \zeta\omega_n x_0}{\omega_d} \right)^2 \frac{1}{X_0^2} = 1$$

or

$$X_0^2 = \frac{\omega_d^2 x_0^2 + \dot{x}^2 + \zeta^2 \omega_n^2 x_0^2 + 2\zeta\omega_n x_0 \dot{x}}{\omega_d^2}$$

or

$$X_0 = \frac{\sqrt{x_0^2 \omega_n^2 + \dot{x}^2 + 2x_0 \dot{x} \zeta \omega_n}}{\omega_n \sqrt{1 - \zeta^2}} \quad (2.72e)$$

Similarly, Eq. (2.70f) can be used to find the expressions of X and ϕ . The final expressions can be summarized as

$$X = X_0 = \sqrt{(C'_1)^2 + (C'_2)^2} = \frac{\sqrt{x_0^2 \omega_n^2 + \dot{x}_0^2 + 2x_0 \dot{x}_0 \zeta \omega_n}}{\sqrt{1 - \zeta^2} \omega_n} \quad (2.73)$$

$$\phi_0 = \tan^{-1} \left(\frac{C'_1}{C'_2} \right) = \tan^{-1} \left(\frac{x_0 \omega_n \sqrt{1 - \zeta^2}}{\dot{x}_0 + \zeta \omega_n x_0} \right) \quad (2.74)$$

$$\phi = \tan^{-1} \left(\frac{C'_2}{C'_1} \right) = \tan^{-1} \left(\frac{\dot{x}_0 + \zeta \omega_n x_0}{x_0 \omega_n \sqrt{1 - \zeta^2}} \right) \quad (2.75)$$

The motion described by Eq. (2.72a) is a damped harmonic motion of angular frequency $\sqrt{1 - \zeta^2} \omega_n$, but because of the factor $e^{-\zeta \omega_n t}$, the amplitude decreases exponentially with time, as shown in Fig. 2.22. The quantity

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n \quad (2.76)$$

is called the *frequency of damped vibration*. It can be seen that the frequency of damped vibration ω_d is always less than the undamped natural frequency ω_n . The decrease in the frequency of damped vibration with increasing amount of damping, given by Eq. (2.76), is shown graphically in Fig. 2.23. The underdamped case is very important in the study of mechanical vibrations, as it is the only case that leads to an oscillatory motion [2.10].

Case 2. *Critically damped system* ($\zeta = 1$ or $c = c_c$ or $c/2m = \sqrt{k/m}$). In this case, the two roots s_1 and s_2 in Eq. (2.68) are equal:

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n \quad (2.77)$$

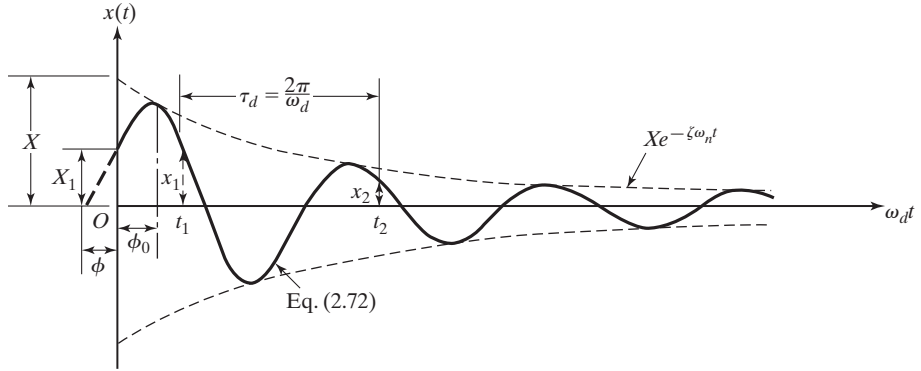
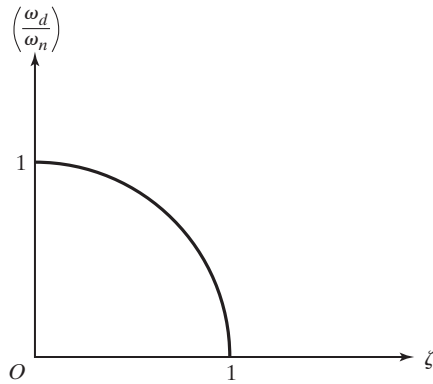


FIGURE 2.22 Underdamped solution.

FIGURE 2.23 Variation of ω_d with damping.

Because of the repeated roots, the solution of Eq. (2.59) is given by [2.6]¹

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \quad (2.78)$$

The application of the initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$ for this case gives

$$\begin{aligned} C_1 &= x_0 \\ C_2 &= \dot{x}_0 + \omega_n x_0 \end{aligned} \quad (2.79)$$

¹Equation (2.78) can also be obtained by making ζ approach unity in the limit in Eq. (2.70d). As $\zeta \rightarrow 1$, $\omega_d = \sqrt{1 - \zeta^2} \omega_n \rightarrow 0$; hence $\cos \omega_d t \rightarrow 1$ and $\sin \omega_d t \rightarrow \omega_d t$. Thus (2.70d) yields

$$x(t) = e^{-\omega_n t}(C'_1 + C'_2 \omega_d t) = (C_1 + C_2 t)e^{-\omega_n t}$$

where $C_1 = C'_1$ and $C_2 = C'_2 \omega_d$ are new constants.

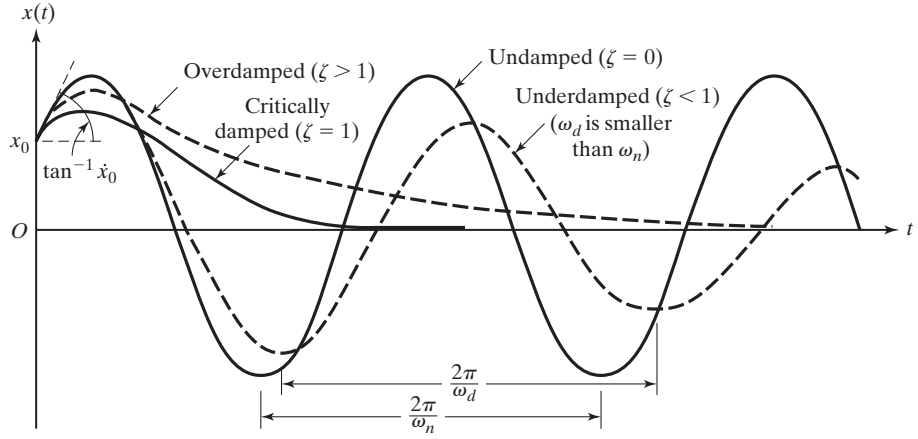


FIGURE 2.24 Comparison of motions with different types of damping.

and the solution becomes

$$x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t]e^{-\omega_n t} \quad (2.80)$$

It can be seen that the motion represented by Eq. (2.80) is *aperiodic* (i.e., nonperiodic). Since $e^{-\omega_n t} \rightarrow 0$ as $t \rightarrow \infty$, the motion will eventually diminish to zero, as indicated in Fig. 2.24.

Case 3. *Overdamped system* ($\zeta > 1$ or $c > c_c$ or $c/2m > \sqrt{k/m}$). As $\sqrt{\zeta^2 - 1} > 0$, Eq. (2.68) shows that the roots s_1 and s_2 are real and distinct and are given by

$$\begin{aligned} s_1 &= (-\zeta + \sqrt{\zeta^2 - 1})\omega_n < 0 \\ s_2 &= (-\zeta - \sqrt{\zeta^2 - 1})\omega_n < 0 \end{aligned}$$

with $s_2 \ll s_1$. In this case, the solution, Eq. (2.69), can be expressed as

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \quad (2.81)$$

For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, the constants C_1 and C_2 can be obtained:

$$\begin{aligned} C_1 &= \frac{x_0 \omega_n (\zeta + \sqrt{\zeta^2 - 1}) + \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}} \\ C_2 &= \frac{-x_0 \omega_n (\zeta - \sqrt{\zeta^2 - 1}) - \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}} \end{aligned} \quad (2.82)$$

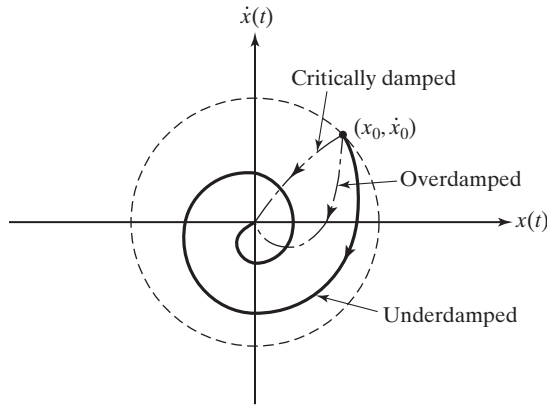


FIGURE 2.25 Phase plane of a damped system.

Equation (2.81) shows that the motion is aperiodic regardless of the initial conditions imposed on the system. Since roots s_1 and s_2 are both negative, the motion diminishes exponentially with time, as shown in Fig. 2.24.

Note the following aspects of these systems:

1. The graphical representation of different types of the characteristics roots s_1 and s_2 , and the corresponding responses (solutions) of the system are presented in Section 2.7. The representation of the roots s_1 and s_2 with varying values of the system parameters c , k , and m in the complex plane (known as the root locus plots) is considered in Section 2.8.
2. A critically damped system will have the smallest damping required for aperiodic motion; hence the mass returns to the position of rest in the shortest possible time without overshooting. The property of critical damping is used in many practical applications. For example, large guns have dashpots with critical damping value, so that they return to their original position after recoil in the minimum time without vibrating. If the damping provided were more than the critical value, some delay would be caused before the next firing.
3. The free damped response of a single-degree-of-freedom system can be represented in phase-plane or state space as indicated in Fig. 2.25.

EXAMPLE 2.11

Response of an Underdamped System due to Initial Conditions

The parameters of a single-degree-of-freedom system are given by $m = 1$ kg, $c = 5$ N-s/m, and $k = 16$ N/m. Find the response of the system for the following initial conditions:

- a. $x(0) = 0.1$ m and $\dot{x}(0) = 2$ m/s
- b. $x(0) = -0.1$ m and $\dot{x}(0) = 2$ m/s

Solution: If we express the response of the system as [Eq. (2. 70e)]

$$x(t) = X_0 e^{-\zeta \omega_n t} \sin(\omega_d t + \phi_0) \quad (\text{E.1})$$

with X_0 and ϕ_0 given by Eqs. (2.73) and (2.74), respectively:

$$X_0 = \frac{\sqrt{x_0^2 \omega_n^2 + \dot{x}_0^2 + 2\zeta \omega_n x_0 \dot{x}_0}}{\omega_d} \quad (\text{E.2})$$

$$\phi_0 = \tan^{-1} \left(\frac{x_0 \omega_d}{\dot{x}_0 + \zeta \omega_n x_0} \right) \quad (\text{E.3})$$

Here

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{1}} = 4 \text{ rad/s}, \quad c_c = 2\sqrt{km} = 2\sqrt{16(1)} = 8 \text{ N-s/m}, \quad \zeta = \frac{c}{c_c} = \frac{5}{8} = 0.625,$$

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n = \sqrt{1 - 0.625^2}(4) = 0.7806(4) = 3.1225 \text{ rad/s}$$

a. Using the known quantities, Eqs. (E.2) and (E.3) give

$$X_0 = \frac{\sqrt{(0.1)^2 4^2 + 2^2 + 2(0.625)(4)(0.1)(2)}}{3.1225} = \frac{2.2716}{3.1225} = 0.7275 \text{ m}$$

$$\begin{aligned} \phi_0 &= \tan^{-1} \left(\frac{0.1(3.1225)}{2 + 0.625(4)(0.1)} \right) = \tan^{-1} \left(\frac{0.3122}{2.25} \right) = \tan^{-1}(0.1387) \\ &= 7.8997^\circ \text{ or } 0.1379 \text{ rad} \end{aligned}$$

$$\text{Since } \sin \phi_0 = \frac{x_0}{X_0} = \frac{0.1}{0.7275} = 0.1374 > 0$$

$$\text{and } \cos \phi_0 = \frac{\dot{x} + \zeta \omega_n x_0}{X_0 \omega_d} = \frac{2 + 0.625(4)(0.1)}{0.7275(3.1225)} = 0.9905 > 0, \phi_0 \text{ falls in the first quadrant}$$

with $\phi_0 = 7.8997^\circ$ or 0.1379 rad. Thus the response of the system is given by

$$x(t) = 0.7275 e^{-0.25t} \sin(3.1225t + 0.1379) \text{ m}$$

b. Using the known quantities, Eqs. (E.2) and (E.3) give

$$X_0 = \frac{\sqrt{(-0.1)^2 4^2 + 2^2 + 2(0.625)(4)(-0.1)(2)}}{3.1225} = \frac{1.7776}{3.1225} = 0.5693 \text{ m}$$

$$\begin{aligned} \phi_0 &= \tan^{-1} \left(\frac{-0.1(3.1225)}{2 + 0.625(4)(-0.1)} \right) = \tan^{-1} \left(\frac{-0.3122}{1.75} \right) = \tan^{-1}(-0.1784) \\ &= -10.1167^\circ (-0.1766 \text{ rad}) \text{ or } 349.8833^\circ (6.1066 \text{ rad}) \end{aligned}$$

$$\text{Since } \sin \phi_0 = \frac{x_0}{X_0} = \frac{-0.1}{0.5693} = -0.1756 < 0$$

$$\text{and } \cos \phi_0 = \frac{\dot{x}_0 + \zeta \omega_n x_0}{X_0 \omega_d} = \frac{2 + 0.625(4)(-0.1)}{0.5693(3.1225)} = 0.9867 > 0, \phi_0 \text{ falls in the fourth quad-}$$

rant with $\phi_0 = 349.8833^\circ$ or 6.1066 rad. Thus the response of the system can be expressed as

$$x(t) = 0.5693 e^{-0.25t} \sin(3.1225t + 6.1066) \text{ m}$$

■

EXAMPLE 2.12

Identification of the Characteristics of the System from the Response

The response of a single-degree-of-freedom system that is initially displaced and released is given by

$$x(t) = 0.05 e^{-6t} \sin(5t + 1.3333) \text{ m} \quad (\text{E.1})$$

Determine the damping ratio, natural frequency, and the initial displacement of the system.

Solution: Because the response given by Eq. (E.1) is oscillatory, due to the presence of the term, $\sin(5t + 1.3333)$, the system is underdamped. Also, the system is initially subjected to displacement only and hence the initial velocity, $\dot{x}_0 = 0$. By comparing Eq. (E.1) to Eq. (2.70e), the phase angle can be identified as $\phi_0 = 1.3333$ rad or 76.3923° or

$$\tan^{-1}(1.3333) = \tan^{-1}\left(\frac{x_0 \omega_n \sqrt{1 - \zeta^2}}{\zeta \omega_n x_0}\right) = \tan^{-1}\left(\frac{\sqrt{1 - \zeta^2}}{\zeta}\right) \quad (\text{E.2})$$

Equation (E.2) gives

$$\frac{\sqrt{1 - \zeta^2}}{\zeta} = 1.3333 \quad (\text{E.3})$$

By squaring both sides of Eq. (E.3) and rearranging the resulting equation, we obtain $2.7777 \zeta^2 = 1$ or $\zeta = 0.6$.

Equating the coefficient of t in the sine term to $\sqrt{1 - \zeta^2} \omega_n$, we obtain, $\sqrt{1 - \zeta^2} \omega_n = 8$ or

$$\omega_n = \frac{8}{\sqrt{1 - \zeta^2}} = \frac{8}{\sqrt{1 - (0.6)^2}} = \frac{8}{0.8} = 10 \text{ rad/s}$$

Equating the number 0.05 before the exponential term in Eq. (E.1) to X_0 , we obtain

$$0.05 = X_0 = \frac{\sqrt{x_0^2 \omega_n^2 + \dot{x}_0^2 + 2x_0 \dot{x}_0 \zeta \omega_n}}{\sqrt{1 - \zeta^2} \omega_n} \quad (\text{E.4})$$

Using $\dot{x}_0 = 0$, Eq. (E.4) yields

$$0.05 = \frac{x_0 \omega_n}{\sqrt{1 - \zeta^2} \omega_n} = \frac{x_0}{\sqrt{1 - \zeta^2}} = \frac{x_0}{\sqrt{1 - 0.6^2}} = \frac{x_0}{0.8}$$

Thus $x_0 = 0.05(0.8) = 0.04 \text{ m}$

Verification:

The exponential term given is e^{-6t} ; it should be equal to $e^{-\zeta \omega_n t}$. In the present case, the relation $-\zeta \omega_n = -6$ is satisfied.

■

2.6.3 Logarithmic Decrement

The logarithmic decrement represents the rate at which the amplitude of a free-damped vibration decreases. It is defined as the natural logarithm of the ratio of any two successive amplitudes. Let t_1 and t_2 denote the times corresponding to two consecutive amplitudes (displacements), measured one cycle apart for an underdamped system, as in Fig. 2.22. Using Eq. (2.70f), we can form the ratio

$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta \omega_n t_1} \cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta \omega_n t_2} \cos(\omega_d t_2 - \phi_0)} \quad (2.83)$$

But $t_2 = t_1 + \tau_d$, where $\tau_d = 2\pi/\omega_d$ is the period of damped vibration. Hence $\cos(\omega_d t_2 - \phi_0) = \cos(2\pi + \omega_d t_1 - \phi_0) = \cos(\omega_d t_1 - \phi_0)$, and Eq. (2.83) can be written as

$$\frac{x_1}{x_2} = \frac{e^{-\zeta \omega_n t_1}}{e^{-\zeta \omega_n (t_1 + \tau_d)}} = e^{\zeta \omega_n \tau_d} \quad (2.84)$$

The logarithmic decrement δ can be obtained from Eq. (2.84) and using Eq.(2.67),

$$\delta = \ln \frac{x_1}{x_2} = \zeta \omega_n \tau_d = \zeta \omega_n \frac{2\pi}{\sqrt{1 - \zeta^2} \omega_n} = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} = \frac{2\pi}{\omega_d} \cdot \frac{c}{2m} \quad (2.85)$$

For small damping, Eq. (2.85) can be approximated:

$$\delta \simeq 2\pi \zeta \quad \text{if} \quad \zeta \ll 1 \quad (2.86)$$

Figure 2.26 shows the variation of the logarithmic decrement δ with ζ as given by Eqs. (2.85) and (2.86). It can be noticed that for values up to $\zeta = 0.3$, the two curves are difficult to distinguish.

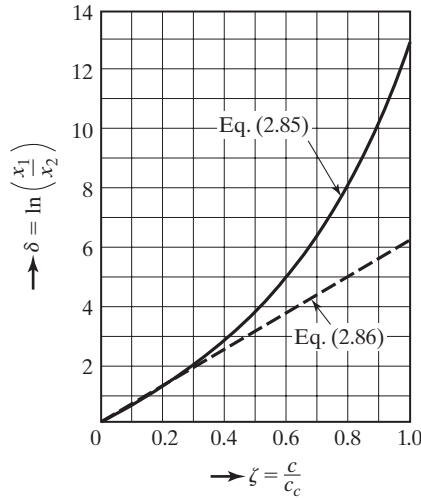


FIGURE 2.26 Variation of logarithmic decrement with damping.

The logarithmic decrement is dimensionless and is actually another form of the dimensionless damping ratio ζ . Once δ is known, ζ can be found by solving Eq. (2.85):

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \quad (2.87)$$

If we use Eq. (2.86) instead of Eq. (2.85), we have

$$\zeta \simeq \frac{\delta}{2\pi} \quad (2.88)$$

If the damping in the given system is not known, we can determine it experimentally by measuring any two consecutive displacements x_1 and x_2 . By taking the natural logarithm of the ratio of x_1 and x_2 , we obtain δ . By using Eq. (2.87), we can compute the damping ratio ζ . In fact, the damping ratio ζ can also be found by measuring two displacements separated by any number of complete cycles. If x_1 and x_{m+1} denote the amplitudes corresponding to times t_1 and $t_{m+1} = t_1 + m\tau_d$, where m is an integer, we obtain

$$\frac{x_1}{x_{m+1}} = \frac{x_1 x_2 x_3}{x_2 x_3 x_4} \cdots \frac{x_m}{x_{m+1}} \quad (2.89)$$

Since any two successive displacements separated by one cycle satisfy the equation

$$\frac{x_j}{x_{j+1}} = e^{\zeta \omega_n \tau_d} \quad (2.90)$$

Eq. (2.89) becomes

$$\frac{x_1}{x_{m+1}} = (e^{\zeta \omega_n \tau_d})^m = e^{m\zeta \omega_n \tau_d} \quad (2.91)$$

Equations (2.91) and (2.85) yield

$$\delta = \frac{1}{m} \ln \left(\frac{x_1}{x_{m+1}} \right) \quad (2.92)$$

which can be substituted into Eq. (2.87) or (2.88) to obtain the viscous damping ratio ζ .

2.6.4 Energy Dissipated in Viscous Damping

In a viscously damped system, the rate of change of energy with time (dW/dt) is given by

$$\frac{dW}{dt} = \text{force} \times \text{velocity} = Fv = -cv^2 = -c \left(\frac{dx}{dt} \right)^2 \quad (2.93)$$

using Eq. (2.58). The negative sign in Eq. (2.93) denotes that energy dissipates with time. Assume a simple harmonic motion as $x(t) = X \sin \omega_d t$, where X is the amplitude of motion and the energy dissipated in a complete cycle is given by²

²In the case of a damped system, simple harmonic motion $x(t) = X \cos \omega_d t$ is possible only when the steady-state response is considered under a harmonic force of frequency ω_d (see Section 3.4). The loss of energy due to the damper is supplied by the excitation under steady-state forced vibration [2.7].

$$\begin{aligned}\Delta W &= \int_{t=0}^{(2\pi/\omega_d)} c \left(\frac{dx}{dt} \right)^2 dt = \int_0^{2\pi} cX^2 \omega_d \cos^2 \omega_d t \cdot d(\omega_d t) \\ &= \pi c \omega_d X^2\end{aligned}\quad (2.94)$$

This shows that the energy dissipated is proportional to the square of the amplitude of motion. Note that it is not a constant for given values of damping and amplitude, since ΔW is also a function of the frequency ω_d .

Equation (2.94) is valid even when there is a spring of stiffness k parallel to the viscous damper. To see this, consider the system shown in Fig. 2.27. The total force resisting motion can be expressed as

$$F = -kx - c\dot{x} = -kx - c\dot{x} \quad (2.95)$$

If we assume simple harmonic motion

$$x(t) = X \sin \omega_d t \quad (2.96)$$

as before, Eq. (2.95) becomes

$$F = -kX \sin \omega_d t - c\omega_d X \cos \omega_d t \quad (2.97)$$

The energy dissipated in a complete cycle will be

$$\begin{aligned}\Delta W &= \int_{t=0}^{2\pi/\omega_d} Fv \, dt \\ &= \int_0^{2\pi/\omega_d} kX^2 \omega_d \sin \omega_d t \cdot \cos \omega_d t \cdot d(\omega_d t) \\ &\quad + \int_0^{2\pi/\omega_d} c\omega_d X^2 \cos^2 \omega_d t \cdot d(\omega_d t) = \pi c \omega_d X^2\end{aligned}\quad (2.98)$$

which can be seen to be identical with Eq. (2.94). This result is to be expected, since the spring force will not do any net work over a complete cycle or any integral number of cycles.

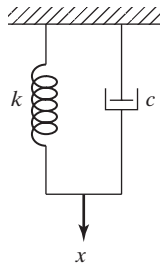


FIGURE 2.27

Spring and damper
in parallel.

We can also compute the fraction of the total energy of the vibrating system that is dissipated in each cycle of motion ($\Delta W/W$), as follows. The total energy of the system W can be expressed either as the maximum potential energy ($\frac{1}{2}kX^2$) or as the maximum kinetic energy ($\frac{1}{2}mv_{\max}^2 = \frac{1}{2}mX^2\omega_d^2$), the two being approximately equal for small values of damping. Thus

$$\frac{\Delta W}{W} = \frac{\pi c \omega_d X^2}{\frac{1}{2} m \omega_d^2 X^2} = 2 \left(\frac{2\pi}{\omega_d} \right) \left(\frac{c}{2m} \right) = 2\delta \simeq 4\pi\zeta = \text{constant} \quad (2.99)$$

using Eqs. (2.85) and (2.88). The quantity $\Delta W/W$ is called the *specific damping capacity* and is useful in comparing the damping capacity of engineering materials. Another quantity known as the *loss coefficient* is also used for comparing the damping capacity of engineering materials. The loss coefficient is defined as the ratio of the energy dissipated per radian and the total strain energy:

$$\text{loss coefficient} = \frac{(\Delta W/2\pi)}{W} = \frac{\Delta W}{2\pi W} \quad (2.100)$$

2.6.5 Torsional Systems with Viscous Damping

The methods presented in Sections 2.6.1 through 2.6.4 for linear vibrations with viscous damping can be extended directly to viscously damped torsional (angular) vibrations. For this, consider a single-degree-of-freedom torsional system with a viscous damper, as shown in Fig. 2.28(a). The viscous damping torque is given by (Fig. 2.28(b)):

$$T = -c_t \dot{\theta} \quad (2.101)$$

where c_t is the torsional viscous damping constant, $\dot{\theta} = d\theta/dt$ is the angular velocity of the disc, and the negative sign denotes that the damping torque is opposite the direction of angular velocity. The equation of motion can be derived as

$$J_0 \ddot{\theta} + c_t \dot{\theta} + k_t \theta = 0 \quad (2.102)$$

where J_0 = mass moment of inertia of the disc, k_t = spring constant of the system (restoring torque per unit angular displacement), and θ = angular displacement of the disc. The solution of Eq. (2.102) can be found exactly as in the case of linear vibrations. For example, in the underdamped case, the frequency of damped vibration is given by

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n \quad (2.103)$$

where

$$\omega_n = \sqrt{\frac{k_t}{J_0}} \quad (2.104)$$

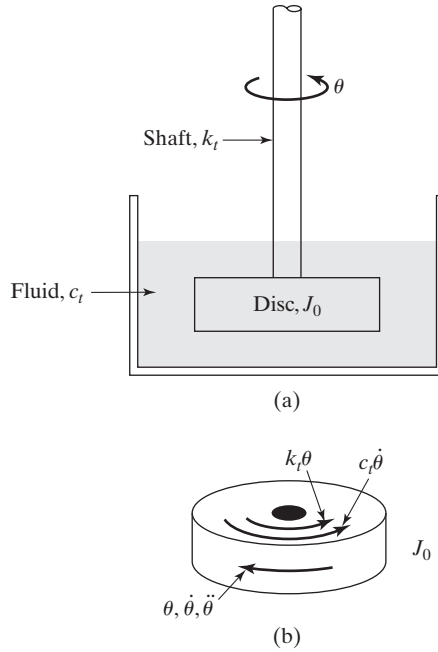


FIGURE 2.28 Torsional viscous damper.

and

$$\zeta = \frac{c_t}{c_{tc}} = \frac{c_t}{2J_0\omega_n} = \frac{c_t}{2\sqrt{k_t J_0}} \quad (2.105)$$

where c_{tc} is the critical torsional damping constant.

EXAMPLE 2.13

Response of Anvil of a Forging Hammer

The anvil of a forging hammer weighs 5000 N and is mounted on a foundation that has a stiffness of 5×10^6 N/m and a viscous damping constant of 10,000 N-s/m. During a particular forging operation, the tup (i.e., the falling weight or the hammer), weighing 1000 N, is made to fall from a height of 2 m onto the anvil (Fig. 2.29(a)). If the anvil is at rest before impact by the tup, determine the response of the anvil after the impact. Assume that the coefficient of restitution between the anvil and the tup is 0.4.

Solution: First, we use the principle of conservation of momentum and the definition of the coefficient of restitution to find the initial velocity of the anvil. Let the velocities of the tup just before and just after impact with the anvil be v_{t1} and v_{t2} , respectively. Similarly, let v_{a1} and v_{a2} be the velocities of the anvil just before and just after the impact, respectively (Fig. 2.29(b)). Note that

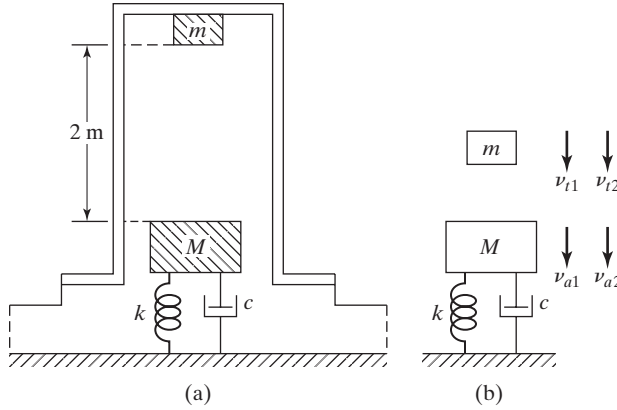


FIGURE 2.29 Forging hammer.

the displacement of the anvil is measured from its static equilibrium position and all velocities are assumed to be positive when acting downward. The principle of conservation of momentum gives

$$M(v_{a2} - v_{a1}) = m(v_{t1} - v_{t2}) \quad (\text{E.1})$$

where $v_{a1} = 0$ (anvil is at rest before the impact) and v_{t1} can be determined by equating its kinetic energy just before impact to its potential energy before dropping from a height of $h = 2$ m:

$$\frac{1}{2}mv_{t1}^2 = mgh \quad (\text{E.2})$$

or

$$v_{t1} = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 2} = 6.26099 \text{ m/s}$$

Thus Eq. (E.1) becomes

$$\frac{5000}{9.81}(v_{a2} - 0) = \frac{1000}{9.81}(6.26099 - v_{t2})$$

that is,

$$510.204082 v_{a2} = 638.87653 - 102.040813 v_{t2} \quad (\text{E.3})$$

The definition of the coefficient of restitution (r) yields:

$$r = -\left(\frac{v_{a2} - v_{t2}}{v_{a1} - v_{t1}}\right) \quad (\text{E.4})$$

that is,

$$0.4 = -\left(\frac{v_{a2} - v_{t2}}{0 - 6.26099}\right)$$

that is,

$$v_{a2} = v_{t2} + 2.504396 \quad (\text{E.5})$$

The solution of Eqs. (E.3) and (E.5) gives

$$v_{a2} = 1.460898 \text{ m/s}; \quad v_{i2} = -1.043498 \text{ m/s}$$

Thus the initial conditions of the anvil are given by

$$x_0 = 0; \quad \dot{x}_0 = 1.460898 \text{ m/s}$$

The damping coefficient is equal to

$$\zeta = \frac{c}{2\sqrt{kM}} = \frac{1000}{2\sqrt{(5 \times 10^6)\left(\frac{5000}{9.81}\right)}} = 0.0989949$$

The undamped and damped natural frequencies of the anvil are given by

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{5 \times 10^6}{\left(\frac{5000}{9.81}\right)}} = 98.994949 \text{ rad/s}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 98.994949 \sqrt{1 - 0.0989949^2} = 98.024799 \text{ rad/s}$$

The displacement response of the anvil is given by Eq. (2.72a):

$$x(t) = e^{-\zeta \omega_n t} \left\{ \frac{\dot{x}_0}{\omega_d} \sin \omega_d t \right\}$$

$$= e^{-9.799995 t} \{0.01490335 \sin 98.024799 t\} \text{ m}$$

■

EXAMPLE 2.14

Shock Absorber for a Motorcycle

An underdamped shock absorber is to be designed for a motorcycle of mass 200 kg (Fig. 2.30(a)). When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in Fig. 2.30(b). Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 s and the amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e., $x_{1.5} = x_1/4$). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.

Approach: We use the equation for the logarithmic decrement in terms of the damping ratio, equation for the damped period of vibration, time corresponding to maximum displacement for an underdamped system, and envelope passing through the maximum points of an underdamped system.

Solution: Since $x_{1.5} = x_1/4$, $x_2 = x_{1.5}/4 = x_1/16$. Hence the logarithmic decrement becomes

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln(16) = 2.7726 = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}} \quad (\text{E.1})$$

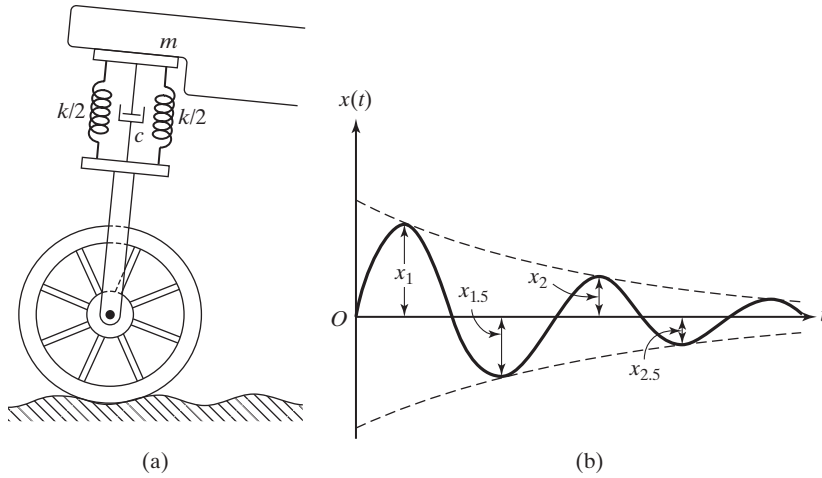


FIGURE 2.30 Shock absorber of a motorcycle.

from which the value of ζ can be found as $\zeta = 0.4037$. The damped period of vibration is given to be 2 s. Hence

$$2 = \tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

$$\omega_n = \frac{2\pi}{2\sqrt{1 - (0.4037)^2}} = 3.4338 \text{ rad/s}$$

The critical damping constant can be obtained:

$$c_c = 2m\omega_n = 2(200)(3.4338) = 1373.54 \text{ N-s/m}$$

Thus the damping constant is given by

$$c = \zeta c_c = (0.4037)(1373.54) = 554.4981 \text{ N-s/m}$$

and the stiffness by

$$k = m\omega_n^2 = (200)(3.4338)^2 = 2358.2652 \text{ N/m}$$

The displacement of the mass will attain its maximum value at time t_1 , given by

$$\sin \omega_d t_1 = \sqrt{1 - \zeta^2}$$

(See Problem 2.127.) This gives

$$\sin \omega_d t_1 = \sin \pi t_1 = \sqrt{1 - (0.4037)^2} = 0.9149$$

or

$$t_1 = \frac{\sin^{-1}(0.9149)}{\pi} = 0.3678 \text{ s}$$

The envelope passing through the maximum points (see Problem 2.127) is given by

$$x = \sqrt{1 - \zeta^2} X e^{-\zeta \omega_n t} \quad (\text{E.2})$$

Since $x = 250$ mm, Eq. (E.2) gives at t_1

$$0.25 = \sqrt{1 - (0.4037)^2} X e^{-(0.4037)(3.4338)(0.3678)}$$

or

$$X = 0.4550 \text{ m}$$

The velocity of the mass can be obtained by differentiating the displacement

$$x(t) = X e^{-\zeta \omega_n t} \sin \omega_d t$$

as

$$\dot{x}(t) = X e^{-\zeta \omega_n t} (-\zeta \omega_n \sin \omega_d t + \omega_d \cos \omega_d t) \quad (\text{E.3})$$

When $t = 0$, Eq. (E.3) gives

$$\begin{aligned} \dot{x}(t=0) &= \dot{x}_0 = X \omega_d = X \omega_n \sqrt{1 - \zeta^2} = (0.4550)(3.4338) \sqrt{1 - (0.4037)^2} \\ &= 1.4294 \text{ m/s} \end{aligned}$$

■

EXAMPLE 2.15

Analysis of Cannon

The schematic diagram of a large cannon is shown in Fig. 2.31 [2.8]. When the gun is fired, high-pressure gases accelerate the projectile inside the barrel to a very high velocity. The reaction force pushes the gun barrel in the direction opposite that of the projectile. Since it is desirable to bring the gun barrel to rest in the shortest time without oscillation, it is made to translate backward against a critically damped spring-damper system called the *recoil mechanism*. In a particular case, the gun barrel and the recoil mechanism have a mass of 500 kg with a recoil spring of stiffness 10,000 N/m. The gun recoils 0.4 m upon firing. Find (1) the critical damping coefficient of the damper, (2) the initial recoil velocity of the gun, and (3) the time taken by the gun to return to a position 0.1 m from its initial position.

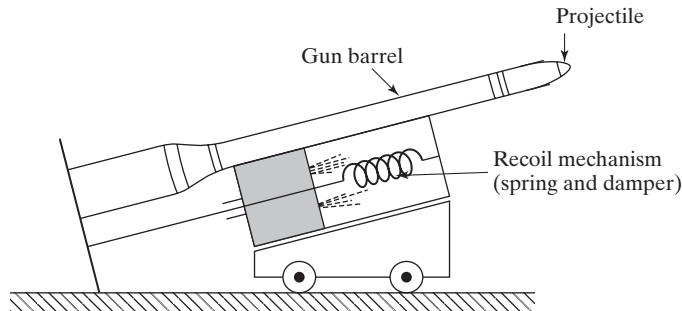


FIGURE 2.31 Recoil of cannon.

Solution:

1. The undamped natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10,000}{500}} = 4.4721 \text{ rad/s}$$

and the critical damping coefficient (Eq. 2.65) of the damper is

$$c_c = 2m\omega_n = 2(500)(4.4721) = 4472.1 \text{ N-s/m}$$

2. The response of a critically damped system is given by Eq. (2.78):

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \quad (\text{E.1})$$

where $C_1 = x_0$ and $C_2 = \dot{x}_0 + \omega_n x_0$. The time t_1 at which $x(t)$ reaches a maximum value can be obtained by setting $\dot{x}(t) = 0$. The differentiation of Eq. (E.1) gives

$$\dot{x}(t) = C_2 e^{-\omega_n t} - \omega_n (C_1 + C_2 t)e^{-\omega_n t}$$

Hence $\dot{x}(t) = 0$ yields

$$t_1 = \left(\frac{1}{\omega_n} - \frac{C_1}{C_2} \right) \quad (\text{E.2})$$

In this case, $x_0 = C_1 = 0$; hence Eq. (E.2) leads to $t_1 = 1/\omega_n$. Since the maximum value of $x(t)$ or the recoil distance is given to be $x_{\max} = 0.4 \text{ m}$, we have

$$x_{\max} = x(t = t_1) = C_2 t_1 e^{-\omega_n t_1} = \frac{\dot{x}_0}{\omega_n} e^{-1} = \frac{\dot{x}_0}{e\omega_n}$$

or

$$\dot{x}_0 = x_{\max} \omega_n e = (0.4)(4.4721)(2.7183) = 4.8626 \text{ m/s}$$

3. If t_2 denotes the time taken by the gun to return to a position 0.1 m from its initial position, we have

$$0.1 = C_2 t_2 e^{-\omega_n t_2} = 4.8626 t_2 e^{-4.4721 t_2} \quad (\text{E.3})$$

The solution of Eq. (E.3) gives $t_2 = 0.8258 \text{ s}$. ■

2.7 Graphical Representation of Characteristic Roots and Corresponding Solutions³

2.7.1 Roots of the Characteristic Equation

The free vibration of a single-degree-of-freedom spring-mass-viscous-damper system shown in Fig. 2.21 is governed by Eq. (2.59):

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (2.106)$$

whose characteristic equation can be expressed as (Eq. (2.61)):

$$ms^2 + cs + k = 0 \quad (2.107)$$

or

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad (2.108)$$

³If necessary, Sections 2.7 and 2.8 can be skipped without losing continuity.

The roots of this characteristic equation, called the *characteristic roots* or, simply, *roots*, help us in understanding the behavior of the system. The roots of Eq. (2.107) or (2.108) are given by (see Eqs. (2.62) and (2.68)):

$$s_1, s_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \quad (2.109)$$

or

$$s_1, s_2 = -\zeta\omega_n \pm i\omega_n \sqrt{1 - \zeta^2} \quad (2.110)$$

2.7.2 Graphical Representation of Roots and Corresponding Solutions

The roots given by Eq. (2.110) can be plotted in a complex plane, also known as the s -plane, by denoting the real part along the horizontal axis and the imaginary part along the vertical axis. Noting that the response of the system is given by

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \quad (2.111)$$

where C_1 and C_2 are constants, the following observations can be made by examining Eqs. (2.110) and (2.111):

1. Because the exponent of a larger real negative number (such as e^{-2t}) decays faster than the exponent of a smaller real negative number (such as e^{-t}), the roots lying farther to the left in the s -plane indicate that the corresponding responses decay faster than those associated with roots closer to the imaginary axis.
2. If the roots have positive real values of s —that is, the roots lie in the right half of the s -plane—the corresponding response grows exponentially and hence will be unstable.
3. If the roots lie on the imaginary axis (with zero real value), the corresponding response will be naturally stable.
4. If the roots have a zero imaginary part, the corresponding response will not oscillate.
5. The response of the system will exhibit an oscillatory behavior only when the roots have nonzero imaginary parts.
6. The farther the roots lie to the left of the s -plane, the faster the corresponding response decreases.
7. The larger the imaginary part of the roots, the higher the frequency of oscillation of the corresponding response of the system.

Figure 2.32 shows some representative locations of the characteristic roots in the s -plane and the corresponding responses [2.15]. The characteristics that describe the behavior of the response of a system include oscillatory nature, frequency of oscillation, and response time. These characteristics are inherent to the system (depend on the values of m , c , and k) and are determined by the characteristic roots of the system but not by the initial conditions. The initial conditions determine only the amplitudes and phase angles.

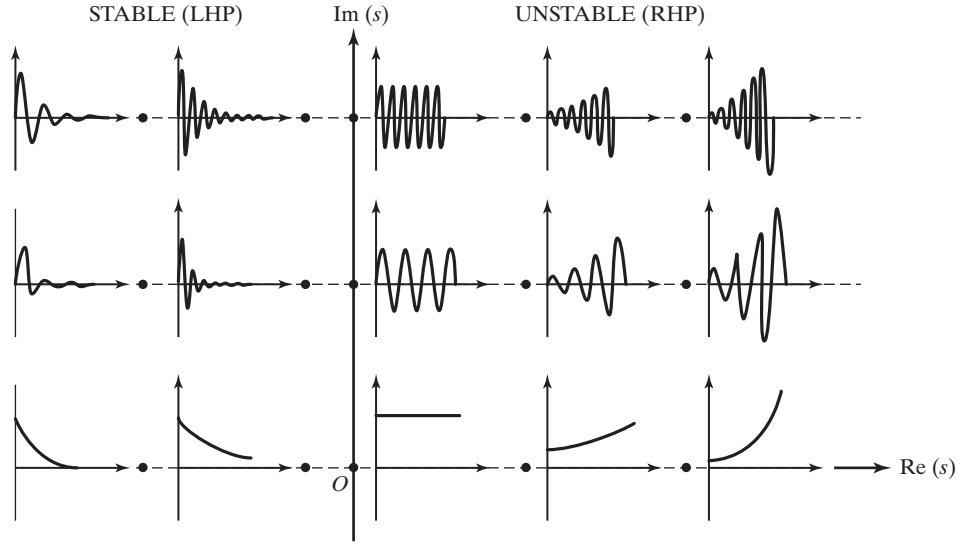


FIGURE 2.32 Locations of characteristic roots (•) and the corresponding responses of the system.

2.8 Parameter Variations and Root Locus Representations

2.8.1 Interpretations of ω_n , ω_d , ζ , and τ in the s -plane

Although the roots s_1 and s_2 appear as complex conjugates, we consider only the roots in the upper half of the s -plane. The root s_1 is plotted as point A with the real value as $\zeta\omega_n$ and the complex value as $\omega_n\sqrt{1-\zeta^2}$, so that the length of OA is ω_n (Fig. 2.33). Thus the roots lying on the circle of radius ω_n correspond to the same natural frequency (ω_n) of the system (PAQ denotes a quarter of the circle). Thus different concentric circles represent systems with different natural frequencies as shown in Fig. 2.34. The horizontal line passing through point A corresponds to the damped natural frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2}$. Thus, lines parallel to the real axis denote systems having different damped natural frequencies, as shown in Fig. 2.35.

It can be seen, from Fig. 2.33, that the angle made by the line OA with the imaginary axis is given by

$$\sin \theta = \frac{\zeta\omega_n}{\omega_n} = \zeta \quad (2.112)$$

or

$$\theta = \sin^{-1} \zeta \quad (2.113)$$

Thus, radial lines passing through the origin correspond to different damping ratios, as shown in Fig. 2.36. Therefore, when $\zeta = 0$, we have no damping ($\theta = 0$), and the damped natural frequency will reduce to the undamped natural frequency. Similarly, when

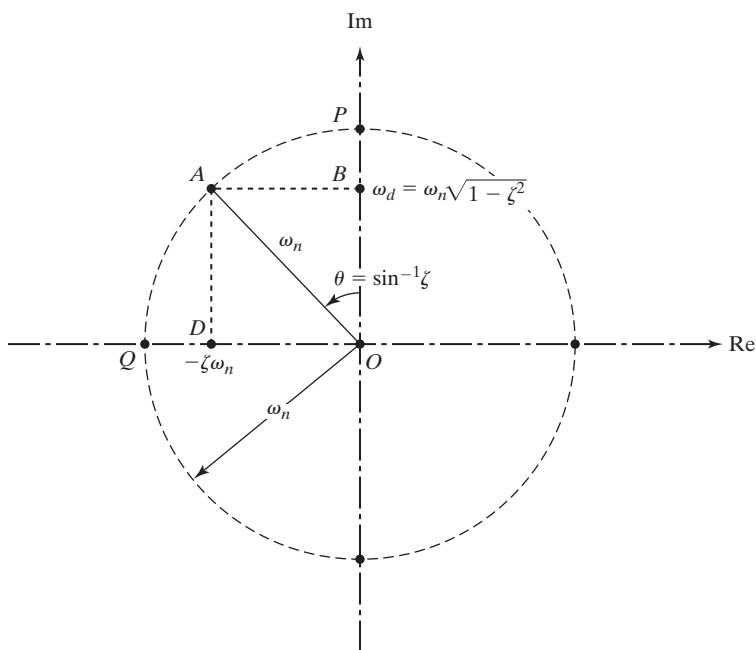


FIGURE 2.33 Interpretations of ω_n , ω_d , and ζ .

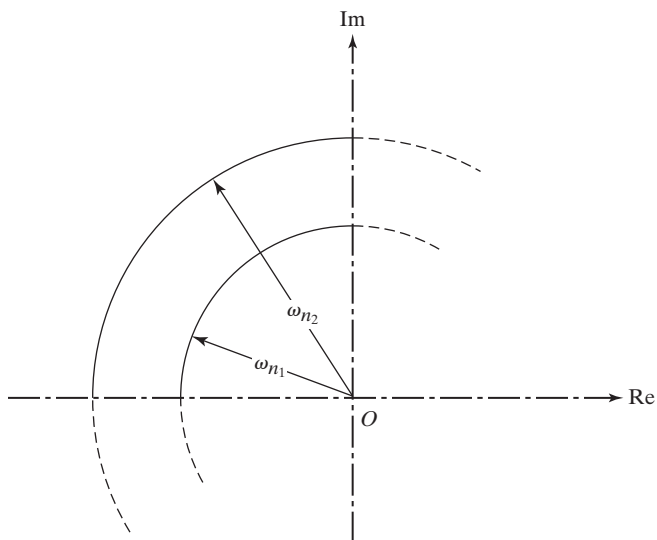
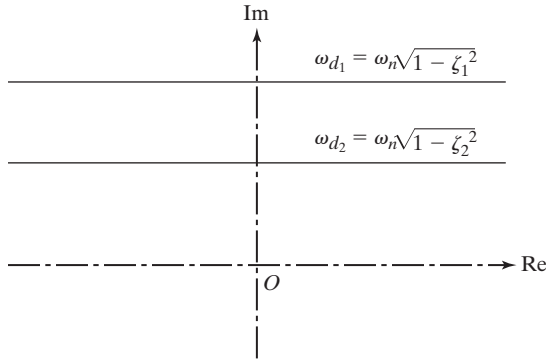
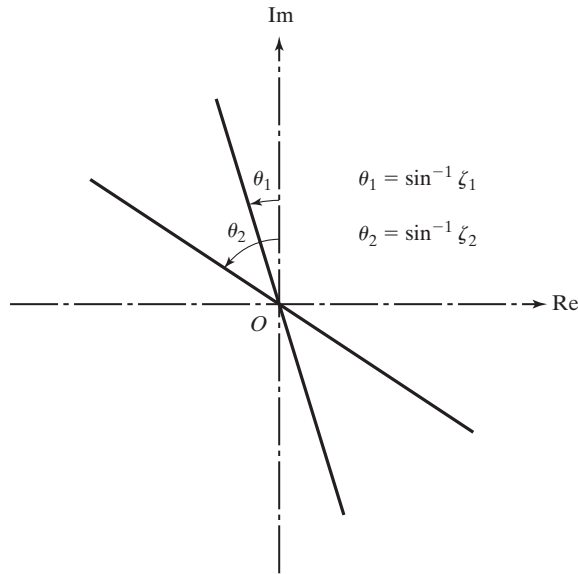


FIGURE 2.34 ω_n in s -plane.

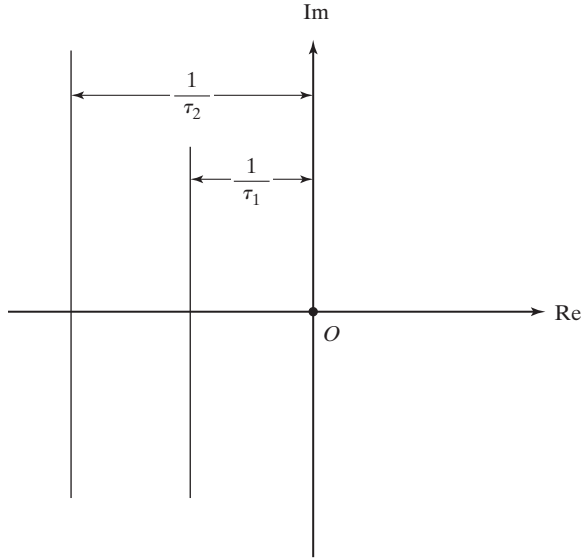
FIGURE 2.35 ω_d in s -plane.FIGURE 2.36 ζ in s -plane.

$\zeta = 1$, we have critical damping and the radical line lies along the negative real axis. The time constant of the system, τ , is defined as

$$\tau = \frac{1}{\zeta \omega_n} \quad (2.114)$$

and hence the distance DO or AB represents the reciprocal of the time constant, $\zeta \omega_n = \frac{1}{\tau}$.

Hence different lines parallel to the imaginary axis denote reciprocals of different time constants (Fig. 2.37).

FIGURE 2.37 τ in s -plane.

2.8.2 Root Locus and Parameter Variations

A plot or graph that shows how changes in one of the parameters of the system will modify the roots of the characteristic equation of the system is known as the root locus plot. The root locus method is a powerful method of analysis and design for stability and transient response of a system. For a vibrating system, the root locus can be used to describe qualitatively the performance of the system as various parameters, such as the mass, damping constant, or spring constant, are changed. In the root locus method, the path or locus of the roots of the characteristic equation is plotted without actually finding the roots themselves. This is accomplished by using a set of rules which lead to a reasonably accurate plot in a relatively short time [2.8]. We study the behavior of the system by varying one parameter, among the damping ratio, spring constant, and mass, at a time in terms of the locations of its characteristic roots in the s -plane.

Variation of the damping ratio: We vary the damping constant from zero to infinity and study the migration of the characteristic roots in the s -plane. For this, we use Eq. (2.109). We notice that negative values of the damping constant ($c < 0$) need not be considered, because they result in roots lying in the positive real half-plane that correspond to an unstable system. Thus we start with $c = 0$ to obtain, from Eq. (2.109),

$$s_{1,2} = \pm \frac{\sqrt{-4mk}}{2m} = \pm i \sqrt{\frac{k}{m}} = \pm i\omega_n \quad (2.115)$$

Thus the locations of the characteristic roots start on the imaginary axis. Because the roots appear in complex conjugate pairs, we concentrate on the upper imaginary half-plane and then locate the roots in the lower imaginary half-plane as mirror images. By keeping the undamped natural frequency (ω_n) constant, we vary the damping constant c . Noting that the real and imaginary parts of the roots in Eq. (2.109) can be expressed as

$$-\sigma = -\frac{c}{2m} = -\zeta\omega_n \quad \text{and} \quad \frac{\sqrt{4mk - c^2}}{2m} = \omega_n\sqrt{1 - \zeta^2} = \omega_d \quad (2.116)$$

for $0 < \zeta < 1$, we find that

$$\sigma^2 + \omega_d^2 = \omega_n^2 \quad (2.117)$$

Since ω_n is held fixed, Eq. (2.117) represents the equation of a circle with a radius $r = \omega_n$ in the σ (real) and ω_d (imaginary) plane. The radius vector $r = \omega_n$ will make an angle θ with the positive imaginary axis with

$$\sin \theta = \frac{\omega_d}{\omega_n} = \alpha \quad (2.118)$$

$$\cos \theta = \frac{\sigma}{\omega_n} = \frac{\zeta\omega_n}{\omega_n} = \zeta \quad (2.119)$$

with

$$\alpha = \sqrt{1 - \zeta^2} \quad (2.120)$$

Thus the two roots trace loci or paths in the form of circular arcs as the damping ratio is increased from zero to unity as shown in Fig. 2.38. The root with positive imaginary part moves in the counterclockwise direction while the root with negative imaginary part moves in the clockwise direction. When the damping ratio (ζ) is equal to one, the two loci meet, denoting that the two roots coincide—that is, the characteristic equation has repeated roots. As we increase the damping ratio beyond the value of unity, the system becomes overdamped and, as seen earlier in Section 2.6, both the roots will become real. From the properties of a quadratic equation, we find that the product of the two roots is equal to the coefficient of the lowest power of s (which is equal to ω_n^2 in Eq. (2.108)).

Since the value of ω_n is held constant in this study, the product of the two roots is a constant. With increasing values of the damping ratio (ζ), one root will increase and the other root will decrease, with the locus of each root remaining on the negative real axis. Thus one root will approach $-\infty$ and the other root will approach zero. The two loci will join or coincide at a point, known as the *breakaway point*, on the negative real axis. The two parts of the loci that lie on the negative real axis, one from point P to $-\infty$ and the other from point P to the origin, are known as *segments*.

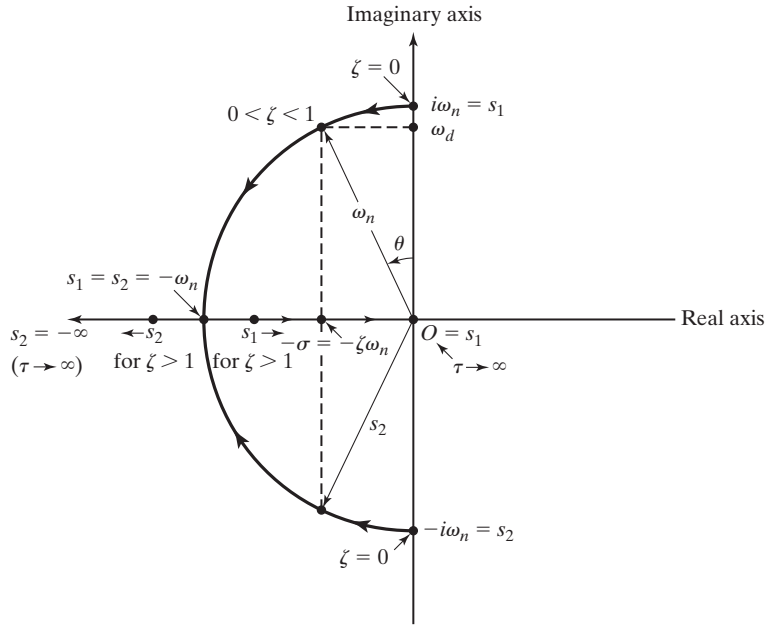


FIGURE 2.38 Root locus plot with variation of damping ratio ζ .

EXAMPLE 2.16

Study of Roots with Variation of c

Plot the root locus diagram of the system governed by the equation

$$3s^2 + cs + 27 = 0 \quad (\text{E.1})$$

by varying the value of $c > 0$.

Solution: The roots of Eq. (E.1) are given by

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 324}}{6} \quad (\text{E.2})$$

We start with a value of $c = 0$. At $c = 0$, the roots are given by $s_{1,2} = \pm 3i$. These roots are shown as dots on the imaginary axis in Fig. 2.39. By using an increasing sequence of values of c , Eq. (E.2) gives the roots as indicated in Table 2.1.

It can be seen that the roots remain complex conjugates as c is increased up to a value of $c = 18$. At $c = 18$, both the roots become real and identical with a value of -3.0 . As c increases beyond a value of 18, the roots remain distinct with negative real values. One root becomes more and more negative and the other root becomes less and less negative. Thus, as $c \rightarrow \infty$, one root approaches $-\infty$ while the other root approaches 0. These trends of the roots are shown in Fig 2.39.

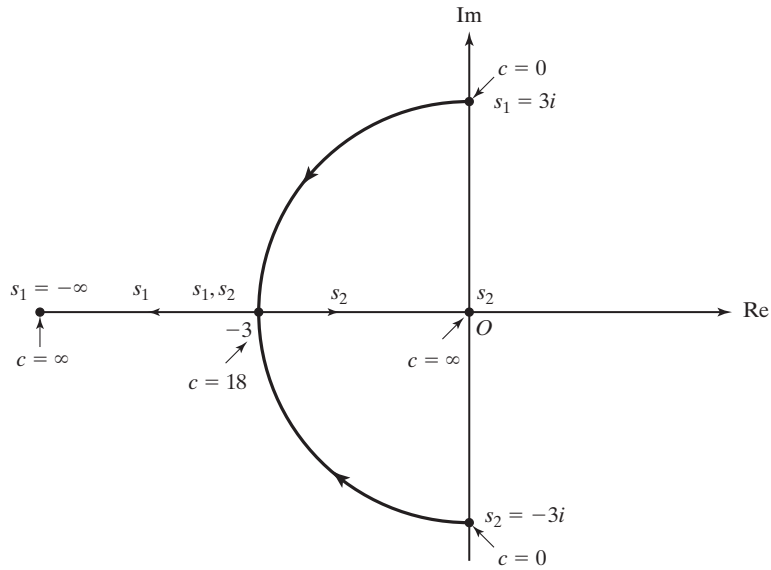


FIGURE 2.39 Root locus plot with variation of damping constant (c).

TABLE 2.1 Roots of Eq. (E.2) of Example 2.16

Value of c	Value of s_1	Value of s_2
0	$+3i$	$-3i$
2	$-0.3333 + 2.9814i$	$-0.3333 - 2.9814i$
4	$-0.6667 + 2.9721i$	$-0.6667 - 2.9721i$
6	$-1.0000 + 2.8284i$	$-1.0000 - 2.8284i$
8	$-1.3333 + 2.6874i$	$-1.3333 - 2.6874i$
10	$-1.6667 + 2.4944i$	$-1.6667 - 2.4944i$
12	$-2.0000 + 2.2361i$	$-2.0000 - 2.2361i$
14	$-2.3333 + 1.8856i$	$-2.3333 - 1.8856i$
16	$-2.6667 + 1.3744i$	$-2.6667 - 1.3744i$
18	-3.0000	-3.0000
20	-1.8803	-4.7863
30	-1.0000	-9.0000
40	-0.7131	-12.6202
50	-0.5587	-16.1079
100	-0.2722	-33.0611
1000	-0.0270	-333.3063

Variation of the spring constant: Since the spring constant does not appear explicitly in Eq. (2.108), we consider a specific form of the characteristic equation, Eq. (2.107), as

$$s^2 + 16s + k = 0 \quad (2.121)$$

The roots of Eq. (2.121) are given by

$$s_{1,2} = \frac{-16 \pm \sqrt{256 - 4k}}{2} = -8 \pm \sqrt{64 - k} \quad (2.122)$$

Since the spring stiffness cannot be negative for real vibration systems, we consider the variation of the values of k from zero to infinity. Equation (2.122) shows that for $k = 64$, both the roots are real and identical. As k is made greater than 64, the roots become complex conjugates. The roots for different values of k are shown in Table 2.2. The variations of the two roots can be plotted (as dots), as shown in Fig. 2.40.

Variation of the mass: To find the migration of the roots with a variation of the mass m , we consider a specific form of the characteristic equation, Eq. (2.107), as

$$ms^2 + 14s + 20 = 0 \quad (2.123)$$

whose roots are given by

$$s_{1,2} = \frac{-14 \pm \sqrt{196 - 80m}}{2m} \quad (2.124)$$

Since negative values as well as zero value of mass need not be considered for physical systems, we vary the value of m in the range $1 \leq m < \infty$. Some values of m and the corresponding roots given by Eq. (2.124) are shown in Table 2.3.

It can be seen that both the roots are negative with values $(-1.6148, -12.3852)$ for $m = 1$ and $(-2, -5)$ for $m = 2$. The larger root is observed to move to the left and the

TABLE 2.2 Roots of Eq. (2.122) for different values of k

Value of k	Value of s_1	Value of s_2
0	0	-16
16	-1.0718	-14.9282
32	-2.3431	-13.6569
48	-4	-12
64	-8	-8
80	$-8 + 4i$	$-8 - 4i$
96	$-8 + 5.6569i$	$-8 - 5.6569i$
112	$-8 + 6.9282i$	$-8 - 6.9282i$
128	$-8 + 8i$	$-8 - 8i$

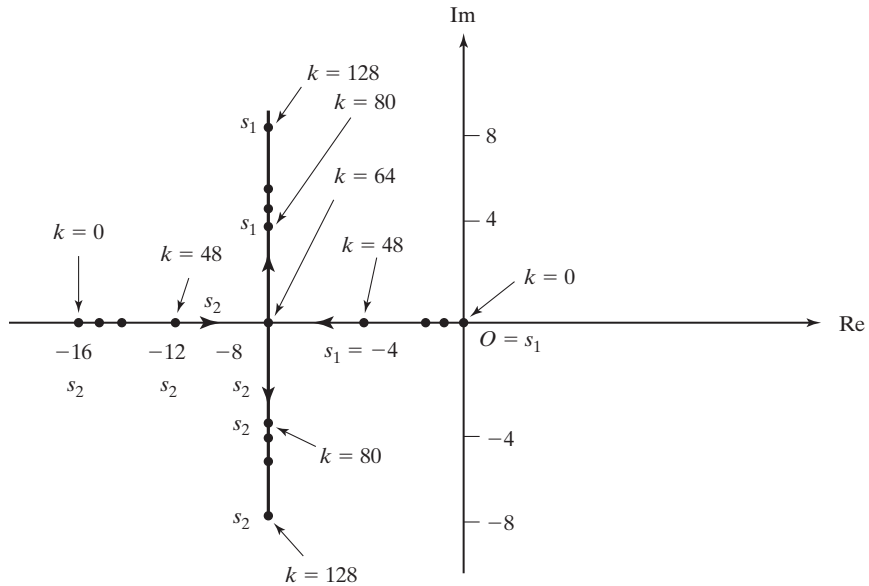


FIGURE 2.40 Root locus plot with variation of spring constant (k).

TABLE 2.3 Roots of Eq. (2.124) for different values of m

Value of m	Value of s_1	Value of s_2
1	-1.6148	-12.3852
2	-2.0	-5.0
2.1	-2.0734	-4.5932
2.4	-2.5	-3.3333
2.45	-2.8571	-2.8571
2.5	$-2.8 + 0.4000i$	$-2.8 - 0.4000i$
3	$-2.3333 + 1.1055i$	$-2.3333 - 1.1055i$
5	$-1.4 + 1.4283i$	$-1.4 - 1.4283i$
8	$-0.8750 + 1.3169i$	$-0.8750 - 1.3169i$
10	$-0.7000 + 1.2288i$	$-0.7000 - 1.2288i$
14	$-0.5000 + 1.0856i$	$-0.5000 - 1.0856i$
20	$-0.3500 + 0.9367i$	$-0.3500 - 0.9367i$
30	$-0.2333 + 0.7824i$	$-0.2333 - 0.7824i$
40	$-0.1750 + 0.6851i$	$-0.1750 - 0.6851i$
50	$-0.1400 + 0.6167i$	$-0.1400 - 0.6167i$
100	$-0.0700 + 0.4417i$	$-0.0700 - 0.4417i$
1000	$-0.0070 + 0.1412i$	$-0.0070 - 0.1412i$

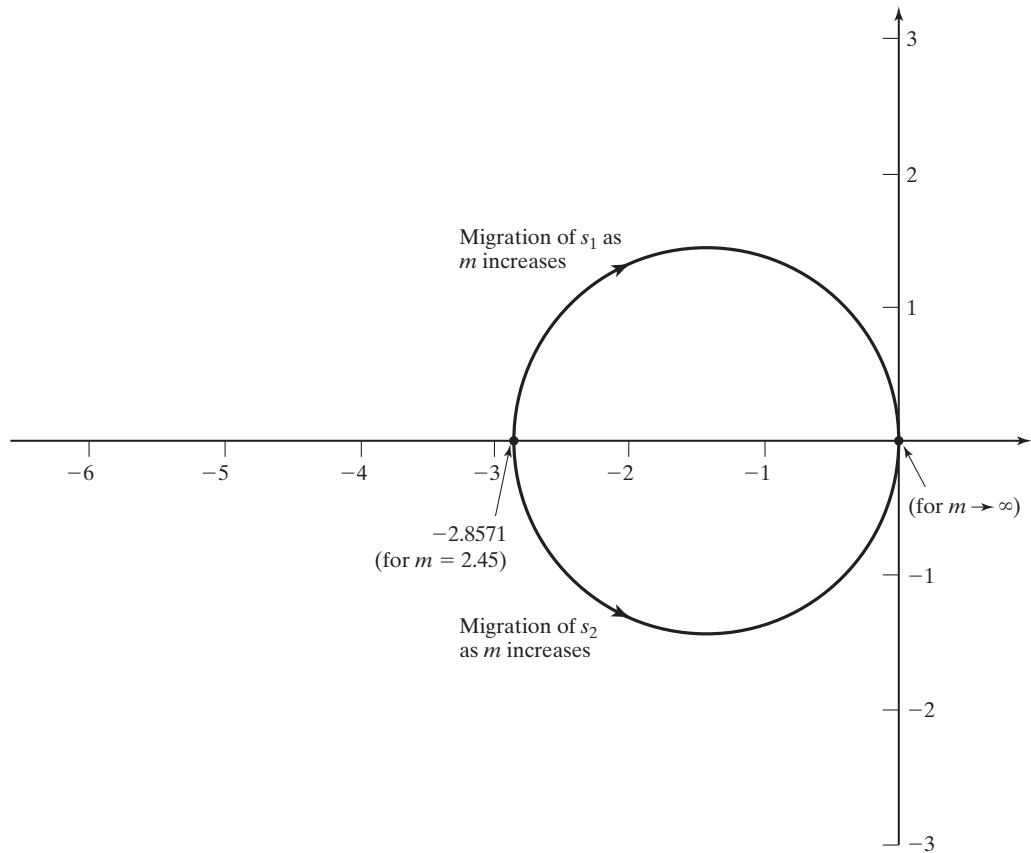


FIGURE 2.41 Root locus plot with variation of mass (m).

smaller root is found to move to the right, as shown in Fig. 2.41. The larger and smaller roots are found to converge to the value -2.8571 as m increases to a value of 2.45 . Beyond this value of $m = 2.45$, the roots become complex conjugate. As the value of m increases from 2.45 to a large value ($\rightarrow \infty$), the loci of the two complex conjugates (roots) are shown by the curve (circle) shown in Fig. 2.41. For $m \rightarrow \infty$, both the complex conjugate roots converge to zero ($s_1, s_2 \rightarrow 0$).

2.9 Free Vibration with Coulomb Damping

In many mechanical systems, *Coulomb* or *dry-friction* dampers are used because of their mechanical simplicity and convenience [2.9]. Also, in vibrating structures, whenever the components slide relative to each other, dry-friction damping appears internally. As stated in Section 1.9, Coulomb damping arises when bodies slide on dry surfaces. Coulomb's law of dry friction states that, when two bodies are in contact, the force required to produce

sliding is proportional to the normal force acting in the plane of contact. Thus the friction force F is given by

$$F = \mu N = \mu W = \mu mg \quad (2.125)$$

where N is the normal force, equal to the weight of the mass ($W = mg$) and μ is the coefficient of sliding or kinetic friction. The value of the coefficient of friction (μ) depends on the materials in contact and the condition of the surfaces in contact. For example, $\mu \simeq 0.1$ for metal on metal (lubricated), 0.3 for metal on metal (unlubricated), and nearly 1.0 for rubber on metal. The friction force acts in a direction opposite to the direction of velocity. Coulomb damping is sometimes called *constant damping*, since the damping force is independent of the displacement and velocity; it depends only on the normal force N between the sliding surfaces.

2.9.1 Equation of Motion

Consider a single-degree-of-freedom system with dry friction as shown in Fig. 2.42(a). Since the friction force varies with the direction of velocity, we need to consider two cases, as indicated in Figs. 2.42(b) and (c).

Case 1. When x is positive and dx/dt is positive or when x is negative and dx/dt is positive (i.e., for the half cycle during which the mass moves from left to right), the equation of motion can be obtained using Newton's second law (see Fig. 2.42(b)):

$$m\ddot{x} = -kx - \mu N \quad \text{or} \quad m\ddot{x} + kx = -\mu N \quad (2.126)$$

This is a second-order nonhomogeneous differential equation. The solution can be verified by substituting Eq. (2.127) into Eq. (2.126):

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t - \frac{\mu N}{k} \quad (2.127)$$

where $\omega_n = \sqrt{k/m}$ is the frequency of vibration and A_1 and A_2 are constants whose values depend on the initial conditions of this half cycle.

Case 2. When x is positive and dx/dt is negative or when x is negative and dx/dt is negative (i.e., for the half cycle during which the mass moves from right to left), the equation of motion can be derived from Fig. 2.42(c) as

$$-kx + \mu N = m\ddot{x} \quad \text{or} \quad m\ddot{x} + kx = \mu N \quad (2.128)$$

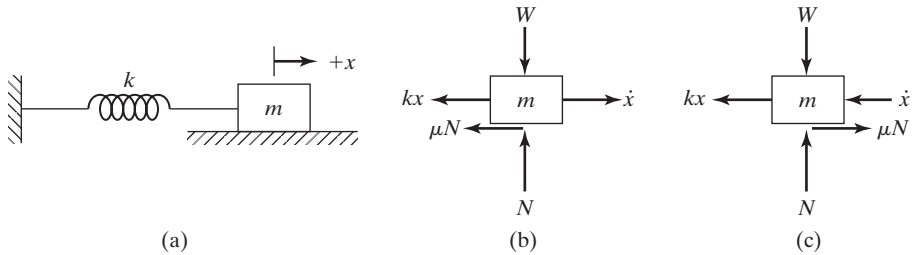


FIGURE 2.42 Spring-mass system with Coulomb damping.

The solution of Eq. (2.128) is given by

$$x(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t + \frac{\mu N}{k} \quad (2.129)$$

where A_3 and A_4 are constants to be found from the initial conditions of this half cycle. The term $\mu N/k$ appearing in Eqs. (2.127) and (2.129) is a constant representing the virtual displacement of the spring under the force μN , if it were applied as a static force. Equations (2.127) and (2.129) indicate that in each half cycle the motion is harmonic, with the equilibrium position changing from $\mu N/k$ to $-(\mu N/k)$ every half cycle, as shown in Fig. 2.43.

2.9.2 Solution

Equations (2.126) and (2.128) can be expressed as a single equation (using $N = mg$):

$$m\ddot{x} + \mu mg \operatorname{sgn}(\dot{x}) + kx = 0 \quad (2.130)$$

where $\operatorname{sgn}(y)$ is called the signum function, whose value is defined as 1 for $y > 0$, -1 for $y < 0$, and 0 for $y = 0$. Equation (2.130) can be seen to be a nonlinear differential equation for which a simple analytical solution does not exist. Numerical methods can be used to solve Eq. (2.130) conveniently (see Example 2.24). Equation (2.130), however, can be solved analytically if we break the time axis into segments separated by $\dot{x} = 0$ (i.e., time intervals with different directions of motion). To find the solution using this procedure, let us assume the initial conditions as

$$\begin{aligned} x(t = 0) &= x_0 \\ \dot{x}(t = 0) &= 0 \end{aligned} \quad (2.131)$$

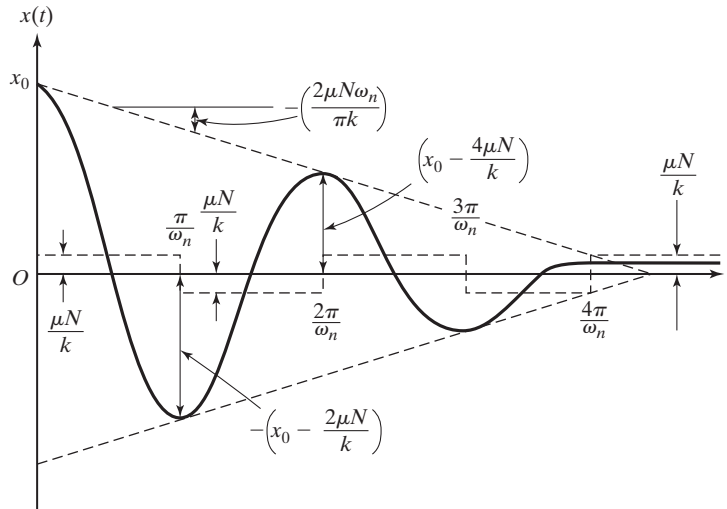


FIGURE 2.43 Motion of the mass with Coulomb damping.

That is, the system starts with zero velocity and displacement x_0 at $t = 0$. Since $x = x_0$ at $t = 0$, the motion starts from right to left. Let x_0, x_1, x_2, \dots denote the amplitudes of motion at successive half cycles. Using Eqs. (2.129) and (2.131), we can evaluate the constants A_3 and A_4 :

$$A_3 = x_0 - \frac{\mu N}{k}, \quad A_4 = 0$$

Thus Eq. (2.129) becomes

$$x(t) = \left(x_0 - \frac{\mu N}{k} \right) \cos \omega_n t + \frac{\mu N}{k} \quad (2.132)$$

This solution is valid for half the cycle only—that is, for $0 \leq t \leq \pi/\omega_n$. When $t = \pi/\omega_n$, the mass will be at its extreme left position and its displacement from equilibrium position can be found from Eq. (2.132):

$$-x_1 = x\left(t = \frac{\pi}{\omega_n}\right) = \left(x_0 - \frac{\mu N}{k} \right) \cos \pi + \frac{\mu N}{k} = -\left(x_0 - \frac{2\mu N}{k} \right)$$

Since the motion started with a displacement of $x = x_0$ and, in a half cycle, the value of x became $-[x_0 - (2\mu N/k)]$, the reduction in magnitude of x in time π/ω_n is $2\mu N/k$.

In the second half cycle, the mass moves from left to right, so Eq. (2.127) is to be used. The initial conditions for this half cycle are

$$x(t = 0) = \text{value of } x \text{ at } t = \frac{\pi}{\omega_n} \text{ in Eq. (2.132)} = -\left(x_0 - \frac{2\mu N}{k} \right)$$

and

$$\begin{aligned} \dot{x}(t = 0) &= \text{value of } \dot{x} \text{ at } t = \frac{\pi}{\omega_n} \text{ in Eq. (2.132)} \\ &= \left\{ \text{value of } -\omega_n \left(x_0 - \frac{\mu N}{k} \right) \sin \omega_n t \text{ at } t = \frac{\pi}{\omega_n} \right\} = 0 \end{aligned}$$

Thus the constants in Eq. (2.127) become

$$-A_1 = -x_0 + \frac{3\mu N}{k}, \quad A_2 = 0$$

so that Eq. (2.127) can be written as

$$x(t) = \left(x_0 - \frac{3\mu N}{k} \right) \cos \omega_n t - \frac{\mu N}{k} \quad (2.133)$$

This equation is valid only for the second half cycle—that is, for $\pi/\omega_n \leq t \leq 2\pi/\omega_n$. At the end of this half cycle the value of $x(t)$ is

$$x_2 = x\left(t = \frac{\pi}{\omega_n}\right) \text{ in Eq. (2.133)} = x_0 - \frac{4\mu N}{k}$$

and

$$\dot{x}\left(t = \frac{\pi}{\omega_n}\right) \text{ in Eq. (2.133)} = 0$$

These become the initial conditions for the third half cycle, and the procedure can be continued until the motion stops. The motion stops when $x_n \leq \mu N/k$, since the restoring force exerted by the spring (kx) will then be less than the friction force μN . Thus the number of half cycles (r) that elapse before the motion ceases is given by

$$x_0 - r \frac{2\mu N}{k} \leq \frac{\mu N}{k}$$

that is,

$$r \geq \left\{ \frac{x_0 - \frac{\mu N}{k}}{\frac{2\mu N}{k}} \right\} \quad (2.134)$$

Note the following characteristics of a system with Coulomb damping:

1. The equation of motion is nonlinear with Coulomb damping, whereas it is linear with viscous damping.
2. The natural frequency of the system is unaltered with the addition of Coulomb damping, whereas it is reduced with the addition of viscous damping.
3. The motion is periodic with Coulomb damping, whereas it can be nonperiodic in a viscously damped (overdamped) system.
4. The system comes to rest after some time with Coulomb damping, whereas the motion theoretically continues forever (perhaps with an infinitesimally small amplitude) with viscous and hysteresis damping.
5. The amplitude reduces linearly with Coulomb damping, whereas it reduces exponentially with viscous damping.
6. In each successive cycle, the amplitude of motion is reduced by the amount $4\mu N/k$, so the amplitudes at the end of any two consecutive cycles are related:

$$X_m = X_{m-1} - \frac{4\mu N}{k} \quad (2.135)$$

As the amplitude is reduced by an amount $4\mu N/k$ in one cycle (i.e., in time $2\pi/\omega_n$), the slope of the enveloping straight lines (shown dotted) in Fig. 2.43 is

$$-\left(\frac{4\mu N}{k}\right) / \left(\frac{2\pi}{\omega_n}\right) = -\left(\frac{2\mu N\omega_n}{\pi k}\right)$$

The final position of the mass is usually displaced from equilibrium ($x = 0$) position and represents a permanent displacement in which the friction force is locked. Slight tapping will usually make the mass come to its equilibrium position.

2.9.3 Torsional Systems with Coulomb Damping

If a constant frictional torque acts on a torsional system, the equation governing the angular oscillations of the system can be derived, similar to Eqs. (2.126) and (2.128), as

$$J_0\ddot{\theta} + k_t\theta = -T \quad (2.136)$$

and

$$J_0\ddot{\theta} + k_t\theta = T \quad (2.137)$$

where T denotes the constant damping torque (similar to μN for linear vibrations). The solutions of Eqs. (2.136) and (2.137) are similar to those for linear vibrations. In particular, the frequency of vibration is given by

$$\omega_n = \sqrt{\frac{k_t}{J_0}} \quad (2.138)$$

and the amplitude of motion at the end of the r th half cycle (θ_r) is given by

$$\theta_r = \theta_0 - r \frac{2T}{k_t} \quad (2.139)$$

where θ_0 is the initial angular displacement at $t = 0$ (with $\dot{\theta} = 0$ at $t = 0$). The motion ceases when

$$r \geq \left\{ \begin{array}{l} \theta_0 - \frac{T}{k_t} \\ \frac{2T}{k_t} \end{array} \right\} \quad (2.140)$$

EXAMPLE 2.17

Coefficient of Friction from Measured Positions of Mass

A metal block, placed on a rough surface, is attached to a spring and is given an initial displacement of 10 cm from its equilibrium position. After five cycles of oscillation in 2 s, the final position of the metal block is found to be 1 cm from its equilibrium position. Find the coefficient of friction between the surface and the metal block.

Solution: Since five cycles of oscillation were observed to take place in 2 s, the period (τ_n) is $2/5 = 0.4$ s, and hence the frequency of oscillation is $\omega_n = \sqrt{\frac{k}{m}} = \frac{2\pi}{\tau_n} = \frac{2\pi}{0.4} = 15.708$ rad/s. Since the amplitude of oscillation reduces by

$$\frac{4\mu N}{k} = \frac{4\mu mg}{k}$$

in each cycle, the reduction in amplitude in five cycles is

$$5\left(\frac{4\mu mg}{k}\right) = 0.10 - 0.01 = 0.09 \text{ m}$$

or

$$\mu = \frac{0.09k}{20mg} = \frac{0.09\omega_n^2}{20g} = \frac{0.09(15.708)^2}{20(9.81)} = 0.1132$$

■

EXAMPLE 2.18

Pulley Subjected to Coulomb Damping

A steel shaft of length 1 m and diameter 50 mm is fixed at one end and carries a pulley of mass moment of inertia 25 kg-m^2 at the other end. A band brake exerts a constant frictional torque of 400 N-m around the circumference of the pulley. If the pulley is displaced by 6° and released, determine (1) the number of cycles before the pulley comes to rest and (2) the final settling position of the pulley.

Solution:

1. The number of half cycles that elapse before the angular motion of the pulley ceases is given by Eq. (2.140):

$$r \geq \left\{ \frac{\theta_0 - \frac{T}{k_t}}{\frac{2T}{k_t}} \right\} \quad (\text{E.1})$$

where θ_0 = initial angular displacement = $6^\circ = 0.10472$ rad, k_t = torsional spring constant of the shaft given by

$$k_t = \frac{GJ}{l} = \frac{(8 \times 10^{10}) \left\{ \frac{\pi}{32} (0.05)^4 \right\}}{1} = 49,087.5 \text{ N-m/rad}$$

and T = constant friction torque applied to the pulley = 400 N-m. Equation (E.1) gives

$$r \geq \frac{0.10472 - \left(\frac{400}{49,087.5} \right)}{\left(\frac{800}{49,087.5} \right)} = 5.926$$

Thus the motion ceases after six half cycles.

2. The angular displacement after six half cycles is given by Eq. (2.139):

$$\theta = 0.10472 - 6 \times 2 \left(\frac{400}{49,087.5} \right) = 0.006935 \text{ rad} = 0.39734^\circ$$

Thus the pulley stops at 0.39734° from the equilibrium position on the same side of the initial displacement.

■

2.10 Free Vibration with Hysteretic Damping

Consider the spring-viscous-damper arrangement shown in Fig. 2.44(a). For this system, the force F needed to cause a displacement $x(t)$ is given by

$$F = kx + c\dot{x} \quad (2.141)$$

For a harmonic motion of frequency ω and amplitude X ,

$$x(t) = X \sin \omega t \quad (2.142)$$

Equations (2.141) and (2.142) yield

$$\begin{aligned} F(t) &= kX \sin \omega t + cX\omega \cos \omega t \\ &= kx \pm c\omega \sqrt{X^2 - (X \sin \omega t)^2} \\ &= kx \pm c\omega \sqrt{X^2 - x^2} \end{aligned} \quad (2.143)$$

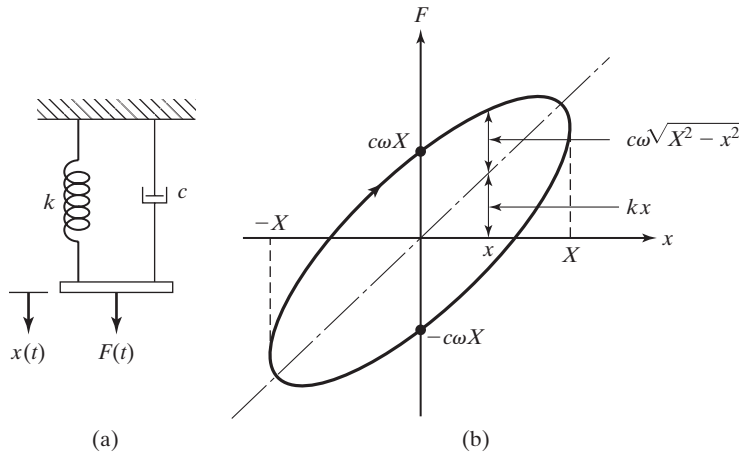


FIGURE 2.44 Spring-viscous-damper system.

When F versus x is plotted, Eq. (2.143) represents a closed loop, as shown in Fig. 2.44(b). The area of the loop denotes the energy dissipated by the damper in a cycle of motion and is given by

$$\begin{aligned}\Delta W &= \oint F dx = \int_0^{2\pi/\omega} (kX \sin \omega t + cX\omega \cos \omega t)(\omega X \cos \omega t) dt \\ &= \pi \omega c X^2\end{aligned}\quad (2.144)$$

Equation (2.144) has been derived in Section 2.6.4 also (see Eq. (2.98)).

As stated in Section 1.9, the damping caused by the friction between the internal planes that slip or slide as the material deforms is called hysteresis (or solid or structural) damping. This causes a hysteresis loop to be formed in the stress-strain or force-displacement curve (see Fig. 2.45(a)). The energy loss in one loading and unloading cycle is equal to the area enclosed by the hysteresis loop [2.11–2.13]. The similarity between Figs. 2.44(b) and 2.45(a) can be used to define a hysteresis damping constant. It was found experimentally that the energy loss per cycle due to internal friction is independent of the frequency but approximately proportional to the square of the amplitude. In order to achieve this observed behavior from Eq. (2.144), the damping coefficient c is assumed to be inversely proportional to the frequency as

$$c = \frac{h}{\omega} \quad (2.145)$$

where h is called the hysteresis damping constant. Equations (2.144) and (2.145) give

$$\Delta W = \pi h X^2 \quad (2.146)$$

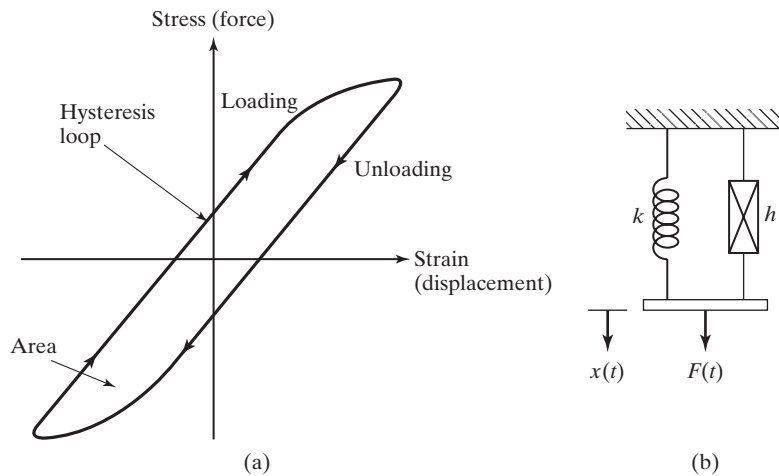


FIGURE 2.45 Hysteresis loop.

Complex Stiffness. In Fig. 2.44(a), the spring and the damper are connected in parallel, and for a general harmonic motion, $x = Xe^{i\omega t}$, the force is given by

$$F = kXe^{i\omega t} + c\omega iXe^{i\omega t} = (k + i\omega c)x \quad (2.147)$$

Similarly, if a spring and a hysteresis damper are connected in parallel, as shown in Fig. 2.45(b), the force-displacement relation can be expressed as

$$F = (k + ih)x \quad (2.148)$$

where

$$k + ih = k\left(1 + i\frac{h}{k}\right) = k(1 + i\beta) \quad (2.149)$$

is called the complex stiffness of the system and $\beta = h/k$ is a constant indicating a dimensionless measure of damping.

Response of the System. In terms of β , the energy loss per cycle can be expressed as

$$\Delta W = \pi k\beta X^2 \quad (2.150)$$

Under hysteresis damping, the motion can be considered to be nearly harmonic (since ΔW is small), and the decrease in amplitude per cycle can be determined using energy balance. For example, the energies at points P and Q (separated by half a cycle) in Fig. 2.46 are related as

$$\frac{kX_j^2}{2} - \frac{\pi k\beta X_j^2}{4} - \frac{\pi k\beta X_{j+0.5}^2}{4} = \frac{kX_{j+0.5}^2}{2}$$

or

$$\frac{X_j}{X_{j+0.5}} = \sqrt{\frac{2 + \pi\beta}{2 - \pi\beta}} \quad (2.151)$$

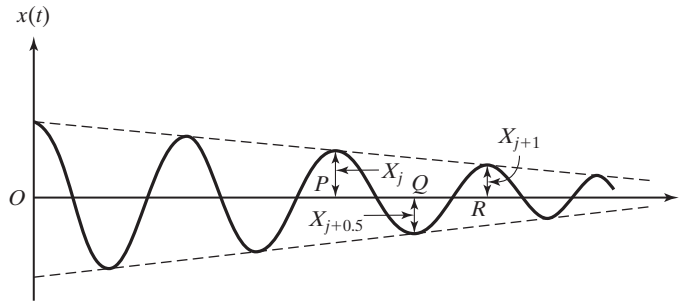


FIGURE 2.46 Response of a hysteretically damped system.

Similarly, the energies at points Q and R give

$$\frac{X_{j+0.5}}{X_{j+1}} = \sqrt{\frac{2 + \pi\beta}{2 - \pi\beta}} \quad (2.152)$$

Multiplication of Eqs. (2.151) and (2.152) gives

$$\frac{X_j}{X_{j+1}} = \frac{2 + \pi\beta}{2 - \pi\beta} = \frac{2 - \pi\beta + 2\pi\beta}{2 - \pi\beta} \simeq 1 + \pi\beta = \text{constant} \quad (2.153)$$

The hysteresis logarithmic decrement can be defined as

$$\delta = \ln\left(\frac{X_j}{X_{j+1}}\right) \simeq \ln(1 + \pi\beta) \simeq \pi\beta \quad (2.154)$$

Since the motion is assumed to be approximately harmonic, the corresponding frequency is defined by [2.10]:

$$\omega = \sqrt{\frac{k}{m}} \quad (2.155)$$

The equivalent viscous damping ratio ζ_{eq} can be found by equating the relation for the logarithmic decrement δ :

$$\begin{aligned} \delta &\simeq 2\pi\zeta_{\text{eq}} \simeq \pi\beta = \frac{\pi h}{k} \\ \zeta_{\text{eq}} &= \frac{\beta}{2} = \frac{h}{2k} \end{aligned} \quad (2.156)$$

Thus the equivalent damping constant c_{eq} is given by

$$c_{\text{eq}} = c_c \cdot \zeta_{\text{eq}} = 2\sqrt{mk} \cdot \frac{\beta}{2} = \beta\sqrt{mk} = \frac{\beta k}{\omega} = \frac{h}{\omega} \quad (2.157)$$

Note that the method of finding an equivalent viscous damping coefficient for a structurally damped system is valid only for harmonic excitation. The above analysis assumes that the system responds approximately harmonically at the frequency ω .

EXAMPLE 2.19

Estimation of Hysteretic Damping Constant

The experimental measurements on a structure gave the force-deflection data shown in Fig. 2.47. From this data, estimate the hysteretic damping constant β and the logarithmic decrement δ .

Solution:

Approach: We equate the energy dissipated in a cycle (area enclosed by the hysteresis loop) to ΔW of Eq. (2.146).

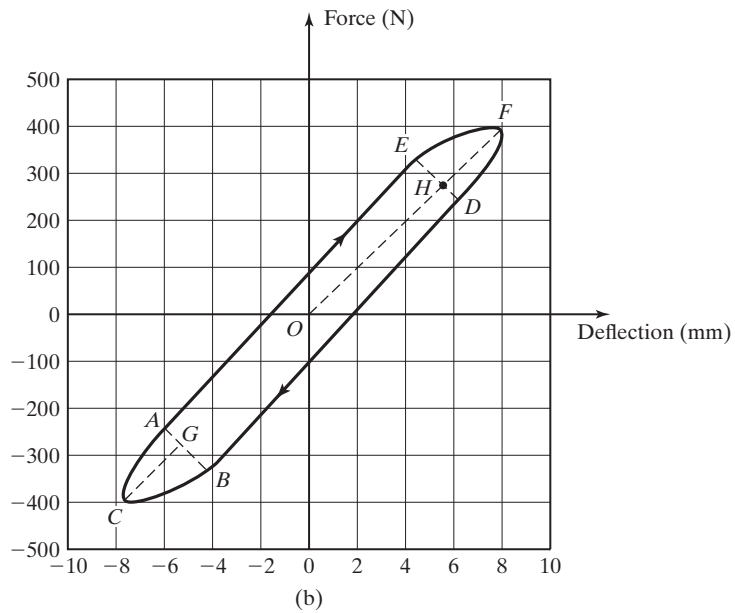
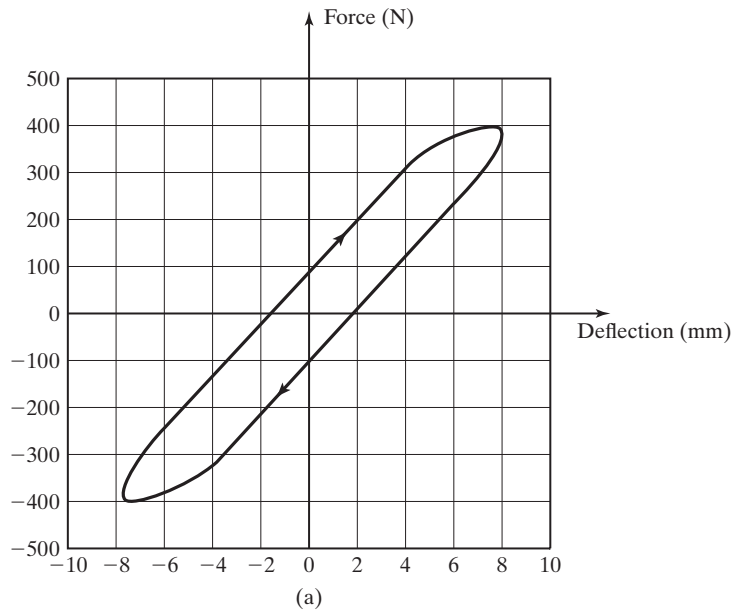


FIGURE 2.47 Force-deflection curve.

The energy dissipated in each full load cycle is given by the area enclosed by the hysteresis curve. Each square in Fig. 2.47 denotes $100 \times 2 = 200$ N-mm. The area enclosed by the loop can be found as area ACB + area $ABDE$ + area $DFE \simeq \frac{1}{2}(AB)(CG) + (AB)(AE) + \frac{1}{2}(DE)(FH) = \frac{1}{2}(1.25)(1.8) + (1.25)(8) + \frac{1}{2}(1.25)(1.8) = 12.25$ square units. This area represents an energy of $12.25 \times 200/1000 = 2.5$ N-m. From Eq. (2.146), we have

$$\Delta W = \pi h X^2 = 2.5 \text{ N-m} \quad (\text{E.1})$$

Since the maximum deflection X is 0.008 m and the slope of the force-deflection curve (given approximately by the slope of the line OF) is $k = 400/8 = 50$ N/mm = 50,000 N/m, the hysteretic damping constant h is given by

$$h = \frac{\Delta W}{\pi X^2} = \frac{2.5}{\pi(0.008)^2} = 12,433.95 \quad (\text{E.2})$$

and hence

$$\beta = \frac{h}{k} = \frac{12,433.95}{50,000} = 0.248679$$

The logarithmic decrement can be found as

$$\delta \simeq \pi\beta = \pi(0.248679) = 0.78125 \quad (\text{E.3})$$

■

EXAMPLE 2.20

Response of a Hysteretically Damped Bridge Structure

A bridge structure is modeled as a single-degree-of-freedom system with an equivalent mass of 5×10^5 kg and an equivalent stiffness of 25×10^6 N/m. During a free-vibration test, the ratio of successive amplitudes was found to be 1.04. Estimate the structural damping constant (β) and the approximate free-vibration response of the bridge.

Solution: Using the ratio of successive amplitudes, Eq. (2.154) yields the hysteresis logarithmic decrement (δ) as

$$\delta = \ln\left(\frac{X_j}{X_{j+1}}\right) = \ln(1.04) = \ln(1 + \pi\beta)$$

or

$$1 + \pi\beta = 1.04 \quad \text{or} \quad \beta = \frac{0.04}{\pi} = 0.0127$$

The equivalent viscous damping coefficient (c_{eq}) can be determined from Eq. (2.157) as

$$c_{eq} = \frac{\beta k}{\omega} = \frac{\beta k}{\sqrt{\frac{k}{m}}} = \beta \sqrt{km} \quad (\text{E.1})$$

Using the known values of the equivalent stiffness (k) and the equivalent mass (m) of the bridge, Eq. (E.1) yields

$$c_{\text{eq}} = (0.0127)\sqrt{(25 \times 10^6)(5 \times 10^5)} = 44.9013 \times 10^3 \text{ N-s/m}$$

The equivalent critical damping constant of the bridge can be computed using Eq. (2.65) as

$$c_c = 2\sqrt{km} = 2\sqrt{(25 \times 10^6)(5 \times 10^5)} = 7071.0678 \times 10^3 \text{ N-s/m}$$

Since $c_{\text{eq}} < c_c$, the bridge is underdamped, and hence its free-vibration response is given by Eq. (2.72a) as

$$x(t) = e^{-\zeta\omega_n t} \left\{ x_0 \cos \sqrt{1 - \zeta^2} \omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1 - \zeta^2} \omega_n} \sin \sqrt{1 - \zeta^2} \omega_n t \right\}$$

where

$$\zeta = \frac{c_{\text{eq}}}{c_c} = \frac{40.9013 \times 10^3}{7071.0678 \times 10^3} = 0.0063$$

and x_0 and \dot{x}_0 denote the initial displacement and initial velocity given to the bridge at the start of free vibration. ■

2.11 Stability of Systems

Stability is one of the most important characteristics for any vibrating system. Although many definitions can be given for the term *stability* depending on the kind of system or the point of view, we consider our definition for linear and time-invariant systems (i.e., systems for which the parameters m , c , and k do not change with time). A system is defined to be *asymptotically stable* (called *stable* in controls literature) if its free-vibration response approaches zero as time approaches infinity. A system is considered to be *unstable* if its free-vibration response grows without bound (approaches infinity) as time approaches infinity. Finally, a system is said to be *stable* (called *marginally stable* in controls literature) if its free-vibration response neither decays nor grows, but remains constant or oscillates as time approaches infinity. It is evident that an unstable system whose free-vibration response grows without bounds can cause damage to the system, adjacent property, or human life. Usually, dynamic systems are designed with limit stops to prevent their responses from growing with no limit.

As will be seen in Chapters 3 and 4, the total response of a vibrating system, subjected to external forces/excitations, is composed of two parts—one the forced response and the other the free-vibration response. For such systems, the definitions of asymptotically stable, unstable, and stable systems given above are still applicable. This implies that, for stable systems, only the forced response remains as the free-vibration response approaches zero as time approaches infinity.

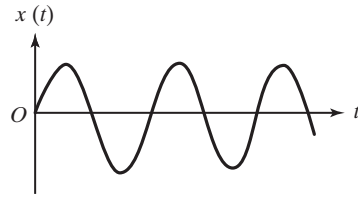
Stability can be interpreted in terms of the roots of the characteristic roots of the system. As seen in Section 2.7, the roots lying in the left half-plane (LHP) yield either pure exponential decay or damped sinusoidal free-vibration responses. These responses decay

to zero as time approaches infinity. Thus, systems whose characteristic roots lie in the left half of the s -plane (with a negative real part) will be asymptotically stable. The roots lying in the right half-plane yield either pure exponentially increasing or exponentially increasing sinusoidal free-vibration responses. These free-vibration responses approach infinity as time approaches infinity. Thus, systems whose characteristic roots lie in the right half of the s -plane (with positive real part) will be unstable. Finally, the roots lying on the imaginary axis of the s -plane yield pure sinusoidal oscillations as free-vibration response. These responses neither increase nor decrease in amplitude as time grows. Thus, systems whose characteristic roots lie on the imaginary axis of the s -plane (with zero real part) will be stable.⁴

Notes:

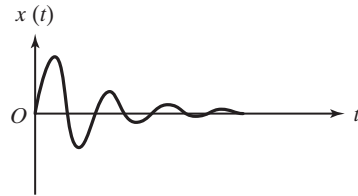
1. It is evident, from the definitions given, that the signs of the coefficients of the characteristic equation, Eq. (2.107), determine the stability behavior of a system. For example, from the theory of polynomial equations, if there is any number of negative terms or if any term in the polynomial in s is missing, then one of the roots will be positive, which results in an unstable behavior of the system. This aspect is considered further in Section 3.11 as well as in Section 5.8 in the form of the Routh-Hurwitz stability criterion.
2. In an unstable system, the free-vibration response may grow without bound with no oscillations or it may grow without bound with oscillations. The first behavior is called *divergent instability* and the second is called *flutter instability*. These cases are also known as *self-excited vibration* (see Section 3.11).
3. If a linear model of a system is asymptotically stable, then it is not possible to find a set of initial conditions for which the response approaches infinity. On the other hand, if the linear model of the system is unstable, it is possible that certain initial conditions might make the response approach zero as time increases. As an example, consider a system governed by the equation of motion $\ddot{x} - x = 0$ with characteristic roots given by $s_{1,2} = \mp 1$. Thus the response is given by $x(t) = C_1 e^{-t} + C_2 e^t$, where C_1 and C_2 are constants. If the initial conditions are specified as $x(0) = 1$ and $\dot{x}(0) = -1$, we find that $C_1 = 1$ and $C_2 = 0$ and hence the response becomes $x(t) = e^{-t}$, which approaches zero as time increases to infinity.
4. Typical responses corresponding to different types of stability are shown in Figs. (2.48)(a)–(d).
5. Stability of a system can also be explained in terms of its energy. According to this scheme, a system is considered to be asymptotically stable, stable, or unstable if its energy decreases, remains constant, or increases, respectively, with time. This idea forms the basis for Lyapunov stability criterion [2.14, 2.16, 2.17].
6. Stability of a system can also be investigated based on how sensitive the response or motion is to small perturbations (or variations) in the parameters (m , c , and k) and/or small perturbations in the initial conditions.

⁴Strictly speaking, the statement is true only if the roots that lie on the imaginary axis appear with multiplicity one. If such roots appear with multiplicity $n > 1$, the system will be unstable because the free-vibration response of such systems will be of the form $Ct^n \sin(\omega t + \phi)$.



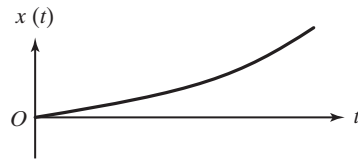
Stable system

(a)



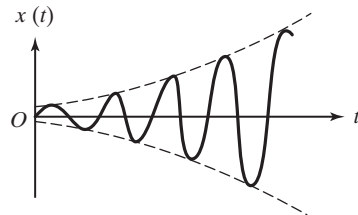
Asymptotically stable system

(b)



Unstable system (with divergent instability)

(c)



Unstable system (with flutter instability)

(d)

FIGURE 2.48 Different types of stability.**EXAMPLE 2.21****Stability of a System**

Consider a uniform rigid bar, of mass m and length l , pivoted at one end and connected symmetrically by two springs at the other end, as shown in Fig. 2.49. Assuming that the springs are unstretched when the bar is vertical, derive the equation of motion of the system for small angular displacements (θ) of the bar about the pivot point, and investigate the stability behavior of the system.

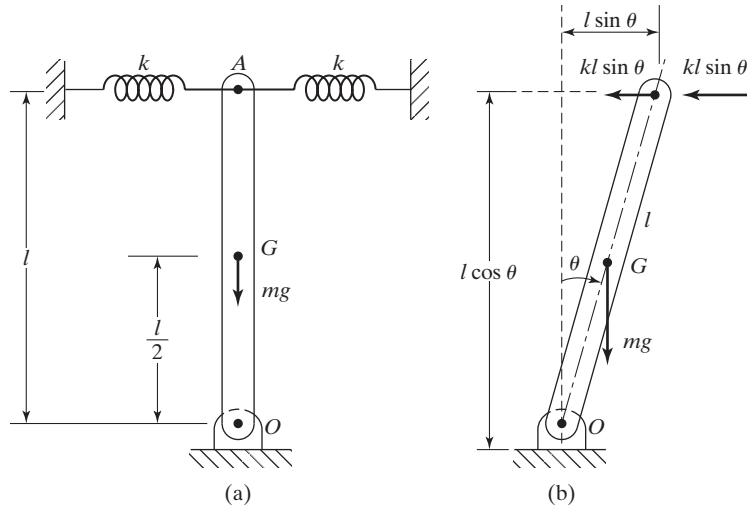


FIGURE 2.49 Stability of a rigid bar.

Solution: When the bar is displaced by an angle θ , the spring force in each spring is $kl \sin \theta$; the total spring force is $2kl \sin \theta$. The gravity force $W = mg$ acts vertically downward through the center of gravity, G . The moment about the point of rotation O due to the angular acceleration $\ddot{\theta}$ is $J_0 \ddot{\theta} = (ml^2/3) \ddot{\theta}$. Thus the equation of motion of the bar, for rotation about the point O , can be written as

$$\frac{ml^2}{3} \ddot{\theta} + (2kl \sin \theta)l \cos \theta - W \frac{l}{2} \sin \theta = 0 \quad (\text{E.1})$$

For small oscillations, Eq. (E.1) reduces to

$$\frac{ml^2}{3} \ddot{\theta} + 2kl^2 \theta - \frac{Wl}{2} \theta = 0 \quad (\text{E.2})$$

or

$$\ddot{\theta} + \alpha^2 \theta = 0 \quad (\text{E.3})$$

where

$$\alpha^2 = \left(\frac{12kl^2 - 3Wl}{2ml^2} \right) \quad (\text{E.4})$$

The characteristic equation is given by

$$s^2 + \alpha^2 = 0 \quad (\text{E.5})$$

and hence the solution of Eq. (E.2) depends on the sign of α^2 as indicated below.

Case 1. When $(12kl^2 - 3Wl)/2ml^2 > 0$, the solution of Eq. (E.2) represents a stable system with stable oscillations and can be expressed as

$$\theta(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \quad (\text{E.6})$$

where A_1 and A_2 are constants and

$$\omega_n = \left(\frac{(12kl^2 - 3Wl)}{2ml^2} \right)^{1/2} \quad (\text{E.7})$$

Case 2. When $(12kl^2 - 3Wl)/2ml^2 = 0$, Eq. (E.2) reduces to $\ddot{\theta} = 0$ and the solution can be obtained directly by integrating twice as

$$\theta(t) = C_1 t + C_2 \quad (\text{E.8})$$

For the initial conditions $\theta(t = 0) = \theta_0$ and $\dot{\theta}(t = 0) = \dot{\theta}_0$, the solution becomes

$$\theta(t) = \dot{\theta}_0 t + \theta_0 \quad (\text{E.9})$$

Equation (E.9) shows that the system is unstable with the angular displacement increasing linearly at a constant velocity $\dot{\theta}_0$. However, if $\dot{\theta}_0 = 0$, Eq. (E.9) denotes a stable or static equilibrium position with $\theta = \theta_0$ —that is, the pendulum remains in its original position, defined by $\theta = \theta_0$.

Case 3. When $(12kl^2 - 3Wl)/2ml^2 < 0$, the solution of Eq. (E.2) can be expressed as

$$\theta(t) = B_1 e^{\alpha t} + B_2 e^{-\alpha t} \quad (\text{E.10})$$

where B_1 and B_2 are constants. For the initial conditions $\theta(t = 0) = \theta_0$ and $\dot{\theta}(t = 0) = \dot{\theta}_0$, Eq. (E.10) becomes

$$\theta(t) = \frac{1}{2\alpha} [(\alpha\theta_0 + \dot{\theta}_0)e^{\alpha t} + (\alpha\theta_0 - \dot{\theta}_0)e^{-\alpha t}] \quad (\text{E.11})$$

Equation (E.11) shows that $\theta(t)$ increases exponentially with time; hence the motion is unstable. The physical reason for this is that the restoring moment due to the spring ($2kl^2\theta$), which tries to bring the system to the equilibrium position, is less than the nonrestoring moment due to gravity $[-W(l/2)\theta]$, which tries to move the mass away from the equilibrium position.

■

2.12 Examples Using MATLAB

EXAMPLE 2.22

Variations of Natural Frequency and Period with Static Deflection

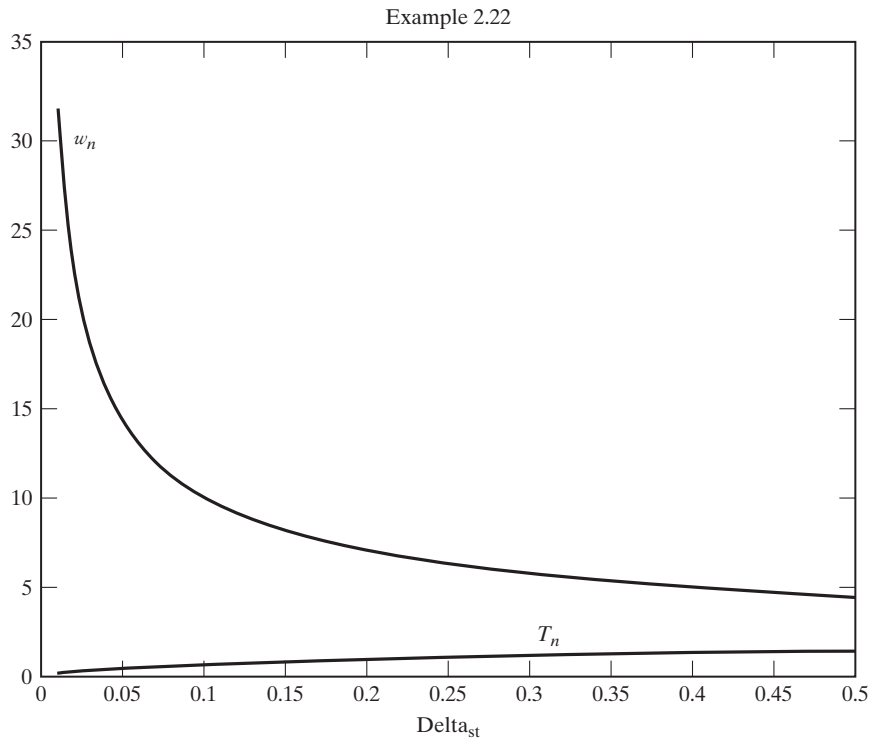
Plot the variations of the natural frequency and the time period with static deflection of an undamped system using MATLAB.

Solution: The natural frequency (ω_n) and the time period (τ_n) are given by Eqs. (2.28) and (2.30):

$$\omega_n = \left(\frac{g}{\delta_{st}} \right)^{1/2}, \quad \tau_n = 2\pi \left(\frac{\delta_{st}}{g} \right)^{1/2}$$

Using $g = 9.81 \text{ m/s}^2$, ω_n and τ_n are plotted over the range of $\delta_{st} = 0$ to 0.5 using a MATLAB program.

```
% Ex2_22.m
g = 9.81;
for i = 1: 101
    t(i) = 0.01 + (0.5-0.01) * (i-1)/100;
    w(i) = (g/t(i))^0.5;
    tao(i) = 2 * pi * (t(i)/g)^0.5;
end
plot(t,w);
gtext('w_n');
hold on;
plot(t, tao);
gtext('T_n');
xlabel('Delta_s_t');
title('Example 2.17');
```



Variations of natural frequency and time period.

EXAMPLE 2.23

Free-Vibration Response of a Spring-Mass System

A spring-mass system with a mass of 3500 kg and stiffness 85,000 N/m is subject to an initial displacement of $x_0 = 7.5 \text{ cm}$ and an initial velocity of $\dot{x}_0 = 10 \text{ cm/s}$. Plot the time variations of the mass's displacement, velocity, and acceleration using MATLAB.

Solution: The displacement of an undamped system can be expressed as (see Eq. (2.23)):

$$x(t) = A_0 \sin(\omega_n t + \phi_0) \quad (\text{E.1})$$

where

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{85,000}{3500}} = 4.928 \text{ rad/s}$$

$$A_0 = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} = \left[(0.075)^2 + \left(\frac{0.1}{4.928} \right)^2 \right]^{1/2} = 0.0776 \text{ m}$$

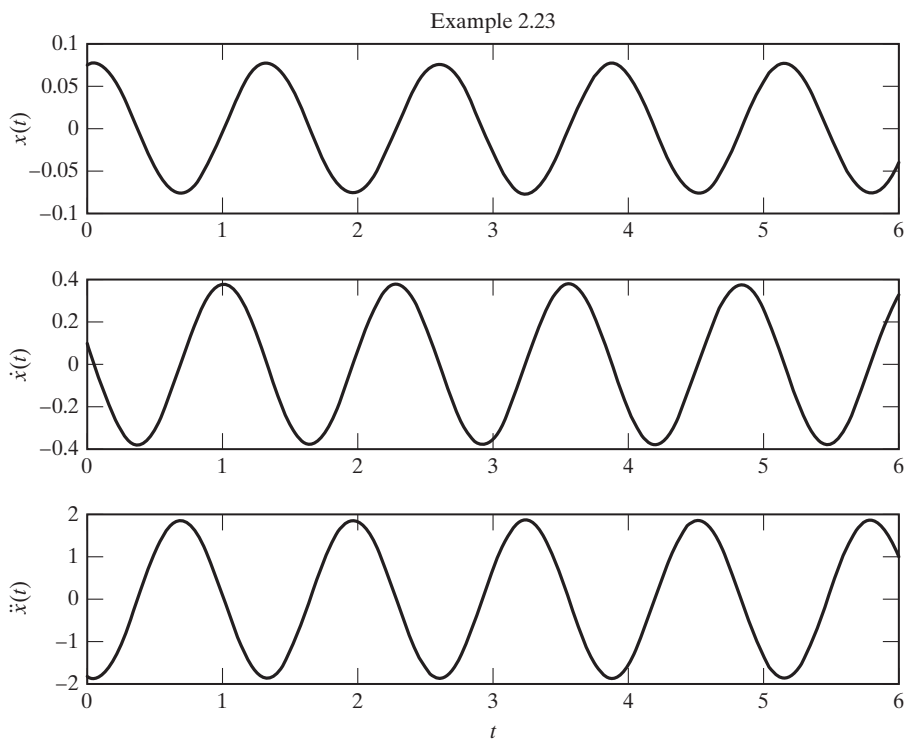
$$\phi_0 = \tan^{-1} \left(\frac{x_0 \omega_n}{\dot{x}_0} \right) = \tan^{-1} \left(\frac{(0.075)(4.928)}{0.1} \right) = 74.86^\circ = 1.3065 \text{ rad}$$

Thus Eq. (E.1) yields

$$x(t) = 0.0776 \sin(4.928t + 1.3096) \text{ m} \quad (\text{E.2})$$

$$\dot{x}(t) = 0.3824 \cos(4.928t + 1.3096) \text{ m/s} \quad (\text{E.3})$$

$$\ddot{x}(t) = -1.885 \sin(4.92t + 1.3096) \text{ m/s}^2 \quad (\text{E.4})$$



Response of an undamped system.

Equations (E.2)–(E.4) are plotted using MATLAB in the range $t = 0$ to 6 seconds.

```
% Ex2_23.m
for i = 1: 101
    t(i) = 6 * (i-1)/100;
    x(i) = 0.0776 * sin(4.928 * t(i) + 1.3096);
    x1(i) = 0.3824 * cos(4.928 * t(i) + 1.3096);
    x2(i) = -1.885 * sin(4.928 * t(i) + 1.3096);
end
subplot (311);
plot (t,x);
ylabel ('x(t)');
title ('Example 2.23');
subplot (312);
plot (t,x1);
ylabel ('x^(.)(t)');
subplot (313);
plot (t,x2);
xlabel ('t');
ylabel ('x^(.)(t)');
```

■

EXAMPLE 2.24

Free-Vibration Response of a System with Coulomb Damping

Find the free-vibration response of a spring-mass system subject to Coulomb damping for the following initial conditions: $x(0) = 0.5$ m, $\dot{x}(0) = 0$.

Data: $m = 10$ kg, $k = 200$ N/m, $\mu = 0.5$

Solution: The equation of motion can be expressed as

$$m\ddot{x} + \mu mg \operatorname{sgn}(\dot{x}) + kx = 0 \quad (\text{E.1})$$

In order to solve the second-order differential equation, Eq. (E.1), using the Runge-Kutta method (see Appendix F), we rewrite Eq. (E.1) as a set of two first-order differential equations as follows:

$$\begin{aligned} x_1 &= x, & x_2 &= \dot{x}_1 = \dot{x} \\ \dot{x}_1 &= x_2 \equiv f_1(x_1, x_2) \end{aligned} \quad (\text{E.2})$$

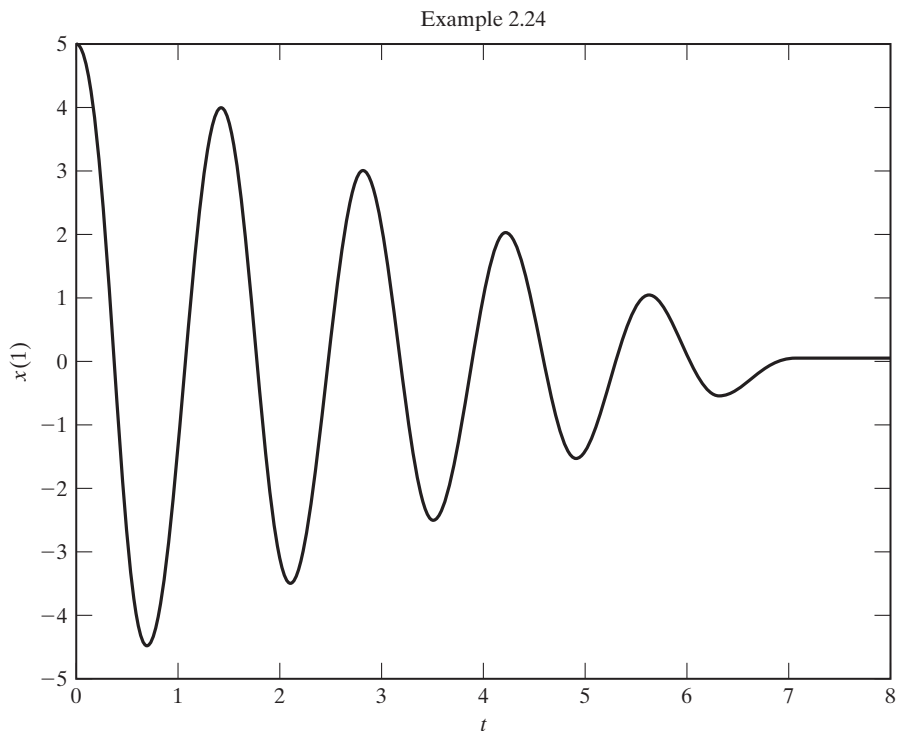
$$\dot{x}_2 = -\mu g \operatorname{sgn}(x_2) - \frac{k}{m}x_1 \equiv f_2(x_1, x_2) \quad (\text{E.3})$$

Equations (E.2) and (E.3) can be expressed in matrix notation as

$$\dot{\vec{X}} = \vec{f}(\vec{X}) \quad (\text{E.4})$$

where

$$\vec{X} = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}, \quad \vec{f} = \begin{Bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{Bmatrix}, \quad \vec{X}(t=0) = \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix}$$



Solution of Eq. (E4):

The MATLAB program **ode23** is used to find the solution of Eq. (E.4) as shown below.

```
% Ex2_24.m
% This program will use dfunc1.m
tspan = [0: 0.05: 8];
x0 = [5.0; 0.0];
[t, x] = ode23 ('dfunc1', tspan, x0);
plot (t, x(:, 1));
xlabel ('t');
ylabel ('x(1)');
title ('Example 2.19');

% dfunc1.m
function f = dfunc1 (t, x)
f = zeros (2, 1);
f(1) = x(2);
f(2) = -0.5 * 9.81 * sign(x(2)) - 200 * x(1) / 10;
```

■

EXAMPLE 2.25**Free-Vibration Response of a Viscously Damped System Using MATLAB**

Develop a general-purpose MATLAB program, called **Program2.m**, to find the free-vibration response of a viscously damped system. Use the program to find the response of a system with the following data:

$$m = 450.0, \quad k = 26519.2, \quad c = 1000.0, \quad x_0 = 0.539657, \quad \dot{x}_0 = 1.0$$

Solution: **Program2.m** is developed to accept the following input data:

m = mass
 k = spring stiffness
 c = damping constant
 x_0 = initial displacement
 \dot{x}_0 = initial velocity
 n = number of time steps at which values of $x(t)$ are to be found
 delt = time interval between consecutive time steps (Δt)

The program gives the following output:

step number i , time (i), $x(i)$, $\dot{x}(i)$, $\ddot{x}(i)$

The program also plots the variations of x , \dot{x} , and \ddot{x} with time.

```
>> program2
Free vibration analysis of a single degree of freedom analysis

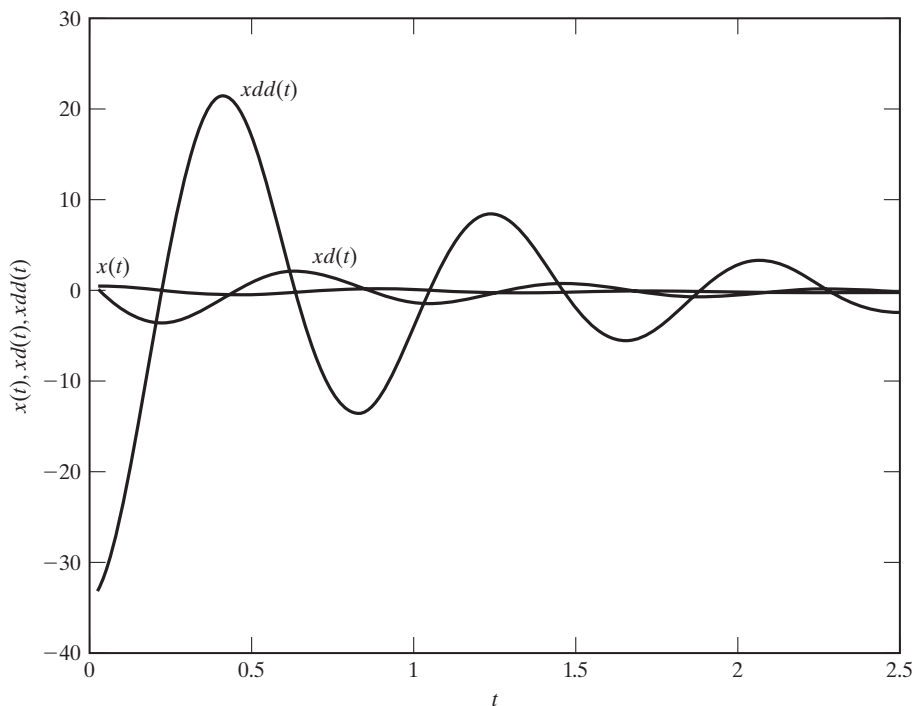
Data:

m=          4.50000000e+002
k=          2.65192000e+004
c=          1.00000000e+003
x0=          5.39657000e-001
xd0=         1.00000000e+000
n=           100
delt=        2.50000000e-002

system is under damped

Results:

   i   time(i)           x(i)           xd(i)           xdd(i)
   1  2.500000e-002    5.540992e-001    1.596159e-001    -3.300863e+001
   2  5.000000e-002    5.479696e-001    -6.410545e-001    -3.086813e+001
   3  7.500000e-002    5.225989e-001    -1.375559e+000    -2.774077e+001
   4  1.000000e-001    4.799331e-001    -2.021239e+000    -2.379156e+001
   5  1.250000e-001    4.224307e-001    -2.559831e+000    -1.920599e+001
   6  1.500000e-001    3.529474e-001    -2.977885e+000    -1.418222e+001
   .
   .
   .
  96  2.400000e+000    2.203271e-002    2.313895e-001    -1.812621e+000
  97  2.425000e+000    2.722809e-002    1.834092e-001    -2.012170e+000
  98  2.450000e+000    3.117018e-002    1.314707e-001    -2.129064e+000
  99  2.475000e+000    3.378590e-002    7.764312e-002    -2.163596e+000
 100  2.500000e+000    3.505350e-002    2.395118e-002    -2.118982e+000
```



Variations of x , \dot{x} , and \ddot{x} .

CHAPTER SUMMARY

We considered the equations of motion and their solutions for the free vibration of undamped and damped single-degree-of-freedom systems. Four different methods—namely, Newton’s second law of motion, D’Alembert’s principle, the principle of virtual displacements, and the principle of conservation of energy—were presented for deriving the equation of motion of undamped systems. Both translational and torsional systems were considered. The free-vibration solutions have been presented for undamped systems. The equation of motion, in the form of a first-order differential equation, was considered for a mass-damper system (with no spring), and the idea of time constant was introduced.

The free-vibration solution of viscously damped systems was presented along with the concepts of underdamped, overdamped, and critically damped systems. The free-vibration solutions of systems with Coulomb and hysteretic damping were also considered. The graphical representation of characteristic roots in the complex plane and the corresponding solutions were explained. The effects of variation of the parameters m , c , and k on the characteristic roots and their representations using root locus plots were also considered. The identification of the stability status of a system was also explained.

Now that you have finished this chapter, you should be able to answer the review questions and solve the problems that follow.

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REVIEW QUESTIONS

2.1 Give brief answers to the following:

- 1. Suggest a method for determining the damping constant of a highly damped vibrating system that uses viscous damping.
- 2. Can you apply the results of Section 2.2 to systems where the restoring force is not proportional to the displacement—that is, where k is not a constant?

3. State the parameters corresponding to m , c , k , and x for a torsional system.
4. What effect does a decrease in mass have on the frequency of a system?
5. What effect does a decrease in the stiffness of the system have on the natural period?
6. Why does the amplitude of free vibration gradually diminish in practical systems?
7. Why is it important to find the natural frequency of a vibrating system?
8. How many arbitrary constants must a general solution to a second-order differential equation have? How are these constants determined?
9. Can the energy method be used to find the differential equation of motion of all single-degree-of-freedom systems?
10. What assumptions are made in finding the natural frequency of a single-degree-of-freedom system using the energy method?
11. Is the frequency of a damped free vibration smaller or greater than the natural frequency of the system?
12. What is the use of the logarithmic decrement?
13. Is hysteresis damping a function of the maximum stress?
14. What is critical damping, and what is its importance?
15. What happens to the energy dissipated by damping?
16. What is equivalent viscous damping? Is the equivalent viscous-damping factor a constant?
17. What is the reason for studying the vibration of a single-degree-of-freedom system?
18. How can you find the natural frequency of a system by measuring its static deflection?
19. Give two practical applications of a torsional pendulum.
20. Define these terms: damping ratio, logarithmic decrement, loss coefficient, and specific damping capacity.
21. In what ways is the response of a system with Coulomb damping different from that of systems with other types of damping?
22. What is complex stiffness?
23. Define the hysteresis damping constant.
24. Give three practical applications of the concept of center of percussion.
25. What is the order of the equation of motion given by $m\ddot{v} + c\dot{v} = 0$?
26. Define the time constant.
27. What is a root locus plot?
28. What is the significance of $c < 0$?
29. What is a time-invariant system?

2.2 Indicate whether each of the following statements is true or false:

1. The amplitude of an undamped system will not change with time.
2. A system vibrating in air can be considered a damped system.
3. The equation of motion of a single-degree-of-freedom system will be the same whether the mass moves in a horizontal plane or an inclined plane.
4. When a mass vibrates in a vertical direction, its weight can always be ignored in deriving the equation of motion.
5. The principle of conservation of energy can be used to derive the equation of motion of both damped and undamped systems.
6. The damped frequency can in some cases be larger than the undamped natural frequency of the system.
7. The damped frequency can be zero in some cases.

8. The natural frequency of vibration of a torsional system is given by $\sqrt{\frac{k}{m}}$, where k and m denote the torsional spring constant and the polar mass moment of inertia, respectively.
9. Rayleigh's method is based on the principle of conservation of energy.
10. The final position of the mass is always the equilibrium position in the case of Coulomb damping.
11. The undamped natural frequency of a system is given by $\sqrt{g/\delta_{st}}$, where δ_{st} is the static deflection of the mass.
12. For an undamped system, the velocity leads the displacement by $\pi/2$.
13. For an undamped system, the velocity leads the acceleration by $\pi/2$.
14. Coulomb damping can be called constant damping.
15. The loss coefficient denotes the energy dissipated per radian per unit strain energy.
16. The motion diminishes to zero in both underdamped and overdamped cases.
17. The logarithmic decrement can be used to find the damping ratio.
18. The hysteresis loop of the stress-strain curve of a material causes damping.
19. The complex stiffness can be used to find the damping force in a system with hysteresis damping.
20. Motion in the case of hysteresis damping can be considered harmonic.
21. In the s -plane, the locus corresponding to constant natural frequency will be a circle.
22. The characteristic equation of a single-degree-of-freedom system can have one real root and one complex root.

2.3 Fill in the blanks with proper words:

1. The free vibration of an undamped system represents interchange of _____ and _____ energies.
2. A system undergoing simple harmonic motion is called a(n) _____ oscillator.
3. The mechanical clock represents a(n) _____ pendulum.
4. The center of _____ can be used advantageously in a baseball bat.
5. With viscous and hysteresis damping, the motion _____ forever, theoretically.
6. The damping force in Coulomb damping is given by _____.
7. The _____ coefficient can be used to compare the damping capacity of different engineering materials.
8. Torsional vibration occurs when a(n) _____ body oscillates about an axis.
9. The property of _____ damping is used in many practical applications, such as large guns.
10. The logarithmic decrement denotes the rate at which the _____ of a free damped vibration decreases.
11. Rayleigh's method can be used to find the _____ frequency of a system directly.
12. Any two successive displacements of the system, separated by a cycle, can be used to find the _____ decrement.
13. The damped natural frequency (ω_d) can be expressed in terms of the undamped natural frequency (ω_n) as _____.
14. The time constant denotes the time at which the initial response reduces by _____ %.
15. The term e^{-2t} decays _____ than the term e^{-t} as time t increases.
16. In the s -plane, lines parallel to real axis denote systems having different _____ frequencies.

2.4 Select the most appropriate answer out of the multiple choices given:

1. The natural frequency of a system with mass m and stiffness k is given by:
 - a. $\frac{k}{m}$
 - b. $\sqrt{\frac{k}{m}}$
 - c. $\sqrt{\frac{m}{k}}$
2. In Coulomb damping, the amplitude of motion is reduced in each cycle by:
 - a. $\frac{\mu N}{k}$
 - b. $\frac{2\mu N}{k}$
 - c. $\frac{4\mu N}{k}$
3. The amplitude of an undamped system subject to an initial displacement 0 and initial velocity \dot{x}_0 is given by:
 - a. \dot{x}_0
 - b. $\dot{x}_0 \omega_n$
 - c. $\frac{\dot{x}_0}{\omega_n}$
4. The effect of the mass of the spring can be accounted for by adding the following fraction of its mass to the vibrating mass:
 - a. $\frac{1}{2}$
 - b. $\frac{1}{3}$
 - c. $\frac{4}{3}$
5. For a viscous damper with damping constant c , the damping force is:
 - a. $c\dot{x}$
 - b. cx
 - c. $c\ddot{x}$
6. The relative sliding of components in a mechanical system causes:
 - a. dry-friction damping
 - b. viscous damping
 - c. hysteresis damping
7. In torsional vibration, the displacement is measured in terms of a(n):
 - a. linear coordinate
 - b. angular coordinate
 - c. force coordinate
8. The damping ratio, in terms of the damping constant c and critical damping constant (c_c), is given by:
 - a. $\frac{c_c}{c}$
 - b. $\frac{c}{c_c}$
 - c. $\sqrt{\frac{c}{c_c}}$
9. The amplitude of an underdamped system subject to an initial displacement x_0 and initial velocity 0 is given by:
 - a. x_0
 - b. $2x_0$
 - c. $x_0 \omega_n$
10. The phase angle of an undamped system subject to an initial displacement x_0 and initial velocity 0 is given by:
 - a. x_0
 - b. $2x_0$
 - c. 0
11. The energy dissipated due to viscous damping is proportional to the following power of the amplitude of motion:
 - a. 1
 - b. 2
 - c. 3
12. For a critically damping system, the motion will be:
 - a. periodic
 - b. aperiodic
 - c. harmonic
13. The energy dissipated per cycle in viscous damping with damping constant c during the simple harmonic motion $x(t) = X \sin \omega_d t$, is given by:
 - a. $\pi c \omega_d X^2$
 - b. $\pi \omega_d X^2$
 - c. $\pi c \omega_d X$
14. For a vibrating system with a total energy W and a dissipated energy ΔW per cycle, the specific damping capacity is given by:
 - a. $\frac{W}{\Delta W}$
 - b. $\frac{\Delta W}{W}$
 - c. ΔW
15. If the characteristic roots have positive real values, the system response will be:
 - a. stable
 - b. unstable
 - c. asymptotically stable

16. The frequency of oscillation of the response of a system will be higher if the imaginary part of the roots is:
 - a. smaller
 - b. zero
 - c. larger
17. If the characteristic roots have a zero imaginary part, the response of the system will be:
 - a. oscillatory
 - b. nonoscillatory
 - c. steady
18. The shape of the root locus of a single-degree-of-freedom system for $0 \leq \zeta \leq 1$ is:
 - a. circular
 - b. horizontal line
 - c. radial line
19. The shape of the root locus of a single-degree-of-freedom system as k is varied is:
 - a. vertical and horizontal lines
 - b. circular arc
 - c. radial lines

2.5 Match the following for a single-degree-of-freedom system with $m = 1$, $k = 2$, and $c = 0.5$:

1. Natural frequency, ω_n	a. 1.3919
2. Linear frequency, f_n	b. 2.8284
3. Natural time period, τ_n	c. 1.1287
4. Damped frequency, ω_d	d. 0.2251
5. Critical damping constant, c_c	e. 0.1768
6. Damping ratio, ζ	f. 4.4429
7. Logarithmic decrement, δ	g. 1.4142

2.6 Match the following for a mass $m = 5$ kg moving with velocity $v = 10$ m/s:

Damping force	Type of damper
1. 20 N	a. Coulomb damping with a coefficient of friction of 0.3
2. 1.5 N	b. Viscous damping with a damping coefficient 1 N-s/m
3. 30 N	c. Viscous damping with a damping coefficient 2 N-s/m
4. 25 N	d. Hysteretic damping with a hysteretic damping coefficient of 12 N/m at a frequency of 4 rad/s
5. 10 N	e. Quadratic damping (force = av^2) with damping constant $a = 0.25$ N-s ² /m ²

2.7 Match the following characteristics of the s -plane:

Locus	Significance
1. Concentric circles	a. Different values of damped natural frequency
2. Lines parallel to real axis	b. Different values of reciprocals of time constant
3. Lines parallel to imaginary axis	c. Different values of damping ratio
4. Radial lines through origin	d. Different values of natural frequency

2.8 Match the following terms related to stability of systems:

Type of system	Nature of free-vibration response as time approaches infinity
1. Asymptotically stable	a. Neither decays nor grows
2. Unstable	b. Grows with oscillations
3. Stable	c. Grows without oscillations
4. Divergent instability	d. Approaches zero
5. Flutter instability	e. Grows without bound

PROBLEMS

Section 2.2 Free Vibration of an Undamped Translational System

- 2.1** An industrial press is mounted on a rubber pad to isolate it from its foundation. If the rubber pad is compressed 5 mm by the self weight of the press, find the natural frequency of the system.
- 2.2** A spring-mass system has a natural period of 0.21 s. What will be the new period if the spring constant is (a) increased by 50% and (b) decreased by 50%?
- 2.3** A spring-mass system has a natural frequency of 10 Hz. When the spring constant is reduced by 800 N/m, the frequency is altered by 45%. Find the mass and spring constant of the original system.
- 2.4** A helical spring, when fixed at one end and loaded at the other, requires a force of 100 N to produce an elongation of 10 mm. The ends of the spring are now rigidly fixed, one end vertically above the other, and a mass of 10 kg is attached at the middle point of its length. Determine the time taken to complete one vibration cycle when the mass is set vibrating in the vertical direction.
- 2.5** An air-conditioning chiller unit weighing 10 kN is to be supported by four air springs (Fig. 2.50). Design the air springs such that the natural frequency of vibration of the unit lies between 5 rad/s and 10 rad/s.

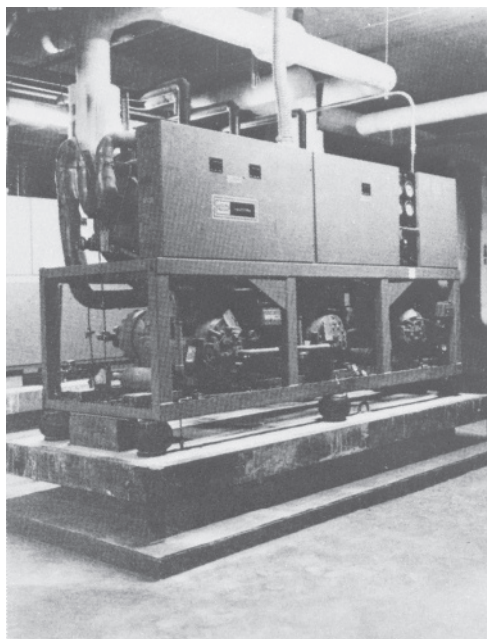


FIGURE 2.50 Air-conditioning chiller unit.
(Courtesy of *Sound and Vibration*.)

- 2.6 The maximum velocity attained by the mass of a simple harmonic oscillator is 10 cm/s, and the period of oscillation is 2 s. If the mass is released with an initial displacement of 2 cm, find (a) the amplitude, (b) the initial velocity, (c) the maximum acceleration, and (d) the phase angle.
- 2.7 Three springs and a mass are attached to a rigid, weightless bar PQ as shown in Fig. 2.51. Find the natural frequency of vibration of the system.

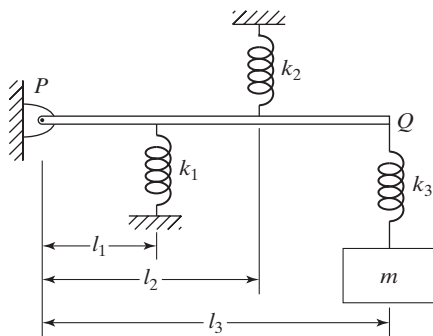


FIGURE 2.51 Rigid bar with springs and mass attached.

- 2.8 An automobile having a mass of 2000 kg deflects its suspension springs 0.02 m under static conditions. Determine the natural frequency of the automobile in the vertical direction by assuming damping to be negligible.
- 2.9 Find the natural frequency of vibration of a spring-mass system arranged on an inclined plane, as shown in Fig. 2.52.

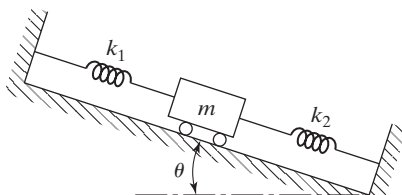


FIGURE 2.52 Spring-mass system on inclined plane.

- 2.10 A loaded mine cart, with a mass of 2000 kg, is being lifted by a frictionless pulley and a wire rope, as shown in Fig. 2.53. Find the natural frequency of vibration of the cart in the given position.

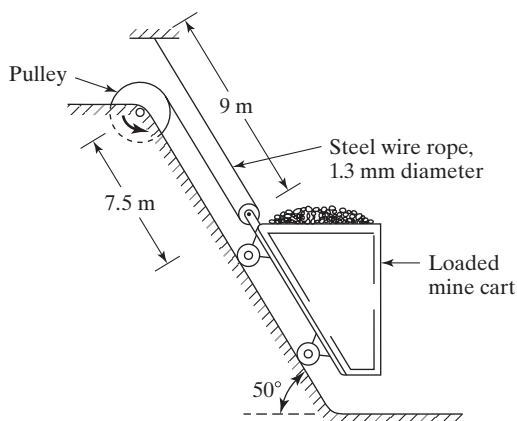


FIGURE 2.53 Mine cart on inclined plane.

- 2.11** A rotating machine weighing 1000 N (including the foundation block) is isolated by supporting it on six identical helical springs, as shown in Fig. 2.54. Design the springs so that the unit can be used in an environment in which the vibratory frequency ranges from 0 to 5 Hz. **Hint:** Design the spring for a natural frequency of at least 10 Hz.

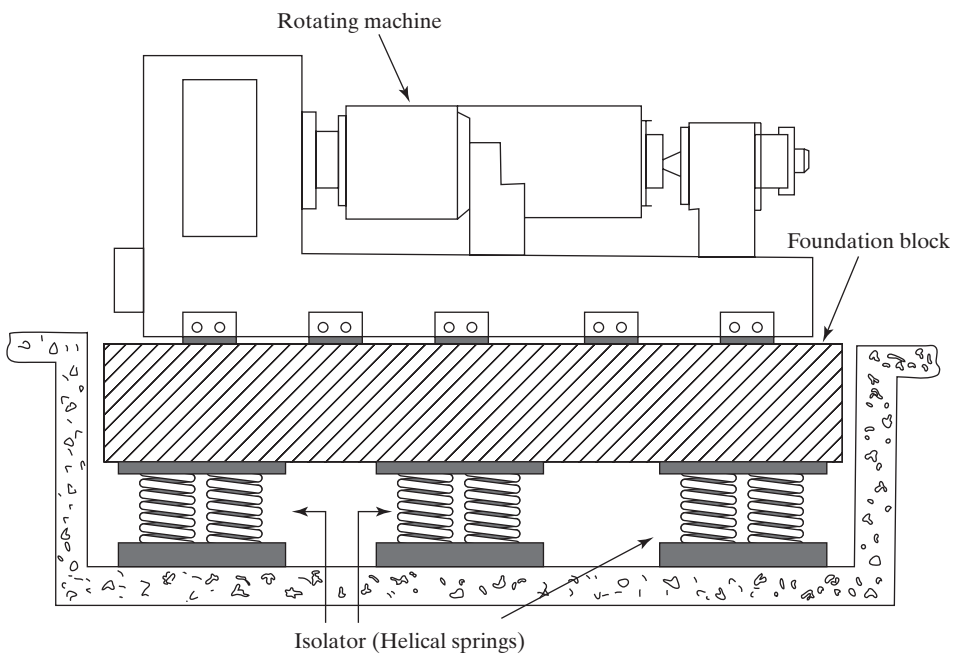


FIGURE 2.54 Isolated rotating machine.

- 2.12** Find the natural frequency of the system shown in Fig. 2.55 with and without the springs k_1 and k_2 in the middle of the elastic beam.

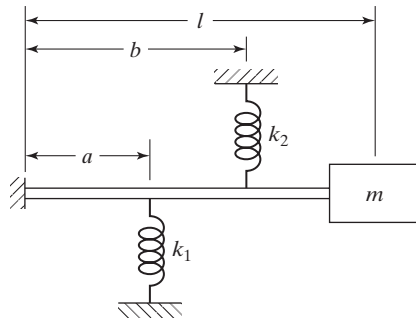


FIGURE 2.55 Elastic beam with springs and mass attached.

- 2.13** Find the natural frequency of the pulley system shown in Fig. 2.56 by neglecting the friction and the masses of the pulleys.

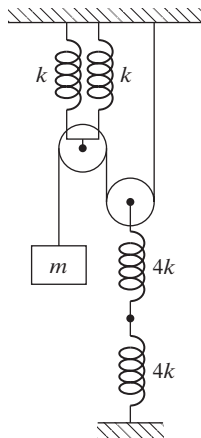


FIGURE 2.56 Pulley system with springs and mass.

- 2.14** A weight W is supported by three frictionless and massless pulleys and a spring of stiffness k , as shown in Fig. 2.57. Find the natural frequency of vibration of weight W for small oscillations.
- 2.15** A rigid block of mass M is mounted on four elastic supports, as shown in Fig. 2.58. A mass m drops from a height l and adheres to the rigid block without rebounding. If the spring constant of each elastic support is k , find the natural frequency of vibration of the system (a) without the mass m , and (b) with the mass m . Also find the resulting motion of the system in case (b).

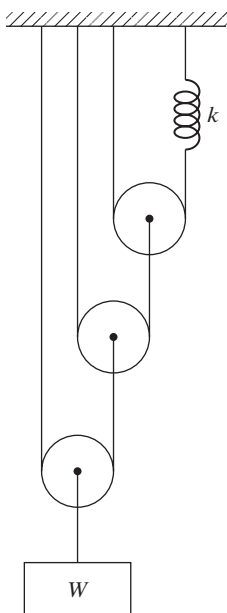


FIGURE 2.57 Three pulleys with spring and mass.

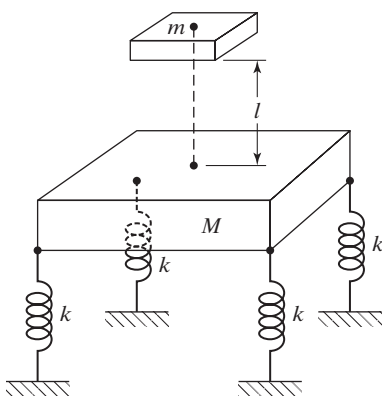


FIGURE 2.58 Mass dropping on a spring-supported rigid block.

- 2.16** A sledgehammer strikes an anvil with a velocity of 15 m/s (Fig. 2.59). The hammer and the anvil have a mass of 6 kg and 50 kg, respectively. The anvil is supported on four springs, each of stiffness $k = 17.5 \text{ kN/m}$. Find the resulting motion of the anvil (a) if the hammer remains in contact with the anvil and (b) if the hammer does not remain in contact with the anvil after the initial impact.

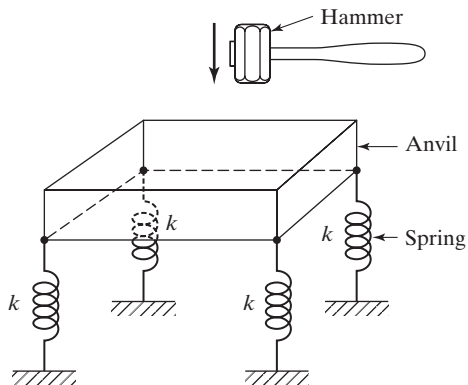


FIGURE 2.59 Hammer striking an anvil.

- 2.17 Derive the expression for the natural frequency of the system shown in Fig. 2.60. Note that the load W is applied at the tip of beam 1 and midpoint of beam 2.

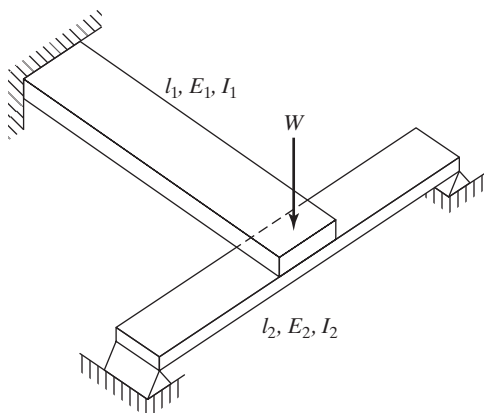


FIGURE 2.60 Load applied to a two-beam system.

- 2.18 A heavy machine weighing 9810 N is being lowered vertically down by a winch at a uniform velocity of 2 m/s. The steel cable supporting the machine has a diameter of 0.01 m. The winch is suddenly stopped when the steel cable's length is 20 m. Find the period and amplitude of the ensuing vibration of the machine.
- 2.19 The natural frequency of a spring-mass system is found to be 2 Hz. When an additional mass of 1 kg is added to the original mass m , the natural frequency is reduced to 1 Hz. Find the spring constant k and the mass m .
- 2.20 A heavy machine tool is transported by a helicopter. The crate containing the machine tool weighs 12,000 N and is supported by a steel cable, of length 5 m and diameter d m, as shown

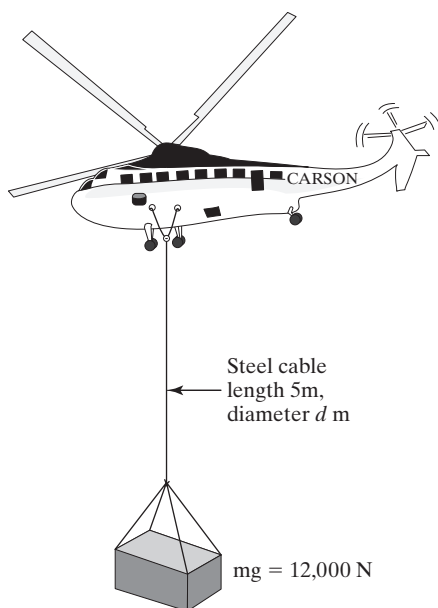


FIGURE 2.61 Helicopter carrying a machine tool.
(Courtesy of Carson Helicopters Inc.)

in Fig. 2.61. If the natural time period of the crate is found to be 0.1 s, find the diameter of the steel cable.

- 2.21** Four weightless rigid links and a spring are arranged to support a weight W in two different ways, as shown in Fig. 2.62. Determine the natural frequencies of vibration of the two arrangements.

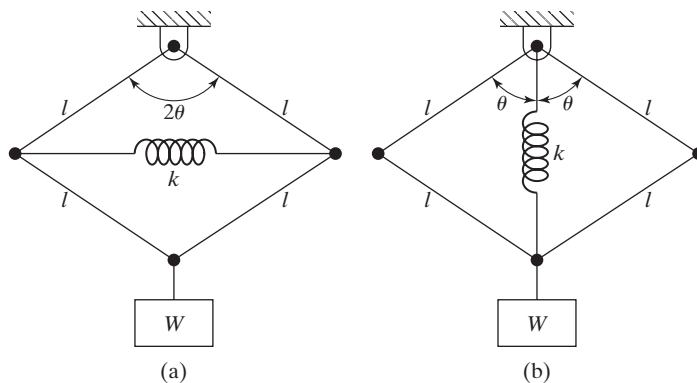


FIGURE 2.62 Two arrangements to support a weight.

- 2.22** A scissors jack is used to lift a load W . The links of the jack are rigid and the collars can slide freely on the shaft against the springs of stiffnesses k_1 and k_2 (see Fig. 2.63). Find the natural frequency of vibration of the weight in the vertical direction.

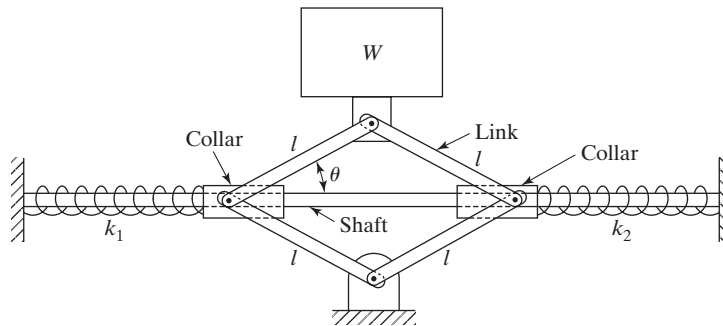


FIGURE 2.63 Weight lifted by scissors jack.

- 2.23** A weight is suspended using six rigid links and two springs in two different ways, as shown in Fig. 2.64. Find the natural frequencies of vibration of the two arrangements.

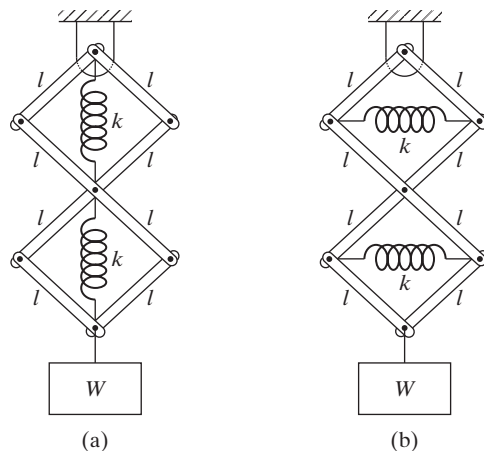


FIGURE 2.64 Weight suspended to rigid links and springs.

- 2.24** Figure 2.65 shows a small mass m restrained by four linearly elastic springs, each of which has an unstretched length l , and an angle of orientation of 45° with respect to the x -axis. Determine the equation of motion for small displacements of the mass in the x direction.
- 2.25** A mass m is supported by two sets of springs oriented at 30° and 120° with respect to the X -axis, as shown in Fig. 2.66. A third pair of springs, each with a stiffness of k_3 , is to be designed so as to make the system have a constant natural frequency while vibrating in any direction x . Determine the necessary spring stiffness k_3 and the orientation of the springs with respect to the X -axis.

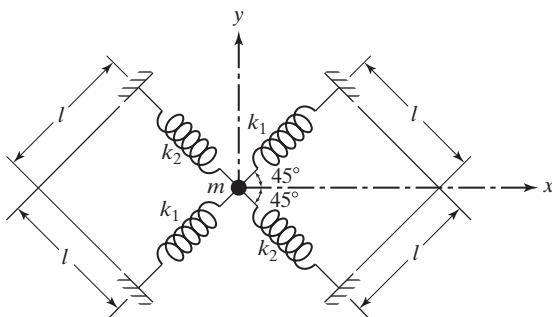


FIGURE 2.65 Mass restrained by four springs.

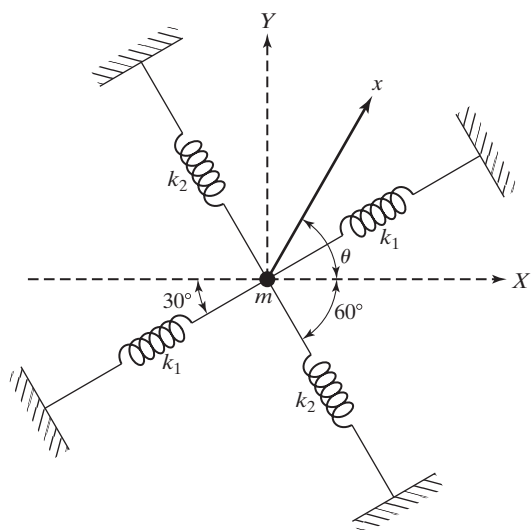


FIGURE 2.66 Mass connected by two sets of springs.

- 2.26** A mass m is attached to a cord that is under a tension T , as shown in Fig. 2.67. Assuming that T remains unchanged when the mass is displaced normal to the cord, (a) write the differential equation of motion for small transverse vibrations and (b) find the natural frequency of vibration.

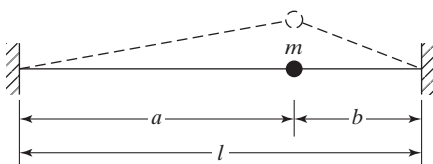


FIGURE 2.67 Mass attached to a cord.

- 2.27** A bungee jumper, of mass 70 kg, ties one end of an elastic rope of length 65 m and stiffness 1.75 kN/m to a bridge and the other end to himself and jumps from the bridge (Fig. 2.68). Assuming the bridge to be rigid, determine the vibratory motion of the jumper about his static equilibrium position.

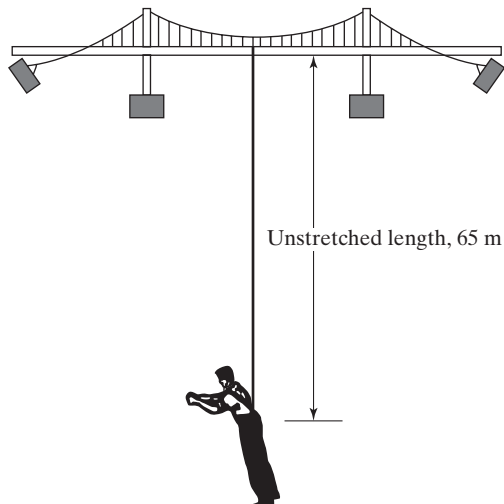


FIGURE 2.68 Bungee jumper jumping from a bridge.

- 2.28** An acrobat, of mass 50 kg, walks on a tightrope, as shown in Fig. 2.69. If the natural frequency of vibration in the given position, in vertical direction, is 10 rad/s, find the tension in the rope.

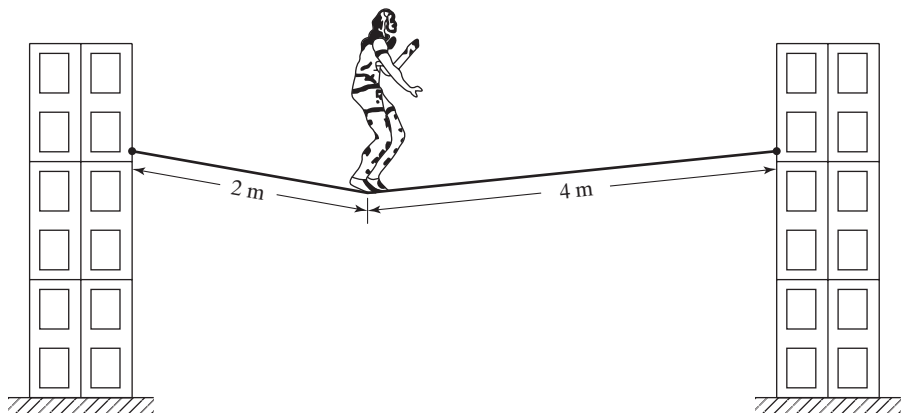


FIGURE 2.69 Acrobat walking on a tight rope.

- 2.29** The schematic diagram of a centrifugal governor is shown in Fig. 2.70. The length of each rod is l , the mass of each ball is m , and the free length of the spring is h . If the shaft speed is ω , determine the equilibrium position and the frequency for small oscillations about this position.

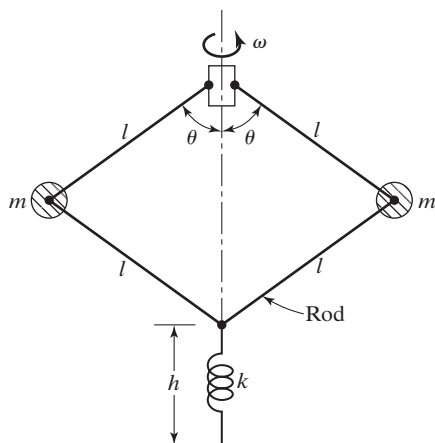


FIGURE 2.70 Centrifugal governor.

- 2.30** In the Hartnell governor shown in Fig. 2.71, the stiffness of the spring is 10^4 N/m and the weight of each ball is 25 N. The length of the ball arm is 20 cm, and that of the sleeve arm is 12 cm. The distance between the axis of rotation and the pivot of the bell crank lever is 16 cm. The spring is compressed by 1 cm when the ball arm is vertical. Find (a) the speed of the governor at which the ball arm remains vertical and (b) the natural frequency of vibration for small displacements about the vertical position of the ball arms.

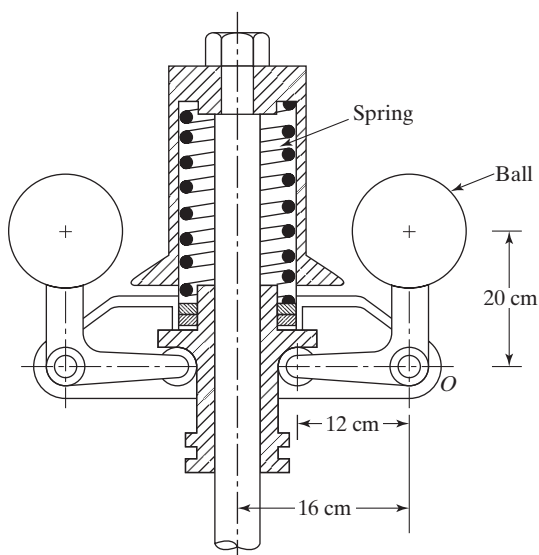


FIGURE 2.71 Hartnell governor.

- 2.31** A square platform $PQRS$ and a car that it is supporting have a combined mass of M . The platform is suspended by four elastic wires from a fixed point O , as indicated in Fig. 2.72. The vertical distance between the point of suspension O and the horizontal equilibrium position of the platform is h . If the side of the platform is a and the stiffness of each wire is k , determine the period of vertical vibration of the platform.

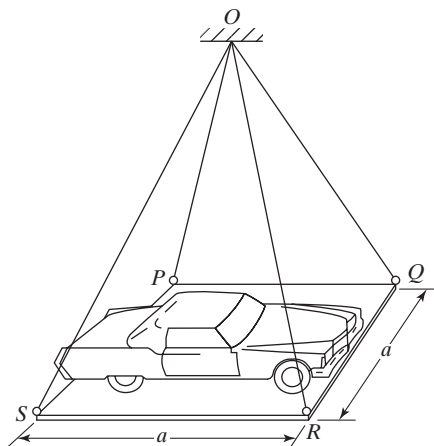


FIGURE 2.72 Car suspended by four wires.

- 2.32** The inclined manometer, shown in Fig. 2.73, is used to measure pressure. If the total length of mercury in the tube is L , find an expression for the natural frequency of oscillation of the mercury.

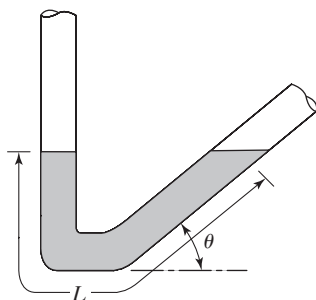


FIGURE 2.73 Inclined manometer.

- 2.33** The crate, of mass 250 kg, hanging from a helicopter (shown in Fig. 2.74(a)) can be modeled as shown in Fig. 2.74(b). The rotor blades of the helicopter rotate at 300 rpm. Find the diameter of the steel cables so that the natural frequency of vibration of the crate is at least twice the frequency of the rotor blades.

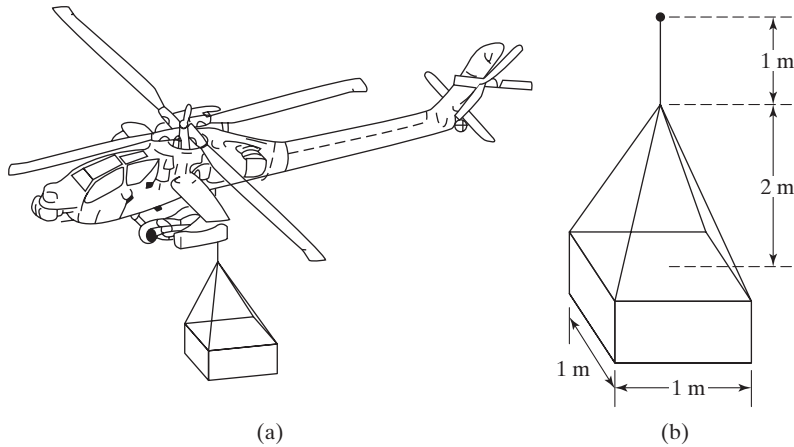


FIGURE 2.74 Crate hanging from a helicopter.

- 2.34** A pressure-vessel head is supported by a set of steel cables of length 2 m as shown in Fig. 2.75. The time period of axial vibration (in vertical direction) is found to vary from 5 s to 4.0825 s when an additional mass of 5000 kg is added to the pressure-vessel head. Determine the equivalent cross-sectional area of the cables and the mass of the pressure-vessel head.

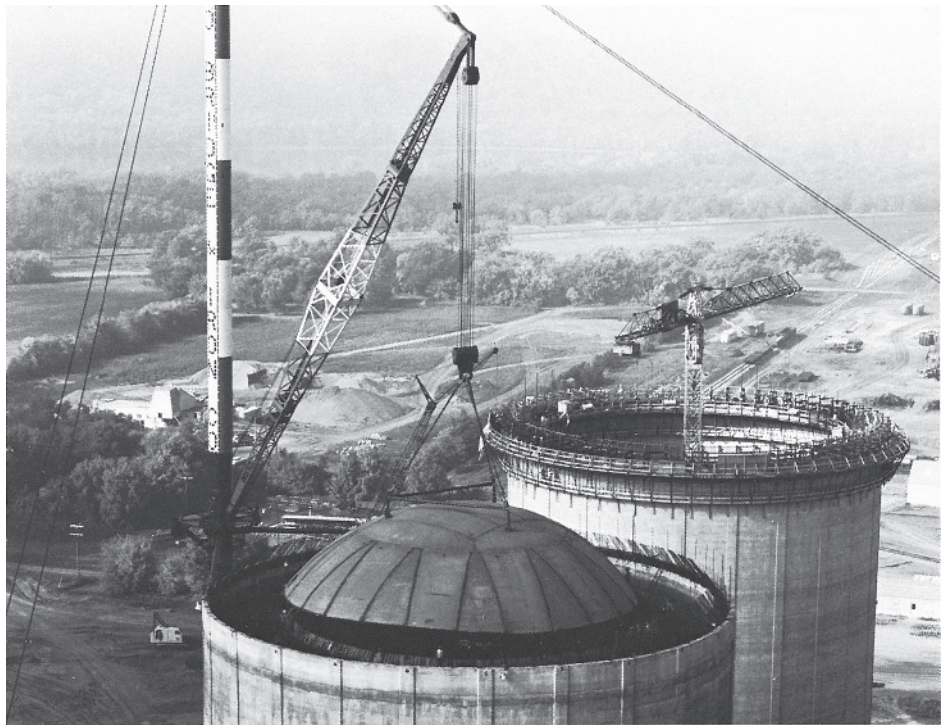


FIGURE 2.75 Pressure vessel supported by steel cables. (Photo courtesy of CBI Industries, Inc.)

- 2.35** A flywheel is mounted on a vertical shaft, as shown in Fig. 2.76. The shaft has a diameter d and length l and is fixed at both ends. The flywheel has a weight of W and a radius of gyration of r . Find the natural frequency of the longitudinal, the transverse, and the torsional vibration of the system.

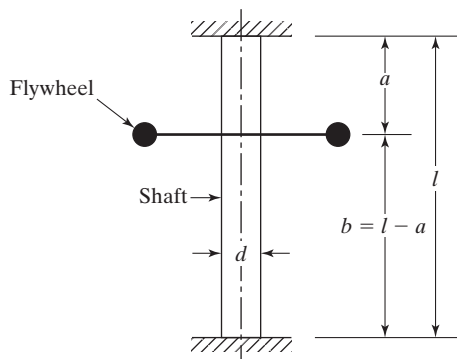


FIGURE 2.76 Flywheel mounted on a shaft.

- 2.36** A TV antenna tower is braced by four cables, as shown in Fig. 2.77. Each cable is under tension and is made of steel with a cross-sectional area of 322 mm^2 . The antenna tower can be modeled as a steel beam of square section of side 25 mm for estimating its mass and stiffness. Find the tower's natural frequency of bending vibration about the y -axis.

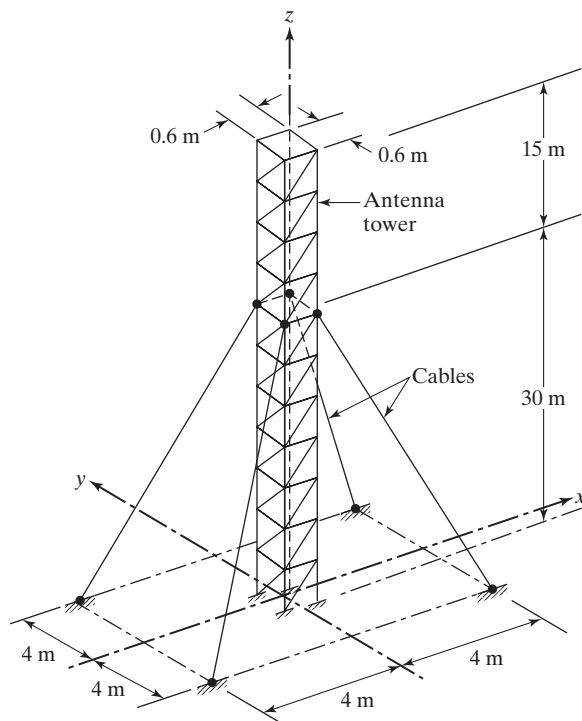


FIGURE 2.77 TV antenna tower braced by cables.

- 2.37** Figure 2.78(a) shows a steel traffic sign, of thickness 3 mm fixed to a steel post. The post is 2 m high with a cross section $50 \text{ mm} \times 6 \text{ mm}$, and it can undergo torsional vibration (about the z -axis) or bending vibration (either in the zx -plane or the yz -plane). Determine the mode of vibration of the post in a storm during which the wind velocity has a frequency component of 1.25 Hz.

Hints:

1. Neglect the weight of the post in finding the natural frequencies of vibration.
2. Torsional stiffness of a shaft with a rectangular section (see Fig. 2.78(b)) is given by

$$k_t = 5.33 \frac{ab^3 G}{l} \left[1 - 0.63 \frac{b}{a} \left(1 - \frac{b^4}{12a^4} \right) \right]$$

where G is the shear modulus.

3. Mass moment of inertia of a rectangular block about axis OO (see Fig. 2.78(c)) is given by

$$I_{OO} = \frac{\rho l}{3} (b^3 h + h^3 b)$$

where ρ is the density of the block.

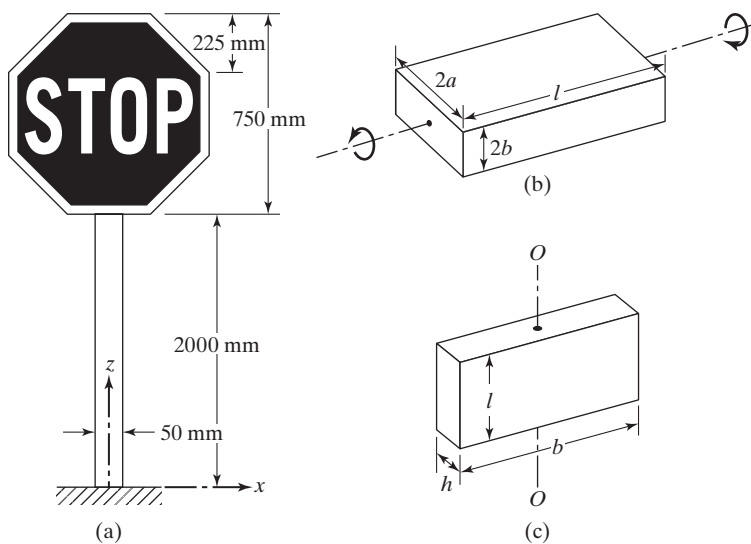


FIGURE 2.78 Traffic sign.

- 2.38** A building frame is modeled by four identical steel columns, each of weight w , and a rigid floor of weight W , as shown in Fig. 2.79. The columns are fixed at the ground and have a bending rigidity of EI each. Determine the natural frequency of horizontal vibration of the building frame by assuming the connection between the floor and the columns to be (a) pivoted as shown in Fig. 2.79(a) and (b) fixed against rotation as shown in Fig. 2.79(b). Include the effect of self weights of the columns.

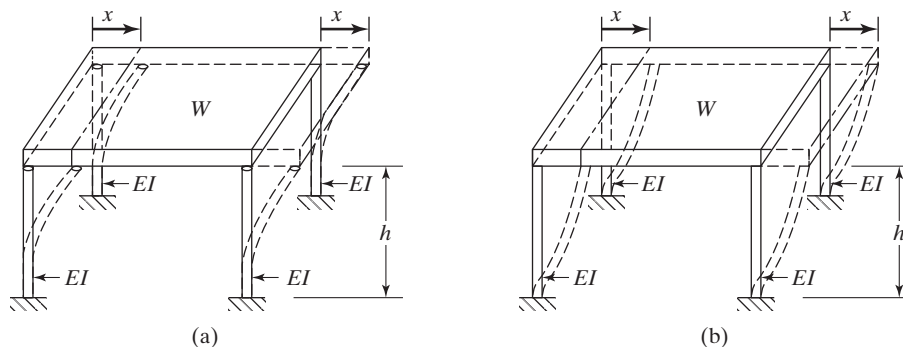


FIGURE 2.79 Building frame.

- 2.39** A pick-and-place robot arm, shown in Fig. 2.80, carries an object of mass 5 kg. Find the natural frequency of the robot arm in the axial direction for the following data: $l_1 = 0.3$ m, $l_2 = 0.25$ m, $l_3 = 0.2$ m; $E_1 = E_2 = E_3 = 69$ GPa; $D_1 = 50$ mm, $D_2 = 38$ mm, $D_3 = 25$ mm; $d_1 = 45$ mm, $d_2 = 32$ mm, $d_3 = 20$ mm.

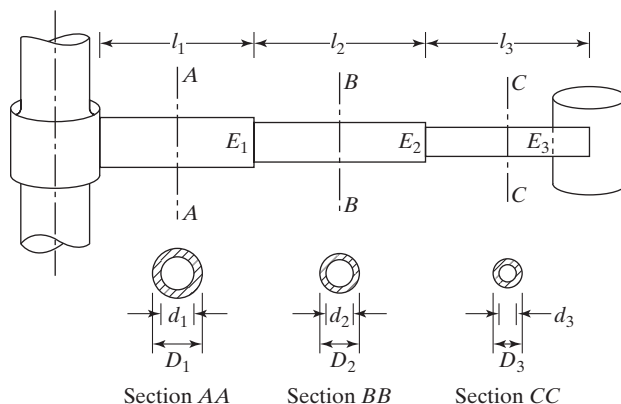


FIGURE 2.80 Robot arm carrying an object.

- 2.40** A helical spring of stiffness k is cut into two halves and a mass m is connected to the two halves as shown in Fig. 2.81(a). The natural time period of this system is found to be 0.5 s. If an identical spring is cut so that one part is one-fourth and the other part three-fourths of the original length, and the mass m is connected to the two parts as shown in Fig. 2.81(b), what would be the natural period of the system?

- 2.41*** Figure 2.82 shows a metal block supported on two identical cylindrical rollers rotating in opposite directions at the same angular speed. When the center of gravity of the block is initially displaced by a distance x , the block will be set into simple harmonic motion. If

*The asterisk denotes a design problem or a problem with no unique answer.

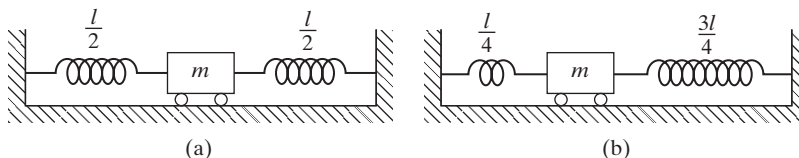


FIGURE 2.81 Mass connected to springs in two ways.

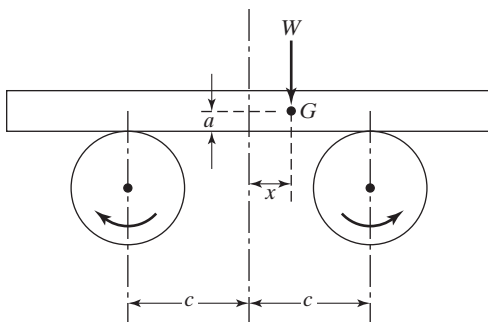


FIGURE 2.82 Metal block on two rollers.

the frequency of motion of the block is found to be ω , determine the coefficient of friction between the block and the rollers.

- 2.42*** If two identical springs of stiffness k each are attached to the metal block of Problem 2.41 as shown in Fig. 2.83, determine the coefficient of friction between the block and the rollers.

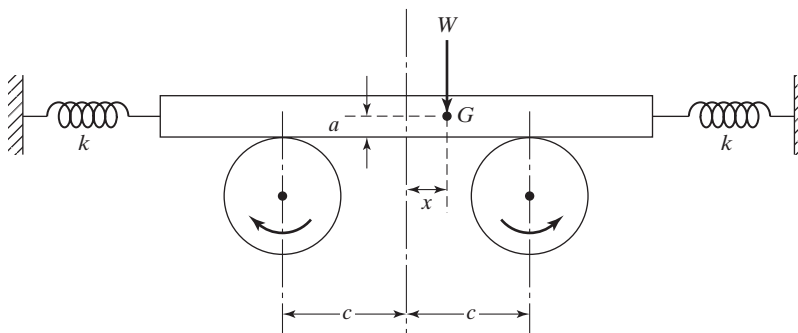


FIGURE 2.83 Spring-connected metal block on rollers.

- 2.43** An electromagnet of mass 1500 kg is at rest while holding an automobile of mass 900 kg in a junkyard. The electric current is turned off, and the automobile is dropped. Assuming that the crane and the supporting cable have an equivalent spring constant of 1.75×10^6 N/m, find the following: (a) the natural frequency of vibration of the electromagnet, (b) the resulting motion of the electromagnet, and (c) the maximum tension developed in the cable during the motion.

- 2.44** Derive the equation of motion of the system shown in Fig. 2.84, using the following methods:
 (a) Newton's second law of motion, (b) D'Alembert's principle, (c) principle of virtual work,
 and (d) principle of conservation of energy.

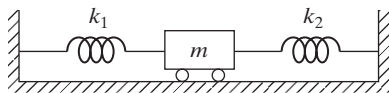


FIGURE 2.84 Spring-mass system.

- 2.45–2.46** Draw the free-body diagram and derive the equation of motion using Newton's second law of motion for each of the systems shown in Figs. 2.85 and 2.86.

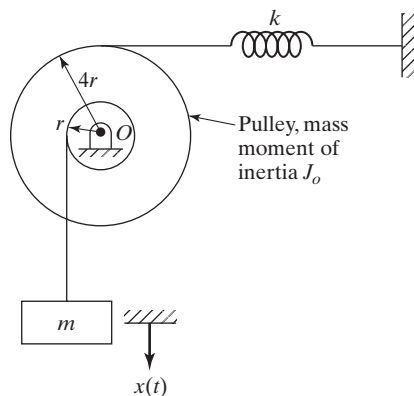


FIGURE 2.85 Pulley connected to mass and spring.

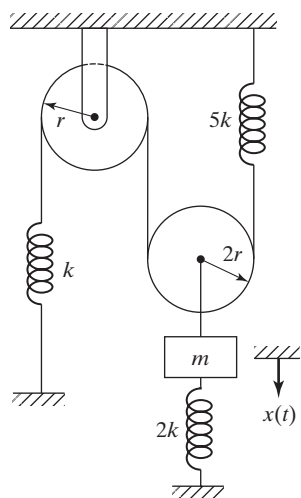


FIGURE 2.86 Pulleys connected to springs and mass.

2.47–2.48 Derive the equation of motion using the principle of conservation of energy for each of the systems shown in Figs. 2.85 and 2.86.

2.49 Determine the equivalent spring constant and the natural frequency of vibration of the system shown in Fig. 2.87.

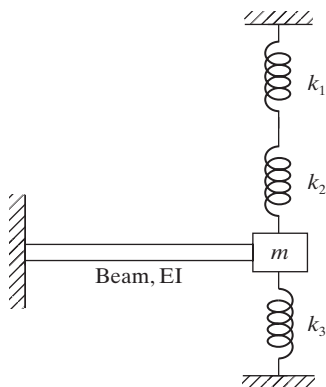


FIGURE 2.87 Beam connected to mass and springs.

2.50–2.53 Find the natural frequency of vibration in bending of the system shown in Figs. 2.88(a)–(d) by modeling the system as a single-degree-of-freedom system. Assume that the mass is 50 kg and the beam has a square cross section of $5\text{ cm} \times 5\text{ cm}$, and is made of steel with a Young's modulus of 207 GPa.

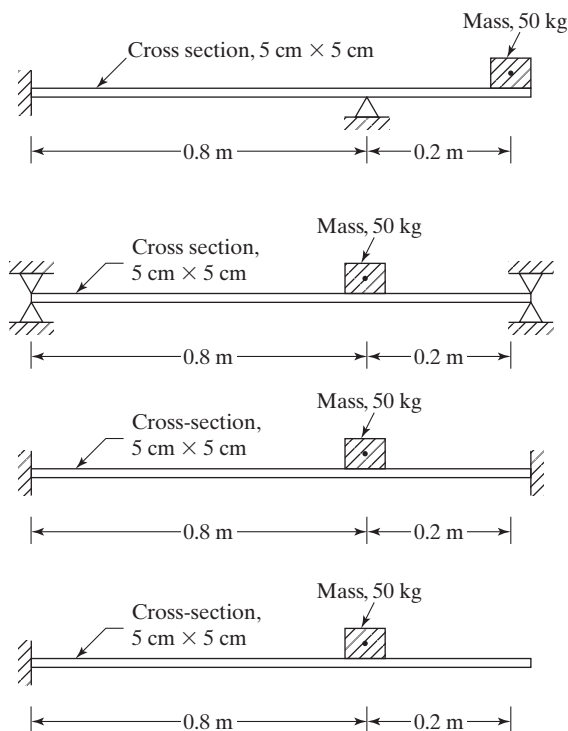


FIGURE 2.88 Different arrangements of beam supporting a mass.

- 2.54** A steel beam of length 1 m carries a mass of 50 kg at its free end, as shown in Fig. 2.89. Find the natural frequency of transverse vibration of the system by modeling it as a single-degree-of-freedom system.

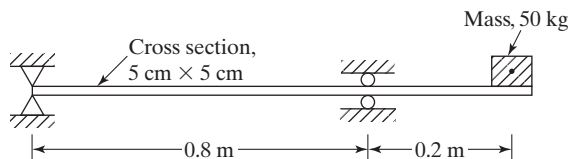


FIGURE 2.89 Beam carrying a mass.

- 2.55–2.58** Determine the natural frequency of vibration, in bending, of the system shown in Figs. 2.90(a)–(d) by modeling the system as a single-degree-of-freedom system. Assume that the mass is $m = 50\text{ kg}$, spring stiffness is $k = 10,000$ and the beam has a square cross section of $5\text{ cm} \times 5\text{ cm}$, and is made of steel with a Young's modulus of 207 GPa.

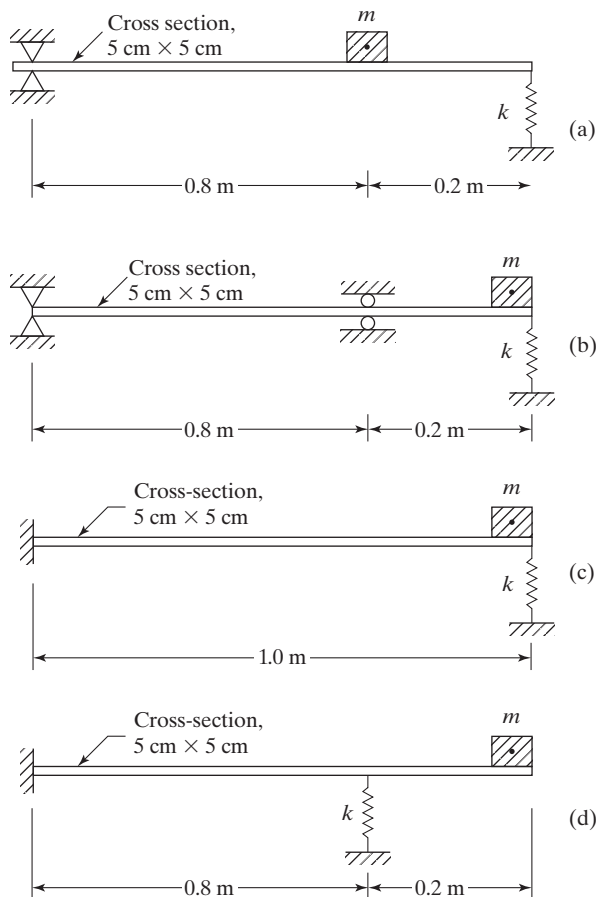


FIGURE 2.90 Different arrangements of beam supported on a spring.

- 2.59** An undamped single-degree-of-freedom system consists of a mass 5 kg and a spring of stiffness 2000 N/m. Find the response of the system using Eq. (2.21) when the mass is subjected to the following initial conditions:
- $x_0 = 20 \text{ mm}$, $\dot{x}_0 = 200 \text{ mm/s}$
 - $x_0 = -20 \text{ mm}$, $\dot{x}_0 = 200 \text{ mm/s}$
 - $x_0 = 20 \text{ mm}$, $\dot{x}_0 = -200 \text{ mm/s}$
 - $x_0 = -20 \text{ mm}$, $\dot{x}_0 = -200 \text{ mm/s}$
- 2.60** An undamped single-degree-of-freedom system consists of a mass 10 kg and a spring of stiffness 1000 N/m. Determine the response of the system using Eq. (2.21) when the mass is subjected to the following initial conditions:
- $x_0 = 10 \text{ mm}$, $\dot{x}_0 = 100 \text{ mm/s}$
 - $x_0 = -10 \text{ mm}$, $\dot{x}_0 = 100 \text{ mm/s}$
 - $x_0 = 10 \text{ mm}$, $\dot{x}_0 = -100 \text{ mm/s}$
 - $x_0 = -10 \text{ mm}$, $\dot{x}_0 = -100 \text{ mm/s}$
- 2.61** Describe how the phase angle ϕ_0 in Eq. (2.23) is to be computed for different combinations of positive and negative values of the initial displacement (x_0) and the initial velocity (\dot{x}_0).
- 2.62** Find the response of the system described in Problem 2.59 using Eq. (2.23).
- 2.63** Find the response of the system described in Problem 2.60 using Eq. (2.23).
- 2.64** Find the response of the system described in Example 2.1 using Eq. (2.23).
- 2.65** The trunk of the tree shown in Fig. 2.91 can be assumed to be a uniform cylinder of diameter $d = 0.25 \text{ m}$ with a density of $\rho = 800 \text{ kg/m}^3$ and Young's modulus of $E = 1.2 \text{ GPa}$, and the crown of the tree has a mass of $m_c = 100 \text{ kg}$.
- If the maximum deflection of the trunk, δ , due to a transverse force (F_0) applied at the top of the trunk is equal to 40% of the buckling length needed for buckling to occur when the weight of the crown of the tree acts along the axial direction at the top of the trunk [2.15].
 - Find the natural frequency of vibration of the tree in sway motion.
- State the assumptions made in the solution.
- 2.66** A bird of mass 2 kg sits at the end of a horizontal branch of a tree as shown in Fig. 2.92. The branch has a length of 4 m from the trunk of the tree with a diameter of 0.1 m. If the density of the branch is 700 kg/m^3 and the Young's modulus is 10 GPa, determine the following [2.15]:
- Equation of motion of the bird considering the weights of the bird and a uniformly distributed weight of the branch.
 - Natural frequency of vibration of the branch with the bird.

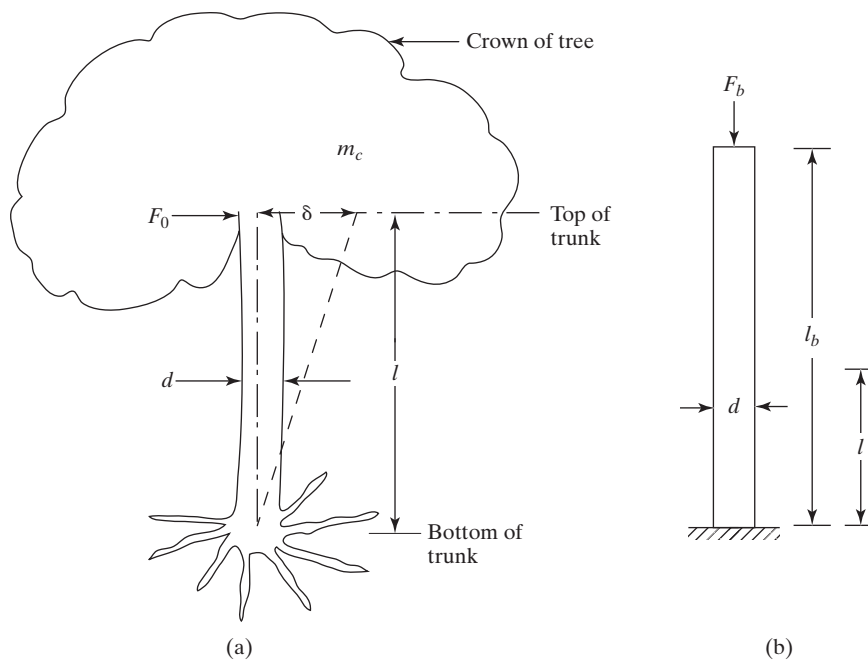


FIGURE 2.91 Trunk of a tree subjected to force.

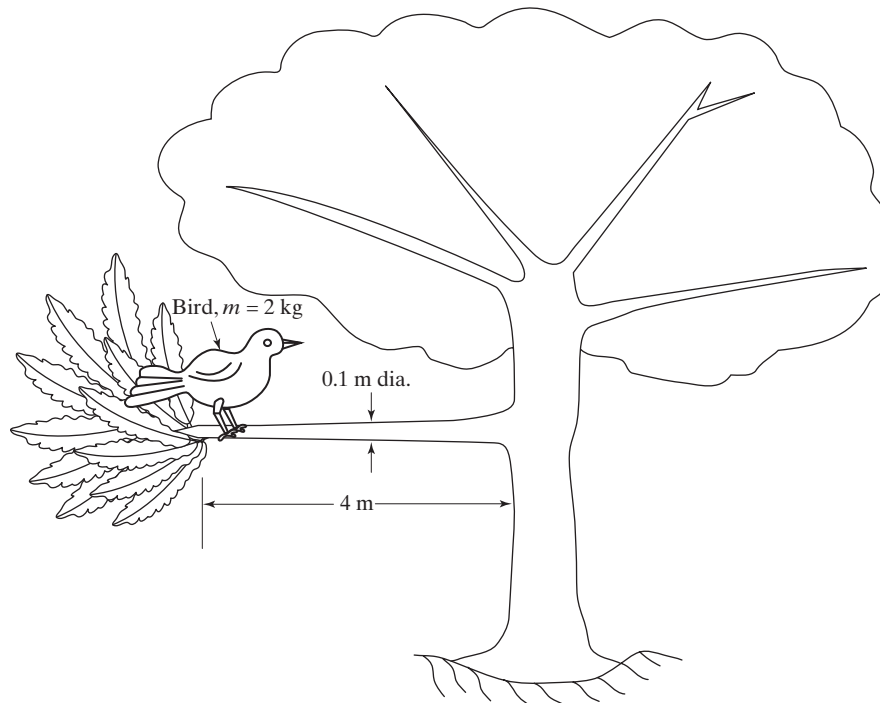


FIGURE 2.92 Bird sitting on branch.

- 2.67** A bird of mass $m = 2 \text{ kg}$ sits at the top of a slender vertical branch of a tree as shown in Fig. 2.93. The height of the branch from the trunk of the tree is 2 m and the diameter of the branch is $d \text{ m}$. The density of the branch is 700 kg/m^3 and the Young's modulus is 10 GPa .
- Find the minimum diameter of the branch to avoid buckling under the weight of the bird (by neglecting the weight of the branch). Consider the branch as a fixed free column.
 - Find the natural frequency of vibration of the system (bird on the top of the branch) by treating the branch as a cantilever beam using the diameter found in part (a).

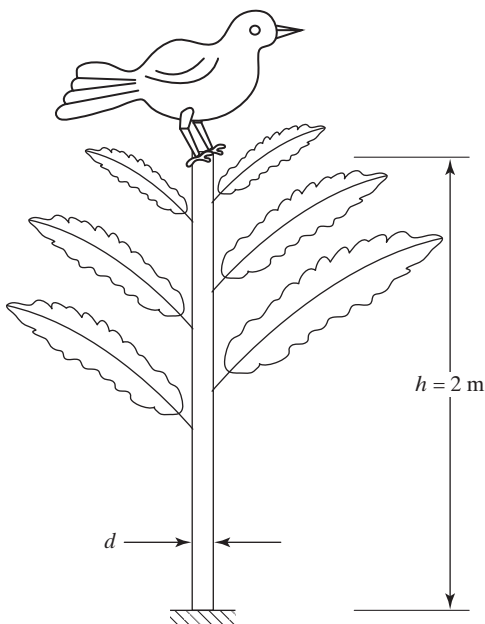


FIGURE 2.93 Bird sitting at top of vertical branch.

- 2.68** Determine the displacement, velocity, and acceleration of the mass of a spring-mass system with $k = 500 \text{ N/m}$, $m = 2 \text{ kg}$, $x_0 = 0.1 \text{ m}$, and $\dot{x}_0 = 5 \text{ m/s}$.
- 2.69** Determine the displacement (x), velocity (\dot{x}), and acceleration (\ddot{x}) of a spring-mass system with $\omega_n = 10 \text{ rad/s}$ for the initial conditions $x_0 = 0.05 \text{ m}$ and $\dot{x}_0 = 1 \text{ m/s}$. Plot $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ from $t = 0$ to 5 s .
- 2.70** The free-vibration response of a spring-mass system is observed to have a frequency of 2 rad/s , an amplitude of 10 mm , and a phase shift of 1 rad from $t = 0$. Determine the initial conditions that caused the free vibration. Assume the damping ratio of the system as 0.1 .
- 2.71** An automobile is found to have a natural frequency of 20 rad/s without passengers and 17.32 rad/s with passengers of mass 500 kg . Find the mass and stiffness of the automobile by treating it as a single-degree-of-freedom system.

- 2.72** A spring-mass system with mass 2 kg and stiffness 3200 N/m has an initial displacement of $x_0 = 0$. What is the maximum initial velocity that can be given to the mass without the amplitude of free vibration exceeding a value of 0.1 m?
- 2.73** A helical spring, made of music wire of diameter d , has a mean coil diameter (D) of 14 mm and N active coils (turns). It is found to have a frequency of vibration (f) of 193 Hz and a spring rate k of 4.6 N/mm. Determine the wire diameter d and the number of coils N , assuming the shear modulus G is 80 GPa and density ρ is 8000 kg/m³. The spring rate (k) and frequency (f) are given by

$$k = \frac{d^4 G}{8D^3 N}, \quad f = \frac{1}{2} \sqrt{\frac{kg}{W}}$$

where W is the weight of the helical spring and g is the acceleration due to gravity.

- 2.74** Solve Problem 2.73 if the material of the helical spring is changed from music wire to aluminum with $G = 26$ GPa and $\rho = 2690$ kg/m³.
- 2.75** A steel cantilever beam is used to carry a machine at its free end. To save weight, it is proposed to replace the steel beam by an aluminum beam of identical dimensions. Find the expected change in the natural frequency of the beam-machine system.
- 2.76** An oil drum of diameter 1 m and a mass of 500 kg floats in a bath of salt water of density $\rho_w = 1050$ kg/m³. Considering small displacements of the drum in the vertical direction (x), determine the natural frequency of vibration of the system.
- 2.77** The equation of motion of a spring-mass system is given by (units: SI system)

$$500\ddot{x} + 1000\left(\frac{x}{0.025}\right)^3 = 0$$

- Determine the static equilibrium position of the system.
 - Derive the linearized equation of motion for small displacements (x) about the static equilibrium position.
 - Find the natural frequency of vibration of the system for small displacements.
 - Find the natural frequency of vibration of the system for small displacements when the mass is 600 (instead of 500).
- 2.78** A deceleration of 10 m/s² is caused when brakes are applied to a vehicle traveling at a speed of 100 km/hour. Determine the time taken and the distance traveled before the vehicle comes to a complete stop.
- 2.79** A steel hollow cylindrical post is welded to a steel rectangular traffic sign as shown in Fig. 2.94 with the following data:
 Dimensions: $l = 2$ m, $r_o = 0.050$ m, $r_i = 0.045$ m, $b = 0.75$ m, $d = 0.40$ m, $t = 0.005$ m;
 material properties: ρ (specific weight) = 76.50 kN/m³, $E = 207$ GPa, $G = 79.3$ GPa
 Find the natural frequencies of the system in transverse vibration in the yz - and xz -planes by considering the masses of both the post and the sign.

Hint: Consider the post as a cantilever beam in transverse vibration in the appropriate plane.

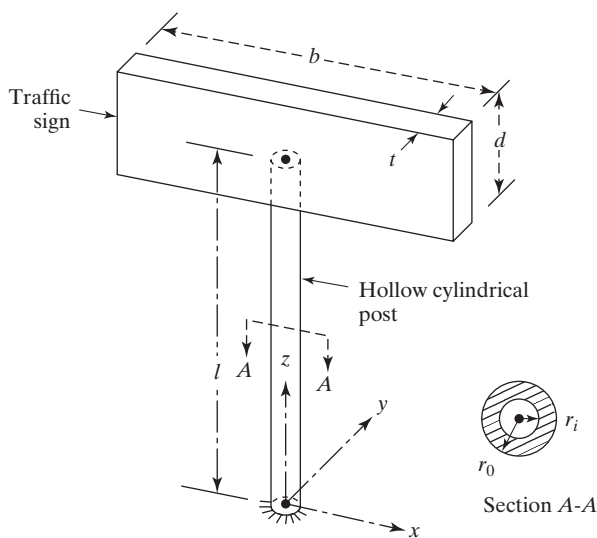


FIGURE 2.94 Traffic sign on cylindrical post.

2.80 Solve Problem 2.79 by changing the material from steel to bronze for both the post and the sign. Material properties of bronze: ρ (specific weight) = 80.1 kN/m^3 , $E = 111.0 \text{ GPa}$, $G = 41.4 \text{ GPa}$.

2.81 A heavy disk of mass moment of inertia J is attached at the free end of a stepped circular shaft as shown in Fig. 2.95. By modeling the system as a single-degree-of-freedom torsional system, determine the natural frequency of torsional vibration.

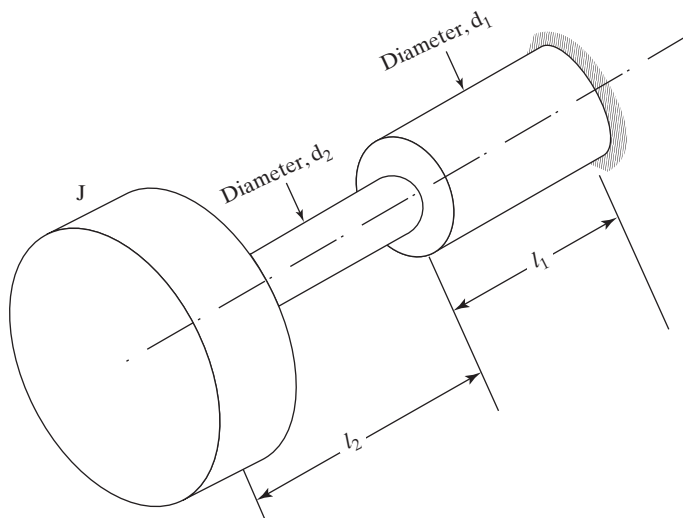


FIGURE 2.95 Heavy disk at end of a stepped shaft.

Section 2.3 Free Vibration of an Undamped Torsional System

- 2.82** A simple pendulum of length 1 m with a bob of mass 1 kg is placed on Mars and was given an initial angular displacement of 5° . If the acceleration due to gravity on Mars, g_{Mars} , is $0.376 g_{\text{Earth}}$, determine the following:
- Natural frequency of the pendulum.
 - Maximum angular velocity of the pendulum.
 - Maximum angular acceleration of the pendulum.
- 2.83** A simple pendulum of length 1 m with a bob of mass 1 kg is placed on Moon and was given an initial angular displacement of 5° . If the acceleration due to gravity on Moon, g_{Moon} , is 1.6263 m/s^2 , determine the following:
- Natural frequency of the pendulum.
 - Maximum angular velocity of the pendulum.
 - Maximum angular acceleration of the pendulum.
- 2.84** A simple pendulum is set into oscillation from its rest position by giving it an angular velocity of 1 rad/s. It is found to oscillate with an amplitude of 0.5 rad. Find the natural frequency and length of the pendulum.
- 2.85** A pulley 250 mm in diameter drives a second pulley 1000 mm in diameter by means of a belt (see Fig. 2.96). The moment of inertia of the driven pulley is $0.2 \text{ kg}\cdot\text{m}^2$. The belt connecting these pulleys is represented by two springs, each of stiffness k . For what value of k will the natural frequency be 6 Hz?
- 2.86** Derive an expression for the natural frequency of the simple pendulum shown in Fig. 1.10. Determine the period of oscillation of a simple pendulum having a mass $m = 5 \text{ kg}$ and a length $l = 0.5 \text{ m}$.

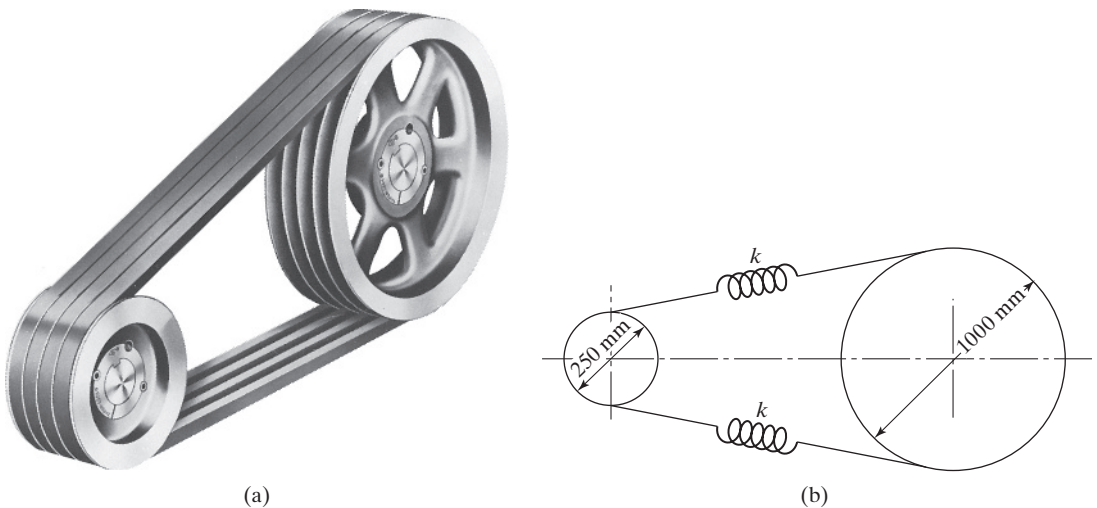


FIGURE 2.96 Pulleys and belt drive. (Photo courtesy of Reliance Electric Company.)

- 2.87** A mass m is attached at the end of a bar of negligible mass and is made to vibrate in three different configurations, as indicated in Fig. 2.97. Find the configuration corresponding to the highest natural frequency.

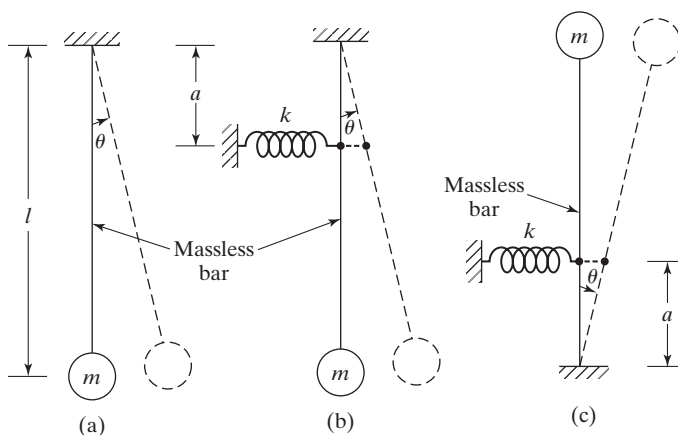


FIGURE 2.97 Different configurations of bar carrying end mass.

- 2.88** Figure 2.98 shows a spacecraft with four solar panels. Each panel has the dimensions $1.5 \text{ m} \times 1 \text{ m} \times 0.025 \text{ m}$ with a density of 2690 kg/m^3 and is connected to the body of the spacecraft by aluminum rods of length 0.3 m and diameter 25 mm . Assuming that the body of the spacecraft is very large (rigid), determine the natural frequency of vibration of each panel about the axis of the connecting aluminum rod.

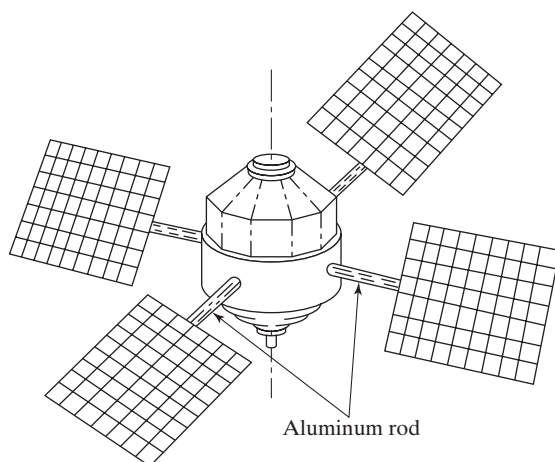


FIGURE 2.98 Spacecraft with solar panels.

- 2.89** One of the blades of an electric fan is removed (as shown by dotted lines in Fig. 2.99). The steel shaft AB , on which the blades are mounted, is equivalent to a uniform shaft of diameter 25 mm and length 150 mm. Each blade can be modeled as a uniform slender rod of mass 1 kg and length 300 mm. Determine the natural frequency of vibration of the remaining three blades about the y -axis.

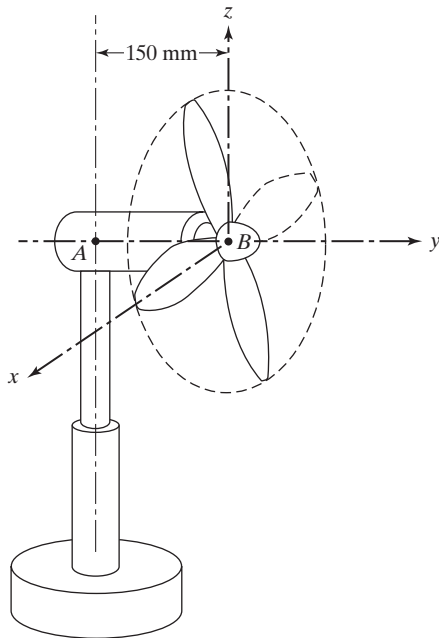


FIGURE 2.99 Electric fan with one blade missing.

- 2.90** A heavy ring of mass moment of inertia $1.0 \text{ kg}\cdot\text{m}^2$ is attached at the end of a two-layered hollow shaft of length 2 m (Fig. 2.100). If the two layers of the shaft are made of steel and brass, determine the natural time period of torsional vibration of the heavy ring.

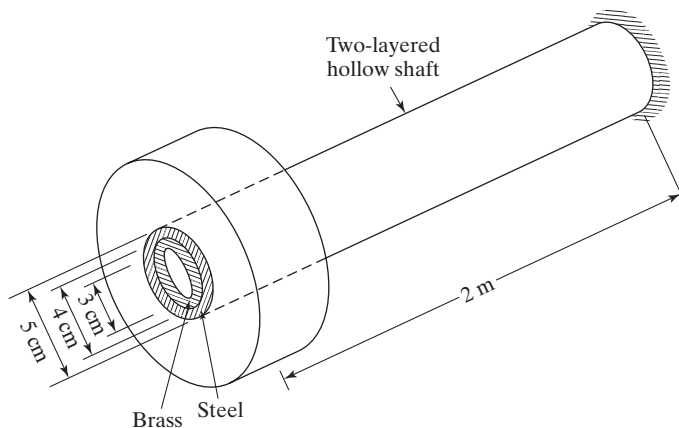


FIGURE 2.100 Heavy ring at end of hollow shaft.

- 2.91** Find the natural frequency of the pendulum shown in Fig. 2.101 when the mass of the connecting bar is not negligible compared to the mass of the pendulum bob.

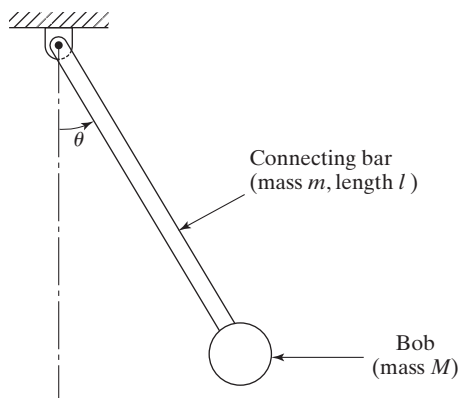


FIGURE 2.101 Pendulum.

- 2.92** A steel shaft of 0.05 m diameter and 2 m length is fixed at one end and carries at the other end a steel disc of 1 m diameter and 0.1 m thickness, as shown in Fig. 2.14. Find the system's natural frequency of torsional vibration.
- 2.93** A uniform slender rod of mass m and length l is hinged at point A and is attached to four linear springs and one torsional spring, as shown in Fig. 2.102. Find the natural frequency of the system if $k = 2000$ N/m, $k_t = 1000$ N-m/rad, $m = 10$ kg, and $l = 5$ m.

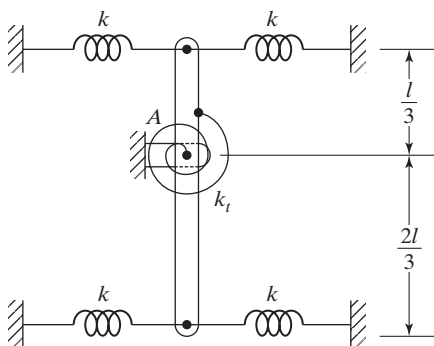


FIGURE 2.102 Slender rod connected to springs.

- 2.94** A cylinder of mass m and mass moment of inertia J_0 is free to roll without slipping but is restrained by two springs of stiffnesses k_1 and k_2 , as shown in Fig. 2.103. Find its natural frequency of vibration. Also find the value of a that maximizes the natural frequency of vibration.

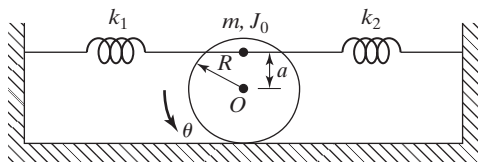


FIGURE 2.103 Cylinder restrained by springs.

2.95 If the pendulum of Problem 2.86 is placed in a rocket moving vertically with an acceleration of 5 m/s^2 , what will be its period of oscillation?

2.96 Find the equation of motion of the uniform rigid bar OA of length l and mass m shown in Fig. 2.104. Also find its natural frequency.

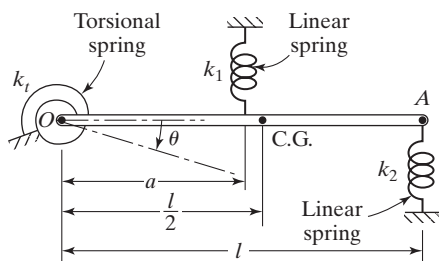


FIGURE 2.104 Rigid bar connected to springs.

2.97 A uniform circular disc is pivoted at point O , as shown in Fig. 2.105. Find the natural frequency of the system. Also find the maximum frequency of the system by varying the value of b .

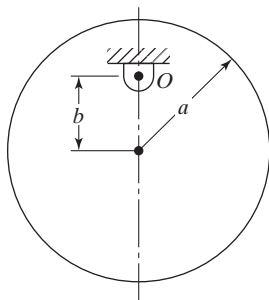


FIGURE 2.105 Circular disc as pendulum.

2.98 Derive the equation of motion of the system shown in Fig. 2.106, using the following methods: (a) Newton's second law of motion, (b) D'Alembert's principle, and (c) principle of virtual work.

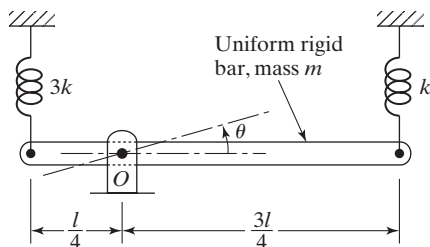


FIGURE 2.106 Rigid bar undergoing angular motion.

- 2.99** Find the natural frequency of the traffic sign system described in Problem 2.79 in torsional vibration about the z -axis by considering the masses of both the post and the sign.

Hint: The spring stiffness of the post in torsional vibration about the z -axis is given by $k_t = \frac{\pi G}{2l}(r_0^4 - r_i^4)$. The mass moment of inertia of the sign about the z -axis is given by $I_0 = \frac{1}{12}m_0(d^2 + b^2)$, where m_0 is the mass of the sign.

- 2.100** Solve Problem 2.99 by changing the material from steel to bronze for both the post and the sign. Material properties of bronze: ρ (specific weight) = 80.1 kN/m^3 , $E = 111.0 \text{ GPa}$, $G = 41.4 \text{ GPa}$.

- 2.101** A mass m_1 is attached at one end of a uniform bar of mass m_2 whose other end is pivoted at point O as shown in Fig. 2.107. Determine the natural frequency of vibration of the resulting pendulum for small angular displacements.

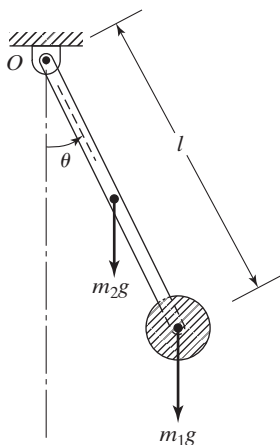


FIGURE 2.107 Uniform bar with end mass.

- 2.102** The angular motion of the forearm of a human hand carrying a mass m_0 is shown in Fig. 2.108. During motion, the forearm can be considered to rotate about the joint (pivot point) O with muscle forces modeled in the form of a force by triceps ($c_1\dot{x}$) and a force in biceps ($-c_2\theta$), where c_1 and c_2 are constants and \dot{x} is the velocity with which triceps are stretched (or contracted). Approximating the forearm as a uniform bar of mass m and length l , derive the equation of motion of the forearm for small angular displacements θ . Also find the natural frequency of the forearm.

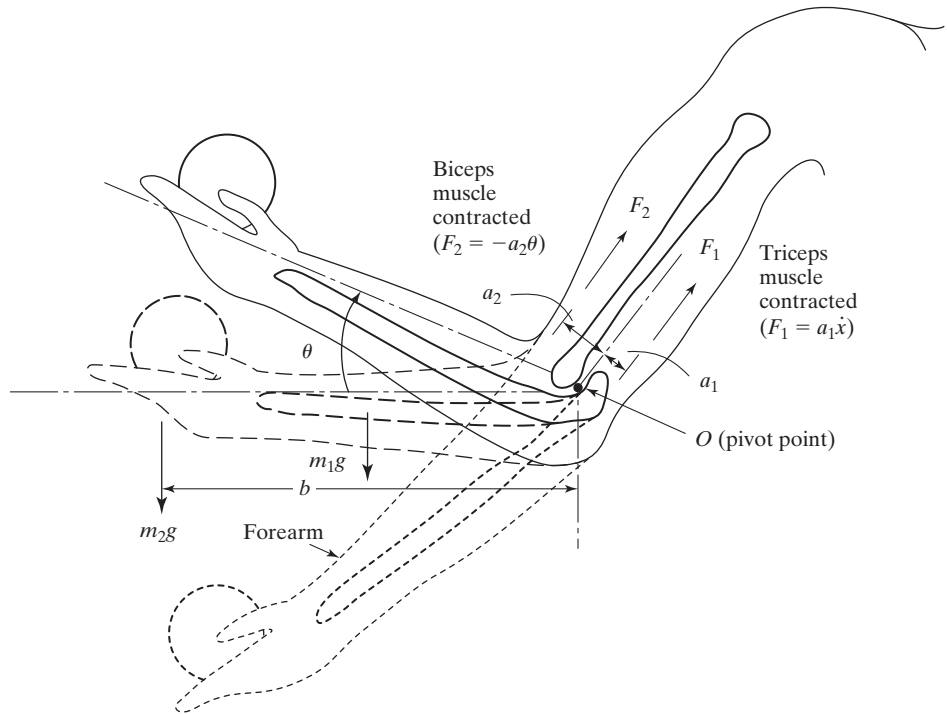


FIGURE 2.108 Motion of arm.

Section 2.4 Response of First-Order Systems and Time Constant

- 2.103** Find the free-vibration response and the time constant, where applicable, of systems governed by the following equations of motion:

- $100\dot{\nu} + 20\nu = 0, \quad \nu(0) = \nu(t=0) = 10$
- $100\dot{\nu} + 20\nu = 10, \quad \nu(0) = \nu(t=0) = 10$
- $100\dot{\nu} - 20\nu = 0, \quad \nu(0) = \nu(t=0) = 10$
- $500\dot{\omega} + 50\omega = 0, \quad \omega(0) = \omega(t=0) = 0.5$

Hint: The time constant can also be defined as the value of time at which the step response of a system rises to 63.2% (100.0% – 36.8%) of its final value.

- 2.104** A viscous damper, with damping constant c , and a spring, with spring stiffness k , are connected to a massless bar AB as shown in Fig. 2.109. The bar AB is displaced by a distance of

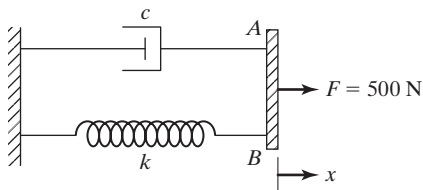


FIGURE 2.109 Spring-damper system subjected to force.

$x = 0.1$ m when a constant force $F = 500$ N is applied. The applied force F is then abruptly released from its displaced position. If the displacement of the bar AB is reduced from its initial value of 0.1 m at $t = 0$ to 0.01 m at $t = 10$ s, find the values of c and k .

- 2.105** The equation of motion of a rocket, of mass m , traveling vertically under a thrust F and air resistance or drag D is given by

$$m\dot{v} = F - D - mg$$

If $m = 1000$ kg, $F = 50,000$ N, $D = 2000 v$, and $g = 9.81$ m/s², find the time variation of the velocity of the rocket, $v(t) = \frac{dx(t)}{dt}$, using the initial conditions $x(0) = 0$ and $v(0) = 0$, where $x(t)$ is the distance traveled by the rocket in time t .

Section 2.5 Rayleigh's Energy Method

- 2.106** Determine the effect of self weight on the natural frequency of vibration of the pinned-pinned beam shown in Fig. 2.110.

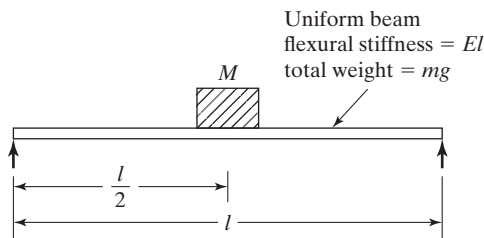


FIGURE 2.110 Pinned-pinned beam.

- 2.107** Use Rayleigh's method to solve Problem 2.7.
- 2.108** Use Rayleigh's method to solve Problem 2.13.
- 2.109** Find the natural frequency of the system shown in Fig. 2.54.
- 2.110** Use Rayleigh's method to solve Problem 2.26.

2.111 Use Rayleigh's method to solve Problem 2.93.

2.112 Use Rayleigh's method to solve Problem 2.96.

2.113 A wooden rectangular prism of density ρ_w , height h , and cross section $a \times b$ is initially depressed in an oil tub and made to vibrate freely in the vertical direction (see Fig. 2.111). Use Rayleigh's method to find the natural frequency of vibration of the prism. Assume the density of oil is ρ_0 . If the rectangular prism is replaced by a uniform circular cylinder of radius r , height h , and density ρ_w , will there be any change in the natural frequency?

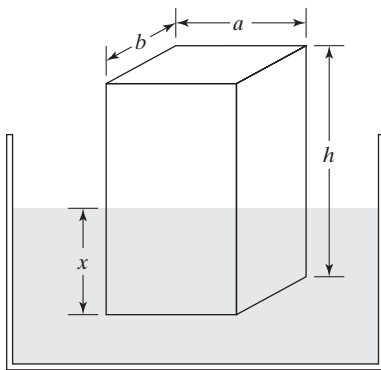


FIGURE 2.111 Wooden prism vibrating in oil tub.

2.114 Use the energy method to find the natural frequency of the system shown in Fig. 2.103.

2.115 Use the energy method to find the natural frequency of vibration of the system shown in Fig. 2.85.

2.116 A cylinder of mass m and mass moment of inertia J is connected to a spring of stiffness k and rolls on a rough surface as shown in Fig. 2.112. If the translational and angular displacements of the cylinder are x and θ from its equilibrium position, determine the following:

- Equation of motion of the system for small displacements in terms of x using the energy method.
- Equation of motion of the system for small displacements in terms of θ using the energy method.
- Find the natural frequencies of the system using the equation of motion derived in parts (a) and (b). Are the resulting natural frequencies same?

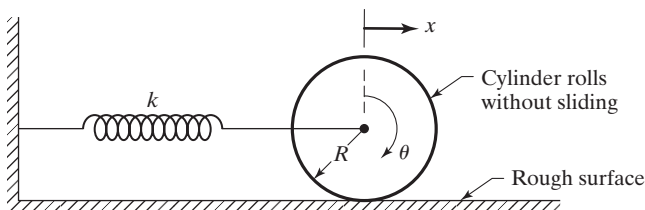


FIGURE 2.112 Cylinder connected to a spring.

Section 2.6 Free Vibration with Viscous Damping

- 2.117** Consider the differential equation of motion for the free vibration of a damped single-degree-of-freedom system given by

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (\text{E.1})$$

Show that Eq. (E.1) remains the same irrespective of the units used by considering the following data and systems of units [2.17]:

- a. SI units: $m = 2 \text{ kg}$, $c = 800 \text{ N-s/m}$, $k = 4000 \text{ N/m}$.
 - b. Metric engineering units: (mass – $\text{kg}_f\text{-s}^2/\text{m}$, force – kg_f , length – m, time – s); $1 \text{ kg}_f\text{-s}^2/\text{m} = 9.807 \text{ kg}$, $1 \text{ kg}_f = 9.807 \text{ N}$, 1 kg .
 - c. Metric absolute units (cgs system): (mass – gram, force – dyne (gram-cm/s^2), length – cm, time – s), $1 \text{ g} = 0.001 \text{ kg}$, $1 \text{ dyne} = 10^{-5} \text{ N}$, $1 \text{ cm} = 0.01 \text{ m}$.
- 2.118** A damped single-degree-of-freedom system has $m = 5 \text{ kg}$, $c = 500 \text{ N-s/m}$, and $k = 5000 \text{ N/m}$. Determine the undamped and damped natural frequencies of vibration and the damping ratio of the system.
- 2.119** A damped single-degree-of-freedom system has $m = 5 \text{ kg}$, $c = 500 \text{ N-s/m}$, and $k = 50,000 \text{ N/m}$. Determine the undamped and damped natural frequencies of vibration and the damping ratio of the system.
- 2.120** A damped single-degree-of-freedom system has $m = 5 \text{ kg}$, $c = 1000 \text{ N-s/m}$, and $k = 50,000 \text{ N/m}$. Determine the undamped and damped natural frequencies of vibration and the damping ratio of the system.
- 2.121** Find the variation of the displacement with time, $x(t)$, of a damped single-degree-of-freedom system with $\zeta = 0.1$ for the following initial conditions:
- a. $x(t = 0) = x_0 = 0.2 \text{ m}$, $\dot{x}_0 = 0$
 - b. $x(t = 0) = x_0 = -0.2 \text{ m}$, $\dot{x}_0 = 0$
 - c. $x_0 = 0$, $\dot{x}_0 = 0.2 \text{ m/s}$
 - d. Plot the variations of displacement found in parts (a), (b), and (c) on the same graph in the range of 0 to 5 s.
- 2.122** Find the variation of the displacement with time, $x(t)$, of a damped single-degree-of-freedom system with $\zeta = 1.0$ for the following initial conditions:
- a. $x(t = 0) = x_0 = 0.2 \text{ m}$, $\dot{x}_0 = 0$
 - b. $x(t = 0) = x_0 = -0.2 \text{ m}$, $\dot{x}_0 = 0$
 - c. $\dot{x}_0 = 0.2 \text{ m/s}$, $x_0 = 0$
 - d. Plot the variations of displacement found in parts (a), (b), and (c) on the same graph in the range of 0 to 5 s.

- 2.123** Find the variation of the displacement with time, $x(t)$, of a damped single-degree-of-freedom system with $\zeta = 2.0$ for the following initial conditions:
- $x(t = 0) = x_0 = 0.2 \text{ m}$, $\dot{x}_0 = 0$
 - $x(t = 0) = x_0 = -0.2 \text{ m}$, $\dot{x}_0 = 0$
 - $\dot{x}_0 = 0.2 \text{ m/s}$, $x_0 = 0$
 - Plot the variations of displacement found in parts (a), (b), and (c) on the same graph in the range of 0 to 5 s.
- 2.124** A heavy disk of mass moment of inertia J is attached at the middle of a circular shaft of length l and diameter d as shown in Fig. 2.113. By modeling the system as a single-degree-of-freedom torsional system, determine the natural frequency of torsional vibration.

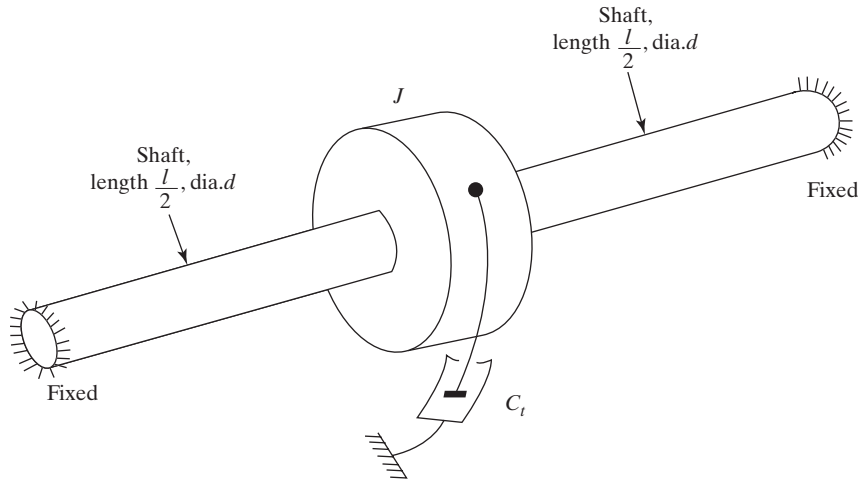


FIGURE 2.113 Heavy disc at middle of shaft.

- 2.125** A simple pendulum is found to vibrate at a frequency of 0.5 Hz in a vacuum and 0.45 Hz in a viscous fluid medium. Find the damping constant, assuming the mass of the bob of the pendulum as 1 kg.
- 2.126** The ratio of successive amplitudes of a viscously damped single-degree-of-freedom system is found to be 18:1. Determine the ratio of successive amplitudes if the amount of damping is (a) doubled, and (b) halved.
- 2.127** Assuming that the phase angle is zero, show that the response $x(t)$ of an underdamped single-degree-of-freedom system reaches a maximum value when

$$\sin \omega_d t = \sqrt{1 - \zeta^2}$$

and a minimum value when

$$\sin \omega_d t = -\sqrt{1 - \zeta^2}$$

Also show that the equations of the curves passing through the maximum and minimum values of $x(t)$ are given, respectively, by

$$x = \sqrt{1 - \zeta^2} X e^{-\zeta \omega_n t}$$

and

$$x = -\sqrt{1 - \zeta^2} X e^{-\zeta \omega_n t}$$

- 2.128** Derive an expression for the time at which the response of a critically damped system will attain its maximum value. Also find the expression for the maximum response.
- 2.129** A shock absorber is to be designed to limit its overshoot to 15% of its initial displacement when released. Find the damping ratio ζ_0 required. What will be the overshoot if ζ is made equal to (a) $\frac{3}{4}\zeta_0$, and (b) $\frac{5}{4}\zeta_0$?
- 2.130** The free-vibration responses of an electric motor of weight 500 N mounted on different types of foundations are shown in Figs. 2.114(a) and (b). Identify the following in each case: (i) the nature of damping provided by the foundation, (ii) the spring constant and damping coefficient of the foundation, and (iii) the undamped and damped natural frequencies of the electric motor.

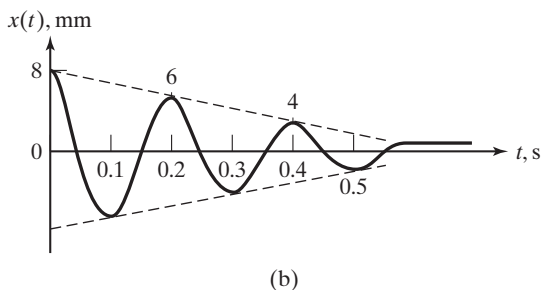
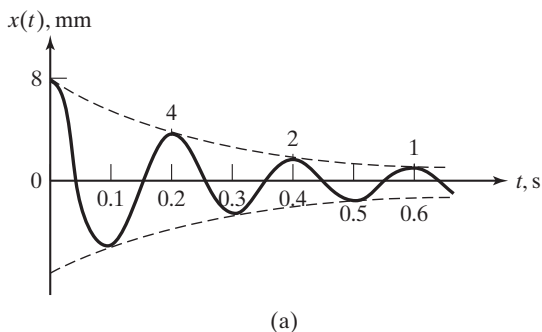


FIGURE 2.114 Free vibration responses of damped systems.

- 2.131** For a spring-mass-damper system, $m = 50$ kg and $k = 5000$ N/m. Find the following: (a) critical damping constant c_c , (b) damped natural frequency when $c = c_c/2$, and (c) logarithmic decrement.

- 2.132** A railroad car of mass 2000 kg traveling at a velocity $v = 10$ m/s is stopped at the end of the tracks by a spring-damper system, as shown in Fig. 2.115. If the stiffness of the spring is $k = 80$ N/mm and the damping constant is $c = 20$ N-s/mm, determine (a) the maximum displacement of the car after engaging the springs and damper and (b) the time taken to reach the maximum displacement.

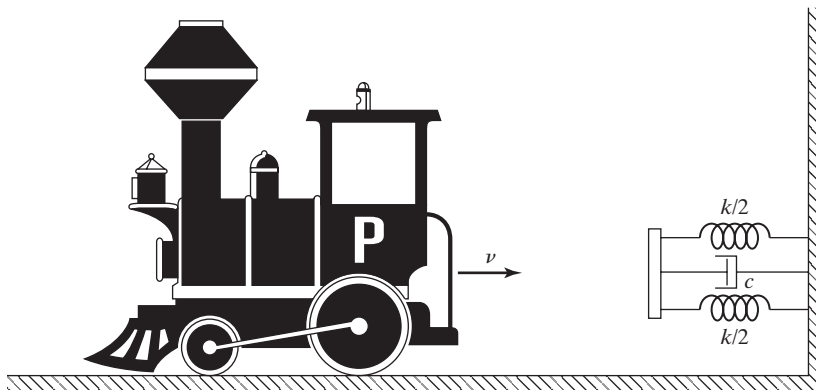


FIGURE 2.115 Railroad car stopped by spring-damper system.

- 2.133** A torsional pendulum has a natural frequency of 200 cycles/min when vibrating in a vacuum. The mass moment of inertia of the disc is 0.2 kg-m^2 . It is then immersed in oil and its natural frequency is found to be 180 cycles/min. Determine the damping constant. If the disc, when placed in oil, is given an initial displacement of 2° , find its displacement at the end of the first cycle.
- 2.134** A boy riding a bicycle can be modeled as a spring-mass-damper system with an equivalent weight, stiffness, and damping constant of 800 N, 50,000 N/m, and 1000 N-s/m, respectively. The differential setting of the concrete blocks on the road caused the level surface to decrease suddenly, as indicated in Fig. 2.116. If the speed of the bicycle is 5 m/s (18 km/hr), determine the displacement of the boy in the vertical direction. Assume that the bicycle is free of vertical vibration before encountering the step change in the vertical displacement.

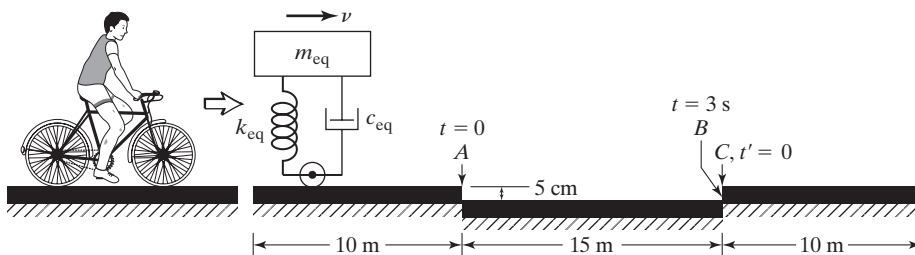


FIGURE 2.116 Bicycle on uneven concrete blocks.

- 2.135** A wooden rectangular prism of mass 10 kg, height 1 m, and cross section $30\text{ cm} \times 60\text{ cm}$ floats and remains vertical in a tub of oil. The frictional resistance of the oil can be assumed to be equivalent to a viscous damping coefficient ζ . When the prism is depressed by a distance of 15 cm from its equilibrium and released, it is found to reach a depth of 14 cm at the end of its first cycle of oscillation. Determine the value of the damping coefficient of the oil.
- 2.136** A body vibrating with viscous damping makes five complete oscillations per second, and in 50 cycles its amplitude diminishes to 10%. Determine the logarithmic decrement and the damping ratio. In what proportion will the period of vibration be decreased if damping is removed?
- 2.137** The maximum permissible recoil distance of a gun is specified as 0.5 m. If the initial recoil velocity is to be between 8 m/s and 10 m/s, find the mass of the gun and the spring stiffness of the recoil mechanism. Assume that a critically damped dashpot is used in the recoil mechanism and the mass of the gun has to be at least 500 kg.
- 2.138** A viscously damped system has a stiffness of 5000 N/m, critical damping constant of 0.2 N-s/mm, and a logarithmic decrement of 2.0. If the system is given an initial velocity of 1 m/s, determine the maximum displacement of the system.
- 2.139** Explain why an overdamped system never passes through the static equilibrium position when it is given (a) an initial displacement only and (b) an initial velocity only.
- 2.140–2.142** Derive the equation of motion and find the natural frequency of vibration of each of the systems shown in Figs. 2.117–2.119.

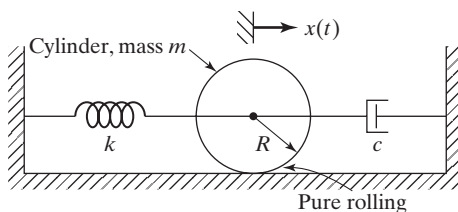


FIGURE 2.117 Roller connected to spring and damper.

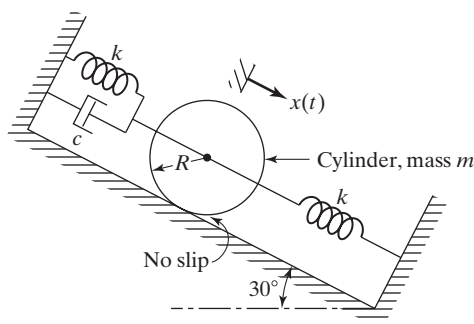


FIGURE 2.118 Roller with spring and damper on inclined plane.

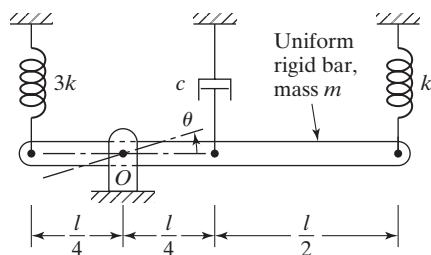


FIGURE 2.119 Rigid bar undergoing angular motion.

2.143–2.145 Using the principle of virtual work, derive the equation of motion for each of the systems shown in Figs. 2.117–2.119.

2.146 A wooden rectangular prism of cross section $40 \text{ cm} \times 60 \text{ cm}$, height 120 cm , and mass 40 kg floats in a fluid as shown in Fig. 2.111. When disturbed, it is observed to vibrate freely with a natural period of 0.5 s . Determine the density of the fluid.

2.147 The system shown in Fig. 2.120 has a natural frequency of 5 Hz for the following data: $m = 10 \text{ kg}$, $J_0 = 5 \text{ kg}\cdot\text{m}^2$, $r_1 = 10 \text{ cm}$, $r_2 = 25 \text{ cm}$. When the system is disturbed by giving it an initial displacement, the amplitude of free vibration is reduced by 80% in 10 cycles. Determine the values of k and c .

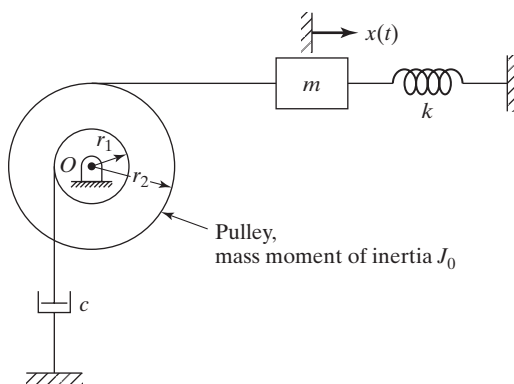


FIGURE 2.120 System with pulley, dashpot and spring.

2.148 The rotor of a dial indicator is connected to a torsional spring and a torsional viscous damper to form a single-degree-of-freedom torsional system. The scale is graduated in equal divisions, and the equilibrium position of the rotor corresponds to zero on the scale. When a torque of $2 \times 10^{-3} \text{ N}\cdot\text{m}$ is applied, the angular displacement of the rotor is found to be 50° with the pointer showing 80 divisions on the scale. When the rotor is released from this position, the pointer swings first to -20 divisions in one second and then to 5 divisions in another second. Find (a) the mass moment of inertia of the rotor, (b) the undamped natural time period of the rotor, (c) the torsional damping constant, and (d) the torsional spring stiffness.

- 2.149** Determine the values of ζ and ω_d for the following viscously damped systems:
- $m = 10 \text{ kg}$, $c = 150 \text{ N-s/m}$, $k = 1000 \text{ N/m}$
 - $m = 10 \text{ kg}$, $c = 200 \text{ N-s/m}$, $k = 1000 \text{ N/m}$
 - $m = 10 \text{ kg}$, $c = 250 \text{ N-s/m}$, $k = 1000 \text{ N/m}$
- 2.150** Determine the free-vibration response of the viscously damped systems described in Problem 2.149 when $x_0 = 0.1 \text{ m}$ and $\dot{x}_0 = 10 \text{ m/s}$.
- 2.151** Find the energy dissipated during a cycle of simple harmonic motion given by $x(t) = 0.2 \sin \omega_d t \text{ m}$ by a viscously damped single-degree-of-freedom system with the following parameters:
- $m = 10 \text{ kg}$, $c = 50 \text{ N-s/m}$, $k = 1000 \text{ N/m}$
 - $m = 10 \text{ kg}$, $c = 150 \text{ N-s/m}$, $k = 1000 \text{ N/m}$

- 2.152** The equation of motion of a spring-mass-damper system, with a hardening-type spring, is given by (in SI units)

$$100\ddot{x} + 500\dot{x} + 10,000x + 400x^3 = 0$$

- Determine the static equilibrium position of the system.
 - Derive the linearized equation of motion for small displacements (x) about the static equilibrium position.
 - Find the natural frequency of vibration of the system for small displacements.
- 2.153** The equation of motion of a spring-mass-damper system, with a softening-type spring, is given by (in SI units)

$$100\ddot{x} + 500\dot{x} + 10,000x - 400x^3 = 0$$

- Determine the static equilibrium position of the system.
 - Derive the linearized equation of motion for small displacements (x) about the static equilibrium position.
 - Find the natural frequency of vibration of the system for small displacements.
- 2.154** The needle indicator of an electronic instrument is connected to a torsional viscous damper and a torsional spring. If the rotary inertia of the needle indicator about its pivot point is 25 kg-m^2 and the spring constant of the torsional spring is 100 N-m/rad , determine the damping constant of the torsional damper if the instrument is to be critically damped.
- 2.155** Find the responses of systems governed by the following equations of motion for the initial conditions $x(0) = 0$, $\dot{x}(0) = 1$:
- $2\ddot{x} + 8\dot{x} + 16x = 0$
 - $3\ddot{x} + 12\dot{x} + 9x = 0$
 - $2\ddot{x} + 8\dot{x} + 8x = 0$

- 2.156** Find the responses of systems governed by the following equations of motion for the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$:
- $2\ddot{x} + 8\dot{x} + 16x = 0$
 - $3\ddot{x} + 12\dot{x} + 9x = 0$
 - $2\ddot{x} + 8\dot{x} + 8x = 0$

- 2.157** Find the responses of systems governed by the following equations of motion for the initial conditions $x(0) = 1$, $\dot{x}(0) = -1$:
- $2\ddot{x} + 8\dot{x} + 16x = 0$
 - $3\ddot{x} + 12\dot{x} + 9x = 0$
 - $2\ddot{x} + 8\dot{x} + 8x = 0$
- 2.158** A spring-mass system is found to vibrate with a frequency of 120 cycles per minute in air and 100 cycles per minute in a liquid. Find the spring constant k , the damping constant c , and the damping ratio ζ when vibrating in the liquid. Assume $m = 10$ kg.
- 2.159** Find the frequency of oscillation and time constant for the systems governed by the following equations:
- $\ddot{x} + 2\dot{x} + 9x = 0$
 - $\ddot{x} + 8\dot{x} + 9x = 0$
 - $\ddot{x} + 6\dot{x} + 9x = 0$
- 2.160** The mass moment of inertia of a nonhomogeneous and/or complex-shaped body of revolution about the axis of rotation can be determined by first finding its natural frequency of torsional vibration about its axis of rotation. In the torsional system shown in Fig. 2.121, the body of revolution (or rotor), of rotary inertia J , is supported on two frictionless bearings and connected to a torsional spring of stiffness k_t . By giving an initial twist (angular displacement) of θ_0 and releasing the rotor, the period of the resulting vibration is measured as τ .
- Find an expression for the mass moment of inertia of the rotor (J) in terms of τ and k_t .
 - Determine the value of J if $\tau = 0.5$ s and $k_t = 5000$ N-m/rad.

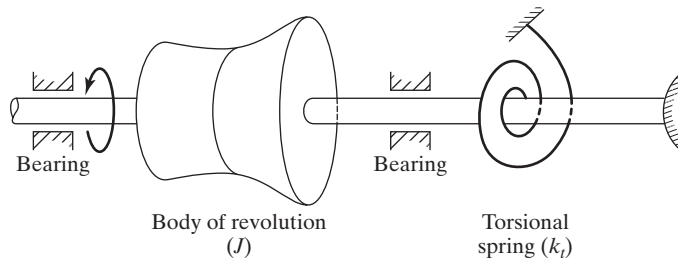


FIGURE 2.121 Body of revolution.

- 2.161** A damped system has the following parameters: $m = 2$ kg, $c = 3$ N-s/m, and $k = 40$ N/m. Determine the natural frequency, damping ratio, and the type of response of the system in free vibration. Find the amount of damping to be added or subtracted to make the system critically damped.
- 2.162** The response of a damped single-degree-of-freedom system is given by

$$x(t) = 0.05 e^{-10t} + 10.5t e^{-10t} \text{ m} \quad (\text{E.1})$$

Determine the natural frequency, damping constant, initial displacement, and initial velocity of the system.

Section 2.7 Graphical Representation of Characteristic Roots and Corresponding Solutions

2.163 The characteristic roots of a single-degree-of-freedom system are given below. Find all the applicable features of the system among the characteristic equation, time constant, undamped natural frequency, damped frequency, and damping ratio.

- a. $s_{1,2} = -4 \pm 5i$
- b. $s_{1,2} = 4 \pm 5i$
- c. $s_{1,2} = -4, -5$
- d. $s_{1,2} = -4, -4$

2.164 Show the characteristic roots indicated in Problem 2.163 (a)–(d) in the s -plane and describe the nature of the response of the system in each case.

2.165 The characteristic equation of a single-degree-of-freedom system, given by Eq. (2.107), can be rewritten as

$$s^2 + as + b = 0 \quad (\text{E.1})$$

where $a = c/m$ and $b = k/m$ can be considered as the parameters of the system. Identify regions that represent a stable, unstable, and marginally stable system in the parameter plane—i.e., the plane in which a and b are denoted along the vertical and horizontal axes, respectively.

Section 2.8 Parameter Variations and Root Locus Representations

2.166 Consider the characteristic equation: $2s^2 + cs + 18 = 0$. Draw the root locus of the system for $c \geq 0$.

2.167 Consider the characteristic equation: $2s^2 + 12s + k = 0$. Draw the root locus of the system for $k \geq 0$.

2.168 Consider the characteristic equation: $ms^2 + 12s + 4 = 0$. Draw the root locus of the system for $m \geq 0$.

Section 2.9 Free Vibration with Coulomb Damping

2.169 A single-degree-of-freedom system consists of a mass of 20 kg and a spring of stiffness 4000 N/m. The amplitudes of successive cycles are found to be 50, 45, 40, 35, ... mm. Determine the nature and magnitude of the damping force and the frequency of the damped vibration.

2.170 A mass of 20 kg slides back and forth on a dry surface due to the action of a spring having a stiffness of 10 N/mm. After four complete cycles, the amplitude has been found to be 100 mm. What is the average coefficient of friction between the two surfaces if the original amplitude was 150 mm? How much time has elapsed during the four cycles?

2.171 A 10-kg mass is connected to a spring of stiffness 3000 N/m and is released after giving an initial displacement of 100 mm. Assuming that the mass moves on a horizontal surface, as

shown in Fig. 2.42(a), determine the position at which the mass comes to rest. Assume the coefficient of friction between the mass and the surface to be 0.12.

- 2.172** A weight of 25 N is suspended from a spring that has a stiffness of 1000 N/m. The weight vibrates in the vertical direction under a constant damping force. When the weight is initially pulled downward a distance of 10 cm from its static equilibrium position and released, it comes to rest after exactly two complete cycles. Find the magnitude of the damping force.
- 2.173** A mass of 20 kg is suspended from a spring of stiffness 10,000 N/m. The vertical motion of the mass is subject to Coulomb friction of magnitude 50 N. If the spring is initially displaced downward by 5 cm from its static equilibrium position, determine (a) the number of half cycles elapsed before the mass comes to rest, (b) the time elapsed before the mass comes to rest, and (c) the final extension of the spring.
- 2.174** The Charpy impact test is a dynamic test in which a specimen is struck and broken by a pendulum (or hammer) and the energy absorbed in breaking the specimen is measured. The energy values serve as a useful guide for comparing the impact strengths of different materials. As shown in Fig. 2.122, the pendulum is suspended from a shaft, is released from a particular position, and is allowed to fall and break the specimen. If the pendulum is made to oscillate freely (with no specimen), find (a) an expression for the decrease in the angle of swing for each cycle caused by friction, (b) the solution for $\theta(t)$ if the pendulum is released

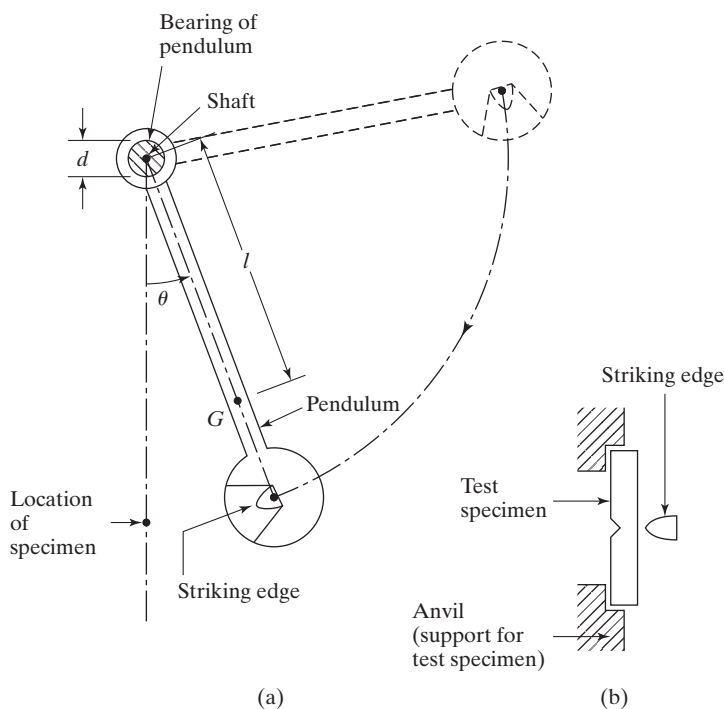


FIGURE 2.122 Charpy impact test.

from an angle θ_0 , and (c) the number of cycles after which the motion ceases. Assume the mass of the pendulum is m and the coefficient of friction between the shaft and the bearing of the pendulum is μ .

2.175 Find the equivalent viscous-damping constant for Coulomb damping for sinusoidal vibration.

2.176 A single-degree-of-freedom system consists of a mass, a spring, and a damper in which both dry friction and viscous damping act simultaneously. The free-vibration amplitude is found to decrease by 1% per cycle when the amplitude is 20 mm and by 2% per cycle when the amplitude is 10 mm. Find the value of $(\mu N/k)$ for the dry-friction component of the damping.

2.177 A metal block, placed on a rough surface, is attached to a spring and is given an initial displacement of 10 cm from its equilibrium position. It is found that the natural time period of motion is 1.0 s and that the amplitude reduces by 0.5 cm in each cycle. Find (a) the kinetic coefficient of friction between the metal block and the surface and (b) the number of cycles of motion executed by the block before it stops.

2.178 The mass of a spring-mass system with $k = 10,000$ N/m and $m = 5$ kg is made to vibrate on a rough surface. If the friction force is $F = 20$ N and the amplitude of the mass is observed to decrease by 50 mm in 10 cycles, determine the time taken to complete the 10 cycles.

2.179 The mass of a spring-mass system vibrates on a dry surface inclined at 30° to the horizontal as shown in Fig. 2.123.

a. Derive the equation of motion.

b. Find the response of the system for the following data:

$$m = 20 \text{ kg}, \quad k = 1000 \text{ N/m}, \quad \mu = 0.1, \quad x_0 = 0.1 \text{ m}, \quad \dot{x}_0 = 5 \text{ m/s}.$$

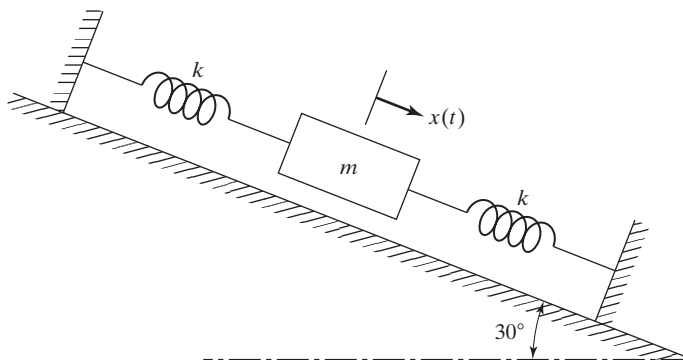


FIGURE 2.123 Spring-mass system on inclined plane.

2.180 The mass of a spring-mass system is initially displaced by 10 cm from its unstressed position by applying a force of 25 N, which is equal to five times the weight of the mass. If the mass is released from this position, how long will the mass vibrate and at what distance will it stop from the unstressed position? Assume a coefficient of friction of 0.2.

Section 2.10 Free Vibration with Hysteretic Damping

- 2.181** The experimentally observed force-deflection curve for a composite structure is shown in Fig. 2.124. Find the hysteresis damping constant, the logarithmic decrement, and the equivalent viscous-damping ratio corresponding to this curve.

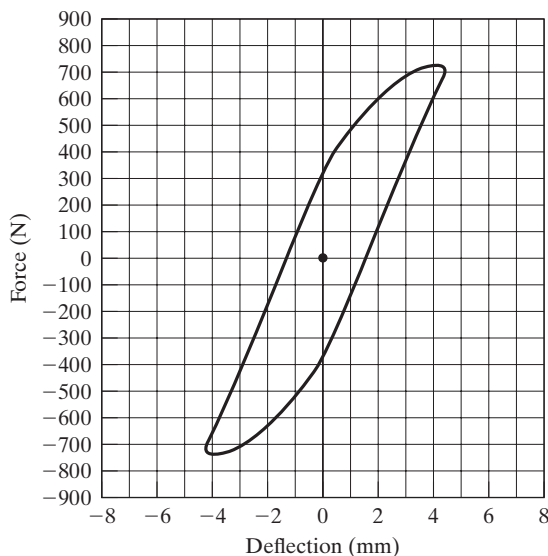


FIGURE 2.124 Force-deflection curve.

- 2.182** A panel made of fiber-reinforced composite material is observed to behave as a single-degree-of-freedom system of mass 1 kg and stiffness 2 N/m. The ratio of successive amplitudes is found to be 1.1. Determine the value of the hysteresis-damping constant β , the equivalent viscous-damping constant c_{eq} , and the energy loss per cycle for an amplitude of 10 mm.
- 2.183** A built-up cantilever beam having a bending stiffness of 200 N/m supports a mass of 2 kg at its free end. The mass is displaced initially by 30 mm and released. If the amplitude is found to be 20 mm after 100 cycles of motion, estimate the hysteresis-damping constant β of the beam.
- 2.184** A mass of 5 kg is attached to the top of a helical spring, and the system is made to vibrate by giving to the mass an initial deflection of 25 mm. The amplitude of the mass is found to reduce to 10 mm after 100 cycles of vibration. Assuming a spring rate of 200 N/m for the helical spring, find the value of the hysteretic-damping coefficient (h) of the spring.

Section 2.11 Stability of Systems

- 2.185** Consider the equation of motion of a simple pendulum:

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0 \quad (\text{E.1})$$

- a. Linearize Eq. (E.1) about an arbitrary angular displacement θ_0 of the pendulum.
- b. Investigate the stability of the pendulum about $\theta_0 = 0$ and $\theta_0 = \pi$ using the linearized equation of motion.

2.186 Figure 2.125 shows a uniform rigid bar of mass m and length l , pivoted at one end (point O) and carrying a circular disk of mass M and mass moment of inertia J (about its rotational axis) at the other end (point P). The circular disk is connected to a spring of stiffness k and a viscous damper of damping constant c as indicated.

- a. Derive the equation of motion of the system for small angular displacements of the rigid bar about the pivot point O and express it in the form:

$$m_0 \ddot{\theta} + c_0 \dot{\theta} + k_0 \theta = 0$$

- b. Derive conditions corresponding to the stable, unstable, and marginally stable behavior of the system.

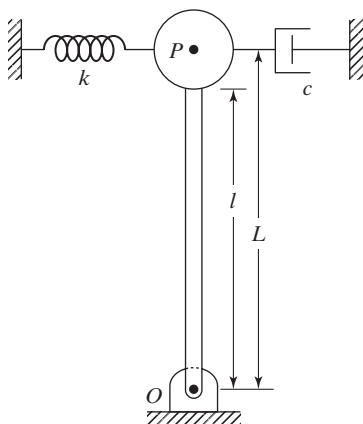


FIGURE 2.125 Angular displacement of a rigid bar.

Section 2.12 Examples Using MATLAB

2.187 Find the free-vibration response of a spring-mass system subject to Coulomb damping using MATLAB for the following data:

$$m = 5 \text{ kg}, \quad k = 100 \text{ N/m}, \quad \mu = 0.5, \quad x_0 = 0.4 \text{ m}, \quad \dot{x}_0 = 0.$$

2.188 Plot the response of a critically damped system (Eq. 2.80) for the following data using MATLAB:

- a. $x_0 = 10 \text{ mm}, 50 \text{ mm}, 100 \text{ mm}; \dot{x}_0 = 0, \omega_n = 10 \text{ rad/s}.$
- b. $x_0 = 0, \dot{x}_0 = 10 \text{ mm/s}, 50 \text{ mm/s}, 100 \text{ mm/s}; \omega_n = 10 \text{ rad/s}.$

2.189 Plot Eq. (2.81) as well as each of the two terms of Eq. (2.81) as functions of t using MATLAB for the following data:

$$\omega_n = 10 \text{ rad/s}, \quad \zeta = 2.0, \quad x_0 = 20 \text{ mm}, \quad \dot{x}_0 = 50 \text{ mm/s}$$

2.190–2.193 Using the MATLAB Program2.m, plot the free-vibration response of a viscously damped system with $m = 4$ kg, $k = 2500$ N/m, $x_0 = 100$ mm, $\dot{x}_0 = -10$ m/s, $\Delta t = 0.01$ s, $n = 50$ for the following values of the damping constant:

- a. $c = 0$
- b. $c = 100$ N-s/m
- c. $c = 200$ N-s/m
- d. $c = 400$ N-s/m

2.194 Find the response of the system described in Problem 2.179 using MATLAB.

DESIGN PROJECTS

2.195* A water turbine of mass 1000 kg and mass moment of inertia 500 kg-m^2 is mounted on a steel shaft, as shown in Fig. 2.126. The operational speed of the turbine is 2400 rpm. Assuming the ends of the shaft to be fixed, find the values of l , a , and d , such that the natural frequency of vibration of the turbine in each of the axial, transverse, and circumferential directions is greater than the operational speed of the turbine.

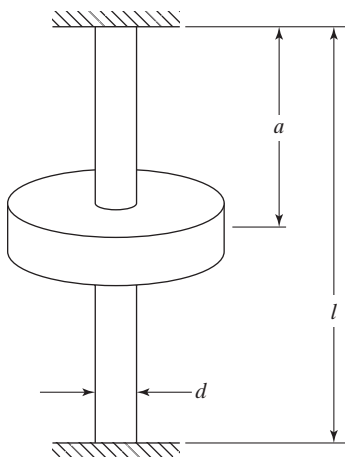


FIGURE 2.126 Water turbine on a shaft.

2.196* Design the columns for each of the building frames shown in Figs. 2.79(a) and (b) for minimum weight such that the natural frequency of vibration is greater than 50 Hz. The mass of the floor (m) is 2000 kg and the length of the columns (l) is 2.5 m. Assume that the columns are made of steel and have a tubular cross section with outer diameter d and wall thickness t .

2.197* One end of a uniform rigid bar of mass m is connected to a wall by a hinge joint O , and the other end carries a concentrated mass M , as shown in Fig. 2.127. The bar rotates about the hinge point O against a torsional spring and a torsional damper. It is proposed to use this

*The asterisk denotes a design-type problem or a problem with no unique answer.

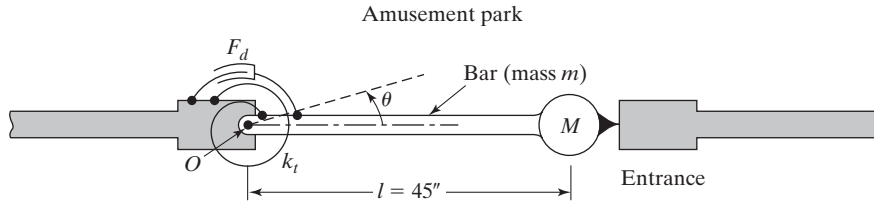


FIGURE 2.127 Amusement park gate.

mechanism, in conjunction with a mechanical counter, to control entrance to an amusement park. Find the masses m and M , the stiffness of the torsional spring (k_t), and the damping force (F_d) necessary to satisfy the following specifications: (1) A viscous damper or a Coulomb damper can be used. (2) The bar has to return to within 5° of closing in less than 2 s when released from an initial position of $\theta = 75^\circ$.

- 2.198*** The lunar excursion module has been modeled as a mass supported by four symmetrically located legs, each of which can be approximated as a spring-damper system with negligible mass (see Fig. 2.128). Design the springs and dampers of the system in order to have the damped period of vibration between 1 s and 2 s.

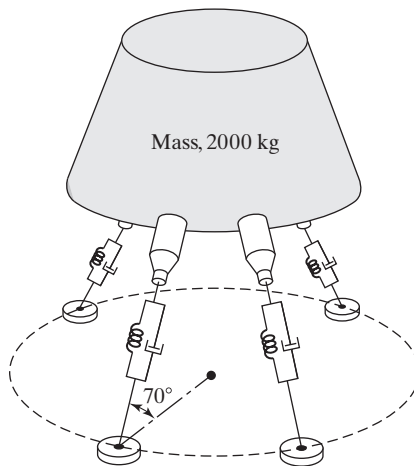


FIGURE 2.128 Lunar excursion module.

- 2.199*** Consider the telescoping boom and cockpit of the firetruck shown in Fig. 2.12(a). Assume that the telescoping boom $PQRS$ is supported by a strut QT , as shown in Fig. 2.129. Determine the cross section of the strut QT so that the natural time period of vibration of the cockpit with the fireperson is equal to 1 s for the following data. Assume that each segment of the telescoping boom and the strut is hollow circular in cross section. In addition, assume that the strut acts as a spring that deforms only in the axial direction.

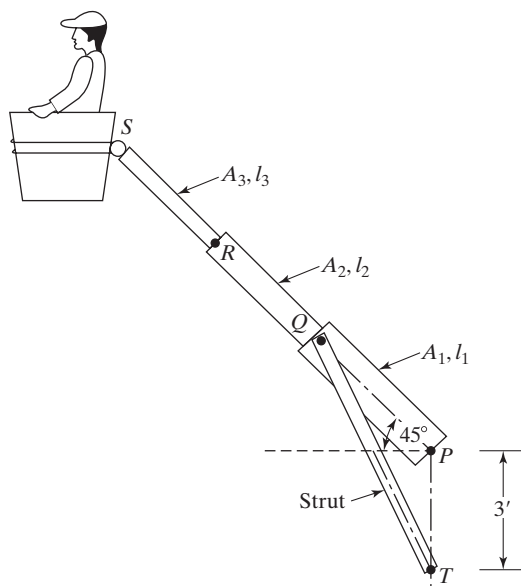


FIGURE 2.129 Telescoping boom and cockpit of fireman.

Data:

Lengths of segments: $PQ = 3.6$ m, $QR = 3$ m, $RS = 2.4$ m, $TP = 0.9$ m

Young's modulus of the telescoping arm and strut = 200 GPa

Outer diameters of sections: $PQ = 5$ cm, $QR = 3.75$ cm, $RS = 2.5$ cm

Inner diameters of sections: $PQ = 4.5$ cm, $QR = 3.25$ cm, $RS = 2$ cm

Weight of the cockpit = 50 kg

Weight of fireperson = 100 kg