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Chapter 1

Linear Transformations

1.1 Linear Transformations

Definition 1.1.1 — linear transformation. let V and W be vector spaces over the field \mathbb{F} . A linear transformation from V into W is a function T from V into W such that

$$T(c\alpha + \beta) = cT\alpha + T\beta,$$

for all $\alpha, \beta \in V$ and $c \in \mathbb{F}$.

■ **Example 1** Let A be a fixed $m \times n$ matrix with entries in the field \mathbb{F} . The function T defined by $T(x) = Ax$ is a linear transformation from $\mathbb{F}^{n \times 1}$ to $\mathbb{F}^{m \times 1}$. The function U defined by $U(\alpha) = \alpha A$ is a linear transformation from $\mathbb{F}^{1 \times m}$ to $\mathbb{F}^{1 \times n}$.

■ **Example 2** Let \mathbb{R} be the field of real numbers and let V be the space of all functions from \mathbb{R} to \mathbb{R} which are continuous. Define T by $(Tf)(x) = \int_0^x f(t) dt$. Then T is a linear transformation from V to V .

Remark 1.1.1 It's important to notice that if T is a linear transformation from V into W , then

$$T(0_V) = 0_W.$$

Remark 1.1.2 Linear transformation is actually defined to preserve linear combinations. That is

$$T(c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n) = c_1T(\alpha_1) + c_2T(\alpha_2) + \dots + c_nT(\alpha_n)$$

Theorem 1.1.1 Let V be a finite dimensional vector space over the field \mathbb{F} , let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be any ordered basis for V , let W be a vector space over the same field \mathbb{F} and let $\beta_1, \beta_2, \dots, \beta_n$ be any vectors in W . Then there is precisely one linear transformation T from V into W such that $T\alpha_j = \beta_j$, for all $j = 1, \dots, n$.

Proof. Given $\alpha \in V$, there is a unique n -tuple (x_1, x_2, \dots, x_n) such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n.$$

We define $T\alpha = x_1\beta_1 + x_2\beta_2 + \dots + x_n\beta_n$. Then T is a well-defined rule for associating with each vector $\alpha \in V$ a vector $T\alpha \in W$. It's clear that $T\alpha_j = \beta_j$ for each j and T is a linear transformation. If U is a linear transformation from V into W with $U\alpha_j = \beta_j, j = 1, 2, \dots, n$, then for the vector $\alpha = \sum_{i=1}^n x_i\alpha_i$ we have

$$U\alpha = U\left(\sum_{i=1}^n x_i\alpha_i\right) = \sum_{i=1}^n x_i(U\alpha_i) = \sum_{i=1}^n x_i\beta_i.$$

So U is exactly the same rule T which we defined. This shows that the linear transformation T with $T\alpha_j = \beta_j$ for each j is unique. ■

Remark 1.1.3 The proof of Theorem 1.1.1 show us the way to actually get the transformation T .

Problem 1.1.1 The vector $\alpha_1 = (1, 2)$, $\alpha_2 = (3, 4)$ form a basis for \mathbb{R}^2 . $\beta_1 = (3, 2, 1)$, $\beta_2 = (6, 5, 4)$ are two vectors in \mathbb{R}^3 . We now want to find a linear transformation T such that $T\alpha_j = \beta_j$. We see that $(1, 0) = -2(3, 2, 1) + (6, 5, 4)$, thus $T(1, 0) = -2(3, 2, 1) + (6, 5, 4) = (0, 1, 2)$. Similarly, we can find $T(0, 1)$. Then we know all about this T .

■ **Example 3** Let T be a linear transformation from \mathbb{F}^m to \mathbb{F}^n . By Theorem 1.1.1 we know that T is uniquely determined by the sequence of vectors $\beta_1, \beta_2, \dots, \beta_m \in \mathbb{F}^n$ where

$$\beta_i = T\epsilon_i, \quad i = 1, 2, \dots, m,$$

Namely, if we have

$$\alpha = (x_1, x_2, \dots, x_m),$$

then

$$T\alpha = x_1\beta_1 + \dots + x_m\beta_m.$$

If B is the $m \times n$ matrix which has row vectors $\beta_1, \beta_2, \dots, \beta_m$, this says that

$$T\alpha = \alpha B.$$

Remark 1.1.4 From Example 3 we show that we can give an explicit and reasonably simple description of all linear transformations from \mathbb{F}^m to \mathbb{F}^n .

Remark 1.1.5 If T is a linear transformation from V into W , then the range of T is a subspace of W .

Remark 1.1.6 The set N consisting of the vectors $\alpha \in V$ such that $T\alpha = 0$ is a subspace of V .

Definition 1.1.2 — null space, rank, nullity. Let V and W be vector spaces over the field \mathbb{F} and let T be a linear transformation from V to W . The null space of T is the set of all vectors $\alpha \in V$ such that $T\alpha = 0$. If V is finite-dimensional, the rank of T is the dimension of range of T and the nullity of T is the dimension of the null space of T .

Theorem 1.1.2 Let V and W be vector spaces over the field \mathbb{F} and let T be a linear transformation from V into W . Suppose that V is finite-dimensional, then

$$\text{rank } T + \text{nullity } T = \dim V$$

Proof. Let $\{\alpha_1, \dots, \alpha_k\}$ be a basis for the null space of T . We can find $\alpha_{k+1}, \dots, \alpha_n \in V$ such that $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis for V . We shall show that $\{T\alpha_{k+1}, \dots, T\alpha_n\}$ is a basis for the range of T . The vector $T\alpha_1, \dots, T\alpha_n$ certainly span the range of T . Since $T\alpha_1, \dots, T\alpha_k$ are zero, $\{T\alpha_{k+1}, \dots, T\alpha_n\}$ spans the range of T . To see they are also independent, suppose we have scalars c_i such that

$$\sum_{i=k+1}^n c_i(T\alpha_i) = 0.$$

Then we have $T(\sum_{i=k+1}^n c_i\alpha_i) = 0$ and accordingly $\alpha = \sum_{i=k+1}^n c_i\alpha_i$ is in the null space of T . We then must have $c_i = 0$ for $i = k+1, \dots, n$. So $\{T\alpha_{k+1}, \dots, T\alpha_n\}$ is a basis for the range, we are done. ■

Theorem 1.1.3 If A is an $m \times n$ matrix with entries in the field \mathbb{F} , then

$$\text{row rank}(A) = \text{column rank}(A).$$

Proof. Let T be the linear transformation from $\mathbb{F}^{n \times 1}$ to $\mathbb{F}^{m \times 1}$ defined by $T(x) = Ax$. Say S is the solution space of the system $Ax = 0$, then nullity $T = \dim S$. So we have

$$\dim S + \text{rank } T = n.$$

Notice that $\text{rank } T$ is actually the dimension of the column space of matrix A . Say the RREF of matrix A is R and the row rank for A and R is r . Let R_1, R_2, \dots, R_n be the columns of matrix R , then there are r of these columns, say R_{p_1}, \dots, R_{p_r} , have a single 1 as their only non-zero entry. Therefore, considering the space S , there are $n - r$ free variables, which means $\dim S = n - r$. So we have $\text{column rank } A = \text{row rank } A$. ■

1.2 The Algebra of Linear Transformation

Theorem 1.2.1 Let V and W be vector spaces over the field \mathbb{F} . Let T and U be linear transformations from V into W . The function $(T + U)$ defined by

$$(T + U)(\alpha) = T\alpha + U\alpha$$

is a linear transformation from V into W . If c is any element of \mathbb{F} , the function (cT) defined by $(cT)(\alpha) = c(T\alpha)$ is a linear transformation from V into W . The set of all linear transformation from V into W , together with the addition and scalar multiplication defined above, is a vector space over the field \mathbb{F} . ■

Proof. Omit. ■

Remark 1.2.1 We denote the space of linear transformations from V into W by $L(V, W)$.

Theorem 1.2.2 Let V be an n dimensional vector space over the field \mathbb{F} , and let W be an m dimensional vector spaces over the field \mathbb{F} . Then the space $L(V, W)$ is finite-dimensional and has dimension mn .

Proof. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\mathcal{B}' = \{\beta_1, \beta_2, \dots, \beta_m\}$ be ordered bases for V and W , respectively. For each pair of integer (p, q) with $1 \leq p \leq n$ and $1 \leq q \leq m$, we define a linear transformation $E^{p,q}$ from V into W by $E^{p,q}(\alpha_i) = \delta_{iq}\beta_p$.

Let T be a linear transformation from V to W . $(A_{1j}, A_{2j}, \dots, A_{mj})$ is the coordinate vector of $T(\alpha_j)$ in the ordered basis \mathcal{B}' , i.e.,

$$T(\alpha_j) = \sum_{i=1}^m A_{ij}\beta_i$$

Our claim is

$$T = \sum_{p=1}^m \sum_{q=1}^n A_{pq} E^{p,q}.$$

Actually, Let U be the linear transformation defined by RHS of the equation, then for each j ,

$$\begin{aligned} U(\alpha_j) &= \sum_{p=1}^m \sum_{q=1}^n A_{pq} E^{p,q} \alpha_j \\ &= \sum_{p=1}^m \sum_{q=1}^n A_{pq} \delta_{jq} \beta_p \\ &= \sum_{p=1}^m A_{pj} \beta_p = T(\alpha_j). \end{aligned}$$

So $T = U$. This means $\{E^{p,q}\}$ spans $L(V, W)$. Furthermore, $\{E^{p,q}\}$ are independent because if $T = \sum_p \sum_q A_{pq} E^{p,q}$ is the zero transformation, then for each j , $T(\alpha_j) = \sum_i A_{ij} \beta_i = 0$. So $A_{ij} = 0$ for every i, j . ■

Theorem 1.2.3 Let V, W, Z be vector spaces over the field \mathbb{F} . Let U be a linear transformation from V into W , and T be a linear transformation from W to Z , then the function (TU) defined by $(TU)(\alpha) = T(U(\alpha))$ is a linear transformation from V to Z .

Proof. Omit. ■

Definition 1.2.1 — linear operator. Let V be a vector space over the field \mathbb{F} . A linear operator on V is a linear transformation from V into V .

Remark 1.2.2 We notice that in Theorem 1.2.3, if $V = W = Z$, then (TU) is also a linear operator on V , i.e., there is a ‘multiplication’ operation defined by composition on $L(V, V)$. In addition, (UT) is also defined, but in general $(UT) - (TU)$ is not zero transformation.

Remark 1.2.3 If T is a linear operator on V , then we can define $T^n = TTT \dots T$ without confusion. Proof is omitted. For convenience, we define $T^0 = I$ (identity transformation).

Lemma 1 Let V be a vector space over the field \mathbb{F} , U, T_1, T_2 be linear operators on V , c is any element in the field \mathbb{F} .

- 1) $U = UI = IU$
- 2) $U(T_1 + T_2) = UT_1 + UT_2; (T_1 + T_2)U = T_1U + T_2U$
- 3) $c(UT) = (cU)T = U(cT)$

Remark 1.2.4 Lemma 1 and Theorem 1.2.3 tell us that $L(V, V)$, together with composition, is what known as a linear algebra with identity.

■ **Example 4** Let A be an $m \times n$ matrix and T be a linear transformation defined by $T(X) = AX$. Let B be an $p \times m$ matrix and U be a linear transformation defined by $U(Y) = BY$. Then

$$\begin{aligned} (UT)(X) &= U(T(X)) \\ &= U(AX) \\ &= B(AX) = BAX. \end{aligned}$$

Remark 1.2.5 The effect of composition of U, T is multiplication of matrices B, A .

Definition 1.2.2 — invertible. A linear transformation T from V into W is invertible if there exist a function U from W into V such that (UT) is the identity transformation on V and (TU) is the identity transformation on W . In this case, U is unique and we denote U by T^{-1} .

Remark 1.2.6 In Definition 1.2.2, T^{-1} exists if and only if

1. T is one-one. ($T\alpha = T\beta \implies \alpha = \beta$)
2. T is onto. (The range of T is W)

Theorem 1.2.4 Let T be a linear transformation from V into W . If T is invertible, then the inverse T^{-1} is a linear transformation from W into V .

Proof. Omit. ■

Remark 1.2.7 We see that $T^{-1}U^{-1}$ is the left and right inverse of UT , therefore the inverse of (UT) is $T^{-1}U^{-1}$.

Definition 1.2.3 — non-singular. We call a linear transformation T non-singular if $T\gamma = 0 \implies \gamma = 0$, i.e., the null space of T is $\{0\}$.

Remark 1.2.8 Evidently, T is one-one if and only if T is non-singular.

Remark 1.2.9 Non-singular linear transformations are those which preserve linear independence, as the following theorem claims.

Theorem 1.2.5 Let T be a linear transformation from V into W . Then T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W .

Proof. Suppose that T is non-singular. Let S be a linearly independent subset of V . If $\alpha_1, \dots, \alpha_k$ are vectors in S , then the vector $T\alpha_1, \dots, T\alpha_k$ are linearly independent. For if

$$c_1(T\alpha_1) + \dots + c_k(T\alpha_k) = 0$$

then

$$T(c_1\alpha_1 + \dots + c_k\alpha_k) = 0$$

therefore

$$c_1\alpha_1 + \dots + c_k\alpha_k = 0.$$

Since α_i are linearly independent, we have for each $j = 1, 2, \dots, k$, $c_j = 0$.

Suppose that T carries independent subsets onto independent subsets. Then T must be non-singular. For if $T\alpha = 0$, and α is not 0. Then an independent set S consisting of α will have its image a dependent set. ■

Theorem 1.2.6 Let V and W be finite-dimensional vector spaces over the field \mathbb{F} such that $\dim V = \dim W$. If T is a linear transformation from V into W , the following are equivalent:

- (i) T is invertible.
- (ii) T is non-singular.
- (iii) T is on-to.

Proof. Let $n = \dim V = \dim W$. Since

$$\text{rank } T + \text{nullity } T = n.$$

So if T is non-singular, then $\text{nullity } T = 0$ and $\text{rank } T = n$, i.e., T is on-to. If T is on-to, then $\text{rank } T = n$ and therefore $\text{nullity } T = 0$ (T is non-singular).

Therefore T is non-singular if and only if $T(V) = W$. So, if either condition (ii) or (iii) holds, the other is satisfied as well and T is invertible. ■

1.3 Isomorphism

Definition 1.3.1 — isomorphism. If V and W are vector spaces over the field \mathbb{F} , any one-one linear transformation T from V onto W is called an isomorphism of V onto W . If there exists an isomorphism of V onto W , we say that V is isomorphic to W .

Remark 1.3.1 If V is isomorphic to W , then W is isomorphic to V .

Theorem 1.3.1 Every n -dimensional vector space over the field \mathbb{F} is isomorphic to the space \mathbb{F}^n .

Proof. Let V be an n -dimensional vector space over the field \mathbb{F} and let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ be a basis for V . We define a function T from V to \mathbb{F}^n , as follows: If α is in V , let $T\alpha$ be the n -tuple (x_1, \dots, x_n) of coordinates of α relative to the ordered basis β . Also, it's easy to verify T is a linear transformation and T is one-one and on-to. ■

Remark 1.3.2 One often identifies isomorphic spaces though the vectors and operations may be quite different.

1.4 Representation of Transformations by Matrices

Let V be an n -dimensional vector space over the field \mathbb{F} and let W be an m -dimensional vector space over \mathbb{F} . Let $\mathcal{B} = \alpha_1, \alpha_2, \dots, \alpha_n$ be an ordered basis for V and $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$ an ordered basis for W . If T is any linear transformation from V to W , then T is determined by its action on vectors α_j . Each of the n vectors $T\alpha_j$ is uniquely expressible as a linear combination

$$T\alpha_j = \sum_{i=1}^m A_{ij}\beta_i$$

of the β_i , the scalars A_{1j}, \dots, A_{mj} being the coordinates of $T\alpha_j$ in the ordered basis \mathcal{B}' . Accordingly, the transformation T is determined by the mn scalars A_{ij} . The $m \times n$ matrix A defined by $A(i, j) = A_{ij}$ is called *the matrix of T relative to the pair of ordered bases \mathcal{B} and \mathcal{B}'* .

If $\alpha = x_1\alpha_1 + \dots + x_n\alpha_n$ is a vector in V , then

$$\begin{aligned} T\alpha &= T\left(\sum_{j=1}^n x_j\alpha_j\right) \\ &= \sum_{j=1}^n x_j T(\alpha_j) \\ &= \sum_{j=1}^n x_j \sum_{i=1}^m A_{ij}\beta_i \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n A_{ij}x_j\right) \beta_i. \end{aligned}$$

If X is the coordinate matrix of α in the ordered basis \mathcal{B} , then the computation above shows that AX is the coordinate matrix of the vector $T\alpha$ in the ordered basis \mathcal{B}' , because the scalar

$$\sum_{j=1}^n A_{ij}x_j$$

is the entry in the i th row of the column matrix AX .

We also observe that if A is any $m \times n$ matrix over the field \mathbb{F} , then

$$T\left(\sum_{j=1}^n x_j\alpha_j\right) = \sum_{i=1}^m \left(\sum_{j=1}^n A_{ij}x_j\right) \beta_i$$

defines a linear transformation T from V into W , the matrix of which is A , relative to $\mathcal{B}, \mathcal{B}'$. We summarize formally:

Theorem 1.4.1 Let V be an n -dimensional vector space over the field \mathbb{F} and W an m -dimensional vector space over \mathbb{F} . Let \mathcal{B} be an ordered basis for V and \mathcal{B}' be an ordered basis for W . For each linear transformation T from V into W , there is an $m \times n$ matrix A with entries in \mathbb{F} such that

$$[T\alpha]_{\mathcal{B}'} = A[\alpha]_{\mathcal{B}}$$

for every vector α in V . Furthermore, $T \rightarrow A$ is a one-one correspondence between the set of all linear transformations from V into W and the set of all $m \times n$ matrices over the field \mathbb{F} .

Definition 1.4.1 The matrix A associated with T in Theorem 1.4.1 is called the *matrix of T relative to the ordered bases $\mathcal{B}, \mathcal{B}'$* .

Remark 1.4.1 Note that A is the matrix whose columns A_1, \dots, A_n are given by

$$A_j = [T\alpha_j]_{\mathcal{B}'}, \quad j = 1, \dots, n.$$

Remark 1.4.2 If T, U are linear transformation from V into W and $A = [A_1, \dots, A_n], B = [B_1, \dots, B_n]$ is the matrix of T, U relative to the ordered bases $\mathcal{B}, \mathcal{B}'$, then $cA + B$ is the matrix of $cT + U$ relative to $\mathcal{B}, \mathcal{B}'$ because

$$\begin{aligned} cA_j + B_j &= c[T\alpha_j]_{\mathcal{B}'} + [U\alpha_j]_{\mathcal{B}'} \\ &= [cT\alpha_j + U\alpha_j]_{\mathcal{B}'} \\ &= [(cT + U)\alpha_j]_{\mathcal{B}'} \end{aligned}$$

Theorem 1.4.2 Let V be an n -dimensional vector space over the field \mathbb{F} and let W be an m -dimensional vector space over \mathbb{F} . For each pair of ordered bases $\mathcal{B}, \mathcal{B}'$ for V, W respectively, the function which assigns to a linear transformation T its matrix relative to $\mathcal{B}, \mathcal{B}'$ is an isomorphism between the space $L(V, W)$ and the space of all $m \times n$ matrices over the field \mathbb{F} .

Proof. Omit. ■

Remark 1.4.3 If we are considering the representation by matrices of linear transformations of a space into itself, i.e., linear operators on a space V . In this case it's convenient to use the same ordered basis in each case, that is, to take $\mathcal{B} = \mathcal{B}'$. We shall then call the representing matrix simply *the matrix of T relative to the ordered basis \mathcal{B}* , denoted by $[T]_{\mathcal{B}}$. Note that we have

$$[T\alpha]_{\mathcal{B}} = [T]_{\mathcal{B}}[\alpha]_{\mathcal{B}}.$$

Let V, W , and Z be vector spaces over the field \mathbb{F} of respective dimensions n, m and p . Let T be a linear transformation from V into W and U a linear transformation from W into Z . Suppose we have ordered bases

$$\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}, \quad \mathcal{B}' = \{\beta_1, \dots, \beta_m\}, \quad \mathcal{B}'' = \{\gamma_1, \dots, \gamma_p\}$$

for the respective spaces V, W and Z .