Contents

Chapter 1

Linear Transformations

1.1 Linear Transformations

Definition 1.1.1 — linear transformation. let V and W be vector spaces over the field $\mathbb{F}$. A linear transformation from V into W is a function T from V into W such that

$$T(c\alpha + \beta) = cT\alpha + T\beta,$$

for all $\alpha, \beta \in V$ and $c \in \mathbb{F}$.

■ **Example 1** Let A be a fixed $m \times n$matrix with entries in the field $\mathbb{F}$. The function T defined by $T(x) = Ax$ is a linear transformation from $\mathbb{F}^{n\times 1}$ to $\mathbb{F}^{m\times 1}$. The function U defined by $U(\alpha) = \alpha A$ is a linear transformation from $F^{1\times m}$ to $F^{1\times n}$

■ **Example 2** Let $\mathbb{R}$ be the field of real numbers and let V be the space of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are continuous. Define T by $(Tf)(x) = \int_0^x f(t)\,\mathrm{d}t$. Then T is a linear transformation from V to V.

Remark 1.1.1 It's important to notice that if T is a linear transformation from V into W, then

$$T(0_V) = 0_W.$$

Remark 1.1.2 Linear transformation is actually defined to preserve linear combinations. That is

$$T(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n) = c_1T(\alpha_1) + c_2T(\alpha_2) + \ldots + c_nT(\alpha_n)$$

Theorem 1.1.1 Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be any ordered basis for V, let W be a vector space over the same field $\mathbb{F}$ and let $\beta_1, \beta_2, \ldots, \beta_n$ be any vectors in W. Then there is precisely one linear transformation T from V into W such that $T\alpha_j = \beta_j$, for all $j = 1, \ldots, n$.

Proof. Given $\alpha \in V$, there is a unique n-tuple $(x_1, x_2, \ldots, x_n)$ such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n.$$

We define $T\alpha = x_1\beta_1 + x_2\beta_2 \ldots + x_n\beta_n$. Then T is a well-defined rule for associating with each vector $\alpha \in V$ a vector $T\alpha \in W$. It's clear that $T\alpha_j = \beta_j$ for each j and T is a linear transformation. If U is a linear transformation from V into W with $U\alpha_j = \beta_j, j = 1, 2, \ldots, n$, then for the vector

$\alpha = \sum_{i=1}^{n} x_i\alpha_i$ we have

$$U\alpha = U\left(\sum_{i=1}^{n} x_i\alpha_i\right) = \sum_{i=1}^{n} x_i\left(U\alpha_i\right) = \sum_{i=1}^{n} x_i\beta_i.$$

So U is exactly the same rule T which we defined. This shows that the linear transformation T with $T\alpha_j = \beta_j$ for each j is unique. ■

Remark 1.1.3 The proof of Theorem 1.1.1 show us the way to actually get the transformation T.

Problem 1.1.1 The vector $\alpha_1 = (1,2)$, $\alpha_2 = (3,4)$ form a basis for $\mathbb{R}^2$. $\beta_1 = (3,2,1)$, $\beta_2 = (6,5,4)$ are two vectors in $\mathbb{R}^3$. We now want to find a linear transformation T such that $T\alpha_j = \beta_j$.
We see that $(1,0) = -2(3,2,1) + (6,5,4)$, thus $T(1,0) = -2(3,2,1) + (6,5,4) = (0,1,2)$. Similarly, we can find $T(0,1)$. Then we know all about this T.

■ **Example 3** Let T be a linear transformation from $\mathbb{F}^m$ to $\mathbb{F}^n$. By Theorem 1.1.1 we know that T is uniquely determined by the sequence of vectors $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{F}^n$ where

$$\beta_i = T\epsilon_i, \qquad i = 1,2,\ldots,m,$$

Namely, if we have

$$\alpha = (x_1, x_2, \ldots, x_m),$$

then

$$T\alpha = x_1\beta_1 + \ldots + x_n\beta_n.$$

If B is the $m \times n$ matrix which has row vectors $\beta_1, \beta_2, \ldots, \beta_m$, this says that

$$T\alpha = \alpha B.$$

Remark 1.1.4 From Example 3 we show that we can give an explicit and reasonably simple description of all linear transformations from $\mathbb{F}^m$ to $\mathbb{F}^n$.

Remark 1.1.5 If T is a linear transformatino from V into W, then the range of T is a subspace of W.

Remark 1.1.6 The set N consisting of the vectors $\alpha \in V$ such that $T\alpha = 0$ is a subspace of V.

Definition 1.1.2 — null space, rank, nullity. Let V and W be vector spaces over the field $\mathbb{F}$ and let T be a linear transformation from V to W. The null space of T is the set of all vectors $\alpha \in V$ such that $T\alpha = 0$. If V is finite-dimensional, the rank of T is the dimension of range of T and the nullity of T is the dimension of the null space of T.

Theorem 1.1.2 Let V and W be vector spaces over the field $\mathbb{F}$ and let T be a linear transformation from V into W. Suppose that V is finite-dimensional, then

$$\operatorname{rank} T + \operatorname{nullity} T = \dim V$$

Proof. Let $\{\alpha_1, \ldots, \alpha_k\}$ be a basis for the null space of T. We can find $\alpha_{k+1}, \ldots, \alpha_n \in V$ such that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for V. We shall show that $\{T\alpha_{k+1}, \ldots, T\alpha_n\}$ is a basis for the range of T. The vector $T\alpha_1, \ldots, T\alpha_n$ certainly span the range of T. Since $T\alpha_1, \ldots, T\alpha_k$ are zero,

$\{T\alpha_{k+1}, \dots, T\alpha_n\}$ spans the range of T. To see they are also independent, suppose we have scalars c_i such that

$$\sum_{i=k+1}^{n} c_i(T\alpha_i) = 0.$$

Then we have $T(\sum_{i=k+1}^{n} c_i\alpha_i) = 0$ and accordingly $\alpha = \sum_{i=k+1}^{n} c_i\alpha_i$ is in the null space of T. We then must have $c_i = 0$ for $i = k+1, \dots, n$. So $\{T\alpha_{k+1}, \dots, T\alpha_n\}$ is a basis for the range, we are done. ■

Theorem 1.1.3 If A is an $m \times n$ matrix with entries in the field $\mathbb{F}$, then

$$\text{row rank}(A) = \text{column rank}(A).$$

Proof. Let T be the linear transformation from $\mathbb{F}^{n\times 1}$ to $\mathbb{F}^{m\times 1}$ defined by $T(x) = Ax$. Say S is the solution space of the system $Ax = 0$, then $\operatorname{nullity} T = \dim S$. So we have

$$\dim S + \operatorname{rank} T = n.$$

Notice that $\operatorname{rank} T$ is actually the dimension of the column space of matrix A. Say the RREF of matrix A is R and the row rank for A and R is r. Let $R_1, R_2, \dots, R_n$ be the columns of matrix R, then there are r of these columns, say $R_{p_1}, \dots, \mathbb{R}_{p_r}$, have a single 1 as their only non-zero entry. Therefore, considering the space S, there are $n - r$ free variables, which means $\dim S = n - r$. So we have column $\operatorname{rank} A = \text{row } \operatorname{rank} A$. ■

1.2 The Algebra of Linear Transformation

Theorem 1.2.1 Let V and W be vector spaces over the field $\mathbb{F}$. Let T and U be linear transformations from V into W. The function $(T + U)$ defined by

$$(T + U)(\alpha) = T\alpha + U\alpha$$

is a linear transformation from V into W. If c is any element of $\mathbb{F}$, the function (cT) defined by $(cT)(\alpha) = c(T\alpha)$ is a linear transformation from V into W. The set of all linear transformation from V into W, together with the addition and scalar multiplication defined above, is a vector space over the field $\mathbb{F}$.

Proof. Omit. ■

Remark 1.2.1 We denote the space of linear transformations from V into W by $L(V, W)$.

Theorem 1.2.2 Let V be an n dimensional vector space over the field $\mathbb{F}$, and let W be an m dimensional vector spaces over the field $\mathbb{F}$. Then the space $L(V, W)$ is finite-dimensional and has dimension mn.

Proof. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\mathcal{B}' = \{\beta_1, \beta_2, \dots, \beta_m\}$ be ordered bases for V and W, respectively. For each pair of integer (p, q) with $1 \le p \le m$ and $1 \le q \le n$, we define a linear transformation $E^{p,q}$ from V into W by $E^{p,q}(\alpha_i) = \delta_{iq}\beta_p$.
Let T be a linear transformation from V to W. $(A_{1j}, A_{2j}, \dots, A_{mj})$ is the coordinate vector of $T(\alpha_j)$ in the ordered basis $\mathcal{B}'$, i.e.,

$$T(\alpha_j) = \sum_{i=1}^{m} A_{ij}\beta_i$$

Our claim is

$$T = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{p,q}.$$

Actually, Let U be the linear transformation defined by RHS of the equation, then for each j,

$$\begin{aligned} U(\alpha_j) &= \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{p,q}\alpha_j \\ &= \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}\delta_{jq}\beta_p \\ &= \sum_{p=1}^{m} A_{pj}\beta_p = T(\alpha_j). \end{aligned}$$

So $T = U$. This means $\{E^{p,q}\}$ spans $L(V,W)$. Furthermore, $\{E^{p,q}\}$ are independent because if $T = \sum\limits_p\sum\limits_q A_{pq}E^{p,q}$ is the zero transformation, then for each j, $T(\alpha_j) = \sum\limits_i A_{ij}\beta_i = 0$. So $A_{ij} = 0$ for every i, j. ■

Theorem 1.2.3 Let V, W, Z be vector spaces over the field $\mathbb{F}$. Let U be a linear transosformation from V into W, and T be a linear transformation from W to Z, then the funtion (TU) defined by $(TU)(\alpha) = T(U(\alpha))$ is a linear transformation from V to Z.

Proof. Omit. ■

Definition 1.2.1 — linear operator. Let V be a vector spaces over the field $\mathbb{F}$. A linear operator on V is a linear transformation from V into V.

Remark 1.2.2 We notice that in Theorem 1.2.3, if $V = W = Z$, then (TU) is also a linear operator on V, i.e., there is a 'multiplication' operation defined by composition on $L(V,V)$. In addition, (UT) is also defined, but in general $(UT) - (TU)$ is not zero transformation.

Remark 1.2.3 If T is a linear operator on V, then we can define $T^n = TTT\ldots T$ without confusion. Proof is omitted. For convenience, we define $T^0 = I$(identity transformation).

Lemma 1 Let V be a vector space over the field $\mathbb{F}$, U, T_1, T_2 be linear operators on V, c is any elements in the field $\mathbb{F}$.

1) $U = UI = IU$
2) $U(T_1 + T_2) = UT_1 + UT_2; (T_1 + T_2)U = T_1U + T_2U$
3) $c(UT) = (cU)T = U(cT)$

Remark 1.2.4 Lemma 1 and Theorem 1.2.3 tell us that $L(V,V)$, together with cosposion, is what known as a linear algebra with identity.

■ **Example 4** Let A be an $m \times n$matrix and T be a linear transformation defined by $T(X) = Ax$. Let B be an $p \times m$ matrix and U be a linear transformation defined by $U(Y) = BY$. Then

$$\begin{aligned} (UT)(X) &= U(T(X)) \\ &= U(AX) \\ &= B(AX) = BAX. \end{aligned}$$

Remark 1.2.5 The effect of cosposition of U, T is multiplication of matrices B, A.

Definition 1.2.2 — invertible. A linear transformation T from V into W is invertible if there exist a function U from W into V such that (UT) is the identity transformation on V and (TU) is the identity transformaton on W. In this case, U is unique and we denote U by T^{-1}.

Remark 1.2.6 In Definition 1.2.2, T^{-1} exists if and only if

1. T is one-one. ($T\alpha = T\beta \implies \alpha = \beta$)
2. T is onto. (The range of T is W)

Theorem 1.2.4 let T be a linear transformation from V into W. If T is invertible, then the inverse T^{-1} is a linear transformation from W into V.

Proof. Omit. ∎

Remark 1.2.7 We see that $T^{-1}U^{-1}$ is the left and right inverse of UT, therefore the inverse of (UT) is $T^{-1}U^{-1}$.

Definition 1.2.3 — non-singular. We call a linear transformation T non-singular if $T\gamma = 0 \implies \gamma = 0$,i.e., the null space of T is 0.

Remark 1.2.8 Evidently, T is one-one if and only if T is non-singular.

Remark 1.2.9 Non-singular linear transformations are those which preserve linear independence, as the following theorem claims.

Theorem 1.2.5 Let T be a linear transformation from V into W. Then T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

Proof. Suppose that T is non-singular. Let S be a linearly independent subset of V. If $\alpha_1, \dots, \alpha_k$ are vectors in S, then the vector $T\alpha_1, \dots, T\alpha_k$ are linearly independent. For if

$$c_1(T\alpha_1) + \ldots + c_k(T\alpha_k) = 0$$

then

$$T(c_1\alpha_1 + \ldots + c_k\alpha_k) = 0$$

therefore

$$c_1\alpha_1 + \ldots + c_k\alpha_k = 0.$$

Since α_i are linearly independent, we have for each $j = 1, 2, \dots, k$, $c_j = 0$.
Suppose that T carries independent subsets onto independent subsets. Then T must be non-singular. For if $T\alpha = 0$, and α is not 0. Then an independent set S consisting of α will have its image a dependent set. ∎

Theorem 1.2.6 Let V and W be finite-dimensional vector spaces over the field $\mathbb{F}$ such that $\dim V = \dim W$. If T is a linear transformation from V into W, the following are equivalent:
(i) T is invertible.
(ii) T is non-singular.
(iii) T is on-to.

Proof. Let $n = \dim V = \dim W$. Since

$$\operatorname{rank} T + \operatorname{nullity} T = n.$$

So if T is non-singular, then $\operatorname{nullity} T = 0$ and $\operatorname{rank} T = n$, i.e., T is on-to. If T is on-to, then $\operatorname{rank} T = n$ and therefore $\operatorname{nullity} T = 0$ (T is non-singular).
Therefore T is non-singular if and only if $T(V) = W$. So, if either condition (ii) or (iii) holds, the other is satisfied as well and T is invertible. ■

1.3 Isomorphism

Definition 1.3.1 — isomorphism. If V and W are vector spaces over the field $\mathbb{F}$, any one-one linear transformation T from V onto W is called an isomorphism of V onto W. If there exists an isomorphism of V onto W, we say that V is isomorphic to W.

Remark 1.3.1 If V is isomorphic to W, then W is isomorphic to V.

Theorem 1.3.1 Every n-dimensional vector space over the field $\mathbb{F}$is isomorphic to the space $\mathbb{F}^n$.

Proof. Let V be an n-dimensional vector space over the field $\mathbb{F}$and let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ be a basis for V. We define a function T from V to $\mathbb{F}^n$, as follows: If α is in V, let $T\alpha$ be the n-tuple $(x_1, \dots, x_n)$ of coordinates of α relative to the ordered basis β. Also, it's easy to verify T is a linear transformation and T is one-one and on-to. ■

Remark 1.3.2 One often identifies isomorphic spaces though the vectors and operations may be quite different.

1.4 Representation of Transformations by Matrices

Let V be an n-dimensional vector space over the field $\mathbb{F}$and let W be an m-dimensional vector space over $\mathbb{F}$. Let $\mathcal{B} = \alpha_1, \alpha_2, \dots, \alpha_n$ be an ordered basis for V and $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$ an ordered basis for W. If T is any linear transformation from V to W, then T is determined by its action on vectors α_j. Each of the n vectors $T\alpha_j$ is uniquely expressible as a linear combination

$$T\alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i$$

of the β_i, the scalars $A_{1j}, \dots, A_{mj}$ being the coordinates of $T\alpha_j$ in the ordered basis $\mathcal{B}'$. Accordingly, the transformation T is determined by the mn scalars A_{ij}. The $m \times n$matrix A defined by $A(i,j) = A_{ij}$ is called *the matrix of T relative to the pair of ordered bases $\mathcal{B}$ and $\mathcal{B}'$*.

If $\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$ is a vector in V, then

$$\begin{aligned} T\alpha &= T\left(\sum_{j=1}^{n} x_j\alpha_j\right) \\ &= \sum_{j=1}^{n} x_j T(\alpha_j) \\ &= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\beta_i \\ &= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij}x_j\right)\beta_i. \end{aligned}$$

If X is the coordinate matrix of α in the ordered basis $\mathcal{B}$, then the computation above shows that AX is the oordinate matrix of the vector $T\alpha$ in the ordered basis $\mathcal{B}'$, bacause the scalar

$$\sum_{j=1}^{n} A_{ij}x_j$$

is the entry in the ith row of the column matrix AX.
We also observe that if A is any $m \times n$ matrix over the field $\mathbb{F}$, then

$$T\left(\sum_{j=1}^{n} x_j\alpha_j\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij}x_j\right)\beta_i$$

defines a linear transformation T from V into W, the matrix of which is A, relative to $\mathcal{B}, \mathcal{B}'$. We summarize formally:

Theorem 1.4.1 Let V be an n-dimensional vector spaces over the field $\mathbb{F}$and W an m-dimensional vector space over $\mathbb{F}$. Let $\mathcal{B}$ be an ordered basis for V and $\mathcal{B}'$ be an ordered basis for W. For each linear transformation T from V into W, there is an $m \times n$ matrix A with entries in $\mathbb{F}$ such that

$$[T\alpha]_{\mathcal{B}'} = A[\alpha]_{\mathcal{B}}$$

for every vector α in V. Furthermore, $T \to A$ is a one-one correspondence between the set of all linear transformations from V into W and the set of all $m \times n$ matrices over the field $\mathbb{F}$.

Definition 1.4.1 The matrix A associated with T in Theorem 1.4.1 is called the *matrix of T relative to the ordered bases $\mathcal{B}, \mathcal{B}'$*.

Remark 1.4.1 Note that A is the matrix whose columns $A_1, \ldots, A_n$ are given by

$$A_j = [T\alpha_j]_{\mathcal{B}'}, \quad j = 1, \ldots, n.$$

Remark 1.4.2 If T, U are linear transformation from V into W and $A = [A_1, \ldots, A_n], B = [B_1, \ldots, B_n]$ is the matrix of T, U relative to the ordered bases $\mathcal{B}, \mathcal{B}'$, then $cA + B$ is the matrix of $cT+U$ relative to $\mathcal{B},\mathcal{B}'$ because

$$\begin{aligned} cA_j+B_j &= c[T\alpha_j]_{\mathcal{B}'}+[U\alpha_j]_{\mathcal{B}'} \\ &= [cT\alpha_j+U\alpha_j]_{\mathcal{B}'} \\ &= [(cT+U)\alpha_j]_{\mathcal{B}'}. \end{aligned}$$

Theorem 1.4.2 Let V be an n-dimensional vector space over the field $\mathbb{F}$and let W be an m-dimensional vector space over $\mathbb{F}$. For each pair of ordered bases $\mathcal{B},\mathcal{B}'$ for V,W respectively, the function which assigns to a linear transformation T its matrix relative to $\mathcal{B},\mathcal{B}'$ is an isomorphism between the space $L(V,W)$ and the space of all $m\times n$ matrices over the field $\mathbb{F}$.

Proof. Omit. ■

Remark 1.4.3 If we are considering the representation by matrices of linear transformations of a space into itself, i.e., linear operators on a space V. In this case it's convenient to use the same ordered basis in each case, that is, to take $\mathcal{B}=\mathcal{B}'$. We shall then call the representing matrix simply *the matrix of T relative to the ordered basis $\mathcal{B}$*, denoted by $[T]_{\mathcal{B}}$. Note that we have

$$[T\alpha]_{\mathcal{B}}=[T]_{\mathcal{B}}[\alpha]_{\mathcal{B}}.$$

Let V,W, and Z be vector spaces over the field $\mathbb{F}$of respective dimensions n,m and$p$. Let T be a linear transformation from V into W and U a linear transformation from W into Z. Suppose we have ordered bases

$$\mathcal{B}=\{\alpha_1,\dots,\alpha_n\},\quad \mathcal{B}'=\{\beta_1,\dots,\beta_m\},\quad \mathcal{B}''=\{\gamma_1,\dots,\gamma_p\}$$

for the respective spaces V,W and Z.Let A be the matrix of T relative to the pair $\mathcal{B},\mathcal{B}'$ and let B be the matrix of U relative to pair $\mathcal{B},\mathcal{B}''$. It is then easy to see that the matrix C of the transformation UT relative to the pair $\mathcal{B},\mathcal{B}''$ is the product of B and A; for , if $\alpha$is any vector in V

$$[T\alpha]_{\mathcal{B}'}=A[\alpha]_{\mathcal{B}}$$

$$[U(T\alpha)]_{\mathcal{B}''}=B[T\alpha]_{\mathcal{B}'}$$

and so

$$[(UT)(\alpha)]_{\mathcal{B}''}=BA[\alpha]_{\mathcal{B}}$$

and hence, by the definition and uniqueness of the representing matrix, we must have $C=BA$. One can also see this by carrying out the computation

$$\begin{aligned} (UT)(\alpha_j) &= U(T\alpha_j) \\ &= U\left(\sum_{k=1}^{m}A_{kj}\beta_k\right) \\ &= \sum_{k=1}^{m}A_{kj}(U\beta_k) \\ &= \sum_{k=1}^{m}A_{kj}\sum_{i=1}^{p}B_{ik}\gamma_i \\ &= \sum_{i=1}^{p}\left(\sum_{k=1}^{m}B_{ik}A_{kj}\gamma_i\right) \end{aligned}$$

so that we must have

$$C_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj}.$$

Theorem 1.4.3 Let V, W, and Z be finite-dimensional vector spaces over the field $\mathbb{F}$; let T be linear transformation from V into W and U a linear transformation from W into Z. If $\mathcal{B}, \mathcal{B}'$, and $\mathcal{B}''$ are ordered bases for the spaces V, W, and Z, respectively, if A is the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$, and B is the matrix of U relative to the pair $\mathcal{B}', \mathcal{B}''$, then the matrix of the composition UT relative to the pair $\mathcal{B}, \mathcal{B}''$ is the product matrix $C = BA$.

Remark 1.4.4 Theorem 1.4.3 gives a proof that matrix multiplication is assciative. (a proof which requires no calculations)

Remark 1.4.5 If T and U are linear operators on a space V and we are representing by a single ordered basis $\mathcal{B}$, then Theorem 1.4.3 assumes the simple form $[UT]_{\mathcal{B}} = [U]_{\mathcal{B}}[T]_{\mathcal{B}}$. Thus in this case, the correspondence with $\mathcal{B}$ determines between operators and matices is not only a vector space isomorphism but also preserves products. A simple consequence of this is that the linear operator T is invertible if and only if $[T]_{\mathcal{B}}$ is an invertible matrix . For identity operator I is represented by the identity matrix in any ordered basis, and thus

$$UT = TU = I$$

is equivalent to

$$[U]_{\mathcal{B}}[T]_{\mathcal{B}} = [T]_{\mathcal{B}}[U]_{\mathcal{B}} = I.$$

Of course, when T is invertible

$$[T^{-1}]_{\mathcal{B}} = [T]_{\mathcal{B}}^{-1}.$$

Let T be a linear operator on the finite-dimensional space V, and let

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\} \quad \mathcal{B}' = \{\alpha_1', \ldots, \alpha_n'\}$$

be two ordered bases for V. How are the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{B}'}$ related? We observe there is a unique and invertible $n \times n$ matrix P such that

$$[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$$

for every vector α in V. It is the matrix $P = [P_1, \ldots, P_n]$ where $P_j = [\alpha_j']_{\mathcal{B}}$. By definition

$$[T\alpha]_{\mathcal{B}} = [T]_{\mathcal{B}}[\alpha]_{\mathcal{B}}.$$

So we have

$$[T\alpha]_{\mathcal{B}} = P[T\alpha]_{\mathcal{B}'}.$$

$$[T]_{\mathcal{B}}P[\alpha]_{\mathcal{B}'} = P[T\alpha]_{\mathcal{B}'}$$

or

$$P^{-1}[T]_{\mathcal{B}}P[\alpha]_{\mathcal{B}'} = [T\alpha]_{\mathcal{B}'}$$

and so it must be that

$$[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P.$$

Remark 1.4.6 There is a unique linear operator U which carries $\mathcal{B}$ onto $\mathcal{B}'$, defined by

$$U\alpha_j = \alpha'_j, \quad j = 1, \ldots, n.$$

This operator U is invertible since it carries a basis for V onto a basis for V. The matrix P is precisely the matrix of the operator U in the ordered basis $\mathcal{B}$. For, P is defined by

$$\alpha'_j = \sum_{i=1}^{n} P_{ij}\alpha_i$$

and since $U\alpha_j = \alpha'_j$, this equation can be written

$$U\alpha_j = \sum_{i=1}^{n} P_{ij}\alpha_i.$$

So $P = [U]_{\mathcal{B}}$, by definition.

Theorem 1.4.4 Let V be a finite-dimensional vector space over the field $\mathbb{F}$, and let

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\} \quad \text{and} \quad \mathcal{B}' = \{\alpha'_1, \ldots, \alpha'_n\}$$

be ordered bases for V. Suppose T is a linear operator on V. If $P = [P_1, \ldots, P_n]$ is the $n \times n$ matrix with columns $P_j = [\alpha'_j]_{\mathcal{B}}$, then

$$[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P.$$

Alternatively, if U is the invertible operator on V defined by $U\alpha_j = \alpha'_j, j = 1, \ldots, n$, then

$$[T]_{\mathcal{B}'} = [U]_{\mathcal{B}}^{-1}[T]_{\mathcal{B}}[U]_{\mathcal{B}}$$

Definition 1.4.2 — similar. Let A and B be $n \times n$ matrices over the field $\mathbb{F}$. We say that B is similar to A over $\mathbb{F}$if there is an ivertible $n \times n$ matrix P over $\mathbb{F}$such that $B = P^{-1}AP$.

Remark 1.4.7 Similarity is an equivalence relation on the set of $n \times n$ matrices over the field $\mathbb{F}$.

1.5 Linear Functionals