Contents

Chapter 1

Linear Transformations

1.1 Linear Transformations

Definition 1.1.1 — linear transformation. let V and W be vector spaces over the field $\mathbb{F}$. A linear transformation from V into W is a function T from V into W such that

$$T(c\alpha + \beta) = cT\alpha + T\beta,$$

for all $\alpha, \beta \in V$ and $c \in \mathbb{F}$.

■ **Example 1** Let A be a fixed $m \times n$matrix with entries in the field $\mathbb{F}$. The function T defined by $T(x) = Ax$ is a linear transformation from $\mathbb{F}^{n\times 1}$ to $\mathbb{F}^{m\times 1}$. The function U defined by $U(\alpha) = \alpha A$ is a linear transformation from $F^{1\times m}$ to $F^{1\times n}$

■ **Example 2** Let $\mathbb{R}$ be the field of real numbers and let V be the space of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are continuous. Define T by $(Tf)(x) = \int_0^x f(t)\,\mathrm{d}t$. Then T is a linear transformation from V to V.

Remark 1.1.1 It's important to notice that if T is a linear transformation from V into W, then

$$T(0_V) = 0_W.$$

Remark 1.1.2 Linear transformation is actually defined to preserve linear combinations. That is

$$T(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n) = c_1T(\alpha_1) + c_2T(\alpha_2) + \ldots + c_nT(\alpha_n)$$

Theorem 1.1.1 Let V be a finite dimensional vector space over the field $\mathbb{F}$, let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be any ordered basis for V, let W be a vector space over the same field $\mathbb{F}$ and let $\beta_1, \beta_2, \ldots, \beta_n$ be any vectors in W. Then there is precisely one linear transformation T from V into W such that $T\alpha_j = \beta_j$, for all $j = 1, \ldots, n$.

Proof. Given $\alpha \in V$, there is a unique n-tuple $(x_1, x_2, \ldots, x_n)$ such that

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n.$$

We define $T\alpha = x_1\beta_1 + x_2\beta_2 \ldots + x_n\beta_n$. Then T is a well-defined rule for associating with each vector $\alpha \in V$ a vector $T\alpha \in W$. It's clear that $T\alpha_j = \beta_j$ for each j and T is a linear transformation. If U is a linear transformation from V into W with $U\alpha_j = \beta_j, j = 1, 2, \ldots, n$, then for the vector $\alpha = \sum_{i=1}^n x_i\alpha_i$ we have

$$U\alpha = U\left(\sum_{i=1}^n x_i\alpha_i\right) = \sum_{i=1}^n x_i\,(U\alpha_i) = \sum_{i=1}^n x_i\beta_i.$$

So U is exactly the same rule T which we defined. This shows that the linear transformation T with $T\alpha_j = \beta_j$ for each j is unique. ■

Remark 1.1.3 The proof of Theorem 1.1.1 show us the way to actually get the transformation T.

Problem 1.1.1 The vector $\alpha_1 = (1,2)$, $\alpha_2 = (3,4)$ form a basis for $\mathbb{R}^2$. $\beta_1 = (3,2,1)$, $\beta_2 = (6,5,4)$ are two vectors in $\mathbb{R}^3$. We now want to find a linear transformation T such that $T\alpha_j = \beta_j$.
We see that $(1,0) = -2(3,2,1) + (6,5,4)$, thus $T(1,0) = -2(3,2,1) + (6,5,4) = (0,1,2)$. Similarly, we can find $T(0,1)$. Then we know all about this T.

Example 3 Let T be a linear transformation from $\mathbb{F}^m$ to $\mathbb{F}^n$. By Theorem 1.1.1 we know that T is uniquely determined by the sequence of vectors $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{F}^n$ where

$$\beta_i = T\epsilon_i, \qquad i = 1, 2, \ldots, m,$$

Namely, if we have

$$\alpha = (x_1, x_2, \ldots, x_m),$$

then

$$T\alpha = x_1\beta_1 + \ldots + x_n\beta_n.$$

If B is the $m \times n$ matrix which has row vectors $\beta_1, \beta_2, \ldots, \beta_m$, this says that

$$T\alpha = \alpha B.$$

Remark 1.1.4 From Example 3 we show that we can give an explicit and reasonably simple description of all linear transformations from $\mathbb{F}^m$ to $\mathbb{F}^n$.

Remark 1.1.5 If T is a linear transformatino from V into W, then the range of T is a subspace of W.

Remark 1.1.6 The set N consisting of the vectors $\alpha \in V$ such that $T\alpha = 0$ is a subspace of V.

Definition 1.1.2 — null space, rank, nullity. Let V and W be vector spaces over the field $\mathbb{F}$ and let T be a linear transformation from V to W. The null space of T is the set of all vectors $\alpha \in V$ such that $T\alpha = 0$. If V is finite-dimensional, the rank of T is the dimension of range of T and the nullity of T is the dimension of the null space of T.

Theorem 1.1.2 Let V and W be vector spaces over the field $\mathbb{F}$ and let T be a linear transformation from V into W. Suppose that V is finite-dimensional, then

$$\operatorname{rank} T + \operatorname{nullity} T = \dim V$$

Proof. Let $\{\alpha_1, \ldots, \alpha_k\}$ be a basis for the null space of T. We can find $\alpha_{k+1}, \ldots, \alpha_n \in V$ such that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for V. We shall show that $\{T\alpha_{k+1}, \ldots, T\alpha_n\}$ is a basis for the range of T. The vector $T\alpha_1, \ldots, T\alpha_n$ certainly span the range of T. Since $T\alpha_1, \ldots, T\alpha_k$ are zero, $\{T\alpha_{k+1}, \ldots, T\alpha_n\}$ spans the range of T. To see they are also independent, suppose we have scalars c_i such that

$$\sum_{i=k+1}^{n} c_i(T\alpha_i) = 0.$$

Then we have $T(\sum_{i=k+1}^{n} c_i\alpha_i) = 0$ and accordingly $\alpha = \sum_{i=k+1}^{n} c_i\alpha_i$ is in the null space of T. We then must have $c_i = 0$ for $i = k+1, \ldots, n$. So $\{T\alpha_{k+1}, \ldots, T\alpha_n\}$ is a basis for the range, we are done. ■

Theorem 1.1.3 If A is an $m \times n$ matrix with entries in the field $\mathbb{F}$, then

$$\text{row rank}(A) = \text{column rank}(A).$$

Proof. Let T be the linear transformation from $\mathbb{F}^{n\times 1}$ to $\mathbb{F}^{m\times 1}$ defined by $T(x) = Ax$. Say S is the solution space of the system $Ax = 0$, then nullity $T = \dim S$. So we have

$$\dim S + \operatorname{rank} T = n.$$

Notice that $\operatorname{rank} T$ is actually the dimension of the column space of matrix A. Say the RREF of matrix A is R and the row rank for A and R is r. Let $R_1, R_2, \dots, R_n$ be the columns of matrix R, then there are r of these columns, say $R_{p_1}, \dots, \mathbb{R}_{p_r}$, have a single 1 as their only non-zero entry. Therefore, considering the space S, there are $n - r$ free variables, which means $\dim S = n - r$. So we have column $\operatorname{rank} A = \operatorname{row} \operatorname{rank} A$. ■

1.2 The Algebra of Linear Transformation

Theorem 1.2.1 Let V and W be vector spaces over the field $\mathbb{F}$. Let T and U be linear transformations from V into W. The function $(T+U)$ defined by

$$(T+U)(\alpha) = T\alpha + U\alpha$$

is a linear transformation from V into W. If c is any element of $\mathbb{F}$, the function (cT) defined by $(cT)(\alpha) = c(T\alpha)$ is a linear transformation from V into W. The set of all linear transformation from V into W, together with the addition and scalar multiplication defined above, is a vector space over the field $\mathbb{F}$.

Proof. Omit. ■

Remark 1.2.1 We denote the space of linear transformations from V into W by $L(V, W)$.

Theorem 1.2.2 Let V be an n dimensional vector space over the field $\mathbb{F}$, and let W be an m dimensional vector spaces over the field $\mathbb{F}$. Then the space $L(V, W)$ is finite-dimensional and has dimension mn.

Proof. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\mathcal{B}' = \{\beta_1, \beta_2, \dots, \beta_m\}$ be ordered bases for V and W, respectively. For each pair of integer (p, q) with $1 \le p \le m$ and $1 \le q \le n$, we define a linear transformation $E^{p,q}$ from V into W by $E^{p,q}(\alpha_i) = \delta_{iq}\beta_p$.
Let T be a linear transformation from V to W. $(A_{1j}, A_{2j}, \dots, A_{mj})$ is the coordinate vector of $T(\alpha_j)$ in the ordered basis $\mathcal{B}'$, i.e.,

$$T(\alpha_j) = \sum_{i=1}^{m} A_{ij}\beta_i$$

Our claim is

$$T = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{p,q}.$$

Actually, Let U be the linear transformation defined by RHS of the equation, then for each j,

$$\begin{aligned} U(\alpha_j) &= \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{p,q}\alpha_j \\ &= \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}\delta_{jq}\beta_p \\ &= \sum_{p=1}^{m} A_{pj}\beta_p = T(\alpha_j). \end{aligned}$$

So $T = U$. This means $\{E^{p,q}\}$ spans $L(V, W)$. Furthermore, $\{E^{p,q}\}$ are independent because if $T = \sum_p \sum_q A_{pq}E^{p,q}$ is the zero transformation, then for each j, $T(\alpha_j) = \sum_i A_{ij}\beta_i = 0$. So $A_{ij} = 0$ for every i, j. ■

Theorem 1.2.3 Let V, W, Z be vector spaces over the field $\mathbb{F}$. Let U be a linear transoformation from V into W, and T be a linear transformation from W to Z, then the funtion (TU) defined by $(TU)(\alpha) = T(U(\alpha))$ is a linear transformation from V to Z.

Proof. Omit. ■

Definition 1.2.1 — linear operator. Let V be a vector spaces over the field $\mathbb{F}$. A linear operator on V is a linear transformation from V into V.

Remark 1.2.2 We notice that in Theorem 1.2.3, if $V = W = Z$, then (TU) is also a linear operator on V, i.e., there is a 'multiplication' operation defined by composition on $L(V, V)$. In addition, (UT) is also defined, but in general $(UT) - (TU)$ is not zero transformation.

Remark 1.2.3 If T is a linear operator on V, then we can define $T^n = TTT \dots T$ without confusion. Proof is omitted. For convenience, we define $T^0 = I$(identity transformation).

Lemma 1 Let V be a vector space over the field $\mathbb{F}$, U, T_1, T_2 be linear operators on V, c is any elements in the field $\mathbb{F}$.

1) $U = UI = IU$
2) $U(T_1 + T_2) = UT_1 + UT_2; (T_1 + T_2)U = T_1U + T_2U$
3) $c(UT) = (cU)T = U(cT)$

Remark 1.2.4 Lemma 1 and Theorem 1.2.3 tell us that $L(V, V)$, together with cosposion, is what known as a linear algebra with identity.

■ **Example 4** Let A be an $m \times n$matrix and T be a linear transformation defined by $T(X) = Ax$. Let B be an $p \times m$ matrix and U be a linear transformation defined by $U(Y) = BY$. Then

$$\begin{aligned}(UT)(X) &= U(T(X)) \\ &= U(AX) \\ &= B(AX) = BAX.\end{aligned}$$

Remark 1.2.5 The effect of cosposition of U, T is multiplication of matrices B, A.

Definition 1.2.2 — invertible. A linear transformation T from V into W is invertible if there exist a function U from W into V such that (UT) is the identity transformation on V and (TU) is the identity transformaton on W. In this case, U is unique and we denote U by T^{-1}.

Remark 1.2.6 In Definition 1.2.2, T^{-1} exists if and only if

1. T is one-one. ($T\alpha = T\beta \implies \alpha = \beta$)
2. T is onto. (The range of T is W)

Theorem 1.2.4 let T be a linear transformation from V into W. If T is invertible, then the inverse T^{-1} is a linear transformation from W into V.

Proof. Omit. ■

Remark 1.2.7 We see that $T^{-1}U^{-1}$ is the left and right inverse of UT, therefore the inverse of (UT) is $T^{-1}U^{-1}$.