Extension of the SWIFT option pricing scheme for European options calibration under Heston stochastic volatility model

Master in Advanced Mathematics and Mathematical Engineering
Master's thesis
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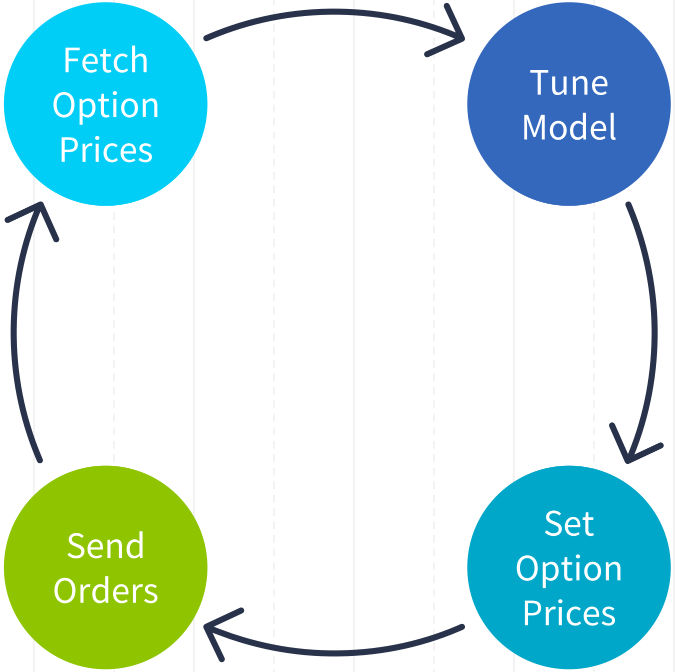
Motivation

Challenges in option contracts trading.

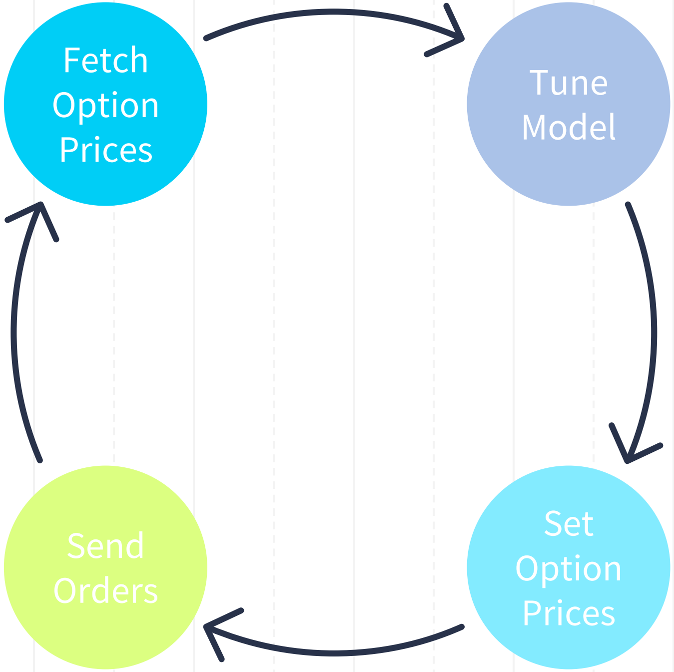
Option Trading Cycle



Option Trading Cycle

Market data

Get the required option and underlying asset data.



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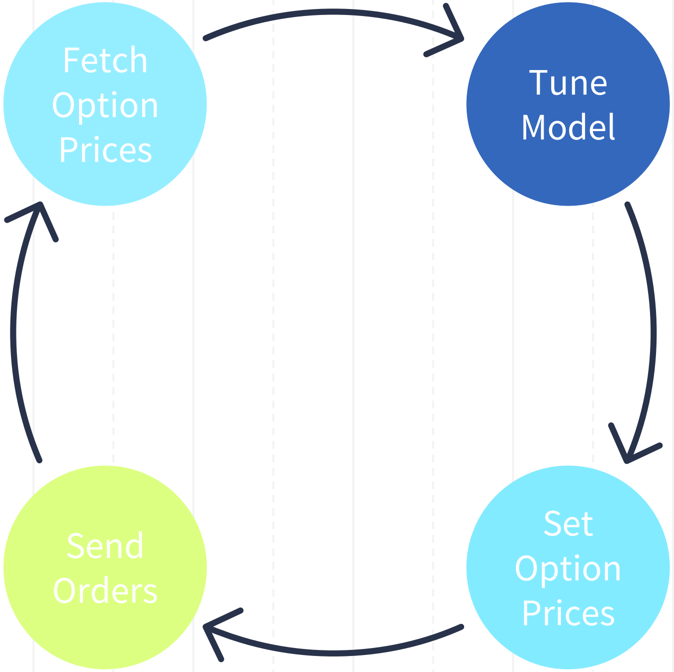

Option calibrator

Choose model parameters that fit the prices of a set of strikes and maturities.

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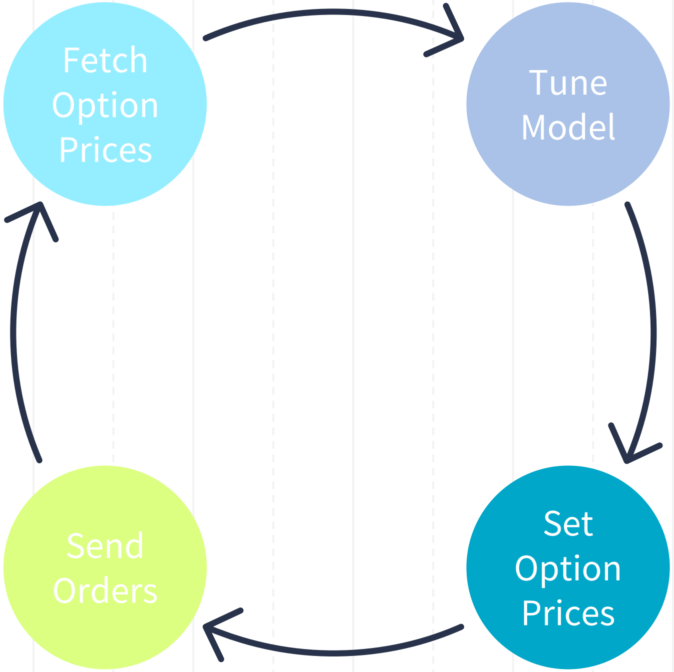

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Choose model parameters that fit the prices of a set of strikes and maturities.

Option pricer
Use the calibrated model all the options of interest.

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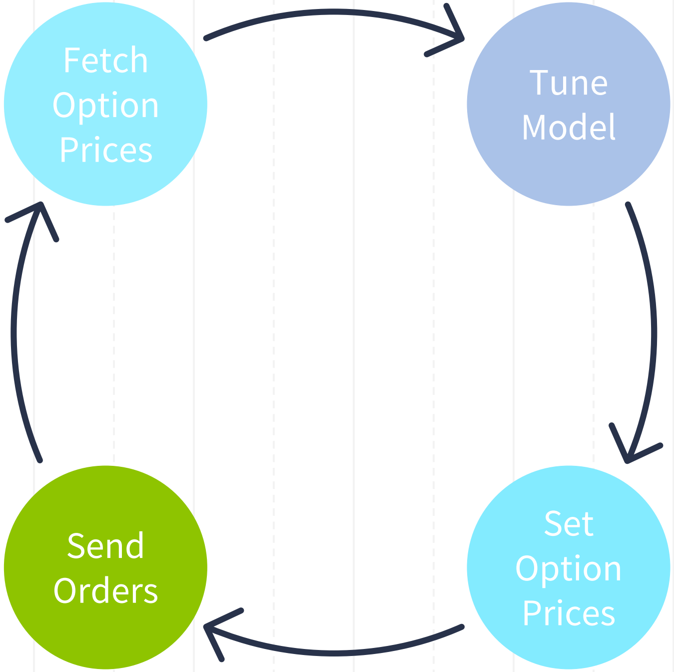

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Investing strategy
Decide how to react to the current theoretic prices and inform the exchange.

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Use the calibrated model all the options of interest.

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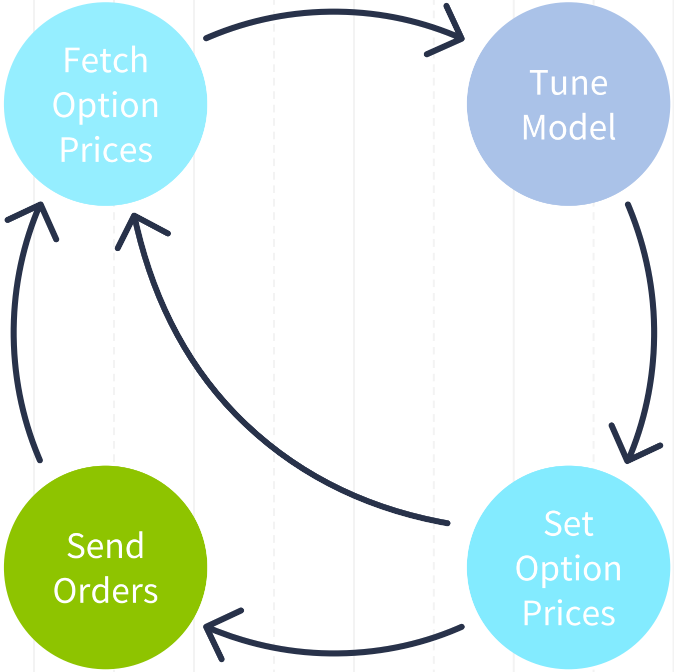

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Option pricer
Use the calibrated model all the options of interest.

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Option Pricing

Risk-neutrality applied to European options.

European Option Price

Risk-neutral assumption

Option payoff

$$e^{-r\tau}\int_{\mathbb{R}} f(y|x)v(y,0)dy$$

State variable choice

Underlying asset model

European Option Price

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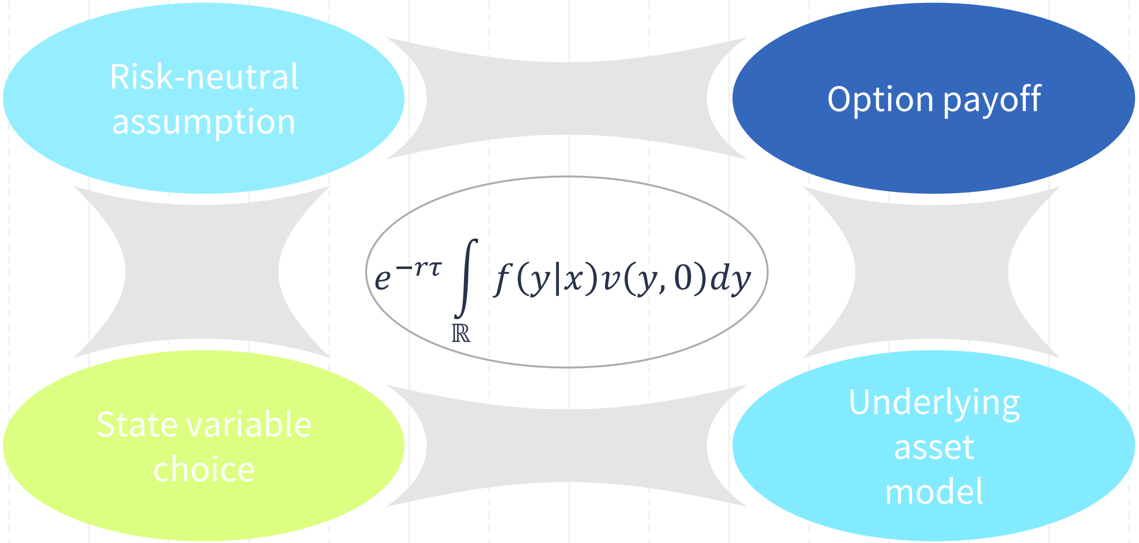

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Underlying asset models

Why is Heston stochastic model relevant?

Black-Scholes-Merton model

Definition

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

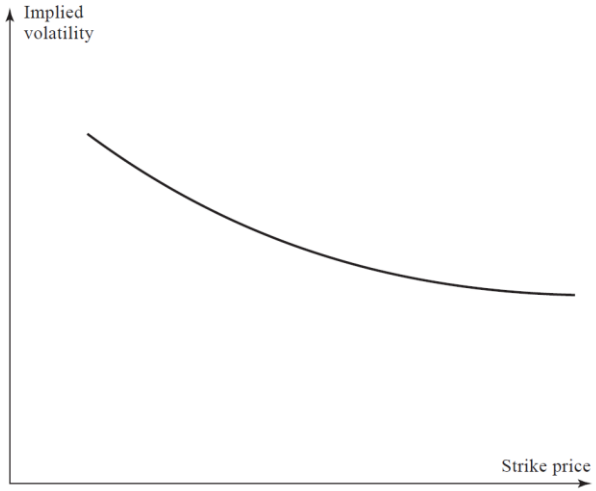
Analytic European option price

$$v^{(C)}(S_t, \tau) = N(d_1)S_t - e^{-r\tau}N(d_2)K$$

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left[ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau\right]$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

Volatility Smile



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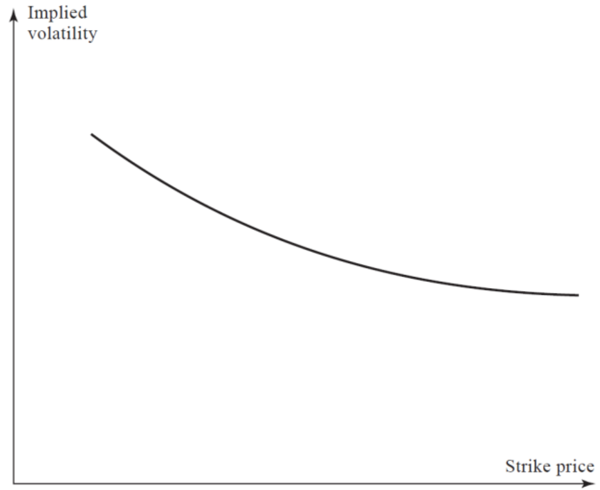

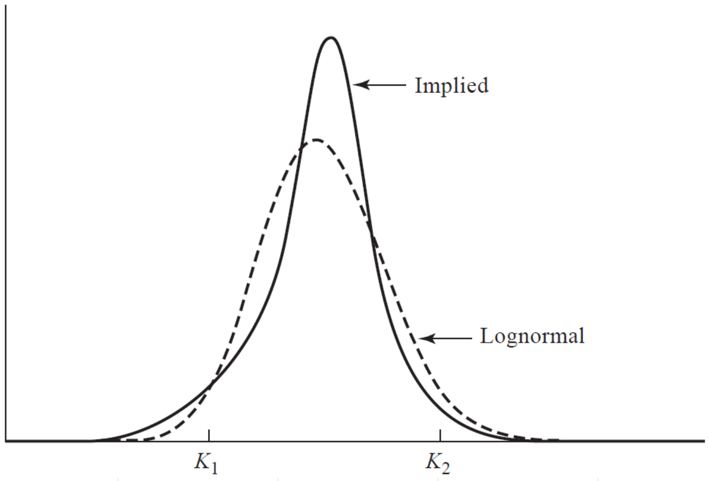

Heston model

Definition

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^{(1)}$$

$$dv_t = \kappa(\bar{v} - v_t)dt + \sigma\sqrt{v_t} dW_t^{(2)}$$

$$dW_t^{(1)} dW_t^{(2)} = \rho dt$$

$$\theta = (v_0, \bar{v}, \rho, \kappa, \sigma)$$

Relation with volatility smile

- v_0: controls position of the volatility surface.
- ρ: controls its skewness.
- κ and σ: control the convexity of the surface.
- $\kappa(v_0 - \bar{v})$: controls the term structure.

Characteristic function expressions

Heston
Discontinuous
Complicated gradient

Schoutens
Continuous
Complicated gradient

Del Baño-Rollin
Discontinuous
Simple gradient

Cui
Continuous
Simple gradient

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Cui's expression

Characteristic function

$$\hat{f}(u) = \exp\left(-iur\tau + \frac{\kappa\bar{v}\sigma\tau iu}{\sigma} - A + \frac{2\kappa\hat{v}}{\sigma^2}\right)$$

Characteristic function gradient

$$\nabla_\theta \hat{f}(u) = h(u)\hat{f}(u)$$

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European Option Pricing

What advantages does SWIFT provide?

Heston formula

$$V(x,\tau) = K\left(e^{x} P_1(\theta; x, \tau) - e^{-ir\tau} P_2(\theta; x, \tau)\right)$$

$$P_1(\theta; x, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} Re\left(\frac{e^{iux}}{iu} \frac{\hat{f}(-u+i)}{\hat{f}(i)}\right) du$$

$$P_2(\theta; x, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} Re\left(\frac{e^{iux}}{iu} \hat{f}(-u)\right) du$$

MRA and Shannon wavelets

Approximation degree
m

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Family of functions
$\phi_{m,k}$

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Function Projection
$$f \approx \sum_{k\in\mathbb{Z}} D_{m,k}\phi_{m,k}$$

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SWIFT

Wavelet projection

Series truncation

Density coefficients approximation

Payoff coefficients approximation

$$v(x,\tau) = e^{-r\tau} \int_{\mathbb{R}} f(y|x) v(y,0) dy$$

SWIFT

Wavelet projection (m)

Series truncation

Density coefficients approximation

Payoff coefficients approximation

$$f_1(y|x) = \sum_{k\in\mathbb{Z}} D_{m,k}\phi_{m,k}$$

$$v_1(x,\tau) = e^{-r\tau}\int_{\mathbb{R}} f_1(y|x)v(y,0)dy$$

SWIFT

- Wavelet projection
- Series truncation (η)
- Density coefficients approximation
- Payoff coefficients approximation

$$f_2(y|x) = \sum_{k=1-\eta}^{\eta} D_{m,k}\phi_{m,k}$$

$$v_2(x,\tau) = e^{-r\tau}\int_{\mathbb{R}} f_2(y|x)v(y,0)dy$$

SWIFT

Wavelet projection

Series truncation

Density coefficients approximation (J_d)

Payoff coefficients approximation

$$f_3(y|x) = \sum_{k=1-\eta}^{\eta} D^*_{m,k}\phi_{m,k}$$

$$v_3(x,\tau) = e^{-r\tau}\sum_{k=1-\eta}^{\eta} D^*_{m,k}V_{m,k}$$

$$V_{m,k} = \int_{\mathbb{R}} \phi_{m,k}v(y,0)dy$$

SWIFT

- Wavelet projection
- Series truncation
- Density coefficients approximation
- Payoff coefficients approximation (J_p)

$$v_4(x,\tau) = e^{-r\tau} \sum_{k=1-\eta}^{\eta} D^*_{m,k} V^*_{m,k}$$

New contributions

Gradient computation

- Invariant parameters assumption.
- Alternative SWIFT formulation for multiple strikes

$$\nabla_\theta v_4(x,\tau) = e^{-r\tau} K \sum_{j=1}^{J_d} Re\{h(u_j 2^m)\hat{f}(u_j 2^m) e^{-iu_j 2^m x} \overline{U}_j(c)\}$$

New contributions

Gradient computation

- Invariant parameters assumption.
- Using an alternative SWIFT formulation for multiple strikes presented in previous studies.

$$\nabla_\theta v_4(x,\tau) = e^{-r\tau} K \sum_{j=1}^{J_d} Re\{h(u_j 2^m)\hat{f}(u_j 2^m)e^{-iu_j 2^m x}\overline{U}_j(c)\}$$

Speed-up techniques

- Extension of previous techniques to the gradient computation.
- Reusing computations through all calibration steps.
- Reusing computations between price and gradient computations.

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Option calibration

Minimization problem approach

What is a calibrated model?

Strike/expiry pairs
(K_i, τ_i)

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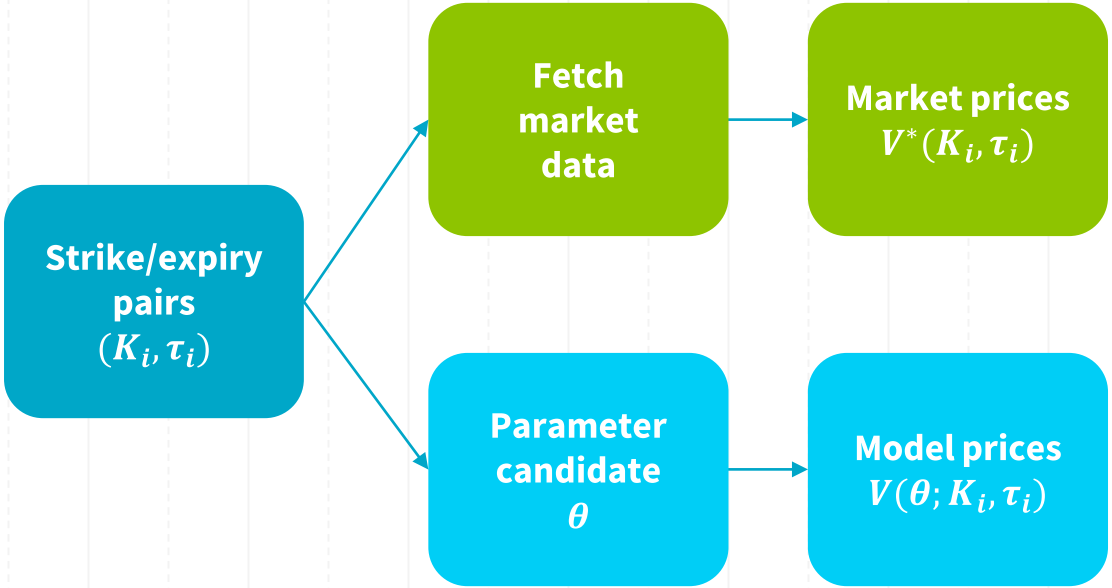
Strike/expiry pairs
(K_i, τ_i)

Fetch market data

Parameter candidate
θ

What is a calibrated model?



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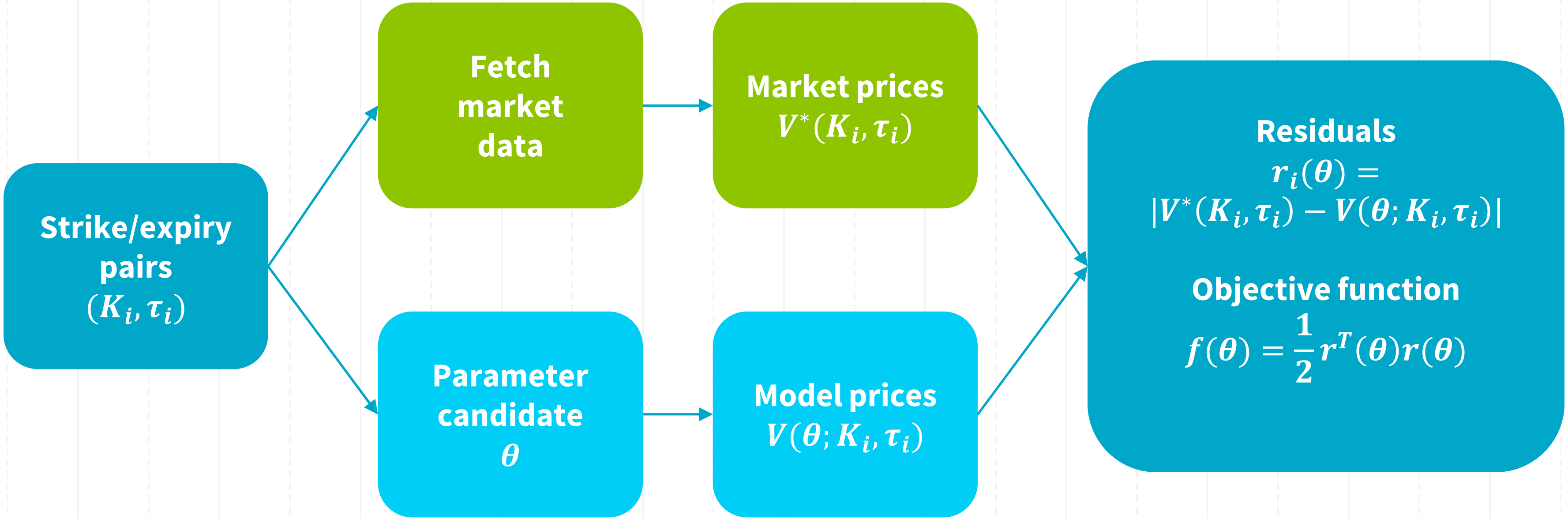

How is the optimum reached?

- A Levenberg-Marquardt method is used.
  - It behaves like steepest descent far from the optimum.
  - It behaves like Gauss-Newton close to the optimum.
- It stops when:
  - ϵ_1: Small value of the objective function.
  - ϵ_2: Small gradient.
  - ϵ_3: Stagnate update.

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Numerical results

Extreme situations, speed, and applicability.

Stress Tests. $\tau = 45$ years.

S	K	$V_{SWIFT}^{m=3}$	$V_{SWIFT}^{m=7}$	$V_{Cui}^{\bar{u}=200}$	$V_{Cui}^{\bar{u}=6}$	$V_{Schoutens}^{\bar{u}=200}$
100	50	65.565	nan	nan	65.565	65.565
100	100	46.911	nan	nan	46.911	46.911
100	200	27.198	nan	nan	27.198	27.198

Speed Tests

	40 strikes x 1 expiry	8 strikes x 5 expiries	8 strikes x 5 expiries (disabling reusable data across strikes)
SWIFT (no speed-ups)	7s	36s	[36s]
SWIFT (with speed-ups)	4.5ms	45ms	170ms
Comparison test	46ms	63ms	[63ms]

Stress Tests. $\tau = 2$ weeks.

S	K	$V_{SWIFT}^{m=3}$	$V_{SWIFT}^{m=7}$	$V_{Cui}^{\bar{u}=200}$
100	50	44.221	50.000	50.000
100	100	0.380	1.045	1.046
100	200	0	0	1.08e-3

Realistic Convergence Tests

	Foreign exchange	Interest rate	Equity
Max absolute error in parameter estimation	1.98e-3	2.66e-4	1.16e-3
Average number of iterations	13.85	6.32	6.78
Average time (ms)	334	194	201
Objective function final value	2.87e-11	2.03e-11	3.64e-11

7 Conclusions and further research

Conclusions

- A complete extension of SWIFT for calibration is provided.
- Cui's Heston characteristic expression numerical problems are discussed.
- The presented calibration method execution time is comparable to the highly efficient reference one.
- Useful for supervised trading.
- Insufficient for high-frequency trading.
- Challenging volatility surfaces are properly calibrated in under 1 second.

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Further research

- Extend SWIFT calibrator to other contracts.
- Investigate alternative optimal parameters.
- Apply the speed-ups to the reference calibrator.

Thanks for you time.

Contact information
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