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Pengolahan Sinyal Digital

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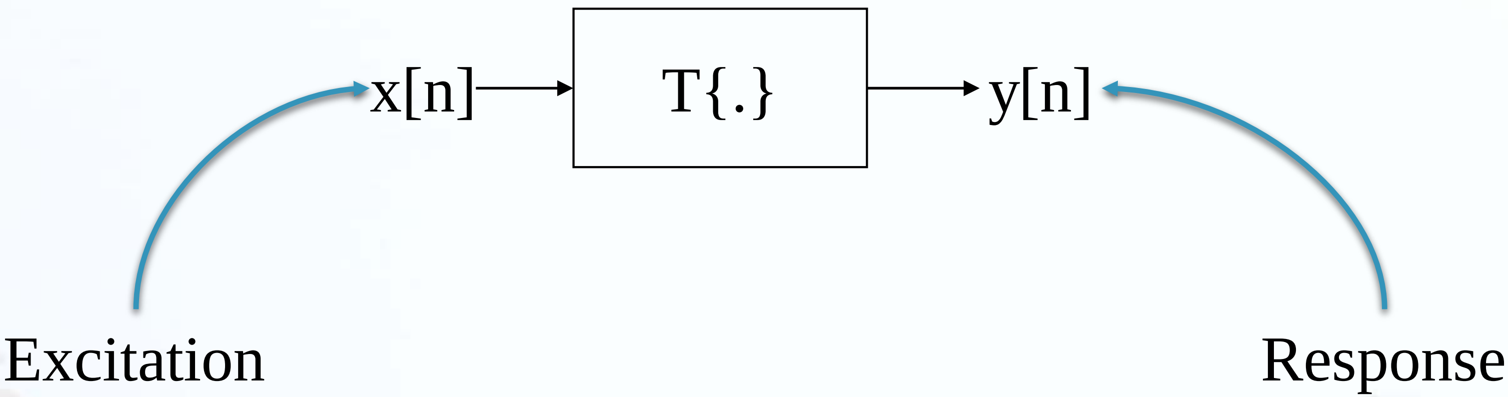
Discrete-Time Systems

Discrete-Time Systems

- Discrete-Time Sequence (described as an operator $T[\cdot]$) is a mathematical operation that maps a given input sequence x[n] into an output sequence y[n]

$$y[n] = T\{x[n]\}$$



Discrete-Time Systems

- Example Discrete-Time Systems
  - Moving (Running) Average

$$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3]$$

  - Maximum

$$y[n] = \max\{x[n], x[n-1], x[n-2]\}$$

  - Ideal Delay System

$$y[n] = x[n - n_o]$$

Memoryless System

- Memoryless System
  - A system is memoryless if the output y[n] at every value of n depends only on the input x[n] at the same value of n
- Example Memoryless Systems
  - Square

$$y[n] = \left(x[n]\right)^2$$

  - Sign

$$y[n] = sign\left\{x[n]\right\}$$

Memoryless System

- Counter Example
  - Ideal Delay System

$$y[n] = x[n - n_o]$$

Linear Systems

- Linear System:
  - Satisfies the principle of superposition
  - A system is linear if and only if

$$T\{x_1[n]+x_2[n]\}=T\left\{x_1[n]\right\}+T\left\{x_2[n]\right\} \quad \text{(additivity)}$$

$$and$$

$$T\left\{ax[n]\right\}=aT\left\{x[n]\right\} \quad \text{(scaling)}$$

Linear Systems

- Examples
  - Ideal Delay System

$$y[n] = x[n - n_o]$$

$$\begin{aligned}
T\{x_1[n] + x_2[n]\} &= x_1[n - n_o] + x_2[n - n_o] \\
T\{x_2[n]\} + T\{x_1[n]\} &= x_1[n - n_o] + x_2[n - n_o] \\
T\{ax[n]\} &= ax_1[n - n_o] \\
aT\{x[n]\} &= ax_1[n - n_o]
\end{aligned}$$

Time-Invariant Systems

- Time-Invariant (shift-invariant) Systems
  - A time shift at the input causes corresponding time-shift at output

$$y[n] = T\{x[n]\} \Rightarrow y[n-n_o] = T\{x[n-n_o]\}$$

- Example
  - Square

$$y[n] = (x[n])^2$$

Delay the input the output is $y_1[n] = (x[n-n_o])^2$

Delay the output gives $y[n-n_o] = (x[n-n_o])^2$

Time-Invariant Systems

- Time-Invariant (shift-invariant) Systems
  - A time shift at the input causes corresponding time-shift at output

$$y[n] = T\{x[n]\} \Rightarrow y[n-n_o] = T\left\{x[n-n_o]\right\}$$

- Counter Example
  - Compressor System

$$y[n] = x[Mn]$$

Delay the input the output is $\quad \mathrm{y}_1\left[n\right] = x[Mn - n_o]$

Delay the output gives $\quad \mathrm{y}\left[\mathrm{n}\text{-}\mathrm{n}_{\mathrm{o}}\right] = x\left[M\left(n-n_o\right)\right]$

Causal System

- Causality
  - A system is causal it's output is a function of only the current and previous samples

- Examples
  - Backward Difference

$$y[n] = x[n] - x[n-1]$$

- Counter Example
  - Forward Difference

$$y[n] = x[n+1] + x[n]$$

Stable System

- Stability (in the sense of bounded-input bounded-output BIBO)
  - A system is stable if and only if every bounded input produces a bounded output

$$\left|x[n]\right| \leq B_x < \infty \Rightarrow \left|y[n]\right| \leq B_y < \infty$$

- Example
  - Square

$$y[n] = \left(x[n]\right)^2$$

if input is bounded by $\left|x[n]\right| \leq B_x < \infty$

output is bounded by $\left|y[n]\right| \leq B_x^2 < \infty$

Stable System

- Counter Example
  - Log

$$y[n] = \log_{10}\left(\left|x[n]\right|\right)$$

even if input is bounded by $\left|x[n]\right| \le B_x < \infty$

output not bounded for $\mathrm{x}\left[\mathrm{n}\right] = 0 \ \Rightarrow \mathrm{y}\left[0\right] = \log_{10}\left(\left|x\left[n\right]\right|\right) = -\infty$

Example

- Determine whether the following systems are linear:
  1. $y(n) = T[x(n)] = 3x^2(n)$
  2. $y(n) = 2x(n - 2) + 5$
  3. $y(n) = x(n + 1) - x(n - 1)$

Example – Tips

- Let $y_1[n] = T[x_1[n]]$ and $y_2[n] = T[x_2[n]]$
  - Determine the response of each system to the linear combination $a_1x_1[n]+a_2x_2[n]$
  - Check whether it is equal to the linear combination $a_1x_1[n] + a_2x_2[n]$
  - a_1 and a_2 are arbitrary constants.

Example – Solution

1. $y(n) = T[x(n)] = 3x^2(n)$:

$$T[a_1x_1(n) + a_2x_2(n)] = 3[a_1x_1(n) + a_2x_2(n)]^2$$

$$= 3a_1^2x_1^2(n) + 3a_2^2x_2^2(n) + 6a_1a_2x_1(n)x_2(n)$$

which is not equal to

$$a_1y_1(n) + a_2y_2(n) = 3a_1^2x_1^2(n) + 3a_2^2x_2^2(n)$$

Hence the given system is nonlinear.

Example – Solution

2. $y(n) = 2x(n-2) + 5$

$$T[a_1x_1(n) + a_2x_2(n)] = 2[a_1x_1(n-2) + a_2x_2(n-2)] + 5$$

$$= a_1y_1(n) + a_2y_2(n) - 5$$

Clearly, the given system is nonlinear even though the input-output relation is a straight-line function.

Example – Solution

3. $y(n) = x(n+1) - x(1-n)$

$$T[a_1x_1(n) + a_2x_2(n)] = a_1x_1(n+1) + a_2x_2(n+1) + a_1x_1(1-n) + a_2x_2(1-n)$$

$$= a_1[x_1(n+1) - x_1(1-n)] + a_2[x_2(n+1) - x_2(1-n)]$$

$$= a_1y_1(n) + a_2y_2(n)$$

Hence the given system is linear

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Linear Time-Invariant Systems

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Linear-Time Invariant System

- A linear system in which an input-output pair, $x(n)$ and $y(n)$, is invariant to a shift k in time
- Special importance for their mathematical tractability
- Most signal processing applications involve LTI systems
- LTI system can be completely characterized by their impulse response

$$\delta[n-k] \longrightarrow \boxed{L[.]} \longrightarrow h_k[n]$$

Linear-Time Invariant System

- Represent any input $x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

$$y[n]=T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}=\sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\}=\sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

- From time invariance we arrive at convolution

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]\, h[n-k]=x[k]*h[k]$$

Properties of LTI Systems

- Convolution is commutative

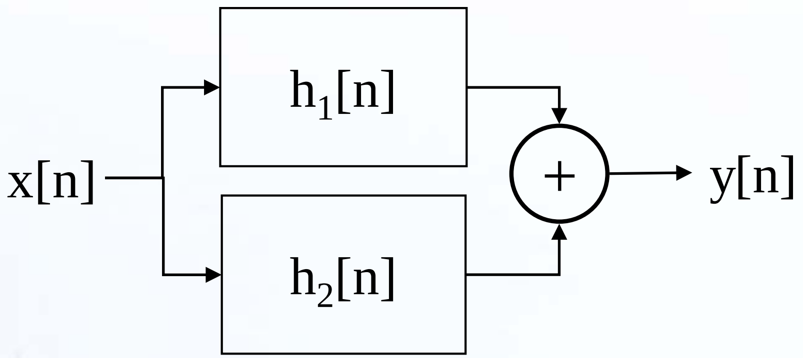
$$x[k]*h[k]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]=h[k]*x[k]$$

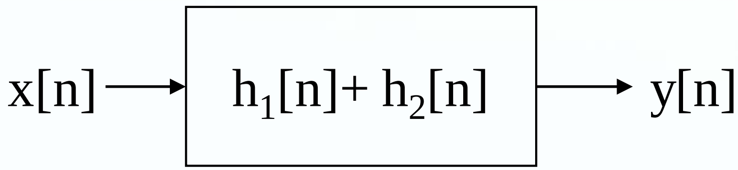
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad h[n] \longrightarrow \boxed{x[n]} \longrightarrow y[n]$$

Properties of LTI Systems

- Convolution is distributive

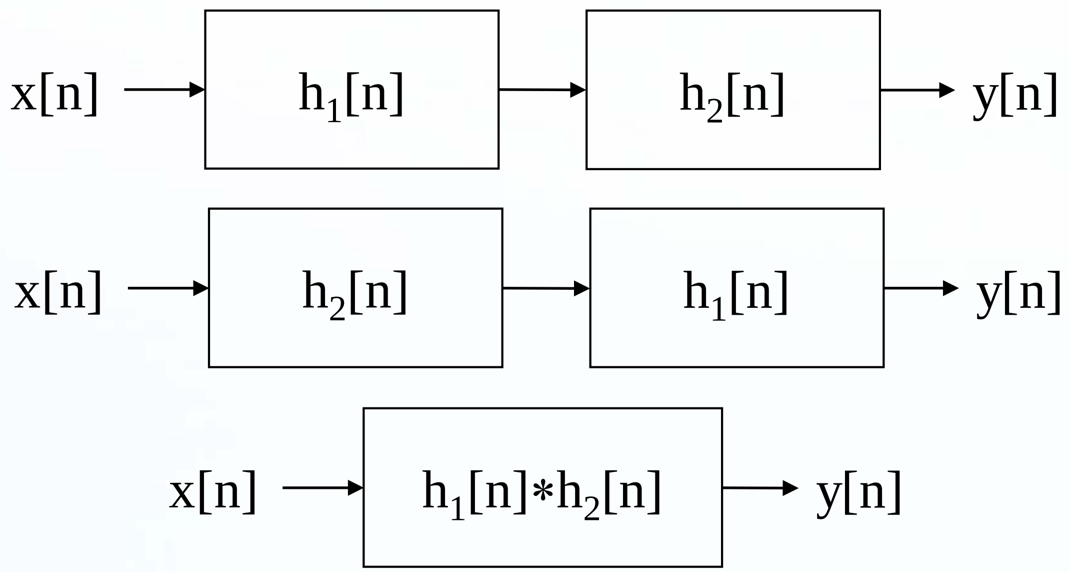
$$x[k]*\left(h_1[k]+h_2[k]\right)=x[k]*h_1[k]+x[k]*h_2[k]$$





Properties of LTI Systems

- Cascade connection of LTI systems



Stable LTI Systems

- An LTI system is (BIBO) stable if and only if
  - Impulse response is absolute summable

$$\sum_{k=-\infty}^{\infty} \left|h[k]\right| < \infty$$

  - Let's write the output of the system as

$$\left|y[n]\right| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \leq \sum_{k=-\infty}^{\infty} \left|h[k]\right|\left|x[n-k]\right|$$

  - If the input is bounded

$$\left|x[n]\right| \leq B_x$$

  - Then the output is bounded by

$$\left|y[n]\right| \leq B_x \sum_{k=-\infty}^{\infty} \left|h[k]\right|$$

  - The output is bounded if the absolute sum is finite

Causal LTI Systems

- An LTI system is causal if and only if

$$h[k] = 0 \quad \text{for k} < 0$$

Example

- Determine whether the following linear systems are time-invariant

1. . $y(n) = L\left[x(n)\right] = 10\sin(0.1\pi n)\,x(n)$
2. . $y(n) = L\left[x(n)\right] = x(n+1) - x(1-n)$
3. . $y(n) = L\left[x(n)\right] = \frac{1}{4}x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2)$

Example

1. Compute the response $y_k(n) = L[x(n-k)]$ to the shifted input sequence
   - Subtracting k from the arguments of every input sequence term on the right-hand side of the linear transformation
2. Compare it to the shifted output sequence $y(n-k)$
   - Replacing every n by $(n-k)$ on the right-hand side of the linear transformation

Example

1. $y(n) = L[x(n)] = 10\sin(0.1\pi n)x(n)$:

   The response due to shifted input is

   $$y_k(n) = L[x(n-k)] = 10\sin(0.1\pi n)x(n-k)$$

   while the shifted output is

   $$y(n-k) = 10\sin[0.1\pi(n-k)]x(n-k) \neq y_k(n)$$

   Hence the given system is not time-invariant

Example

2. $y(n) = L[x(n)] = x(n + 1) - x(1 - n)$

The response due to shifted input is

$$y_k(n) = L[x(n - k)] = x(n + 1 - k) - x(1 - n - k)$$

while the shifted output is

$$y(n-k) = x(n-k) - x(1-[n-k]) = x(n + 1 - k) - x(1-n+k) \neq y_k(n).$$

Hence the given system is not time-invariant.

Example

3. . $y(n)=L\left[x(n)\right]=\frac{1}{4}x(n)+\frac{1}{2}x(n-1)+\frac{1}{4}x(n-2)$

The response due to shifted input is

$$y_k(n)=L\left[x(n-k)\right]=\frac{1}{4}x(n-k)+\frac{1}{2}x(n-k-1)+\frac{1}{4}x(n-k-2)$$

while the shifted output is

$$y(n-k)=\frac{1}{4}x(n-k)+\frac{1}{2}x(n-k-1)+\frac{1}{4}x(n-k-2)=y_k(n)$$

Hence the given system is time-invariant.

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