PENGOLAHAN SINYAL DIGITAL

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THE Z-TRANSFORM

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FOURIER TRANSFORM SHORTCOMINGS

- There are *two* shortcomings to the Fourier transform approach
  - Many useful signals $u(n)$ and $nu(n)$ does not exist in the discrete-time Fourier transform
  - The transient response of a system due to initial conditions or due to changing inputs cannot be computed using the discrete-time Fourier transform approach

THE Z-TRANSFORM

- An essential tool for the analysis of discrete-time systems
- A transformation that maps or transforms a discrete-time signal $x(n)$ into a function $X(z)$ of a complex variable z.

$$X(z) = Z\left[x(n)\right]$$

THE Z-TRANSFORM

- The difference-equation can be converted to a simple algebraic equation which is readily solved for the Z-transform of the output $Y(z)$
- Important qualitative features of discrete-time systems also can be obtained with the help of the Z-transform.
- A discrete-time system is *stable* if and only if every bounded input signal is guaranteed to produce a bounded output signal
  - Stability is an essential characteristic of practical digital filters, and the easiest way to establish stability is with the Z-transform

THE Z-TRANSFORM

- The Z-transform is a powerful tool that is useful for analyzing and solving linear discrete-time systems.

  - Its bilateral (or two-sided) version provides another domain in which a larger class of sequences and systems can be analysed

  - Its unilateral (or one-sided) version can be used to obtain system responses with initial conditions or changing inputs.

THE BILATERAL Z-TRANSFORM

- The *Z-transform* of a discrete-time signal $x(n)$ is a function $X(z)$ of a complex variable *z* defined

$$X(z) \triangleq Z\left[x(n)\right] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- The set of *z* values for which $X(z)$ exists is called the *region of convergence* (*ROC*)

$$R_{x-} < |z| < R_{x+}$$

THE BILATERAL Z-TRANSFORM

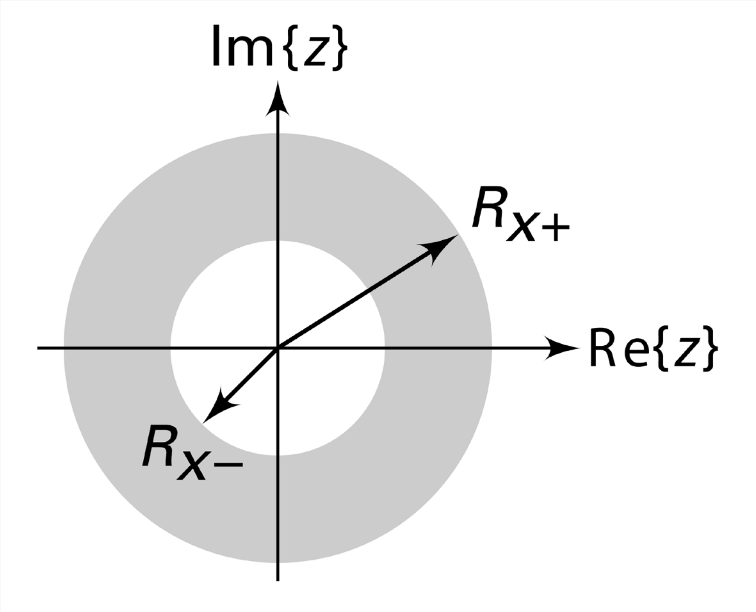
- The inverse *z*-transform of a complex function $X(z)$ is given by

$$x(n) \triangleq Z^{-1}\left[X(z)\right] = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz$$

  - C is a **counterclockwise contour** encircling the origin and lying in the ROC.

THE BILATERAL Z-TRANSFORM

- The complex variable z is called the *complex frequency* given by $z = |z|e^{j\omega}$, where $|z|$ is the magnitude and ω is the real frequency
- The shape of the ROC is an open ring



- R_{x-} may be equal to zero and/or R_{x+} could possibly be ∞.

THE BILATERAL Z-TRANSFORM

- If $R_{x+} < R_{x-}$, then the ROC is a *null space* and the *z*-transform *does not exist*.
- The function $|z| = 1$ (or $z = e^{j\omega}$) is a circle of unit radius in the *z*-plane and is called the *unit circle*.
- If the ROC contains the unit circle, then we can evaluate $X(z)$ on the unit circle.

$$X(z)\big|_{z=e^{j\omega}} = X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega} = \mathcal{F}\left[x(n)\right]$$

  - the discrete-time Fourier transform $X(e^{j\omega})$ may be viewed as a special case of the *z*-transform $X(z)$

A POSITIVE-TIME SEQUENCE

- Let $x_1(n) = a^n u(n),\ 0 < |a| < \infty.$
- Then

$$X_1(z) = \sum_0^\infty a^n z^{-n} = \sum_0^\infty \left(\frac{a}{z}\right)^n = \frac{1}{1-az^{-1}}; \quad \text{if } \left|\frac{a}{z}\right| < 1$$

$$= \frac{z}{z-a}, \qquad |z| > |a| \Rightarrow \mathrm{ROC}_1 : \underbrace{|a|}_{R_{x-}} < |z| < \underbrace{\infty}_{R_{x+}}$$

- $X_1(z)$ is a rational function

A POSITIVE-TIME SEQUENCE

- $X_1(z)$ is a rational function

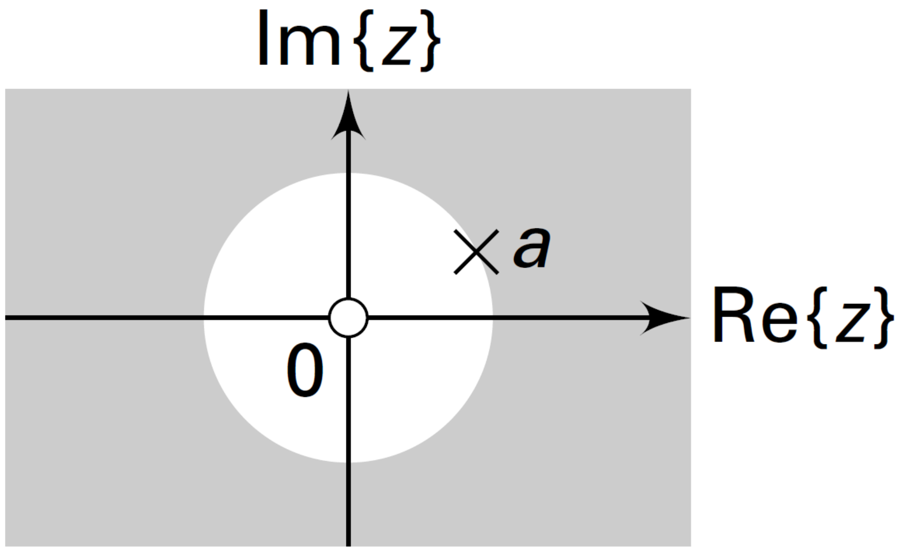
$$X_1(z) \triangleq \frac{B(z)}{A(z)} = \frac{z}{z-a}$$

  - $\mathrm{B(z) = z}$ is the numerator polynomial
  - $\mathrm{A(z) = z{-}a}$ is the denominator polynomial
- The roots of *B*(*z*) are called the *zeros* of *X*(*z*)
- The roots of *A*(*z*) are called the *poles* of *X*(*z*).

A POSITIVE-TIME SEQUENCE

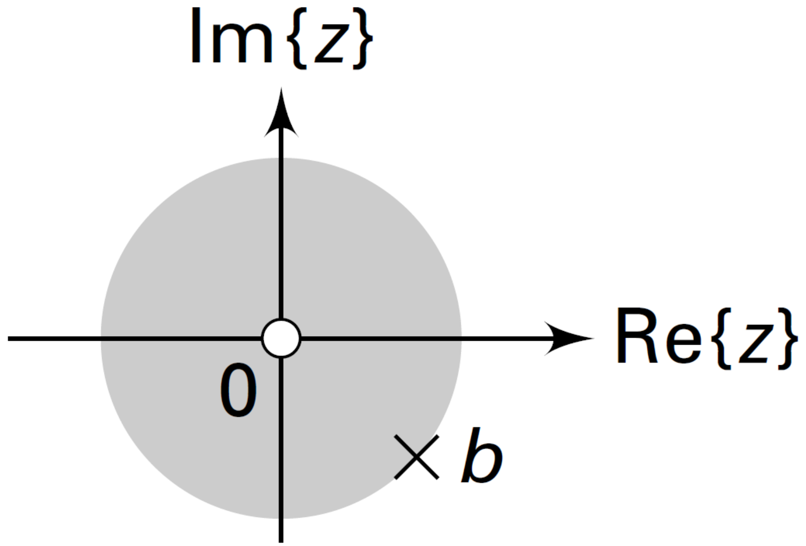
$$X_1(z) = \frac{z}{z-a}, \qquad |z| > |a| \Rightarrow \mathrm{ROC}_1 : \underbrace{|a|}_{R_{x-}} < |z| < \underbrace{\infty}_{R_{x+}}$$

- $X_1(z)$ has a zero at the origin z = 0 and a pole at $z = a$
- Hence $x_1(n)$ can also be represented by a *pole-zero diagram* in the z-plane in which zeros are denoted by O and poles by ×



A NEGATIVE-TIME SEQUENCE

- Let $x_2(n) = -b^n u(-n-1)$, $0 < |b| < \infty$.
- Then



$$X_2(z) = -\sum_{-\infty}^{-1} b^n z^{-n} = -\sum_{-\infty}^{-1}\left(\frac{b}{z}\right)^n = -\sum_{1}^{\infty}\left(\frac{z}{b}\right)^n = 1 - \sum_{0}^{\infty}\left(\frac{z}{b}\right)^n$$

$$= 1 - \frac{1}{1 - z/b} = \frac{z}{z-b}, \quad \mathrm{ROC}_2 : \underbrace{0}_{R_{x-}} < |z| < \underbrace{|b|}_{R_{x+}}$$

COMPARING

- If $b = a$ in both example, then $X_2(z) = X_1(z)$ except for their respective ROCs; that is, $ROC_1 = ROC_2$.
- This implies that the ROC is a distinguishing feature that guarantees the uniqueness of the *z*-transform

A TWO-SIDED SEQUENCE

- Let $x_3(n) = x_1(n) + x_2(n) = a^n u(n) - b^n u(-n-1)$
- Using the preceding two examples

$$X_3(z) = \sum_{n=0}^{\infty} a^n z^{-1} - \sum_{-\infty}^{-1} b^n z^{-1}$$

$$= \left\{ \frac{z}{z-a}, \mathrm{ROC}_1 : |z| > |a| \right\} + \left\{ \frac{z}{z-b}, \mathrm{ROC}_2 : |z| < |b| \right\}$$

$$= \frac{z}{z-a} + \frac{z}{z-b}; \quad \mathrm{ROC}_3 : \ \mathrm{ROC}_1 \cap \mathrm{ROC}_2$$

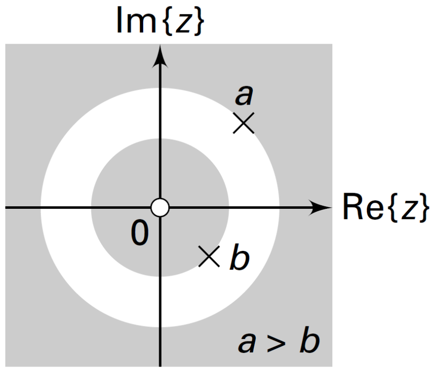
A TWO-SIDED SEQUENCE

$$X_3(z) = \sum_{n=0}^{\infty} a^n z^{-1} - \sum_{-\infty}^{-1} b^n z^{-1}$$

$$= \left\{ \frac{z}{z-a}, \mathrm{ROC}_1 : |z| > |a| \right\} + \left\{ \frac{z}{z-b}, \mathrm{ROC}_2 : |z| < |b| \right\}$$

$$= \frac{z}{z-a} + \frac{z}{z-b}; \quad \mathrm{ROC}_3 : \ \mathrm{ROC}_1 \cap \mathrm{ROC}_2$$

- If |b| < |a|, than ROC3 is a null space, and $X_3(z)$ does not exist.
- If |a| < |b|, then the ROC3 is |a| < |z| < |b|, and $X_3(z)$ exists in this region



PROPERTIES OF THE ROC

- The ROC is **always bounded by a circle** since the convergence condition is on the magnitude $|z|$.
- The sequence $x_1(n) = a^n u(n)$ is a special case of a *right sided sequence*, defined as a sequence $x(n)$ that is zero for some $n < n_0$
  - the ROC for right-sided sequences is **always outside of a circle of radius R_{x-}**.
  - If $n_0 \geq 0$, then the right-sided sequence is also called a *causal* sequence.

PROPERTIES OF THE ROC

- The sequence $x_2(n) = -b^n u(-n-1)$ is a special case of a *left-sided* sequence, defined as a sequence $x(n)$ that is zero for some $n > n_0$.
  - If $n_0 \leq 0$, the resulting sequence is called an *anticausal* sequence.
  - the ROC for left-sided sequences is **always inside of a circle of radius R_{x+}**.
- The sequences that are zero for $n < n_1$ and $n > n_2$ are called *finite-duration sequences*.
  - The ROC for such sequences is **the entire z-plane**.
  - If $n_1 < 0$, then $z = \infty$ is not in the ROC.
  - If $n_2 > 0$, then $z = 0$ is not in the ROC.

PROPERTIES OF THE ROC

- The ROC cannot include a pole since $X(z)$ converges uniformly in there.
- There is at least one pole on the boundary of a ROC of a rational $X(z)$.
- The ROC is one contiguous region; that is, the ROC does not come in pieces.

IMPORTANT PROPERTIES OF THE Z-TRANSFORM

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THE IMPORTANT PROPERTIES OF THE Z-TRANSFORM

- Linearity

$$Z\left[a_1 x_1(n) + a_2 x_2(n)\right] = a_1 X_1(z) + a_2 X_2(z); \quad \text{ROC: ROC}_{x_1} \cap \text{ROC}_{x_2}$$

- Sample shifting

$$Z\left[x(n-n_o)\right] = z^{-n_o} X(z); \quad \text{ROC: ROC}_x$$

- Frequency shifting

$$Z\left[a^n x(n)\right] = X\left(\frac{z}{a}\right); \quad \text{ROC: ROC}_x \text{ scaled by } |a|$$

THE IMPORTANT PROPERTIES OF THE Z-TRANSFORM

- Folding

$$Z\left[x(-n)\right] = X(1/z); \quad \text{ROC: Inverted ROC}_x$$

- Complex conjugation

$$Z\left[x^*(n)\right] = X^*(z^*); \quad \text{ROC: ROC}_x$$

- Differentiation in the z-domain (multiplication-by-a-ramp property)

$$Z\left[nx(n)\right] = -z\frac{dX(z)}{dz}; \quad \text{ROC: ROC}_x$$

THE IMPORTANT PROPERTIES OF THE *Z*-TRANSFORM

- Multiplication

$$Z\left[x_1(n)x_2(n)\right]=\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv;$$

$$\text{ROC: } \text{ROC}_{x_1} \cap \text{ Inverted ROC}_{x_2}$$

  - *C* is a closed contour that encloses the origin and lies in the common ROC
- Convolution

$$Z\left[x_1(n)*x_2(n)\right]=X_1(z)X_2(z); \quad \text{ROC: ROC}_{x_1}\cap\text{ROC}_{x_2}$$

EXAMPLE

- Let $X_1(z) = 2 + 3z^{-1} + 4z^{-2}$ and $X_2(z) = 3 + 4z^{-1} + 5z^{-2} + 6z^{-3}$.
    - Determine $X_3(z) = X_1(z)\, X_2(z)$.

EXAMPLE

- From definition

$$X(z) \triangleq Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- $X_1(z) = 2 + 3z^{-1} + 4z^{-2}$ and $X_2(z) = 3 + 4z^{-1} + 5z^{-2} + 6z^{-3}$
- So

$$x_1(n) = \{2, 3, 4\} \qquad x_2(n) = \{3, 4, 5, 6\}$$

EXAMPLE

- $X_3(z) = X_1(z)\, X_2(z)$ ➔ Convolution

$$Z\left[x_1(n) * x_2(n)\right] = X_1(z)\, X_2(z)$$

- the convolution of these two sequences will give the coefficients of the required polynomial product

```
>> x1 = [2,3,4]; x2 = [3,4,5,6]; x3 = conv(x1,x2)
>> x3 = 6 17 34 43 38 24
```

- Hence

$$X_3(z) = 6+17z^{-1} + 34z^{-2} + 43z^{-3} + 38z^{-4} + 24z^{-5}$$

EXAMPLE

- Let $X_1(z) = z + 2 + 3z^{-1}$ and $X_2(z) = 2z^2 + 4z + 3 + 5z^{-1}$.
  - Determine $X_3(z) = X_1(z)\, X_2(z)$.

EXAMPLE

```
function [y,ny] = conv_m(x,nx,h,nh)
% Modified convolution routine for signal processing
% ----------------------------------------------
% [y,ny] = conv_m(x,nx,h,nh)
% [y,ny] = convolution result
% [x,nx] = first signal
% [h,nh] = second signal
%
nyb = nx(1)+nh(1); nye = nx(length(x)) + nh(length(h));
ny = [nyb:nye]; y = conv(x,h);
```

EXAMPLE

- $X_1(z) = z + 2 + 3z^{-1}$ and $X_2(z) = 2z^2 + 4z + 3 + 5z^{-1}$

$$x_1(n) = \{1, \underset{\uparrow}{2}, 3\} \qquad x_2(n) = \{2, 4, \underset{\uparrow}{3}, 5\}$$

```
>> x1 = [1,2,3]; n1 = [-1:1]; x2 = [2,4,3,5]; n2 = [-2:1];
>> [x3,n3] = conv_m(x1,n1,x2,n2)
x3 = 2 8 17 23 19 15
n3 = -3 -2 -1 0 1 2
```

- Hence

$$X_3(z) = 2z^3 + 8z^2 + 17z + 23 + 19z^{-1} + 15z^{-2}$$

COMMON Z-TRANSFORM PAIRS

Sequence	*Transform*	*ROC*
$\delta(n)$	1	$\forall\, z$
$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u(-n-1)$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-b^n u(-n-1)$	$\dfrac{1}{1-bz^{-1}}$	$\|z\| < \|b\|$
$[a^n \sin\omega_0 n]\, u(n)$	$\dfrac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\| > \|a\|$
$[a^n \cos\omega_0 n]\, u(n)$	$\dfrac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\| > \|a\|$
$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-nb^n u(-n-1)$	$\dfrac{bz^{-1}}{(1-bz^{-1})^2}$	$\|z\| < \|b\|$

EXAMPLE

- Using *z*-transform properties and the *z*-transform table, determine the *z*-transform of

$$x_1(n) = (n-2)(0,5)^{(n-2)} \cos\left[\frac{\pi}{3}(n-2)\right]u(n-2)$$

EXAMPLE

$$x_1(n) = (n-2)(0,5)^{(n-2)} \cos\left[\frac{\pi}{3}(n-2)\right] u(n-2)$$

- Applying the sample-shift property

$$Z\left[x(n-n_o)\right] = z^{-n_o} X(z)$$

$$X(z) = Z\left[x(n)\right] = z^{-2} Z\left[n(0,5)^n \cos\left(\frac{\pi n}{3}\right) u(n)\right]$$

EXAMPLE

$$X(z) = Z\left[x(n)\right] = z^{-2} Z\left[n(0,5)^n \cos\left(\frac{\pi n}{3}\right) u(n)\right]$$

- Applying the multiplication by a ramp property

$$Z\left[nx(n)\right] = -z\frac{dX(z)}{dz}$$

$$\Downarrow$$

$$X(z) = z^{-2}\left\{-z\frac{dZ\left[(0,5)^n \cos\left(\frac{\pi n}{3}\right) u(n)\right]}{dz}\right\}$$

EXAMPLE

$$X(z) = z^{-2}\left\{-z\frac{dZ\left[(0,5)^n \cos\left(\frac{\pi n}{3}\right)u(n)\right]}{dz}\right\}$$

- From table

$$Z\left[(0,5)^n \cos\left(\frac{\pi n}{3}\right)u(n)\right] = \frac{1-\left(0,5\cos\frac{\pi}{3}\right)z^{-1}}{1-2\left(0,5\cos\frac{\pi}{3}\right)z^{-1}+0,25z^{-1}}; \quad |z| > 0,5$$

$$= \frac{1-0,25z^{-1}}{1-0,5z^{-1}+0,25\,\mathrm{z}^{-2}}; \quad |z| > 0,5$$

EXAMPLE

- Hence

$$X(z) = -z^{-1}\frac{d}{dz}\left\{\frac{1-0,25z^{-1}}{1-0,5z^{-1}+0,25\,\mathrm{z}^{-2}}\right\}; \quad |z| > 0,5$$

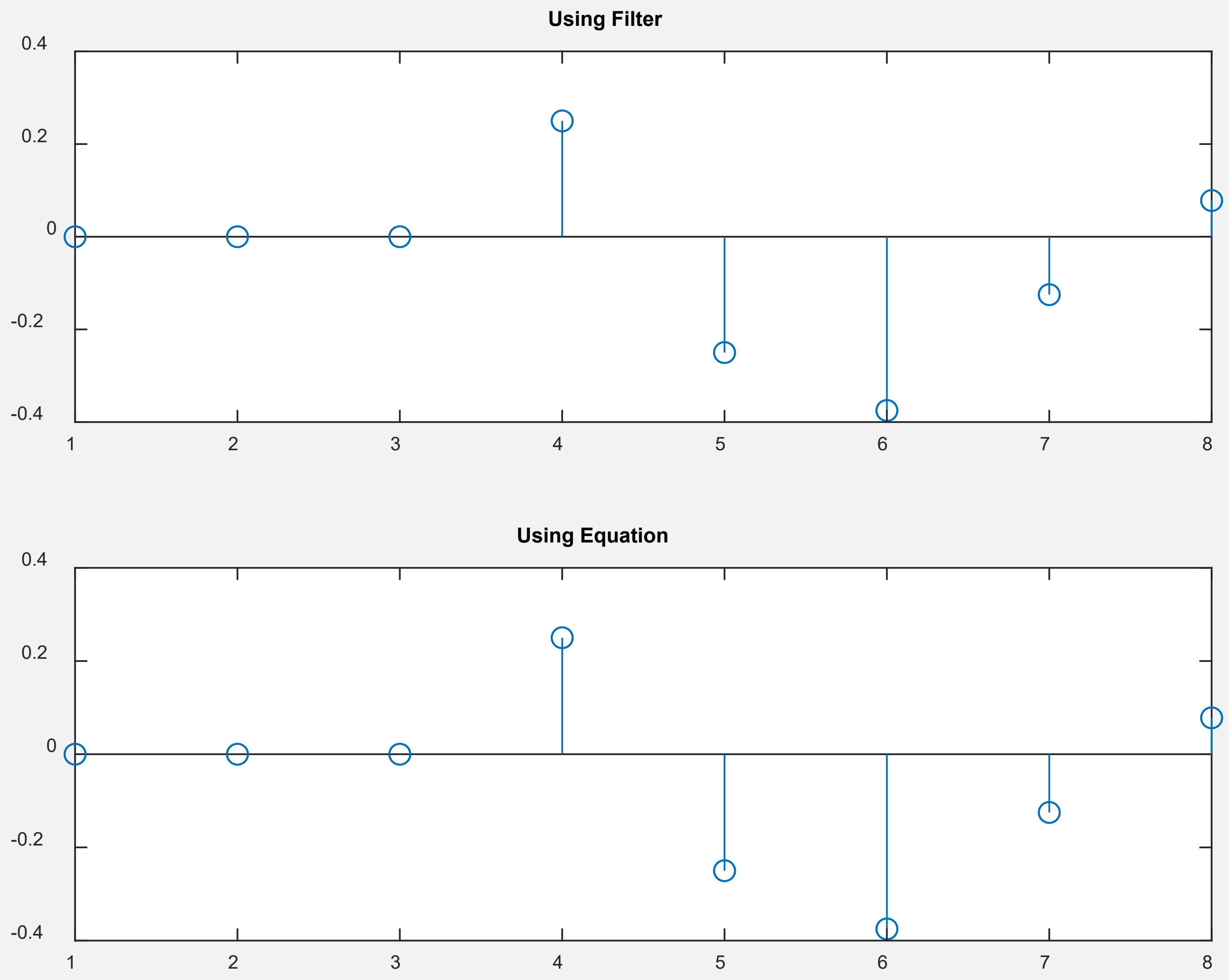
$$= -z^{-1}\left\{\frac{-0,25z^{-2}+0,5z^{-3}-0,0625z^{-4}}{1-z^{-1}+0,75z^{-2}-0,25z^{-3}+0,0625z^{-4}}\right\},$$

$$= \frac{0,25z^{-3}-0,5z^{-4}+0,0625z^{-5}}{1-z^{-1}+0,75z^{-2}-0,25z^{-3}+0,0625z^{-4}}, \qquad |z| > 0,5$$

Example: MATLAB Verification

```
>> b = [0,0,0,0.25,-0.5,0.0625]; a = [1,-1,0.75,-0.25,0.0625];
>> [delta,n]=impseq(0,0,7)
>> x = filter(b,a,delta) % check sequence
>> x = [(n-2).*(1/2).^(n-2).*cos(pi*(n-2)/3)].*stepseq(2,0,7)
```

EXAMPLE: MATLAB VERIFICATION



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