



# PENGOLAHAN SINYAL DIGITAL

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# FREQUENCY DOMAIN REPRESENTATION OF LTI SYSTEMS

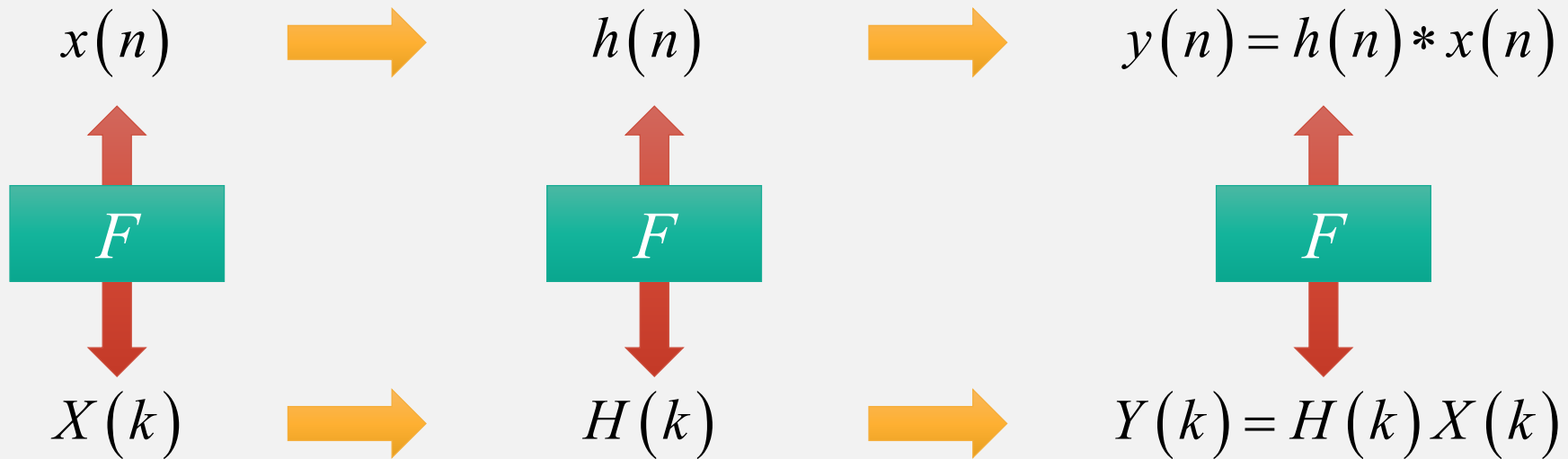
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# FOURIER TRANSFORM

- Fourier transform representation is the most useful signal representation for LTI systems



# RESPONSE TO A COMPLEX EXPONENTIAL

- Let  $x(n) = e^{j\omega_0 n}$  be the input to an LTI system represented by the impulse response  $h(n)$ .

- Then

$$e^{j\omega_0 n} \longrightarrow \boxed{h(n)} \longrightarrow h(n) * e^{j\omega_0 n}$$

$$\begin{aligned} y(n) &= h(n) * e^{j\omega_0 n} = \sum_{-\infty}^{\infty} h(k) e^{j\omega_0 (n-k)} \\ &= \left[ \sum_{-\infty}^{\infty} h(k) e^{-j\omega_0 k} \right] e^{j\omega_0 n} \\ &= \left[ \mathcal{F}[h(n)]_{\omega=\omega_0} \right] e^{j\omega_0 n} \end{aligned}$$

# FREQUENCY RESPONSE

- The discrete-time Fourier transform of an impulse response is called the *frequency response* (or *transfer function*) of an LTI system and is denoted by

$$H(e^{j\omega n}) \triangleq \sum_{-\infty}^{\infty} h(n) e^{-j\omega n}$$

# FREQUENCY RESPONSE

- The LTI system can be represented by

$$x(n) = e^{j\omega_0 n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y(n) = H(e^{j\omega_0}) \times e^{j\omega_0 n}$$

- A linear combination of complex exponentials using the linearity of LTI systems

$$\sum_k A_k e^{j\omega_k n} \longrightarrow \boxed{h(n)} \longrightarrow \sum_k A_k H(e^{j\omega_k}) \times e^{j\omega_k n}$$

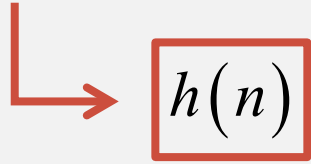
# RESPONSE TO A COMPLEX EXPONENTIAL

- The frequency response  $H(e^{j\omega})$  is a complex function of  $\omega$ .
  - The magnitude  $|H(e^{j\omega})|$  of  $H(e^{j\omega})$  is called the *magnitude (or gain) response* function
  - The angle  $\angle H(e^{j\omega})$  is called the *phase response* function

# RESPONSE TO SINUSOIDAL SEQUENCES

- A LTI system with sinusoidal input

$$x(n) = A \cos(\omega_o n + \theta_o)$$



A red L-shaped arrow points from the bottom of the  $h(n)$  box down and then right to the output equation  $y(n) = A |H(e^{j\omega_o})| \cos(\omega_o n + \theta_o + \angle H(e^{j\omega_o}))$ .

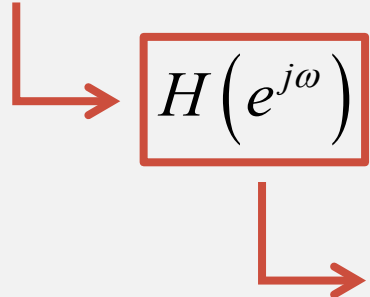
$$y(n) = A |H(e^{j\omega_o})| \cos(\omega_o n + \theta_o + \angle H(e^{j\omega_o}))$$

- The response  $y(n)$  is another sinusoid of the same frequency  $\omega_o$ , with amplitude *gained* by  $|H(e^{j\omega_o})|$  and phase *shifted* by  $\angle H(e^{j\omega_o})$
- This response is called the *steady-state response*, denoted by  $y_{ss}(n)$ .



# RESPONSE TO SINUSOIDAL SEQUENCES

- A linear combination of sinusoidal sequences

$$\sum_k A_k \cos(\omega_k n + \theta_k)$$

$$\sum_k A_k \left| H(e^{j\omega_k}) \right| \cos(\omega_k n + \theta_k + \angle H(e^{j\omega_k}))$$

# RESPONSE TO ARBITRARY SEQUENCES

- Let  $X(e^{j\omega}) = \mathcal{F}\{x(n)\}$   $Y(e^{j\omega}) = \mathcal{F}\{y(n)\}$

- Using the convolution property

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- An LTI system can be represented in the frequency domain by

$$X(e^{j\omega}) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

# RESPONSE TO ARBITRARY SEQUENCES

$$X(e^{j\omega}) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- The output  $y(n)$  is then computed from  $Y(e^{j\omega})$  using the inverse discrete-time Fourier transform
- Requiring an integral operation, which is not a convenient operation in MATLAB
- There is an alternate approach to the computation of output to arbitrary inputs using the z-transform and partial fraction expansion

# EXAMPLE

- Determine the frequency response  $H(e^{j\omega})$  of a system characterized by
$$h(n) = (0.9)^n u(n).$$
- Plot the magnitude and the phase responses.

# EXAMPLE

- Based on frequency response formula

$$H(e^{j\omega n}) \triangleq \sum_{-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$\begin{aligned} H(e^{j\omega n}) &= \sum_{-\infty}^{\infty} h(n) e^{-j\omega n} = \sum_0^{\infty} (0,9) e^{-j\omega n} \\ &= \sum_0^{\infty} (0,9 e^{-j\omega})^n = \frac{1}{1 - 0,9 e^{-j\omega}} \end{aligned}$$

# EXAMPLE

- Hence

$$\begin{aligned} \left| H(e^{j\omega n}) \right| &= \sqrt{\frac{1}{(1 - 0,9 \cos \omega)^2 + (0,9 \sin \omega)^2}} \\ &= \frac{1}{\sqrt{1,81 - 1,9 \cos \omega}} \end{aligned}$$

- and

$$\angle H(e^{j\omega n}) = -\arctan \left[ \frac{0,9 \sin \omega}{1 - 0,9 \cos \omega} \right]$$

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$|a + jb| = \sqrt{a^2 + b^2}$$

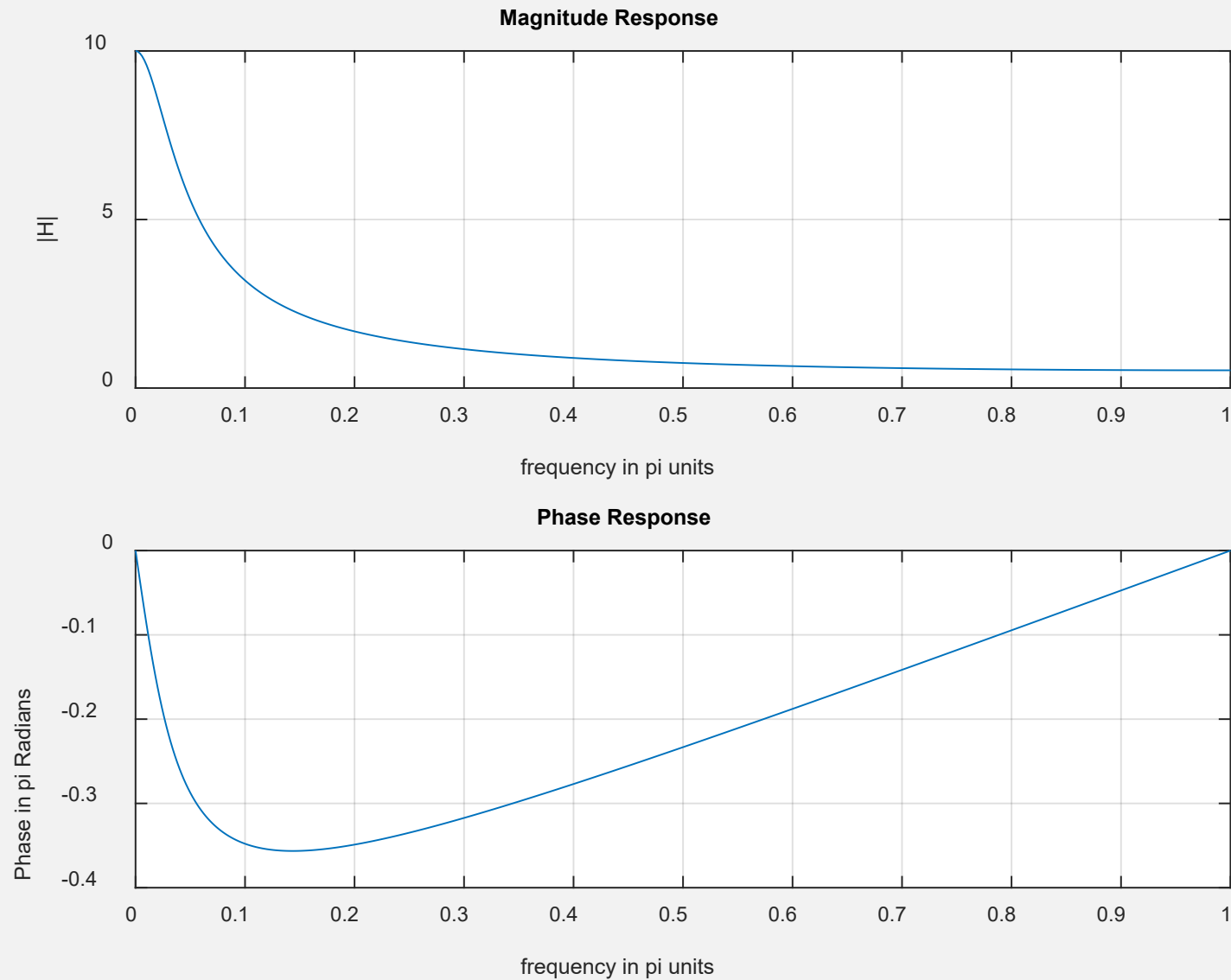
$$\sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

$$\tan \theta = \frac{b}{a}$$

# EXAMPLE: PLOT THE RESPONSES

```
>> w = [0:1:500]*pi/500; % [0, pi] axis divided into 501 points.  
>> H = exp(j*w) ./ (exp(j*w) - 0.9*ones(1,501));  
>> magH = abs(H); angH = angle(H);  
>> subplot(2,1,1); plot(w/pi,magH); grid;  
>> xlabel('frequency in pi units'); ylabel('|H|');  
>> title('Magnitude Response');  
>> subplot(2,1,2); plot(w/pi,angH/pi); grid  
>> xlabel('frequency in pi units');  
>> ylabel('Phase in pi Radians');  
>> title('Phase Response');
```

# EXAMPLE: PLOT THE RESPONSES





# FREQUENCY RESPONSE FUNCTION FROM DIFFERENCE EQUATIONS

- An LTI system is represented by the difference equation

$$y(n) + \sum_{\ell=1}^N a_{\ell} y(n-\ell) = \sum_{m=0}^M b_m x(n-m)$$

- Frequency Response form

$$H(e^{j\omega})e^{j\omega n} + \sum_{\ell=1}^N a_{\ell} H(e^{j\omega})e^{j\omega(n-\ell)} = \sum_{m=0}^M b_m e^{j\omega(n-m)}$$

$$H(e^{j\omega}) = \frac{\sum_{m=0}^M b_m e^{-j\omega m}}{1 + \sum_{\ell=1}^N a_{\ell} e^{-j\omega \ell}}$$

# EXAMPLE

- An LTI system is specified by the difference equation

$$y(n) = 0.8y(n - 1) + x(n)$$

- Determine  $H(e^{j\omega})$ .
- Calculate and plot the steady-state response  $y_{ss}(n)$  to

$$x(n) = \cos(0.05\pi n) u(n)$$

# EXAMPLE

- Rewrite the difference equation as  $y(n) - 0.8y(n - 1) = x(n)$ .
- Using “Frequency Response Function from Difference Equations” formula

$$H(e^{j\omega}) = \frac{1}{1 - 0.8e^{-j\omega}}$$

# EXAMPLE

- In the steady state the input is  $x(n) = \cos(0.05\pi n)$  with frequency  $\omega_o = 0.05\pi$  and  $\theta_o = 0^\circ$ . The response of the system is

$$H(e^{j0,05\pi}) = \frac{1}{1 - 0,8e^{-j0,05\pi}} = 4,0928e^{-j0,5377}$$

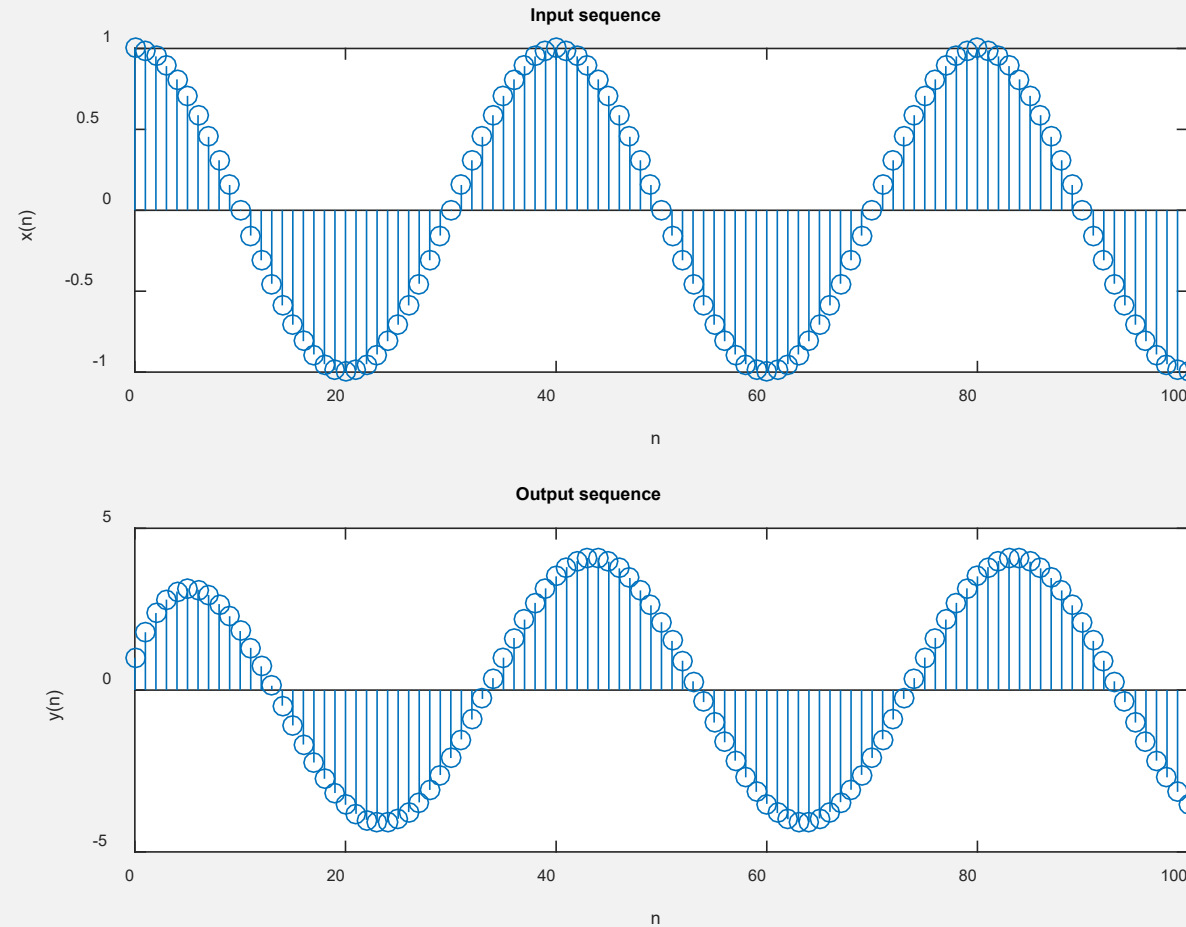
$$\begin{aligned} y_{ss}(n) &= 4,0928 \cos(0.05\pi n - 0,5377) \\ &= 4,0928 \cos[0,05\pi(n - 3,42)] \end{aligned}$$

- This means that at the output the sinusoid is scaled by 4.0928 and shifted by 3.42 samples.

# EXAMPLE: VERIFY WITH MATLAB

```
>> subplot(1,1,1)
>> b = 1; a = [1,-0.8];
>> n=[0:100];x = cos(0.05*pi*n);
>> y = filter(b,a,x);
>> subplot(2,1,1); stem(n,x);
>> xlabel('n'); ylabel('x(n)'); title('Input sequence')
>> subplot(2,1,2); stem(n,y);
>> xlabel('n'); ylabel('y(n)'); title('Output sequence')
```

# EXAMPLE: VERIFY WITH MATLAB



# EXAMPLE

- A 3rd-order lowpass filter is described by the difference equation

$$y(n) = 0,0181x(n) + 0,0543x(n-1) + 0,0543x(n-2) \\ + 0,0181x(n-3) + 1,76y(n-1) - 1,1829y(n-2) + 0,2781y(n-3)$$

- Plot the magnitude and the phase response of this filter, and verify that it is a lowpass filter.

# EXAMPLE

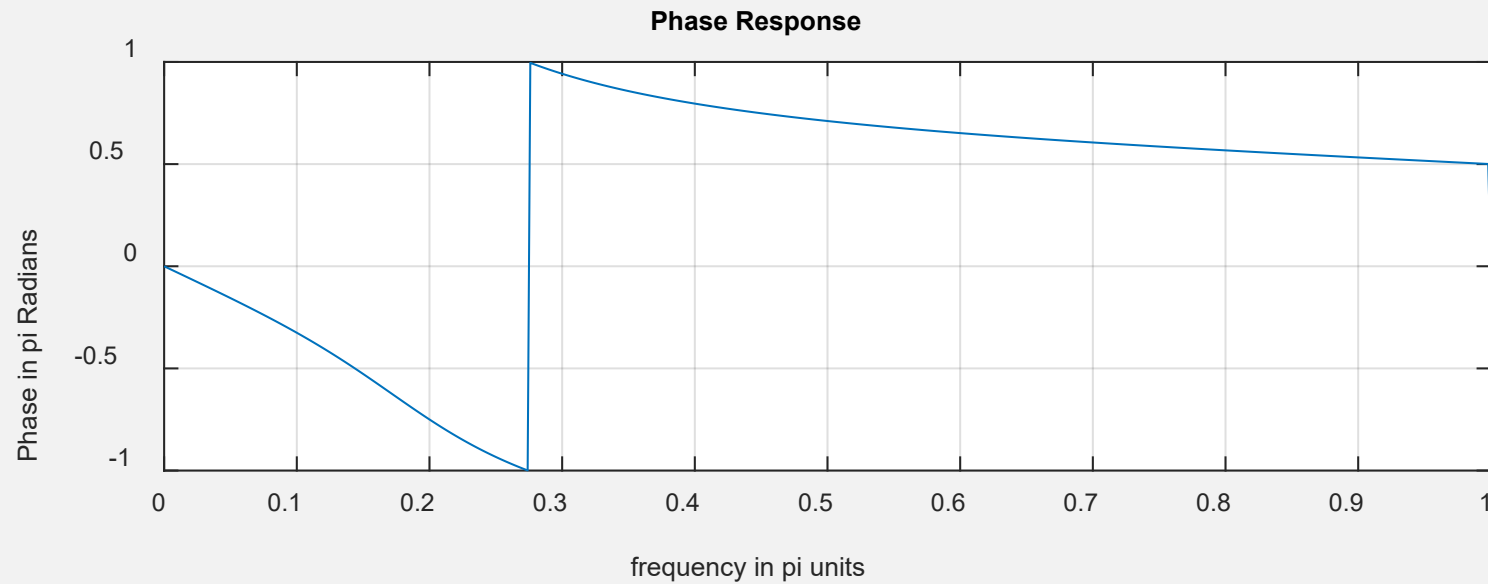
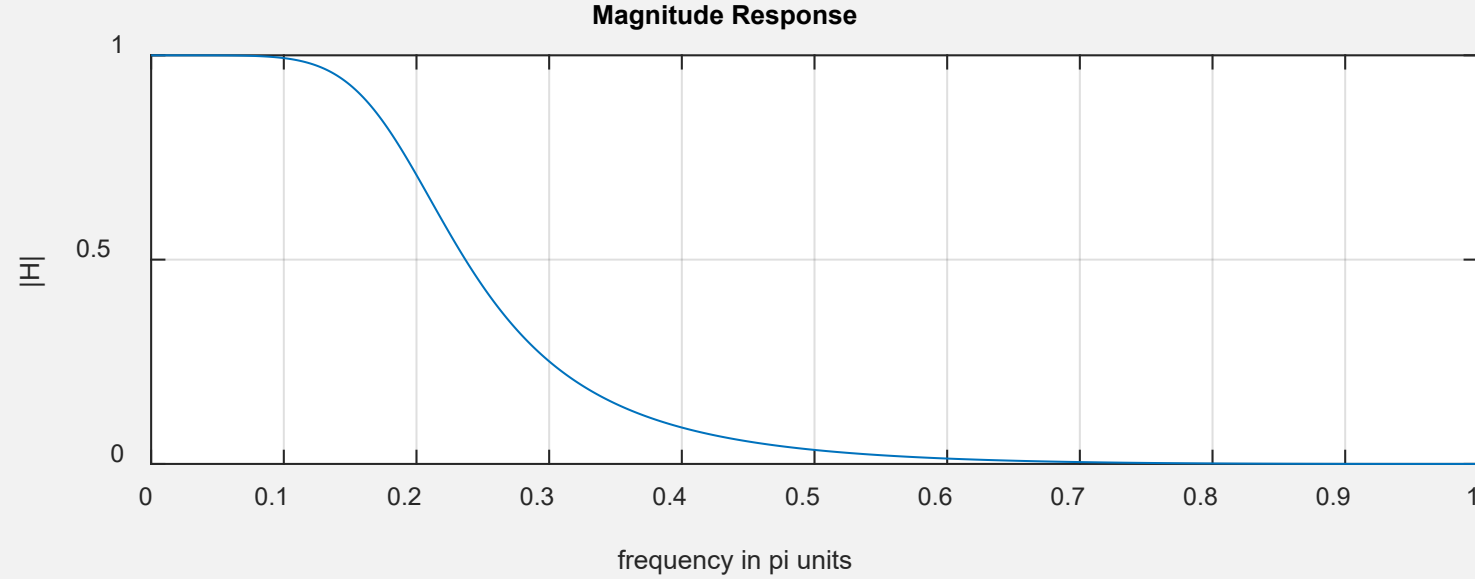
```
>> b = [0.0181, 0.0543, 0.0543, 0.0181]; % filter coefficient array b
>> a = [1.0000, -1.7600, 1.1829, -0.2781]; % filter coefficient array a
>> m = 0:length(b)-1; l = 0:length(a)-1; % index arrays m and l
>> K = 500; k = 0:1:K; % index array k for frequencies
>> w = pi*k/K; % [0, pi] axis divided into 501 points.
>> num = b * exp(-j*m'*w); % Numerator calculations
>> den = a * exp(-j*l'*w); % Denominator calculations
>> H = num ./ den; % Frequency response
>> magH = abs(H); angH = angle(H); % mag and phase responses
```



# EXAMPLE

```
>> subplot(2,1,1); plot(w/pi,magH); grid; axis([0,1,0,1])
>> xlabel('frequency in pi units'); ylabel('|H|');
>> title('Magnitude Response');
>> subplot(2,1,2); plot(w/pi,angH/pi); grid
>> xlabel('frequency in pi units'); ylabel('Phase in pi Radians');
>> title('Phase Response');
```

# EXAMPLE



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