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The z-Transform

Properties of The ROC

- The ROC is **always bounded by a circle** since the convergence condition is on the magnitude $|z|$.
- The sequence $x_1(n) = a^n u(n)$ is a special case of a *right sided sequence*, defined as a sequence $x(n)$ that is zero for some $n < n_0$
  - the ROC for right-sided sequences is **always outside of a circle of radius $\boldsymbol{R_{x-}}$**.
  - If $n_0 \geq 0$, then the right-sided sequence is also called a *causal* sequence.

Properties of The ROC

- The sequence $x_2(n) = -b^n u(-n-1)$ is a special case of a *left-sided* sequence, defined as a sequence $x(n)$ that is zero for some $n > n_0$.
  - If $n_0 \leq 0$, the resulting sequence is called an *anticausal* sequence.
  - the ROC for left-sided sequences is **always inside of a circle of radius R_{x+}**.
- The sequences that are zero for $n < n_1$ and $n > n_2$ are called *finite-duration sequences*.
  - The ROC for such sequences is **the entire *z*-plane**.
  - If $n_1 < 0$, then $z = \infty$ is not in the ROC.
  - If $n_2 > 0$, then $z = 0$ is not in the ROC.

Properties of The ROC

- The ROC cannot include a pole since $X(z)$ converges uniformly in there.
- There is at least one pole on the boundary of a ROC of a rational $X(z)$.
- The ROC is one contiguous region; that is, the ROC does not come in pieces.

Important Properties of the z-Transform

The important properties of the *z*-transform

- Linearity

$$Z\left[a_1x_1(n)+a_2x_2(n)\right]=a_1X_1(z)+a_2X_2(z);\quad \text{ROC: } \text{ROC}_{x_1}\cap \text{ROC}_{x_2}$$

- Sample shifting

$$Z\left[x(n-n_o)\right]=z^{-n_o}X(z);\quad \text{ROC: } \text{ROC}_x$$

- Frequency shifting

$$Z\left[a^n x(n)\right]=X\left(\frac{z}{a}\right);\quad \text{ROC: } \text{ROC}_x \text{ scaled by } |a|$$

The important properties of the *z*-transform

- Folding

$$Z\left[x(-n)\right] = X(1/z); \quad \text{ROC: Inverted ROC}_x$$

- Complex conjugation

$$Z\left[x^*(n)\right] = X^*(z^*); \quad \text{ROC: ROC}_x$$

- Differentiation in the z-domain (*multiplication-by-a-ramp property*)

$$Z\left[nx(n)\right] = -z\frac{dX(z)}{dz}; \quad \text{ROC: ROC}_x$$

The important properties of the *z*-transform

- Multiplication

$$Z\left[x_1(n)x_2(n)\right]=\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv;$$

$$\text{ROC: } \text{ROC}_{x_1} \cap \text{ Inverted ROC}_{x_2}$$

  - C is a closed contour that encloses the origin and lies in the common ROC

- Convolution

$$Z\left[x_1(n)*x_2(n)\right]=X_1(z)X_2(z); \quad \text{ROC: ROC}_{x_1}\cap\text{ROC}_{x_2}$$

Example

- Let $X_1(z) = 2 + 3z^{-1} + 4z^{-2}$ and $X_2(z) = 3 + 4z^{-1} + 5z^{-2} + 6z^{-3}$.
  - Determine $X_3(z) = X_1(z)\, X_2(z)$.

Example

- From definition

$$X(z) \equiv Z\left[x(n)\right] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- $X_1(z) = 2 + 3z^{-1} + 4z^{-2}$ and $X_2(z) = 3 + 4z^{-1} + 5z^{-2} + 6z^{-3}$
- So

$$x_1(n) = \{2, 3, 4\} \qquad x_2(n) = \{3, 4, 5, 6\}$$

Example

- $X_3(z) = X_1(z)\, X_2(z)$ ➔ Convolution

$$Z\left[x_1(n) * x_2(n)\right] = X_1(z)\, X_2(z)$$

- the convolution of these two sequences will give the coefficients of the required polynomial product

  >> x1 = [2,3,4]; x2 = [3,4,5,6]; x3 = conv(x1,x2)

  x3 = 6 17 34 43 38 24

- Hence

$$X_3(z) = 6+17z^{-1} + 34z^{-2} + 43z^{-3} + 38z^{-4} + 24z^{-5}$$

Example

- Let $X_1(z) = z + 2 + 3z^{-1}$ and $X_2(z) = 2z^2 + 4z + 3 + 5z^{-1}$.
  - Determine $X_3(z) = X_1(z)\, X_2(z)$.

```
function [y,ny] = conv_m(x,nx,h,nh)
% Modified convolution routine for signal processing
% ----------------------------------------------------
% [y,ny] = conv_m(x,nx,h,nh)
% [y,ny] = convolution result
% [x,nx] = first signal
% [h,nh] = second signal
%
nyb = nx(1)+nh(1); nye = nx(length(x)) + nh(length(h));
ny = [nyb:nye]; y = conv(x,h);
```

Example

- $X_1(z) = z + 2 + 3z^{-1}$ and $X_2(z) = 2z^2 + 4z + 3 + 5z^{-1}$

$$x_1(n) = \{1, \underset{\uparrow}{2}, 3\} \qquad x_2(n) = \{2, 4, \underset{\uparrow}{3}, 5\}$$

```
>> x1 = [1,2,3]; n1 = [-1:1]; x2 = [2,4,3,5]; n2 = [-2:1];
>> [x3,n3] = conv_m(x1,n1,x2,n2)
x3 = 2 8 17 23 19 15
n3 = -3 -2 -1 0 1 2
```

- Hence

$$X_3(z) = 2z^3 + 8z^2 + 17z + 23 + 19z^{-1} + 15z^{-2}$$

Common z-transform pairs

Sequence	*Transform*	*ROC*
$\delta(n)$	1	$\forall\, z$
$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u(-n-1)$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-b^n u(-n-1)$	$\dfrac{1}{1-bz^{-1}}$	$\|z\| < \|b\|$
$[a^n \sin \omega_0 n]\, u(n)$	$\dfrac{(a \sin \omega_0) z^{-1}}{1-(2a \cos \omega_0) z^{-1} + a^2 z^{-2}}$	$\|z\| > \|a\|$
$[a^n \cos \omega_0 n]\, u(n)$	$\dfrac{1-(a \cos \omega_0) z^{-1}}{1-(2a \cos \omega_0) z^{-1} + a^2 z^{-2}}$	$\|z\| > \|a\|$
$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-nb^n u(-n-1)$	$\dfrac{bz^{-1}}{(1-bz^{-1})^2}$	$\|z\| < \|b\|$

Example

- Using *z*-transform properties and the *z*-transform table, determine the *z*-transform of

$$x_1(n) = (n-2)(0,5)^{(n-2)} \cos\left[\frac{\pi}{3}(n-2)\right] u(n-2)$$

Example

$$x_1(n) = (n-2)(0,5)^{(n-2)}\cos\left[\frac{\pi}{3}(n-2)\right]u(n-2)$$

- Applying the sample-shift property

$$Z\left[x(n-n_o)\right] = z^{-n_o}X(z)$$

$$X(z) = Z\left[x(n)\right] = z^{-2}Z\left[n(0,5)^n\cos\left(\frac{\pi n}{3}\right)u(n)\right]$$

Example

$$X(z) = Z\left[x(n)\right] = z^{-2} Z\left[n(0,5)^n \cos\left(\frac{\pi n}{3}\right) u(n)\right]$$

- Applying the multiplication by a ramp property

$$Z\left[nx(n)\right] = -z\frac{dX(z)}{dz}$$

$$\Downarrow$$

$$X(z) = z^{-2}\left\{-z\frac{dZ\left[(0,5)^n \cos\left(\frac{\pi n}{3}\right) u(n)\right]}{dz}\right\}$$

Example

$$X(z) = z^{-2}\left\{ -z \frac{dZ\left[(0,5)^n \cos\left(\frac{\pi n}{3}\right) u(n)\right]}{dz} \right\}$$

- From table

$$Z\left[(0,5)^n \cos\left(\frac{\pi n}{3}\right) u(n)\right] = \frac{1-\left(0,5\cos\frac{\pi}{3}\right)z^{-1}}{1-2\left(0,5\cos\frac{\pi}{3}\right)z^{-1}+0,25z^{-1}}; \quad |z|>0,5$$

$$= \frac{1-0,25z^{-1}}{1-0,5z^{-1}+0,25z^{-2}}; \quad |z|>0,5$$

Example

- Hence

$$X(z) = -z^{-1}\frac{d}{dz}\left\{\frac{1-0,25z^{-1}}{1-0,5z^{-1}+0,25z^{-2}}\right\}; \quad |z|>0,5$$

$$= -z^{-1}\left\{\frac{-0,25z^{-2}+0,5z^{-3}-0,0625z^{-4}}{1-z^{-1}+0,75z^{-2}-0,25z^{-3}+0,0625z^{-4}}\right\},$$

$$= \frac{0,25z^{-3}-0,5z^{-4}+0,0625z^{-5}}{1-z^{-1}+0,75z^{-2}-0,25z^{-3}+0,0625z^{-4}}, \qquad |z|>0,5$$

Example: MATLAB verification

```
>> b = [0,0,0,0.25,-0.5,0.0625]; a = [1,-1,0.75,-0.25,0.0625];
>> [delta,n]=impseq(0,0,7)
>> x = filter(b,a,delta) % check sequence
>> x = [(n-2).*(1/2).^(n-2).*cos(pi*(n-2)/3)].*stepseq(2,0,7)
```

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