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The Inversion of z-Transform

The Inverse z-Transform

- The inverse *z*-transform of a complex function *X*(*z*) is given by

$$x(n) \triangleq Z^{-1}\left[X(z)\right] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- The inverse *z*-transform computation requires an evaluation of a complex contour integral
  - a complicated procedure
  - use the partial fraction expansion method

The Inverse z-Transform Idea

- $X(z)$ is a rational function of z^{-1}
  - can be expressed as a sum of simple factors using the partial fraction expansion

- The individual sequences corresponding to these factors can be written down using the *z*-transform table.

The inverse *z*-transform procedure

- Given

$$X(z)=\frac{b_o+b_1 z^{-1}+\ldots+b_M z^{-M}}{1+a_1 z^{-1}+\ldots+a_N z^{-N}}, \quad R_{x-}<|z|<R_{z+}$$

- express it as

$$X(z)=\frac{\tilde{b}_o+\tilde{b}_1 z^{-1}+\ldots+\tilde{b}_{N-1} z^{-(N-1)}}{1+a_1 z^{-1}+\ldots+a_N z^{-N}}+\sum_{k=0}^{M-N} C_k z^{-k}$$

Proper rational part   polynomial part if $M \geq N$

- Can be obtained by performing polynomial division if $M \geq N$ using the deconv function.

The inverse *z*-transform procedure

- Perform a partial fraction expansion on the proper rational part of $X(z)$ to obtain

$$X(z)=\sum_{k=1}^{N}\frac{R_k}{1-p_k z^{-1}}+\sum_{k=0}^{M-N}C_k z^{-k}$$

  - p_k is the kth pole of $X(z)$ and R_k is the residue at p_k
- The poles are distinct for which the residues are given by

$$R_k=\frac{\tilde{b}_o+\tilde{b}_1 z^{-1}+\ldots+\tilde{b}_{N-1}z^{-(N-1)}}{1+a_1 z^{-1}+\ldots+a_N z^{-N}}\left(1-p_k z^{-1}\right)\Big|_{z=p_k}$$

The inverse *z*-transform procedure

- If a pole p_k has multiplicity r, then its expansion is given by

$$\sum_{\ell-1}^{r} \frac{R_{k,\ell} z^{-(\ell-1)}}{\left(1-p_k z^{-1}\right)} = \frac{R_{k,1}}{1-p_k z^{-1}} + \frac{R_{k,2} z^{-1}}{\left(1-p_k z^{-1}\right)^2} + \ldots + \frac{R_{k,\mathrm{r}} z^{-(r-1)}}{\left(1-p_k z^{-1}\right)^r}$$

- the residues R_k, are computed using a more general formula

The inverse *z*-transform procedure

- write $x(n)$ as

$$x(n)=\sum_{k=1}^{N} R_k Z^{-1}\left[\frac{1}{1-p_k z^{-1}}\right]+\sum_{k=0}^{M-N} C_k \delta(n-k)$$

- finally, use the relation from Table to complete $x(n)$

$$Z^{-1}\left[\frac{z}{z-p_k}\right]=\begin{cases} p_k^n u(n) & |z_k|\le R_{x-} \\ -p_k^n u(-n-1) & |z_k|\ge R_{x+} \end{cases}$$

Example

- Find the inverse *z*-transform of

$$x(z) = \frac{z}{3z^2 - 4z + 1}$$

Example

$$x(z) = \frac{z}{3z^2 - 4z + 1}$$

- Write

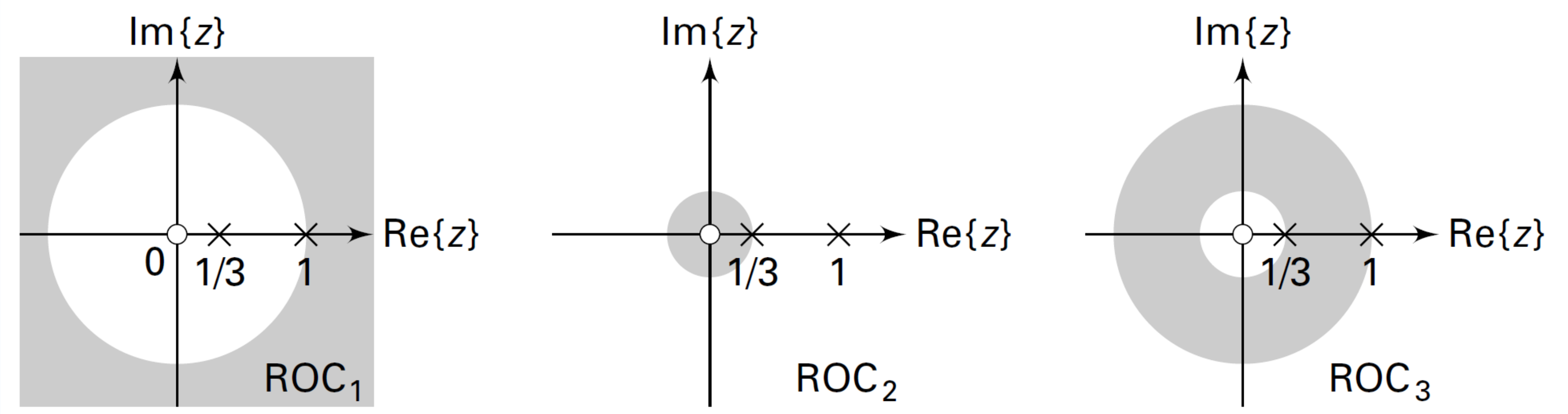
$$X(z) = \frac{z}{3\left(z^2 - \frac{4}{3}z + \frac{1}{3}\right)} = \frac{\frac{1}{3}z^{-1}}{1 - \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}}$$

$$= \frac{\frac{1}{3}z^{-1}}{\left(1 - z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} = \frac{\frac{1}{2}}{1 - z^{-1}} - \frac{\frac{1}{2}}{1 - \frac{1}{3}z^{-1}}$$

$$= \frac{1}{2}\left(\frac{1}{1 - z^{-1}}\right) - \frac{1}{2}\left(\frac{1}{1 - \frac{1}{3}z^{-1}}\right)$$

Example

- $X(z)$ has two poles: $z_1 = 1$ and $z_2 = 1/3$
- there are *three* possible ROCs



Example

1. ROC$_1$: $1 < |z| < \infty$.

   Both poles are on the interior side of the ROC$_1$

   $|z_1| \leq R_{x-} = 1$ and $|z_2| \leq 1$

$$x_1(n) = \frac{1}{2}u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

   a right-sided sequence.

Example

2. ROC$_2$: $1 < |z| < ⅓$.

both poles are on the exterior side of the ROC$_2$

$|z_1| \geq R_{x+} = ⅓$ *and* $|z_2| \geq ⅓$

$$x_2(n) = \frac{1}{2}\{-u(-n-1)\} - \frac{1}{2}\left\{-\left(\frac{1}{3}\right)^n u(-n-1)\right\}$$

$$= \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1) - \frac{1}{2}u(-n-1)$$

a left-sided sequence.

Example

3. ROC$_3$: ⅓ < $|z|$ < 1.

   pole z_1 is on the exterior side of the ROC$_3$: $|z_1| \geq R_{x+} = 1$

   pole z_2 is on the interior side of the ROC$_3$: $|z_2| \leq$ ⅓

$$x_3(n) = -\frac{1}{2}u(-n-1) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

a two-sided sequence.

MATLAB Implementation

- A MATLAB function residuez is available to compute the residue part and the direct (or polynomial) terms of a rational function in z^{-1}.
- A rational function in which the numerator and the denominator polynomials are in *ascending* powers of z^{-1}

$$X(z) = \frac{b_o + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_o + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{B(z)}{A(z)}$$

$$= \sum_{k=1}^{N} \frac{R_k}{1 - p_k z^{-1}} + \sum_{k=0}^{M-N} C_k z^{-k}$$

MATLAB Implementation

[R,p,C]=residuez(b,a)

- Computes the residues, poles, and direct terms of $X(z)$ in which two polynomials $B(z)$ and $A(z)$ are given in two vectors b and a
  - column vector R contains the residues
  - column vector p contains the pole locations
  - row vector C contains the direct terms

MATLAB Implementation

- If p(k) = ... = p(k+r-1) is a pole of multiplicity r, then the expansion includes the term of the form

$$\frac{R_k}{1-p_k z^{-1}}+\frac{R_{k+1}}{\left(1-p_k z^{-1}\right)^2}+\ldots+\frac{R_{k+r-1}}{\left(1-p_k z^{-1}\right)^r}$$

MATLAB Implementation

[b,a]=residuez(R,p,C)

- Three input arguments and two output arguments
- Converts the partial fraction expansion back to polynomials with coefficients in row vectors b and a.

Example - Residue calculations

- Consider the rational function

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

Example - Residue calculations

- Consider the rational function

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

- Rearrange $X(z)$ so that it is a function in ascending powers of z^{-1}.

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{0 + z^{-1}}{3 - 4z^{-1} + z^{-2}}$$

Example - Residue calculations

- using the MATLAB script

$$X(z)=\frac{z^{-1}}{3-4z^{-1}+z^{-2}}=\frac{0+z^{-1}}{3-4z^{-1}+z^{-2}}$$

```
>> b = [0,1]; a = [3,-4,1]; [R,p,C] = residuez(b,a)
R =
0.5000
-0.5000
p =
1.0000
0.3333
c =
[]
```

$$X(z)=\frac{\frac{1}{2}}{1-z^{-1}}-\frac{\frac{1}{2}}{1-\frac{1}{3}z^{-1}}$$

Example - Residue calculations

- convert back to the rational function form

$$X(z) = \frac{\frac{1}{2}}{1 - z^{-1}} - \frac{\frac{1}{2}}{1 - \frac{1}{3}z^{-1}}$$

>> [b,a] = residuez(R,p,C)

b =

0.0000

0.3333

a =

1.0000

-1.3333

0.3333

$$X(z) = \frac{0 + \frac{1}{3}z^{-1}}{1 - \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}} = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{z}{3z^{2} - 4z + 1}$$

Example

- Compute the inverse *z*-transform of

$$X(z) = \frac{1}{\left(1-0,9z^{-1}\right)^2\left(1+0,9z^{-1}\right)}, \quad |z| > 0,9$$

Example

- Evaluate the denominator polynomial as well as the residues using the MATLAB script

$$X(z) = \frac{1}{\left(1-0,9z^{-1}\right)^2\left(1+0,9z^{-1}\right)}, \quad |z| > 0,9$$

>> b = 1; a = poly([0.9,0.9,-0.9])

a = 1.0000 -0.9000 -0.8100 0.7290

>> [R,p,C]=residuez(b,a)

R = 0.2500 0.5000 0.2500

p = 0.9000 0.9000 -0.9000

c = []

Example

- From the residue calculations and using the order of residues

$$X(z) = \frac{0,25}{1-0,9z^{-1}} + \frac{0,5}{(1-0,9z^{-1})^2} + \frac{0,25}{1+0,9z^{-1}}, \quad |z| > 0,9$$

$$= \frac{0,25}{1-0,9z^{-1}} + \frac{0,5}{0,9} z \frac{0,9z^{-1}}{(1-0,9z^{-1})^2} + \frac{0,25}{1+0,9z^{-1}}, \quad |z| > 0,9$$

- Using table and the *z*-transform property of time-shift

$$x(n) = 0,25(0,9)^n u(n) + \frac{5}{9}(n+1)(0,9)^{n+1} u(n+1) + 0,25(-0,9)^n u(n)$$

$$= 0,75(0,9)^n u(n) + 0,5n(0,9)^n u(n) + 0,25(-0,9)^n u(n)$$

Example

- MATLAB verification

```
>> [delta,n] = impseq(0,0,7); x = filter(b,a,delta) % check sequence
x =
Columns 1 through 4
1.00000000000000 0.90000000000000 1.62000000000000 1.45800000000000
Columns 5 through 8
1.96830000000000 1.77147000000000 2.12576400000000 1.91318760000000
>> x = (0.75)*(0.9).^n + (0.5)*n.*(0.9).^n + (0.25)*(-0.9).^n % answer sequence
x =
Columns 1 through 4
1.00000000000000 0.90000000000000 1.62000000000000 1.45800000000000
Columns 5 through 8
1.96830000000000 1.77147000000000 2.12576400000000 1.91318760000000
```

Example

- Determine the inverse *z*-transform of

$$X(z) = \frac{1 + 0{,}4\sqrt{2}z^{-1}}{1 - 0{,}8\sqrt{2}z^{-1} + 0{,}64z^{-2}}$$

- so that the resulting sequence is causal and contains no complex numbers

Example

$$X(z) = \frac{1 + 0,4\sqrt{2}z^{-1}}{1 - 0,8\sqrt{2}z^{-1} + 0,64z^{-2}}$$

- have to find the poles of $X(z)$ in the polar form to determine the ROC of the causal sequence

```
>> b = [1,0.4*sqrt(2)]; a=[1,-0.8*sqrt(2),0.64];
>> [R,p,C] = residuez(b,a)
R =
0.5000 - 1.0000i
0.5000 + 1.0000i
p =
0.5657 + 0.5657i
0.5657 - 0.5657i
C = []
>> Mp=(abs(p))' % pole magnitudes
Mp = 0.8000 0.8000
>> Ap=(angle(p))'/pi % pole angles in pi units
Ap = 0.2500 -0.2500
```

Example

$$X(z)=\frac{1+0,4\sqrt{2}z^{-1}}{1-0,8\sqrt{2}z^{-1}+0,64z^{-2}}$$

- From these calculations

$$X(z)=\frac{0,5-j}{1-0,8e^{+j\frac{\pi}{4}}z^{-1}}+\frac{0,5+j}{1-0,8e^{+j\frac{\pi}{4}}z^{-1}},\quad |z|>0,8$$

- Using table

$$x(n)=(0,5-j)0,8^{n}e^{+j\frac{\pi}{4}n}u(n)+(0,5+j)0,8^{n}e^{-j\frac{\pi}{4}n}u(n)$$

$$=0,8^{n}\left[0,5\left\{e^{+j\frac{\pi}{4}n}+e^{-j\frac{\pi}{4}n}\right\}-j\left\{e^{+j\frac{\pi}{4}n}+e^{-j\frac{\pi}{4}n}\right\}\right]u(n)$$

$$=0,8^{n}\left[\cos\left(\frac{\pi n}{4}\right)+2\sin\left(\frac{\pi n}{4}\right)\right]u(n)$$

Example

$$X(z) = \frac{1 + 0,4\sqrt{2}z^{-1}}{1 - 0,8\sqrt{2}z^{-1} + 0,64z^{-2}}$$

- MATLAB verification

```
>> [delta, n] = impseq(0,0,6);
>> x = filter(b,a,delta) % check sequence
>> x = ((0.8).^n).*(cos(pi*n/4)+2*sin(pi*n/4))
```

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