UNIVERSITAS INDONESIA
Veritas, Probitas, Iustitia

Pengolahan Sinyal Digital

Adhi Harmoko Saputro

Differential Equations

Linear Constant-Coefficient Difference Equations

- In general, an *Nth*-order linear constant coefficient difference equation has the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- The output is not uniquely specified for a given input
  - The initial conditions are required
  - Linearity, time invariance, and causality depend on the initial conditions
  - If initial conditions are assumed to be zero system is linear, time invariant, and causal

Linear Constant-Coefficient Difference Equations

- Example
  - Moving Average

$$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3]$$

  - Difference Equation Representation

$$\sum_{k=0}^{0} a_k y[n-k] = \sum_{k=0}^{3} b_k x[n-k] \quad \text{where} \quad a_k = b_k = 1$$

General Solution

- A general solution can be expressed as the sum of a homogeneous response (natural response), and a particular solution (forced response) of the system:

$$y[n] = y_h[n] + y_p[n]$$

- The concept of *initial rest* of the LTI causal system described by the difference equation here means that $x[n] = 0$ implies $y[n] = 0$

Recursive Solution

- In the discrete-time case, there are an alternative to find the Differential Equations

$$y[n] = \frac{1}{a_o}\left(\sum_{k=0}^{M} b_x x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right)$$

- This is often how digital filters are implemented on a computer or a digital signal processor board
- The response of a differential equation
  - The differential equation is discretized at a given sampling rate to obtain a difference equation
  - The response of the difference equation is computed recursively

Example

- Consider the difference equation

$$y[n]-\frac{5}{6}y[n-1]+\frac{1}{6}y[n-2]=3x[n]-2x[n-1]$$

- Obtaining the recursive form of the difference equation

$$y[n]=\frac{5}{6}y[n-1]-\frac{1}{6}y[n-2]+3x[n]-2x[n-1]$$

- Assuming initial rest and that the input is an impulse, $y[-2] = y[-1] = 0$

Example

- the recursion can be started

$$y[0] = \frac{5}{6}y[-1] - \frac{1}{6}y[-2] + 3x[0] - 2x[-1]$$
$$= \frac{5}{6}(0) - \frac{1}{6}(0) + 3(1) - 2(0) = 3$$

$$y[1] = \frac{5}{6}y[0] - \frac{1}{6}y[-1] + 3x[1] - 2x[0]$$
$$= \frac{5}{6}(3) - \frac{1}{6}(0) + 3(0) - 2(1) = \frac{1}{2}$$

Example

$$y[2]=\frac{5}{6}y[1]-\frac{1}{6}y[0]+3x[2]-2x[1]$$

$$=\frac{5}{6}\left(\frac{1}{2}\right)-\frac{1}{6}(3)+3(0)-2(0)=-\frac{1}{12}$$

Matlab Implemantation

- A function called filter is available to solve difference equations numerically, given the input and the difference equation coefficients.
- In its simplest form this function is invoked by

```
y = filter(b,a,x)
```

- where
  - b = [b0, b1, ..., bM]; a = [a0, a1, ..., aN] are the coefficient arrays

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Matlab Implemantation

- To compute and plot impulse response, MATLAB provides the function impz.
- When invoked by

```
h = impz(b,a,n)
```

- Computes samples of the impulse response of the filter at the sample indices given in n with numerator coefficients in b and denominator coefficients in a.

Matlab Example

- Given the following difference equation

$$y(n) - y(n-1) + 0.9y(n-2) = x(n);$$

- Calculate and plot the impulse response $h(n)$ at $n = -20, \ldots, 100$.
- Calculate and plot the unit step response $s(n)$ at $n = -20, \ldots, 100$.
- Is the system specified by $h(n)$ stable?

Matlab Example

- From the given difference equation the coefficient arrays are
  - b = [1]; a=[1, -1, 0.9];

- MATLAB script:

```
>> b = [1]; a = [1, -1, 0.9]; n = [-20:120];
>> h = impz(b,a,n);
>> subplot(2,1,1); stem(n,h);
>> title('Impulse Response'); xlabel('n');
ylabel('h(n)')
```

Matlab Example

- MATLAB script:

```
n=-20:100;
x = (n-0) >= 0; % x = stepseq(0,-20,120);
s = filter(b,a,x);
subplot(2,1,2); stem(n,s)
title('Step Response'); xlabel('n');
ylabel('s(n)')
```

Matlab Example

- To determine the stability of the system, we have to determine $h(n)$ for all n.
- Although we have not described a method to solve the difference equation
- we can use the plot of the impulse response to observe that $h(n)$ is practically zero for $n > 120$. Hence the sum

$$\sum |h[n]|$$

```
>> sum(abs(h))
ans = 14.8785
```

which implies that the system is stable.

Example

- Let us consider the convolution given in previous example. The input sequence is of finite duration $x(n) = u(n) - u(n-10)$
- while the impulse response is of infinite duration

$$h(n) = (0.9)^n u(n)$$

- Determine $y(n) = x(n) * h(n)$.

Example

- If the LTI system, given by the impulse response *h*(*n*), can be described by a difference equation, then *y*(*n*) can be obtained from the filter function.
- From the *h*(*n*) expression

$$(0.9)h[n-1]=(0.9)(0.9)^{n-1}u[n-1]=(0.9)^{n}u[n-1]$$

$$\begin{aligned} h[n]-(0.9)h[n-1] &= (0.9)^{n}u[n]-(0.9)^{n}u[n-1] \\ &= (0.9)^{n}\left(u[n]-u[n-1]\right)=(0.9)^{n}\delta[n] \\ &= \delta[n] \end{aligned}$$

Example

- The last step follows from the fact that $\delta(n)$ is nonzero only at $n = 0$.
- By definition $h(n)$ is the output of an LTI system when the input is $\delta(n)$.
- Hence substituting $x(n)$ for $\delta(n)$ and $y(n)$ for $h(n)$, the difference equation is

$$y[n]-(0.9)\,y[n-1]=x[n]$$

Example

- MATLAB’s filter function can be used to compute the convolution indirectly.

```
>> b = [1]; a = [1,-0.9];
>> n = -5:50; x = stepseq(0,-5,50) - stepseq(10,-5,50);
>> y = filter(b,a,x);
>> subplot(2,1,2); stem(n,y); title(’Output sequence’)
>> xlabel(’n’); ylabel(’y(n)’); axis([-5,50,-0.5,8])
```

Terima Kasih