UNIVERSITAS INDONESIA
Veritas, Probitas, Iustitia

Pengolahan Sinyal Digital

Adhi Harmoko Saputro

The Discrete-time Fourier Analysis

Discrete-time Fourier Transform

- A linear and time-invariant system can be represented using its response to the unit sample sequence that

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = h(n) * x(n)$$

- A linear and time-invariant system can be represented using the complex exponential signal set

$$\left\{ e^{j\omega n} \right\}$$

Discrete-time Fourier Transform

- General form of DTFT

$$X\left(e^{j\omega}\right) \triangleq \mathcal{F}\left[x(n)\right] = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

- The inverse discrete-time Fourier transform (IDTFT) of $X(e^{j\omega})$ is given by

$$x(n) \triangleq \mathcal{F}^{-1}\left[X\left(e^{j\omega}\right)\right] = \frac{1}{2\pi}\sum_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

Discrete-time Fourier Transform

- The operator $\mathcal{F}[.]$ transforms a discrete signal $x(n)$ into a complex-valued continuous function $X(e^{j\omega})$ of real variable ω, called a digital frequency, which is measured in radians/sample

Example

- Determine the discrete-time Fourier transform of $x(n) = (0.5)^n\, u(n)$.

- The sequence $x(n)$ is absolutely summable; therefore its discrete-time Fourier transform exists.

$$X\left(e^{j\omega}\right)=\sum_{-\infty}^{\infty} x(n)e^{-j\omega n}=\sum_{0}^{\infty}(0,5)^{n}\,e^{-j\omega n}$$

$$=\sum_{0}^{\infty}\left(0,5e^{-j\omega}\right)^{n}=\frac{1}{1-0,5e^{-j\omega}}=\frac{e^{j\omega}}{e^{j\omega}-0,5}$$

The geometric series

- A one-sided exponential sequence of the form

$$\left\{\alpha^n, \quad n \geq 0\right\}$$

- The series converges for $|\alpha| < 1$, while the sum of its components converges to

$$\sum_{n=0}^{\infty} \alpha^n \rightarrow \frac{1}{1-\alpha}, \quad \text{for } |\alpha| < 1$$

Properties of the DTFT

Properties of the DTFT

- **Linearity**: The discrete-time Fourier transform is a linear transformation

$$\mathcal{F}\left[\alpha x_1(n)+\beta x_2(n)\right]=\alpha\mathcal{F}\left[x_1(n)\right]+\beta\mathcal{F}\left[x_2(n)\right]$$

  - for every α, β, $x_1(n)$, and $x_2(n)$.

- **Time shifting**: A shift in the time domain corresponds to the phase shifting.

$$\mathcal{F}\left[x(n-k)\right]=X\left(e^{j\omega}\right)e^{-j\omega k}$$

Properties of the DTFT

- **Frequency shifting**: Multiplication by a complex exponential corresponds to a shift in the frequency domain

$$\mathcal{F}\left[x(n)e^{j\omega_o n}\right] = X\left(e^{j(\omega-\omega_o)}\right)$$

- **Conjugation**: Conjugation in the time domain corresponds to the folding and conjugation in the frequency domain

$$\mathcal{F}\left[x^*(n)\right] = X^*\left(e^{-j\omega}\right)$$

Properties of the DTFT

- **Folding:** Folding in the time domain corresponds to the folding in the frequency domain

$$\mathcal{F}\left[x(-n)\right] = X\left(e^{-j\omega}\right)$$

- **Symmetries in real sequences:** We have already studied the conjugate symmetry of real sequences. These real sequences can be decomposed into their even and odd parts

$$x(n) = x_e(n) + x_o(n)$$

$$\mathcal{F}\left[x_e(n)\right] = \mathrm{Re}\left[X\left(e^{j\omega}\right)\right] \qquad \mathcal{F}\left[x_o(n)\right] = j\,\mathrm{Im}\left[X\left(e^{j\omega}\right)\right]$$

Properties of the DTFT

- **Convolution:** This is one of the most useful properties that makes system analysis convenient in the frequency domain

$$\mathcal{F}\left[x_1(n) * x_2(n)\right] = \mathcal{F}\left[x_1(n)\right]\mathcal{F}\left[x_2(n)\right] = X_1\left(e^{j\omega}\right)X_2\left(e^{j\omega}\right)$$

- **Multiplication:** This is a dual of the convolution property

$$\mathcal{F}\left[x_1(n) \cdot x_2(n)\right] = \mathcal{F}\left[x_1(n)\right] \circledast \mathcal{F}\left[x_2(n)\right]$$
$$\triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1\left(e^{j\theta}\right)X_2\left(e^{j\omega-\theta}\right)d\theta$$

- This convolution-like operation is called a *periodic convolution* and hence denoted by $\circledast$

Properties of the DTFT

- **Energy:** The energy of the sequence $x(n)$ can be written as

$$\varepsilon_x = \sum_{-\infty}^{\infty} \left| x(n) \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X\left(e^{j\omega}\right) \right|^2 d\omega$$

$$= \int_{0}^{\pi} \frac{\left| X\left(e^{j\omega}\right) \right|^2}{\pi} d\omega$$

Example

- Verify the linearity property of DTFT using real-valued finite duration sequences

$$\mathcal{F}\left[\alpha x_1(n)+\beta x_2(n)\right]=\alpha\mathcal{F}\left[x_1(n)\right]+\beta\mathcal{F}\left[x_2(n)\right]$$

- Which $x_1(n)$ and $x_2(n)$ are two random sequences uniformly distributed between [0, 1] over $0 \le n \le 10$.

MATLAB script

```
>> x1 = rand(1,11); x2 = rand(1,11); n = 0:10;
>> alpha = 2; beta = 3; k = 0:500; w = (pi/500)*k;
>> X1 = x1 * (exp(-j*pi/500)).^(n'*k); % DTFT of x1
>> X2 = x2 * (exp(-j*pi/500)).^(n'*k); % DTFT of x2
>> x = alpha*x1 + beta*x2; % Linear combination of x1 & x2
>> X = x * (exp(-j*pi/500)).^(n'*k); % DTFT of x
>> % verification
>> X_check = alpha*X1 + beta*X2; % Linear Combination of X1 & X2
>> error = max(abs(X-X_check)) % Difference
error =7.1054e-015
```

Example

- Verify the sample shift property

$$\mathcal{F}\left[x(n-k)\right] = X\left(e^{j\omega}\right)e^{-j\omega k}$$

- which $x(n)$ is a random sequence uniformly distributed between [0, 1] over $0 \leq n \leq 10$ and let $y(n) = x(n-2)$.

MATLAB script

```
>> x = rand(1,11); n = 0:10;
>> k = 0:500; w = (pi/500)*k;
>> X = x * (exp(-j*pi/500)).^(n'*k); % DTFT of x
>> % signal shifted by two samples
>> y = x; m = n+2;
>> Y = y * (exp(-j*pi/500)).^(m'*k); % DTFT of y
>> % verification
>> Y_check = (exp(-j*2).^w).*X; % multiplication by exp(-j2w)
>> error = max(abs(Y-Y_check)) % Difference
error =5.7737e-015
```

Example

- Verify the frequency shift property

$$\mathcal{F}\left[x(n)e^{j\omega_o n}\right] = X\left(e^{j(\omega-\omega_o)}\right)$$

- Which

$$x(n) = \cos(\pi n/2), \quad 0 \le n \le 100$$

$$y(n) = e^{j\pi n/4}x(n)$$

MATLAB script

```
>> n = 0:100; x = cos(pi*n/2);
>> k = -100:100; w = (pi/100)*k; % frequency between -pi and +pi
>> X = x * (exp(-j*pi/100)).^(n'*k); % DTFT of x
>> y = exp(j*pi*n/4).*x; % signal multiplied by exp(j*pi*n/4)
>> Y = y * (exp(-j*pi/100)).^(n'*k); % DTFT of y
```

MATLAB script

```
% Graphical verification
>> subplot(2,2,1); plot(w/pi,abs(X)); grid; axis([-1,1,0,60])
>> xlabel('frequency in pi units'); ylabel('|X|')
>> title('Magnitude of X')
>> subplot(2,2,2); plot(w/pi,angle(X)/pi); grid; axis([-1,1,-1,1])
>> xlabel('frequency in pi units'); ylabel('radiands/pi')
>> title('Angle of X')
>> subplot(2,2,3); plot(w/pi,abs(Y)); grid; axis([-1,1,0,60])
>> xlabel('frequency in pi units'); ylabel('|Y|')
>> title('Magnitude of Y')
>> subplot(2,2,4); plot(w/pi,angle(Y)/pi); grid; axis([-1,1,-1,1])
>> xlabel('frequency in pi units'); ylabel('radians/pi')
>> title('Angle of Y')
```

Example

- Verify the conjugation property

$$\mathcal{F}\left[x^*(n)\right] = X^*\left(e^{-j\omega}\right)$$

- Which $x(n)$ is a complex-valued random sequence over $-5 \le n \le 10$ with real and imaginary parts uniformly distributed between [0, 1]

MATLAB script

```
>> n = -5:10; x = rand(1,length(n)) + j*rand(1,length(n));
>> k = -100:100; w = (pi/100)*k; % frequency between -pi and +pi
>> X = x * (exp(-j*pi/100)).^(n'*k); % DTFT of x
% conjugation property
>> y = conj(x); % signal conjugation
>> Y = y * (exp(-j*pi/100)).^(n'*k); % DTFT of y
% verification
>> Y_check = conj(fliplr(X)); % conj(X(-w))
>> error = max(abs(Y-Y_check)) % Difference
error = 0
```

Example

- Verify the folding property

$$\mathcal{F}\left[x(-n)\right] = X\left(e^{-j\omega}\right)$$

- Which $x(n)$ be a random sequence over $-5 \le n \le 10$ uniformly distributed between [0, 1].

MATLAB script

```
>> n = -5:10; x = rand(1,length(n));
>> k = -100:100; w = (pi/100)*k; % frequency between -pi and +pi
>> X = x * (exp(-j*pi/100)).^(n'*k); % DTFT of x
% folding property
>> y = fliplr(x); m = -fliplr(n); % signal folding
>> Y = y * (exp(-j*pi/100)).^(m'*k); % DTFT of y
% verification
>> Y_check = fliplr(X); % X(-w)
>> error = max(abs(Y-Y_check)) % Difference
error = 0
```

Example

- Verify the symmetry property of real signals

$$x(n) = x_e(n) + x_o(n)$$

$$\mathcal{F}\left[x_e(n)\right] = \mathrm{Re}\left[X\left(e^{j\omega}\right)\right] \qquad \mathcal{F}\left[x_o(n)\right] = j\,\mathrm{Im}\left[X\left(e^{j\omega}\right)\right]$$

- Which $x(n) = \sin(\pi n/2)$, $-5 \le n \le 10$

MATLAB script

```
>> n = -5:10; x = sin(pi*n/2);
>> k = -100:100; w = (pi/100)*k; % frequency between -pi and +pi
>> X = x * (exp(-j*pi/100)).^(n'*k); % DTFT of x
% signal decomposition
>> [xe,xo,m] = evenodd(x,n); % even and odd parts
>> XE = xe * (exp(-j*pi/100)).^(m'*k); % DTFT of xe
>> XO = xo * (exp(-j*pi/100)).^(m'*k); % DTFT of xo
% verification
>> XR = real(X); % real part of X
>> error1 = max(abs(XE-XR)) % Difference
error1 = 1.8974e-019
>> XI = imag(X); % imag part of X
```

MATLAB script

```
>> error2 = max(abs(XO-j*XI)) % Difference
error2 = 1.8033e-019
% graphical verification
>> subplot(2,2,1); plot(w/pi,XR); grid; axis([-1,1,-2,2])
>> xlabel('frequency in pi units'); ylabel('Re(X)');
>> title('Real part of X')
>> subplot(2,2,2); plot(w/pi,XI); grid; axis([-1,1,-10,10])
>> xlabel('frequency in pi units'); ylabel('Im(X)');
>> title('Imaginary part of X')
```

MATLAB script

```
>> subplot(2,2,3); plot(w/pi,real(XE)); grid; axis([-1,1,-2,2])
>> xlabel('frequency in pi units'); ylabel('XE');
>> title('Transform of even part')
>> subplot(2,2,4); plot(w/pi,imag(XO)); grid; axis([-1,1,-10,10])
>> xlabel('frequency in pi units'); ylabel('XO');
>> title('Transform of odd part')
```

Terima Kasih