Pengolahan Sinyal Digital

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Frequency Domain Representation of LTI Systems

Frequency Domain Representation of LTI Systems

- The Fourier transform representation is the most useful signal representation for LTI systems

Response to A Complex Exponential

- Let $x(n) = e^{j\omega o n}$ be the input to an LTI system represented by the impulse response $h(n)$.

$$e^{j\omega_o n} \longrightarrow \boxed{h(n)} \longrightarrow h(n) * e^{j\omega_o n}$$

- Then

$$y(n) = h(n) * e^{j\omega_o n} = \sum_{-\infty}^{\infty} h(k) e^{j\omega_o (n-k)}$$

$$= \left[\sum_{-\infty}^{\infty} h(k) e^{-j\omega_o k} \right] e^{j\omega_o n}$$

$$= \left[\mathcal{F}\left[h(n)\right]_{\omega=\omega_o} \right] e^{j\omega_o n}$$

Frequency Response

- The discrete-time Fourier transform of an impulse response is called the *frequency response* (or *transfer function*) of an LTI system and is denoted by

$$H\left(e^{j\omega n}\right) \triangleq \sum_{-\infty}^{\infty} h(n) e^{-j\omega n}$$

Frequency Response

- The LTI system can be represented by

$$x(n)=e^{j\omega_o n} \longrightarrow \boxed{H\left(e^{j\omega}\right)} \longrightarrow y(n)=H\left(e^{j\omega_o}\right)\times e^{j\omega_o n}$$

- A linear combination of complex exponentials using the linearity of LTI systems

$$\sum_k A_k e^{j\omega_k n} \longrightarrow \boxed{h(n)} \longrightarrow \sum_k A_k H\left(e^{j\omega_k}\right)\times e^{j\omega_k n}$$

Response to A Complex Exponential

- The frequency response $H(e^{j\omega})$ is a complex function of ω.
  - The magnitude $|H(e^{j\omega})|$ of $H(e^{j\omega})$ is called the *magnitude* (*or gain*) *response* function
  - The angle $\angle H(e^{j\omega})$ is called the *phase response* function

Response to Sinusoidal Sequences

- A LTI system with sinusoidal input

$$x(n) = A\cos(\omega_o n + \theta_o)$$

$$\boxed{h(n)}$$

$$y(n) = A\left|H\left(e^{j\omega_o}\right)\right|\cos\left(\omega_o + \theta_o + \angle H\left(e^{j\omega_o}\right)\right)$$

- The response $y(n)$ is another sinusoid of the same frequency ω_o, with amplitude *gained* by $|H(e^{j\omega_o})|$ and phase *shifted* by $H(e^{j\omega_o})$
- This response is called the *steady-state response*, denoted by $y_{ss}(n)$.

Response to Sinusoidal Sequences

- A linear combination of sinusoidal sequences

$$\sum_k A_k \cos\left(\omega_k n + \theta_k\right)$$

$$\boxed{H\left(e^{j\omega}\right)}$$

$$\sum_k A_k \left|H\left(e^{j\omega_k}\right)\right| \cos\left(\omega_k + \theta_k + \angle H\left(e^{j\omega_k}\right)\right)$$

Response to Arbitrary Sequences

- Let

$$X\left(e^{j\omega}\right) = \mathcal{F}\left|x(n)\right| \qquad Y\left(e^{j\omega}\right) = \mathcal{F}\left|y(n)\right|$$

- Using the convolution property

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right)$$

- An LTI system can be represented in the frequency domain by

$$X\left(e^{j\omega}\right) \longrightarrow \boxed{H\left(e^{j\omega}\right)} \longrightarrow Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right)$$

Response to Arbitrary Sequences

$$X\left(e^{j\omega}\right) \longrightarrow \boxed{H\left(e^{j\omega}\right)} \longrightarrow Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

- The output $y(n)$ is then computed from $Y(e^{j\omega})$ using the inverse discrete-time Fourier transform
- Requiring an integral operation, which is not a convenient operation in MATLAB
- There is an alternate approach to the computation of output to arbitrary inputs using the z-transform and partial fraction expansion

Example

- Determine the frequency response $H(e^{j\omega})$ of a system characterized by $h(n) = (0.9)^n\, u(n)$.
- Plot the magnitude and the phase responses.

Example

- Based on frequency response formula

$$H\left(e^{j\omega n}\right) \triangleq \sum_{-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$H\left(e^{j\omega n}\right) = \sum_{-\infty}^{\infty} h(n) e^{-j\omega n} = \sum_{0}^{\infty} (0,9) e^{-j\omega n}$$

$$= \sum_{0}^{\infty} \left(0,9e^{-j\omega}\right)^{n} = \frac{1}{1-0,9e^{-j\omega}}$$

Example

- Hence

$$\left|H\left(e^{j\omega n}\right)\right| = \sqrt{\frac{1}{\left(1-0,9\cos\omega\right)^2+\left(0,9\sin\omega\right)^2}}$$

$$= \frac{1}{\sqrt{1,81-1,9\cos\omega}}$$

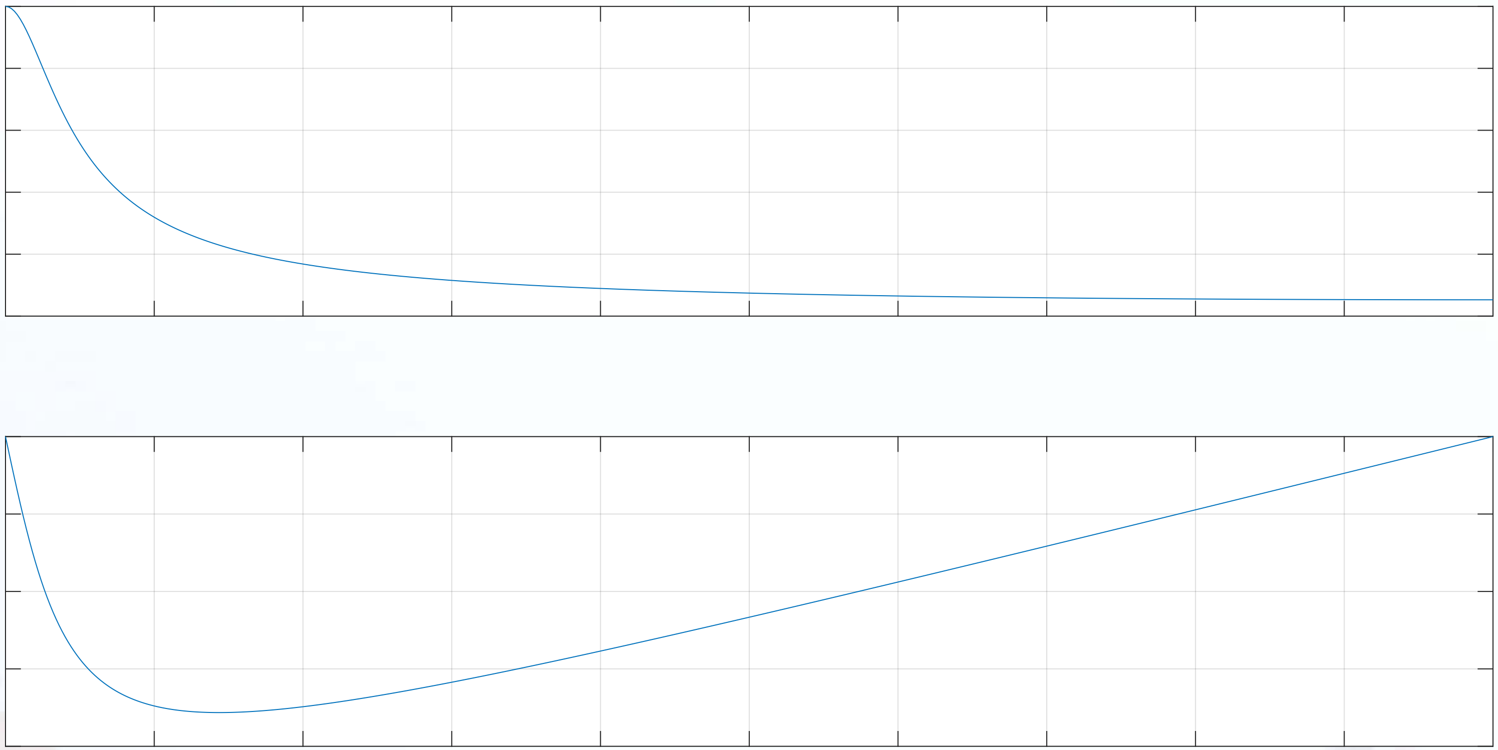
- and

$$\angle H\left(e^{j\omega n}\right) = -\arctan\left[\frac{0,9\sin\omega}{1-0,9\cos\omega}\right]$$

Example: Plot the responses

```
>> w = [0:1:500]*pi/500; % [0, pi] axis divided into 501 points.
>> H = exp(j*w) ./ (exp(j*w) - 0.9*ones(1,501));
>> magH = abs(H); angH = angle(H);
>> subplot(2,1,1); plot(w/pi,magH); grid;
>> xlabel('frequency in pi units'); ylabel('|H|');
>> title('Magnitude Response');
>> subplot(2,1,2); plot(w/pi,angH/pi); grid
>> xlabel('frequency in pi units');
>> ylabel('Phase in pi Radians');
>> title('Phase Response');
```

Example: Plot the responses



Frequency Response Function from Difference Equations

- An LTI system is represented by the difference equation

$$y(n)+\sum_{\ell=1}^{N} a_{\ell} y(n-\ell)=\sum_{m=0}^{M} b_{m} x(n-m)$$

- Frequency Response form

$$H\left(e^{j\omega}\right)e^{j\omega n}+\sum_{\ell=1}^{N} a_{\ell} H\left(e^{j\omega}\right)e^{j\omega(n-\ell)}=\sum_{m=0}^{M} b_{m} e^{j\omega(n-m)}$$

$$H\left(e^{j\omega}\right)=\frac{\sum_{m=0}^{M} b_{m} e^{-j\omega m}}{1+\sum_{\ell=1}^{N} a_{\ell} e^{-j\omega\ell}}$$

Example

- An LTI system is specified by the difference equation

$$y(n) = 0.8y(n-1) + x(n)$$

  - Determine $H(e^{j\omega})$.
  - Calculate and plot the steady-state response $y_{ss}(n)$ to

$$x(n) = \cos(0.05\pi n)\, u(n)$$

Example

- Rewrite the difference equation as $y(n) - 0.8y(n-1) = x(n)$.
- Using “Frequency Response Function from Difference Equations” formula

$$H\left(e^{j\omega}\right) = \frac{1}{1 - 0{,}8\,\mathrm{e}^{-j\omega}}$$

Example

- In the steady state the input is $x(n) = \cos(0.05\pi n)$ with frequency $\omega_o = 0.05\pi$ and $\theta_o = 0°$. The response of the system is

$$H\left(e^{j0,05\pi}\right) = \frac{1}{1-0,8\,\mathrm{e}^{-j0,05\pi}} = 4,0928e^{-j0,5377}$$

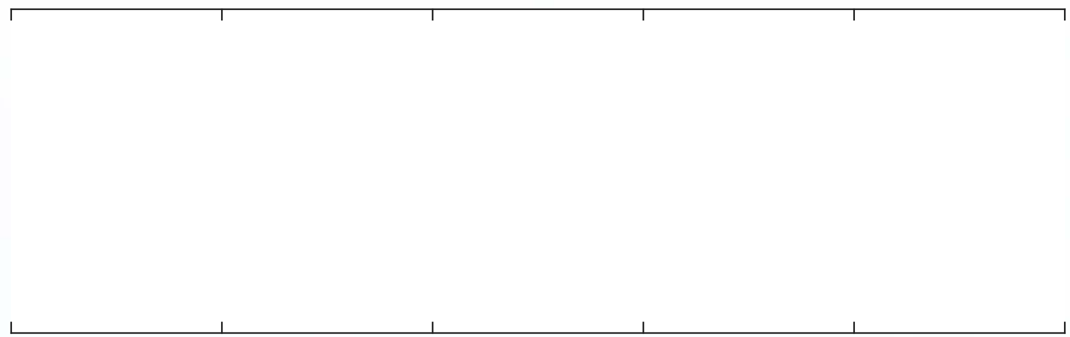
$$\begin{aligned} y_{ss}(n) &= 4,0928\cos\left(0.05\pi n - 0,5377\right) \\ &= 4,0928\cos\left[0,05\pi\left(n-3,42\right)\right] \end{aligned}$$

- This means that at the output the sinusoid is scaled by 4.0928 and shifted by 3.42 samples.

Example: Verify with Matlab

```
>> subplot(1,1,1)
>> b = 1; a = [1,-0.8];
>> n=[0:100];x = cos(0.05*pi*n);
>> y = filter(b,a,x);
>> subplot(2,1,1); stem(n,x);
>> xlabel('n'); ylabel('x(n)'); title('Input sequence')
>> subplot(2,1,2); stem(n,y);
>> xlabel('n'); ylabel('y(n)'); title('Output sequence')
```

Example: Verify with Matlab



Example

- A 3rd-order lowpass filter is described by the difference equation

$$y(n) \;=\; 0,0181x(n) + 0,0543x(n-1) + 0,0543x(n-2)$$
$$+0.0181x(n-3) + 1,76y(n-1) - 1,1829y(n-2) + 0,2781y(n-3)$$

- Plot the magnitude and the phase response of this filter, and verify that it is a lowpass filter.

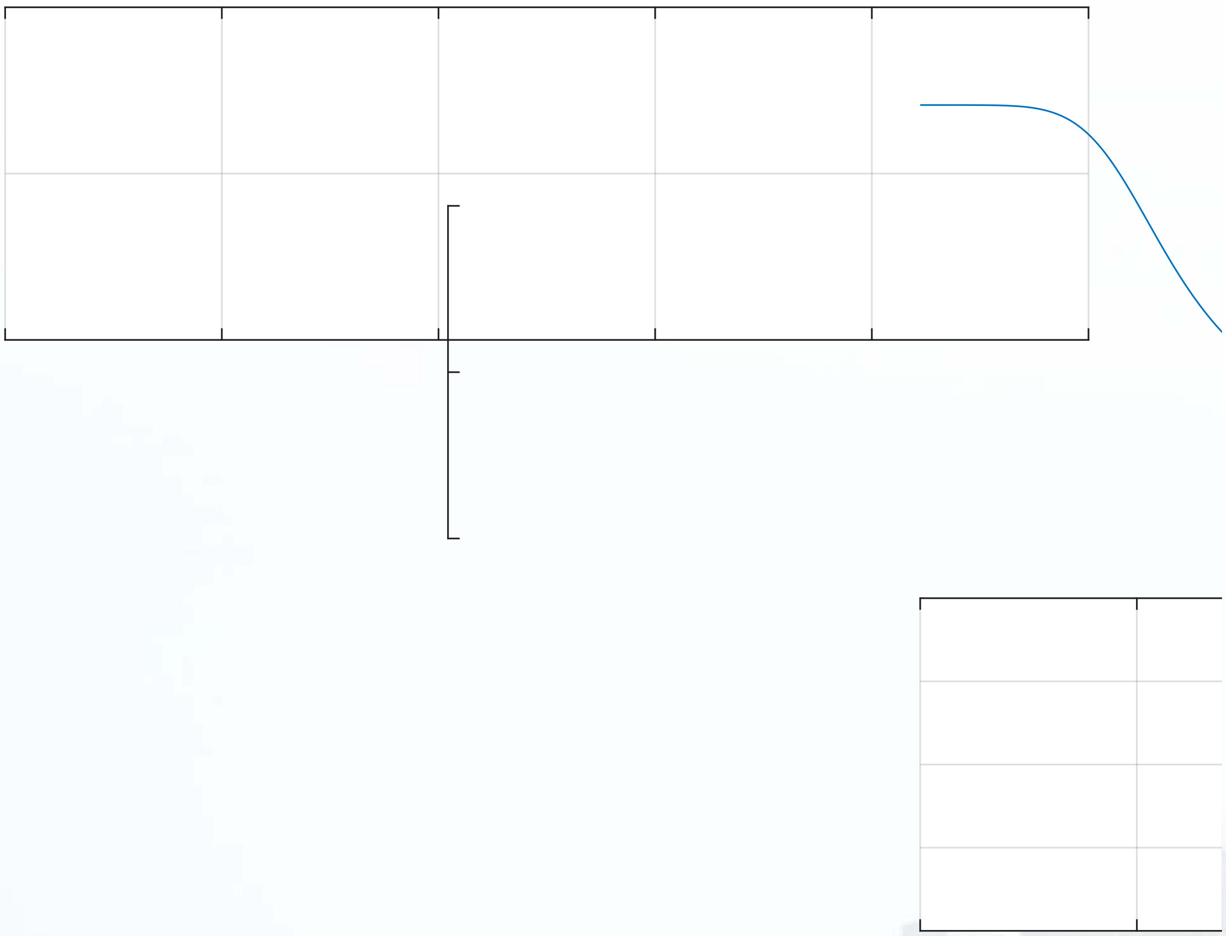
Example

```
>> b = [0.0181, 0.0543, 0.0543, 0.0181]; % filter coefficient array b
>> a = [1.0000, -1.7600, 1.1829, -0.2781]; % filter coefficient array a
>> m = 0:length(b)-1; l = 0:length(a)-1; % index arrays m and l
>> K = 500; k = 0:1:K; % index array k for frequencies
>> w = pi*k/K; % [0, pi] axis divided into 501 points.
>> num = b * exp(-j*m'*w); % Numerator calculations
>> den = a * exp(-j*l'*w); % Denominator calculations
>> H = num ./ den; % Frequency response
>> magH = abs(H); angH = angle(H); % mag and phase responses
```

Example

```
>> subplot(2,1,1); plot(w/pi,magH); grid; axis([0,1,0,1])
>> xlabel('frequency in pi units'); ylabel('|H|');
>> title('Magnitude Response');
>> subplot(2,1,2); plot(w/pi,angH/pi); grid
>> xlabel('frequency in pi units'); ylabel('Phase in pi Radians');
>> title('Phase Response');
```

Example



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