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# Frequency Domain Representation of LTI Systems

# Frequency Domain Representation of LTI Systems

• The Fourier transform representation is the most useful signal representation for LTI systems

#### Response to A Complex Exponential

• Let  $x(n) = e^{j\omega n}$  be the input to an LTI system represented by the impulse response h(n).

$$e^{j\omega_o n} \longrightarrow |h(n)| \longrightarrow h(n) * e^{j\omega_o n}$$

Then

$$y(n) = h(n) * e^{j\omega_o n} = \sum_{-\infty}^{\infty} h(k) e^{j\omega_o (n-k)}$$

$$= \left[\sum_{-\infty}^{\infty} h(k) e^{-j\omega_o k}\right] e^{j\omega_o n}$$

$$= \left[\mathcal{F} \left[h(n)\right]_{\omega=\omega_o}\right] e^{j\omega_o n}$$

### Frequency Response

• The discrete-time Fourier transform of an impulse response is called the *frequency response* (or *transfer function*) of an LTI system and is denoted by

$$H(e^{j\omega n}) \triangleq \sum_{n=0}^{\infty} h(n)e^{-j\omega n}$$

### Frequency Response

The LTI system can be represented by

$$x(n) = e^{j\omega_o n} \longrightarrow H(e^{j\omega}) \longrightarrow y(n) = H(e^{j\omega_o}) \times e^{j\omega_o n}$$

 A linear combination of complex exponentials using the linearity of LTI systems

$$\sum_{k} A_{k} e^{j\omega_{k}n} \longrightarrow h(n) \longrightarrow \sum_{k} A_{k} H(e^{j\omega_{k}}) \times e^{j\omega_{k}n}$$

### Response to A Complex Exponential

- The frequency response  $H(e^{j\omega})$  is a complex function of  $\omega$ .
  - The magnitude  $/H(e^{j\omega})/$  of  $H(e^{j\omega})$  is called the *magnitude* (or gain) response function
  - The angle  $\angle H(e^{j\omega})$  is called the *phase response* function

#### Response to Sinusoidal Sequences

A LTI system with sinusoidal input

$$x(n) = A\cos(\omega_{o}n + \theta_{o})$$

$$h(n)$$

$$y(n) = A |H(e^{j\omega_{o}})|\cos(\omega_{o} + \theta_{o} + \angle H(e^{j\omega_{o}}))$$

- The response y(n) is another sinusoid of the same frequency  $\omega_o$ , with amplitude gained by  $|H(e^{j\omega_0})|$  and phase shifted by  $H(e^{j\omega_0})$
- This response is called the *steady-state response*, denoted by  $y_{ss}(n)$ .

#### Response to Sinusoidal Sequences

A linear combination of sinusoidal sequences

$$\sum_{k} A_{k} \cos(\omega_{k} n + \theta_{k})$$

$$H(e^{j\omega})$$

$$\sum_{k} A_{k} \left| H(e^{j\omega_{k}}) \right| \cos(\omega_{k} + \theta_{k} + \angle H(e^{j\omega_{k}}))$$

## Response to Arbitrary Sequences

• Let

$$X(e^{j\omega}) = \mathcal{F}|x(n)|$$
  $Y(e^{j\omega}) = \mathcal{F}|y(n)|$ 

Using the convolution property

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

• An LTI system can be represented in the frequency domain by

$$X\left(e^{j\omega}\right) \longrightarrow H\left(e^{j\omega}\right) \longrightarrow Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

### Response to Arbitrary Sequences

$$X\left(e^{j\omega}\right) \longrightarrow H\left(e^{j\omega}\right) \longrightarrow Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

- The output y(n) is then computed from  $Y(e^{j\omega})$  using the inverse discrete-time Fourier transform
- Requiring an integral operation, which is not a convenient operation in MATLAB
- There is an alternate approach to the computation of output to arbitrary inputs using the *z*-transform and partial fraction expansion

- Determine the frequency response  $H(e^{j\omega})$  of a system characterized by  $h(n) = (0.9)^n u(n)$ .
- Plot the magnitude and the phase responses.



Based on frequency response formula

$$H(e^{j\omega n}) \triangleq \sum_{-\infty}^{\infty} h(n)e^{-j\omega n}$$

$$H\left(e^{j\omega n}\right) = \sum_{-\infty}^{\infty} h(n)e^{-j\omega n} = \sum_{0}^{\infty} (0,9)e^{-j\omega n}$$
$$= \sum_{0}^{\infty} (0,9e^{-j\omega})^{n} = \frac{1}{1-0,9e^{-j\omega}}$$

Hence

$$|H(e^{j\omega n})| = \sqrt{\frac{1}{(1-0.9\cos\omega)^2 + (0.9\sin\omega)^2}}$$
$$= \frac{1}{\sqrt{1.81 - 1.9\cos\omega}}$$

and

$$\angle H(e^{j\omega n}) = -\arctan \left[ \frac{0.9 \sin \omega}{1 - 0.9 \cos \omega} \right]$$

#### Example: Plot the responses

```
>> w = [0:1:500]*pi/500; % [0, pi] axis divided into 501 points.
>> H = \exp(j^*w) . / (\exp(j^*w) - 0.9^* \text{ones}(1,501));
>> magH = abs(H); angH = angle(H);
>> subplot(2,1,1); plot(w/pi,magH); grid;
>> xlabel('frequency in pi units'); ylabel('|H|');
>> title('Magnitude Response');
>> subplot(2,1,2); plot(w/pi,angH/pi); grid
>> xlabel('frequency in pi units'); ylabel('Phase in pi Radians');
>> title('Phase Response');
```

# Frequency Response Function from Difference Equations

An LTI system is represented by the difference equation

$$y(n) + \sum_{\ell=1}^{N} a_{\ell} y(n-\ell) = \sum_{m=0}^{M} b_{m} x(n-m)$$

Frequency Response form

$$H\left(e^{j\omega}\right)e^{j\omega n} + \sum_{\ell=1}^{N} a_{\ell}H\left(e^{j\omega}\right)e^{j\omega(n-\ell)} = \sum_{m=0}^{M} b_{m}e^{j\omega(n-m)}$$

$$H\left(e^{j\omega}\right) = \frac{\sum_{m=0}^{M} b_{m}e^{-j\omega m}}{1 + \sum_{\ell=1}^{N} a_{\ell}e^{-j\omega\ell}}$$

• An LTI system is specified by the difference equation

$$y(n) = 0.8y(n - 1) + x(n)$$

- Determine  $H(e^{j\omega})$ .
- Calculate and plot the steady-state response  $y_{ss}(n)$  to

$$x(n) = \cos(0.05\pi n) \ u(n)$$

- Rewrite the difference equation as y(n) 0.8y(n-1) = x(n).
- Using "Frequency Response Function from Difference Equations" formula

$$H\left(e^{j\omega}\right) = \frac{1}{1 - 0.8e^{-j\omega}}$$

• In the steady state the input is  $x(n) = \cos(0.05\pi n)$  with frequency  $\omega_o = 0.05\pi$  and  $\theta_o = 0^\circ$ . The response of the system is

$$H(e^{j0,05\pi}) = \frac{1}{1 - 0.8e^{-j0,05\pi}} = 4,0928e^{-j0,5377}$$

$$y_{ss}(n) = 4,0928\cos(0.05\pi n - 0.5377)$$
$$= 4,0928\cos[0.05\pi(n - 3.42)]$$

• This means that at the output the sinusoid is scaled by 4.0928 and shifted by 3.42 samples.

### Example: Verify with Matlab

```
>> subplot(1,1,1)
>> b = 1; a = [1,-0.8];
>> n=[0:100];x = cos(0.05*pi*n);
>> y = filter(b,a,x);
>> subplot(2,1,1); stem(n,x);
>> xlabel('n'); ylabel('x(n)'); title('Input sequence')
>> subplot(2,1,2); stem(n,y);
>> xlabel('n'); ylabel('y(n)'); title('Output sequence')
```

• A 3rd-order lowpass filter is described by the difference equation

$$y(n) = 0.0181x(n) + 0.0543x(n-1) + 0.0543x(n-2) + 0.0181x(n-3) + 1.76y(n-1) - 1.1829y(n-2) + 0.2781y(n-3)$$

• Plot the magnitude and the phase response of this filter, and verify that it is a lowpass filter.

```
>> b = [0.0181, 0.0543, 0.0543, 0.0181]; % filter coefficient array b
>> a = [1.0000, -1.7600, 1.1829, -0.2781]; % filter coefficient array a
>> m = 0:length(b)-1; l = 0:length(a)-1; % index arrays m and l
>> K = 500; k = 0:1:K; % index array k for frequencies
>> w = pi*k/K; % [0, pi] axis divided into 501 points.
>> num = b * exp(-j*m'*w); % Numerator calculations
>> den = a * exp(-j*l'*w); % Denominator calculations
>> H = num ./ den; % Frequency response
>> magH = abs(H); angH = angle(H); % mag and phase responses
```

```
>> subplot(2,1,1); plot(w/pi,magH); grid; axis([0,1,0,1])
>> xlabel('frequency in pi units'); ylabel('|H|');
>> title('Magnitude Response');
>> subplot(2,1,2); plot(w/pi,angH/pi); grid
>> xlabel('frequency in pi units'); ylabel('Phase in pi Radians');
>> title('Phase Response');
```





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