PENGOLAHAN SINYAL DIGITAL

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SYSTEM REPRESENTATION IN THE Z-DOMAIN

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THE SYSTEM FUNCTION

- The system function $H(z)$ is given by

$$H(z) \triangleq \sim Z[h(n)] = \sum_{-\infty}^{\infty} h(n) z^{-n}; \quad R_{h-} < |z| < R_{h+}$$

- Using the convolution property of the z-transform, the output transform $Y(z)$ is given by

$$Y(z) = H(z) X(z) \quad : \quad \mathrm{ROC}_y = \mathrm{ROC}_h \cap \mathrm{ROC}_x$$

THE SYSTEM FUNCTION

- Therefore a linear and time invariant system can be represented in the z-domain by

$$X(z) \quad \rightarrow \quad \boxed{H(z)} \quad \rightarrow \quad Y(z) = H(z)X(z)$$

SYSTEM FUNCTION FROM THE DIFFERENCE EQUATION REPRESENTATION

- LTI systems are described by a difference equation

$$y(n)+\sum_{k=1}^{N} a_k y(n-k)=\sum_{\ell=1}^{M} b_\ell x(n-\ell)$$

- System Function Representation

$$Y(z)+\sum_{k=1}^{N} a_k z^{-k} Y(z)=\sum_{\ell=0}^{M} b_\ell z^{-\ell} X(z)$$

$$H(z)\triangleq\frac{Y(z)}{X(z)}=\frac{\sum_{\ell=0}^{M} b_\ell z^{-\ell}}{1+\sum_{k=1}^{N} a_k z^{-k}}=\frac{B(z)}{A(z)}=\frac{b_o z^{-M}\left(z^M+\ldots+\frac{b_M}{b_o}\right)}{z^{-N}\left(z^N+\ldots+a_N\right)}$$

SYSTEM FUNCTION FROM THE DIFFERENCE EQUATION REPRESENTATION

- After factorization

$$H(z) = b_o z^{N-M} \frac{\prod_{\ell=1}^{N}(z - z_\ell)}{\prod_{k=1}^{N}(z - p_k)}$$

  - z_ℓ are the system zeros
  - p_k are the system poles
- $H(z)$ (an LTI system) can also be represented in the z-domain using a pole-zero plot

TRANSFER FUNCTION REPRESENTATION

- A frequency response function or transfer function $H(z)$ on the unit circle $z = e^{j\omega}$

$$H\left(e^{j\omega}\right) = b_o z e^{j(N-M)\omega} \frac{\prod_{\ell=1}^{N}\left(e^{j\omega} - z_\ell\right)}{\prod_{k=1}^{N}\left(e^{j\omega} - p_k\right)}$$

- The factor $(e^{j\omega} - z_\ell)$ can be interpreted as a *vector* in the complex z-plane from a zero z to the unit circle at $z = e^{j\omega}$
- The factor $(e^{j\omega} - p_k)$ can be interpreted as a vector from a pole p_k to the unit circle at $z = e^{j\omega}$.

TRANSFER FUNCTION REPRESENTATION

- The magnitude response function

$$\left|H\left(e^{j\omega}\right)\right| = \left|b_o\right| \frac{\left|e^{j\omega} - z_1\right| \ldots \left|e^{j\omega} - z_M\right|}{\left|e^{j\omega} - p_1\right| \ldots \left|e^{j\omega} - p_N\right|}$$

  - a product of the lengths of vectors from zeros to the unit circle *divided* by the lengths of vectors from poles to the unit circle and *scaled* by $|b_o|$.

TRANSFER FUNCTION REPRESENTATION

- The phase response function

$$\angle H\left(e^{j\omega}\right)=\left[0 \text{ or } \pi\right]+\left[\left(N-M\right)\omega\right]+\sum_{1}^{M}\angle\left(e^{j\omega}-z_k\right)-\sum_{1}^{N}\angle\left(e^{j\omega}-p_k\right)$$

  - a sum of a constant factor, a linear-phase factor, and a nonlinear-phase factor (angles from the “zero vectors” *minus* the sum of angles from the “pole vectors”).

EXAMPLE

- Given a causal system

$$y(n) = 0.9y(n-1) + x(n)$$

  - Determine $H(z)$ and sketch its pole-zero plot.
  - Plot $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$.
  - Determine the impulse response $h(n)$.

EXAMPLE

- The difference equation can be put in the form

$$y(n) - 0.9y(n - 1) = x(n)$$

$$H(z) = b_o z^{N-M} \frac{\prod_{\ell=1}^{N}(z - z_\ell)}{\prod_{k=1}^{N}(z - p_k)}$$

$$H(z) = \frac{1}{1 - 0,9z^{-1}}; \quad |z| > 0,9$$

- since the system is causal. There is one pole at 0.9 and one zero at the origin

EXAMPLE

- illustrate using the zplane function

```
>> b = [1, 0]; a = [1, -0.9]; zplane(b,a)
```

EXAMPLE

- determine the magnitude and phase of $H(e^{j\omega})$ using freqz function.
- take 100 points along the upper half of the unit circle

```
>> [H,w] = freqz(b,a,100); magH = abs(H); phaH = angle(H);
>> subplot(2,1,1);plot(w/pi,magH);grid
>> xlabel('frequency in pi units'); ylabel('Magnitude');
>> title('Magnitude Response')
>> subplot(2,1,2);plot(w/pi,phaH/pi);grid
>> xlabel('frequency in pi units'); ylabel('Phase in pi units');
>> title('Phase Response')
```

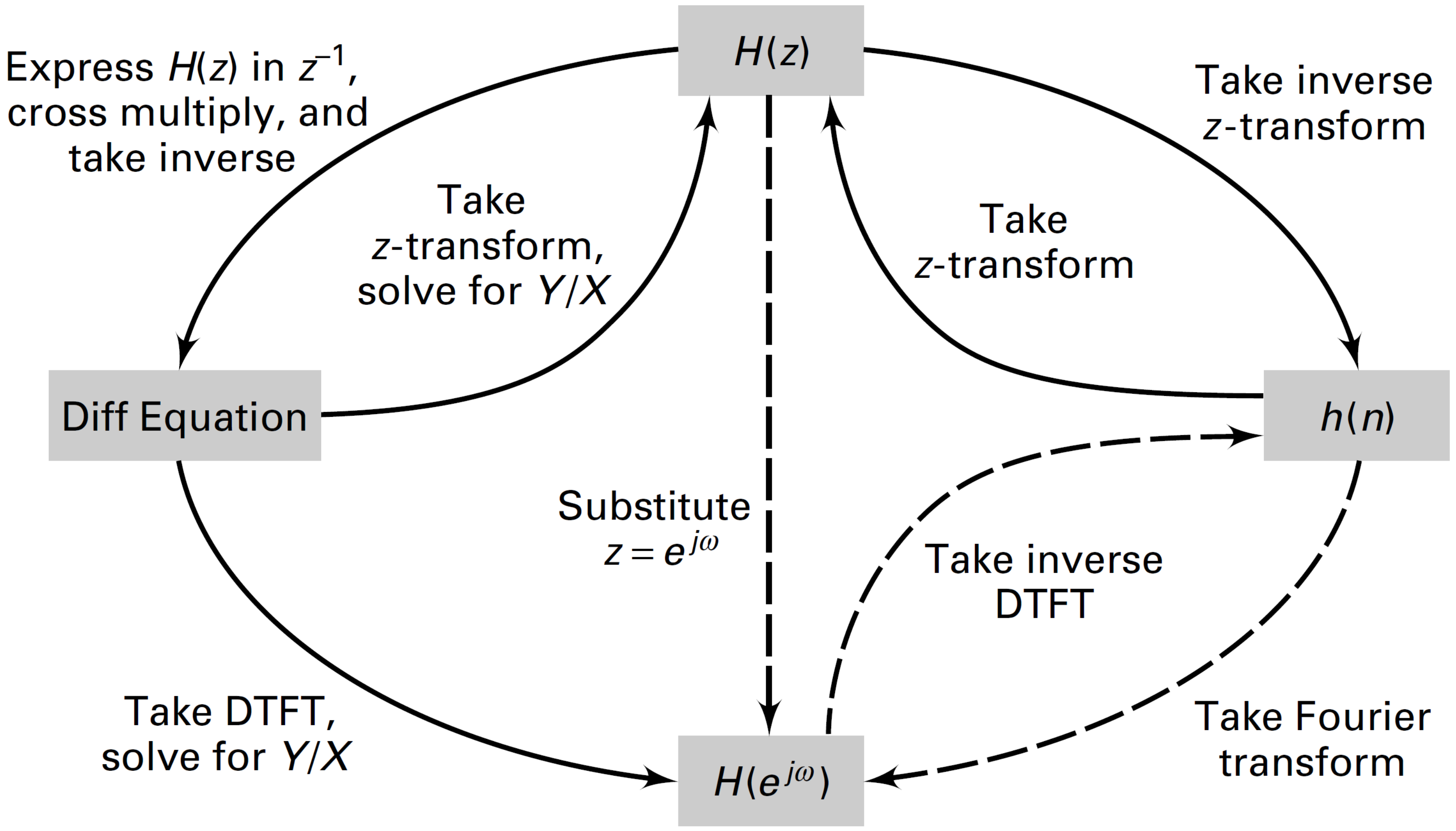
EXAMPLE

- From the *z*-transform in Table

$$h(n) = Z^{-1}\left[\frac{1}{1-0,9z^{-1}}; \quad |z| > 0,9\right]$$

$$= (0,9)^n u(n)$$

RELATIONSHIPS BETWEEN SYSTEM REPRESENTATIONS



Z-DOMAIN LTI STABILITY

- An LTI system is stable if and only if the unit circle is in the ROC of $H(z)$.

Z-DOMAIN CAUSAL LTI STABILITY

- A causal LTI system is stable if and only if the system function $H(z)$ has all its poles inside the unit circle.

EXAMPLE

- A causal LTI system is described by the following difference equation:

$$y(n) = 0.81y(n - 2) + x(n) - x(n - 2)$$

- Determine
  - the system function $H(z)$,
  - the unit impulse response $h(n)$,
  - the unit step response $v(n)$, that is, the response to the unit step $u(n)$, and
  - the frequency response function $H(e^{j\omega})$, and plot its magnitude and phase over $0 \leq \omega \leq \pi$.

EXAMPLE

- Since the system is causal, the ROC will be outside a circle with radius equal to the largest pole magnitude.
- Taking the *z*-transform of both sides of the difference equation

$$H(z) = \frac{1 - z^{-2}}{1 - 0,81z^{-2}} = \frac{1 - z^{-2}}{\left(1 + 0,9z^{-1}\right)\left(1 - 0,9z^{-1}\right)}, \quad |z| > 0,9$$

EXAMPLE

- Using the MATLAB script for the partial fraction expansion

$$H(z) = \frac{1-z^{-2}}{1-0,81z^{-2}} = \frac{1-z^{-2}}{\left(1+0,9z^{-1}\right)\left(1-0,9z^{-1}\right)}, \quad |z| > 0,9$$

```
>> b = [1,0,-1]; a = [1,0,-0.81]; [R,p,C] = residuez(b,a);
R = -0.1173 -0.1173
p = -0.9000 0.9000
C = 1.2346
```

$$H(z) = 1,2346 - 0,1173\frac{1}{1+0,9z^{-1}} - 0,1173\frac{1}{1-0,9z^{-1}}, \quad |z| > 0,9$$

TABLE 4.1 *Some common z-transform pairs*

Sequence	*Transform*	*ROC*
$\delta(n)$	1	$\forall z$
$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u(-n-1)$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-b^n u(-n-1)$	$\dfrac{1}{1-bz^{-1}}$	$\|z\| < \|b\|$
$[a^n \sin \omega_0 n]\, u(n)$	$\dfrac{(a \sin \omega_0) z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\| > \|a\|$
$[a^n \cos \omega_0 n]\, u(n)$	$\dfrac{1-(a \cos \omega_0) z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\| > \|a\|$
$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-nb^n u(-n-1)$	$\dfrac{bz^{-1}}{(1-bz^{-1})^2}$	$\|z\| < \|b\|$

EXAMPLE

$$H(z) = 1,2346 - 0,1173\frac{1}{1+0,9z^{-1}} - 0,1173\frac{1}{1-0,9z^{-1}}, \quad |z| > 0,9$$

- from Table

$$h(n) = 1,2346\,\delta(n) - 0,1173\left\{1+(-1)^n\right\}(0,9)^n u(n)$$

EXAMPLE

- From Table 4.1

$$Z\left[u(n)\right] = U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$$V(z) = H(z)U(z)$$

$$= \left[\frac{\left(1+z^{-1}\right)\left(1-z^{-1}\right)}{\left(1+0,9z^{-1}\right)\left(1-0,9z^{-1}\right)}\right]\left[\frac{1}{1-z^{-1}}\right], \quad |z| > 0,9 \cap |z| > 1$$

$$= \frac{1+z^{-1}}{\left(1+0,9z^{-1}\right)\left(1-0,9z^{-1}\right)}, \quad |z| > 0,9$$

$$= 1,0556\frac{1}{1-0,9z^{-1}} - 0,0556\frac{1}{1+0,9z^{-1}}, \quad |z| > 0,9$$

EXAMPLE

- Finally,

$$v(n) = \left[1,0556(0,9)^n - 0,556(-0,9)^n\right]u(n)$$

- There is a pole-zero cancellation at $z = 1$.
- This has two implications.
  - First, the ROC of $V(z)$ is still $\{|z| > 0.9\}$ and not $\{|z| > 0.9 \cap |z| > 1 = |z| > 1\}$
  - The step response $v(n)$ contains no steady-state term $u(n)$.

EXAMPLE

- The frequency response function $H(e^{j\omega})$
- Substituting $z = e^{j\omega}$ in $H(z)$,

$$H\left(e^{j\omega}\right) = \frac{1 - e^{-j2\omega}}{1 - 0,81e^{-j2\omega}}$$

EXAMPLE

- MATLAB script to compute and plot responses

```
>> w = [0:1:500]*pi/500; H = freqz(b,a,w);
>> magH = abs(H); phaH = angle(H);
>> subplot(2,1,1); plot(w/pi,magH); grid
>> xlabel('frequency in pi units'); ylabel('Magnitude')
>> title('Magnitude Response')
>> subplot(2,1,2); plot(w/pi,phaH/pi); grid
>> xlabel('frequency in pi units'); ylabel('Phase in pi units')
>> title('Phase Response')
```

SOLUTIONS OF THE DIFFERENCE EQUATIONS

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ONE-SIDED Z-TRANSFORM

- The one-sided z-transform of a sequence x(n) is given by

$$Z^{+}\left[x(n)\right] \triangleq Z\left[x(n)u(n)\right] \triangleq X^{+}\left[z\right] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

  - Difference equations generally evolve in the positive n direction.
  - Time frame for these solutions will be $n \geq 0$
  - One form involved finding the particular and the homogeneous solutions
  - The other form involved finding the zero-input (initial condition) and the zero-state responses

ONE-SIDED Z-TRANSFORM

- The sample shifting property is given by

$$Z^{+}\left[x(n-k)\right]=x(-1)z^{1-k}+x(-2)z^{2-k}+\ldots+x(-k)+z^{-k}X^{+}(z)$$

- The result can now be used to solve difference equations with nonzero initial conditions or with changing inputs

$$1+\sum_{k=1}^{N}a_k y(n-k)=\sum_{m=0}^{M}b_m x(n-m),\quad n\geq 0$$

- subject to these initial conditions:

$$\left\{y(i),i=-1,\ldots,-N\right\} \qquad \left\{x(i),i=-1,\ldots,-M\right\}$$

EXAMPLE

- Solve

$$y(n)-\frac{3}{2}y(n-1)+\frac{1}{2}y(n-2)=x(n), \quad n\geq 0$$

- where

$$x(n)=\left(\frac{1}{4}\right)^n u(n)$$

- subject to $y(-1) = 4$ and $y(-2) = 10$.

EXAMPLE

$$y(n)-\frac{3}{2}y(n-1)+\frac{1}{2}y(n-2)=x(n), \quad n\geq 0$$

- Taking the one-sided *z*-transform of both sides of the difference equation

$$Y^+(z)-\frac{3}{2}\left[y(-1)+z^{-1}Y^+(z)\right]+\frac{1}{2}\left[y(-2)+z^{-1}y(-1)+z^{-2}Y^+(z)\right]=\frac{1}{1-\frac{1}{4}z^{-1}}$$

- Substituting the initial conditions and rearranging

$$Y^+(z)\left[1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right]=\frac{1}{1-\frac{1}{4}z^{-1}}+\left(1-2z^{-1}\right)$$

EXAMPLE

$$Y^{+}(z)\left[1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right]=\frac{1}{1-\frac{1}{4}z^{-1}}+\left(1-2z^{-1}\right)$$

$$Y^{+}(z)=\frac{\frac{1}{1-\frac{1}{4}z^{-1}}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}+\frac{1-2z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

Using the partial fraction expansion

$$Y^{+}(z)=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{\frac{2}{3}}{1-z^{-1}}+\frac{\frac{1}{3}}{1-\frac{1}{4}z^{-1}}$$

EXAMPLE

- After inverse transformation the solution is

$$y(n)=\left[\left(\frac{1}{2}\right)^{n}+\frac{2}{3}+\frac{1}{2}\left(\frac{1}{4}\right)^{n}\right]u(n)$$

EXAMPLE

- Homogeneous and particular parts

$$y(n) = \underbrace{\left[\left(\frac{1}{2}\right)^n + \frac{2}{3}\right]u(n)}_{\text{Homogeneous part}} + \underbrace{\frac{1}{2}\left(\frac{1}{4}\right)^n u(n)}_{\text{Particular part}}$$

  - The homogeneous part is due to the *system poles*, and the particular part is due to the *input poles*.

EXAMPLE

- Transient and steady-state responses

$$y(n) = \underbrace{\left[\frac{1}{3}\left(\frac{1}{4}\right)^n \left(\frac{1}{2}\right)^n\right]u(n)}_{\text{Transient response}} + \underbrace{\frac{2}{3}u(n)}_{\text{Steady-state response}}$$

- The transient response is due to poles that are *inside* the unit circle, whereas the steady-state response is due to poles that are *on* the unit circle.

EXAMPLE

- Zero-input (or initial condition) and zero-state responses

$$Y^{+}(z)=\frac{\dfrac{1}{1-\frac{1}{4}z^{-1}}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}+\frac{1-2z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

$$Y_{ZS}(z)=H(z)X(z) \qquad Y_{ZI}(z)=H(z)X_{IC}(z)$$

- $X_{IC}(z)$ can be thought of as an equivalent *initial-condition input* that generates the same output Y_{ZI} as generated by the initial conditions.

$$x_{IC}(n)=\left\{\underset{\uparrow}{1},-2\right\}$$

EXAMPLE

- taking the inverse *z*-transform of each part of

$$Y^{+}(z)=\frac{\dfrac{1}{1-\frac{1}{4}z^{-1}}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}+\frac{1-2z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

- The complete response as

$$y(n)=\underbrace{\left[\frac{1}{3}\left(\frac{1}{4}\right)^{n}-2\left(\frac{1}{2}\right)^{n}+\frac{8}{3}\right]u(n)}_{\text{Zero-state response}}+\underbrace{\left[3\left(\frac{1}{2}\right)^{n}-2\right]u(n)}_{\text{Zero-input response}}$$

MATLAB IMPLEMENTATION

y = filter(b,a,x,xic)

- xic is an equivalent initial-condition input array

MATLAB IMPLEMENTATION

```
>> n = [0:7]; x = (1/4).^n; xic = [1, -2];
>> format long; y1 = filter(b,a,x,xic)
>> y2 = (1/3)*(1/4).^n+(1/2).^n+(2/3)*ones(1,8)
```

MATLAB IMPLEMENTATION

xic = filtic(b,a,Y,X)

- to determine $x_{IC}(n)$ analytically
- b and a are the filter coefficient arrays and Y and X are the initialcondition arrays from the initial conditions on $y(n)$ and $x(n)$

```
>> Y = [4, 10]; xic = filtic(b,a,Y)
```

EXAMPLE

- Solve the difference equation

$$y(n)=\frac{1}{3}\left[x(n)+x(n-1)+x(n-2)+0,95y(n-1)-0,9025y(n-2)\right]$$

- where $x(n) = \cos(\pi n/3)u(n)$ and $y(-1) = -2$, $y(-2) = -3$; $x(-1) = 1$, $x(-2) = 1$
- First determine the solution analytically and then by using MATLAB

EXAMPLE

- Taking a one-sided z-transform of the difference equation

$$Y^+(z)=\frac{1}{3}\left[X^+(z)x(-1)+z^{-1}X^+(z)+x(-2)+z^{-1}x(-1)+z^{-2}X^+(z)\right]$$

$$+0,95\left[y(-1)+z^{-1}Y^+(z)\right]-0,9025\left[y(-2)+z^{-1}y(-1)+z^{-2}Y^+(z)\right]$$

- and substituting the initial conditions

$$Y^+(z)=\frac{\frac{1}{3}+\frac{1}{3}z^{-1}+\frac{1}{3}z^{-2}}{1-0,95z^{-1}+0,9025z^{-2}}X^+(z)+\frac{1,4742+2,1383z^{-1}}{1-0,95z^{-1}+0,9025z^{-2}}$$

- Clearly, $x_{IC}(n)$ = [1,4742, 2,1383].

EXAMPLE

- This simplification and further partial fraction expansion can be done using MATLAB.

```
>> b = [1,1,1]/3; a = [1,-0.95,0.9025];
>> Y = [-2,-3]; X = [1,1]; xic=filtic(b,a,Y,X)
>> bxplus = [1,-0.5]; axplus = [1,-1,1]; % X(z) transform coeff.
>> ayplus = conv(a,axplus) % Denominator of Yplus(z)
>> byplus = conv(b,bxplus)+conv(xic,axplus)
>> [R,p,C] = residuez(byplus,ayplus)
>> Mp = abs(p), Ap = angle(p)/pi
```

EXAMPLE

- Substituting $X^{+}(z)$

$$X^{+}(z)=\frac{1-0,5z^{-1}}{1-z^{-1}+z^{-2}}$$

- obtain $Y^{+}(z)$ as a rational function

$$Y^{+}(z)=\frac{0,0584+j3,9468}{1-e^{-j\pi/3}z^{-1}}+\frac{0,0584-j3,9468}{1-e^{j\pi/3}z^{-1}}$$
$$+\frac{0,8453+j2,0311}{1-0,95e^{j\pi/3}z^{-1}}+\frac{0,8453-j2,0311}{1-0,95e^{-j\pi/3}z^{-1}}$$

EXAMPLE

- From Table

$$y(n) = 0{,}1169\cos(\pi n/3) + 7{,}8937\sin(\pi n/3)$$
$$+(0{,}95)^n\left[1{,}6906\cos(\pi n/3) - 4{,}0623\sin(\pi n/3)\right], \quad n \geq 0$$

EXAMPLE – MATLAB VERIFICATION

```
>> n = [0:7]; x = cos(pi*n/3); y = filter(b,a,x,xic)
% Matlab Verification
>> A=real(2*R(1)); B=imag(2*R(1)); C=real(2*R(3)); D=imag(2*R(4));
>> y=A*cos(pi*n/3)+B*sin(pi*n/3)+((0.95).^n).*(C*cos(pi*n/3)+ D*sin(pi*n/3))
```

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