Examples

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Contents

1 Kalman Filter

1.1 Least Mean-Square Error approach

Let's consider a linear, discrete-time dynamical system with its process equation:

$$\mathbf{x}_{k+1} = \mathbf{F}_{k+1,k}\mathbf{x}_k + \mathbf{w}_k, \tag{1}$$

where$\mathbf{F}_{k+1,k}$ is the transition matrix taking the (often hidden or latent) state$\mathbf{x}_k$ from time$k$ to time$k+1$. The process noise$\mathbf{w}_k$ is assumed to be additive, white, and Gaussian, with zero mean and with covariance matrix defined by

$$E[\mathbf{w}_n\mathbf{w}_k^T] = \begin{cases} \mathbf{Q}_k & for\ n = k \\ \mathbf{0} & for\ n \neq k, \end{cases} \tag{2}$$

(therefore it's a Markov sequence.)
Measurement equation:

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k, \tag{3}$$

where$\mathbf{y}_k$ is the observable variable at time$k$ and$\mathbf{H}_k$ is the measurement matrix. The measurement noise$\mathbf{v}_k$ is assumed to be additive, white, and Gaussian, with zero mean and with covariance matrix defined by

$$E[\mathbf{v}_n \mathbf{v}_k^T] = \begin{cases} \mathbf{R}_k & for\ n = k \\ \mathbf{0} & for\ n \neq k. \end{cases} \tag{4}$$

Moreover, the measurement noise$\mathbf{v}_k$ is uncorrelated with the process noise$\mathbf{w}_k$, i.e.$cov(\mathbf{w}_i, \mathbf{v}_j) = 0$. Also$cov(\mathbf{x}_0, \mathbf{x}_0) \equiv var(\mathbf{x}_0) = \mathbf{P}_0,\ cov(\mathbf{x}_0, \mathbf{w}_k) = cov(\mathbf{x}_0, \mathbf{v}_k) = 0\ for\ all\ k$.

The goal is to find the minimum mean-square error estimate of$\mathbf{x}_k$ given$\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$. Let$\hat{\mathbf{x}}_k$ denote the a posteriori estimate of the signal$\mathbf{x}_k$, given the observations$\mathbf{y}_i,\ i = 1, \ldots, k$. The state-error vector is defined by

$$\tilde{\mathbf{x}}_k \equiv \mathbf{x}_k - \hat{\mathbf{x}}_k. \tag{5}$$

In addition to having zero mean, it has covariance$\mathbf{P}_k$.

Let's define a cost (loss) function for incorrect estimates:

- The cost function is non-negative.
- The cost function is a non-decreasing function of the estimation error$\tilde{\mathbf{x}}_k$.

These two requirements are satisfied by the mean-square error defined by

$$\mathbf{J}_k = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)^2] = E[\tilde{\mathbf{x}}_k^2] \tag{6}$$

To derive an optimal value for the estimate$\hat{\mathbf{x}}_k$, we may invoke two theorems taken from stochastic process theory:

Theorem 1.1 Conditional mean estimator. *If the stochastic processes*$\mathbf{x}_k$ *and*$\mathbf{y}_k$ *are jointly Gaussian, then the optimum estimate*$\hat{\mathbf{x}}_k$ *that minimizes the mean-square error*$\mathbf{J}_k$ *is the conditional mean estimator:*

$$\hat{\mathbf{x}}_k = E[\mathbf{x}_k | \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k].$$

Theorem 1.2 Principle of orthogonality.*Let the stochastic processes*$\mathbf{x}_k$ *and*$\mathbf{y}_k$ *be of zero means; that is,*$E[\mathbf{x}_k] = E[\mathbf{y}_k] = 0$ *for all*k.

Then:

(i) the stochastic processes$\mathbf{x}_k$ *and*$\mathbf{y}_k$ *are jointly Gaussian; or*

(ii) if the optimal estimate$\hat{\mathbf{x}}_k$ *is restricted to be a linear function of the observables and the cost function is the mean-square error,*

(iii) then the optimum estimate$\hat{\mathbf{x}}_k$*, given the observables*$\mathbf{y}_k$*, is the orthogonal projection of*$\mathbf{x}_k$ *on the space spanned by these observables.*

Suppose that a measurement on a linear dynamical system, described by Eqs.(1) and(3), has been made at time$k$. The requirement is to use the information contained in the new measurement$\mathbf{y}_k$ to update the estimate of the unknown state$\mathbf{x}_k$. Let$\hat{\mathbf{x}}_k^-$ denote a priori estimate of the state, which is already available at time$k$. With a linear estimator as the objective, we may express the a posteriori estimate$\hat{\mathbf{x}}_k$ as a linear combination of the a priori estimate and the new measurement, as shown by

$$\hat{\mathbf{x}}_k = \mathbf{G}_k^* \hat{\mathbf{x}}_k^- + \mathbf{G}_k \mathbf{y}_k, \tag{7}$$

where the multiplying matrix factors$\mathbf{G}_k^*$ and$\mathbf{G}_k$ are to be determined. To find these two matrices, we invoke the principle of orthogonality stated under Theorem 1.2:

$$E[\tilde{\mathbf{x}}_k \mathbf{y}_i^T] = 0 \; for \; i = 1, 2, \ldots, k-1 \tag{8}$$

Using(3),(7), and(5),(8), we get

$$E[(\mathbf{x}_k - \mathbf{G}_k^* \hat{\mathbf{x}}_k^- - \mathbf{G}_k \mathbf{H}_k \mathbf{x}_k - \mathbf{G}_k \mathbf{v}_k)\mathbf{y}_i^T] = 0 \; for \; i = 1, 2, \ldots, k-1 \tag{9}$$

The noise is uncorrelated:$E[\mathbf{v}_k \mathbf{y}_i^T] = 0 \; for \; i = 1, 2, \ldots, k-1$
since$E[\mathbf{v}_k(\mathbf{H}_i \mathbf{x}_i + \mathbf{v}_i)^T] = \mathbf{H}_i^T E[\mathbf{v}_k \mathbf{x}_i^T] = \mathbf{H}_i^T E[\mathbf{v}_k(\mathbf{F}_{i-1}\mathbf{x}_{i-1} + \mathbf{w}_{i-1})^T] = 0$
Therefore

$$E[(\mathbf{I} - \mathbf{G}_k \mathbf{H}_k - \mathbf{G}_k^*)\mathbf{x}_k \mathbf{y}_i^T + \mathbf{G}_k^*(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)\mathbf{y}_i^T] = 0 \tag{10}$$

From the principle of orthogonality (both$\hat{\mathbf{x}}_k^-$ and$\hat{\mathbf{x}}_k$ are unbiased estimations) we now note that

$$E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)\mathbf{y}_i^T] = 0 \tag{11}$$

Accordingly,(10) simplifies to

$$(\mathbf{I} - \mathbf{G}_k \mathbf{H}_k - \mathbf{G}_k^*) E[\mathbf{x}_k \mathbf{y}_i^T] = 0 \; for \; i = 1, 2, \ldots, k-1 \tag{12}$$

For arbitrary values of the state$\mathbf{x}_k$ and the observable$\mathbf{y}_i$,(12) can only be satisfied if the scaling factors$\mathbf{G}_k$ and$\mathbf{G}_k^*$ are related as follows:

$$\mathbf{G}_k^* = \mathbf{I} - \mathbf{G}_k \mathbf{H}_k \tag{13}$$

Substituting(13) into(7), we may express the a posteriori estimate of the state at time$k$ as

$$\hat{\mathbf{x}}_k = (\mathbf{I} - \mathbf{G}_k \mathbf{H}_k)\hat{\mathbf{x}}_k^- + \mathbf{G}_k \mathbf{y}_k = \tag{14}$$

$$= \hat{\mathbf{x}}_k^- + \mathbf{G}_k(\mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_k^-), \tag{15}$$

in light of which, the matrix$\mathbf{G}_k$ is called the Kalman gain.

Now let's derive an explicit formula for$\mathbf{G}_k$. Again, from the principle of orthogonality, similar to(11), we have

$$E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)\mathbf{y}_k^T] = 0 \tag{16}$$

Let the error (innovation, i.e. a measure of the"new" information contained in$\mathbf{y}_k$)

$$\tilde{\mathbf{y}}_k = \hat{\mathbf{y}}_k^- - \mathbf{y}_k = \tag{17}$$

$$= \mathbf{H}_k \hat{\mathbf{x}}_k^- - \mathbf{y}_k \tag{18}$$

The parameter$\hat{\mathbf{x}}_k$ depends linearly on$\mathbf{x}_k$, which depends linearly on$\mathbf{y}_k$. Therefore, from(16)

$$E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)\hat{\mathbf{y}}_k^{T-}] = 0 \tag{19}$$

and also (by subtracting(16) from(19))

$$E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)\tilde{\mathbf{y}}_k^T] = 0 \tag{20}$$

Using(3) and(15), we may express the state-error vector$\mathbf{x}_k - \hat{\mathbf{x}}_k$ as

$$\mathbf{x}_k - \hat{\mathbf{x}}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k^- - \mathbf{G}_k(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-) =$$

$$= \mathbf{x}_k - \hat{\mathbf{x}}_k^- - \mathbf{G}_k(\mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-) =$$

$$= \mathbf{x}_k - \mathbf{G}_k\mathbf{H}_k\mathbf{x}_k - \hat{\mathbf{x}}_k^- + \mathbf{G}_k\mathbf{H}_k\hat{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{v}_k =$$

$$= (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\mathbf{x}_k - (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\hat{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{v}_k =$$

$$= (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k^-) - \mathbf{G}_k\mathbf{v}_k =$$

$$= (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\tilde{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{v}_k \tag{21}$$

Hence, substituting(18) and(21) into(20), we get

$$E[((\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\tilde{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{v}_k)(\mathbf{H}_k\hat{\mathbf{x}}_k^- - \mathbf{y}_k)^T] = 0 =$$

$$= E[((\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\tilde{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{v}_k)(\mathbf{H}_k\hat{\mathbf{x}}_k^- - \mathbf{H}_k\mathbf{x}_k - \mathbf{v}_k)^T] =$$

$$= E[((\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\tilde{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{v}_k)(\mathbf{H}_k\tilde{\mathbf{x}}_k^- + \mathbf{v}_k)^T] = 0 \tag{22}$$

Since the measurement noise$\mathbf{v}_k$ is independent of the state$\mathbf{x}_k$ and therefore the error$\tilde{\mathbf{x}}_k^-$, the expectation of(22) reduces to

$$(\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)E[\tilde{\mathbf{x}}_k^- - \tilde{\mathbf{x}}_k^{T-}]\mathbf{H}_k^T - \mathbf{G}_k E[\mathbf{v}_k\mathbf{v}_k^T] = 0 \tag{23}$$

Define the a priori covariance matrix

$$\mathbf{P}_k^- = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T] =$$

$$= E[\tilde{\mathbf{x}}_k^-\tilde{\mathbf{x}}_k^{T-}] \tag{24}$$

Then we rewrite(23) as

$$(\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\mathbf{P}_k^-\mathbf{H}_k^T - \mathbf{G}_k\mathbf{R}_k = 0 \tag{25}$$

Solving this equation for$\mathbf{G}_k$, we get the desired formula

$$\mathbf{P}_k^-\mathbf{H}_k^T - \mathbf{G}_k\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T - \mathbf{G}_k\mathbf{R}_k = 0$$

$$\mathbf{G}_k(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T + \mathbf{R}_k) = \mathbf{P}_k^-\mathbf{H}_k^T$$

$$\mathbf{G}_k = \mathbf{P}_k^-\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T + \mathbf{R}_k)^{-1} \tag{26}$$

Similar for the a posteriori covariance

$$\mathbf{P}_k = E[\tilde{\mathbf{x}}_k\tilde{\mathbf{x}}_k^T] \tag{27}$$

By substituting(13) into(7), we obtain

$$\hat{\mathbf{x}}_k = (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\hat{\mathbf{x}}_k^- + \mathbf{G}_k\mathbf{y}_k$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{G}_k(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-) \tag{28}$$

Subtract$\mathbf{x}_k$ from both sides of the latter equation to obtain

$$\hat{\mathbf{x}}_k - \mathbf{x}_k = \hat{\mathbf{x}}_k^- + \mathbf{G}_k\mathbf{H}_k\mathbf{x}_k + \mathbf{G}_k\mathbf{v}_k - \mathbf{G}_k\mathbf{H}_k\hat{\mathbf{x}}_k^- - \mathbf{x}_k$$

$$\tilde{\mathbf{x}}_k = \tilde{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{H}_k\tilde{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{v}_k$$

$$\tilde{\mathbf{x}}_k = (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\tilde{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{v}_k \tag{29}$$

By substituting(29) into(27) and noting that$E[\tilde{\mathbf{x}}_k^-\mathbf{v}_k^T] = 0$, we obtain

$$\mathbf{P}_k = E[((\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\tilde{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{v}_k)((\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\tilde{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{v}_k)^T] =$$

$$= (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)E[\tilde{\mathbf{x}}_k^-\tilde{\mathbf{x}}_k^{T-}](\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)^T + \mathbf{G}_kE[\mathbf{v}_k\mathbf{v}_k^T]\mathbf{G}_k^T =$$

$$= (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\mathbf{P}_k^-(\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)^T + \mathbf{G}_k\mathbf{R}_k\mathbf{G}_k^T \tag{30}$$

This is so-called"Joseph form" of the covariance update equation. By substituting for$\mathbf{G}_k$ from(26), it can be put into the following form

$$\mathbf{P}_k = (\mathbf{P}_k^- - \mathbf{G}_k\mathbf{H}_k\mathbf{P}_k^-)(\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)^T + \mathbf{G}_k\mathbf{R}_k\mathbf{G}_k^T =$$

$$= \mathbf{P}_k^- - \mathbf{G}_k\mathbf{H}_k\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{H}_k^T\mathbf{G}_k^T + \mathbf{G}_k\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T\mathbf{G}_k^T + \mathbf{G}_k\mathbf{R}_k\mathbf{G}_k^T =$$

$$= (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{H}_k^T\mathbf{G}_k^T + \mathbf{G}_k(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T + \mathbf{R}_k)\mathbf{G}_k^T \tag{31}$$

From(25)

$$\mathbf{G}_k\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T = \mathbf{P}_k^-\mathbf{H}_k^T - \mathbf{G}_k\mathbf{R}_k$$

Substituting this into(31)

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{H}_k^T\mathbf{G}_k^T + (\mathbf{P}_k^-\mathbf{H}_k^T - \mathbf{G}_k\mathbf{R}_k + \mathbf{G}_k\mathbf{R}_k)\mathbf{G}_k^T =$$

$$= (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{H}_k^T\mathbf{G}_k^T + \mathbf{P}_k^-\mathbf{H}_k^T\mathbf{G}_k^T =$$

$$= (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\mathbf{P}_k^- \tag{32}$$

This is the one most often used in computation. It implements the effect that *conditioning on the measurement* has on the covariance matrix of estimation uncertainty.

Note: for better numerical stability (to preserve the positive definiteness) use the representation via the square root matrix.

Let's see how the covariance changes in time.

$$\hat{\mathbf{x}}_k^- = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1}$$

For notational simplicity let$\hat{\mathbf{x}}_{k-1} \equiv \hat{\mathbf{x}}_{k-1}^+$ and$\mathbf{P}_{k-1} \equiv \mathbf{P}_{k-1}^+$.

Subtracting$\mathbf{x}_k$ from both sides to obtain

$$\hat{\mathbf{x}}_k^- - \mathbf{x}_k = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1} - \mathbf{x}_k$$

$$\tilde{\mathbf{x}}_k^- = \mathbf{F}_{k-1}(\hat{\mathbf{x}}_{k-1} - \mathbf{x}_{k-1}) - \mathbf{w}_{k-1} =$$

$$= \mathbf{F}_{k-1}\tilde{\mathbf{x}}_{k-1} - \mathbf{w}_{k-1} \tag{33}$$

for the propagation of the estimation error$\tilde{\mathbf{x}}$. Post-multiply it by$\tilde{\mathbf{x}}_k^{T-}$ and take the expected values. Use the fact that$E[\tilde{\mathbf{x}}_{k-1}\mathbf{w}_{k-1}^T] = 0$ to obtain results

$$E[\tilde{\mathbf{x}}_k^-\tilde{\mathbf{x}}_k^{T-}] = \mathbf{P}_k^- =$$

$$= \mathbf{F}_{k-1}E[\tilde{\mathbf{x}}_{k-1}\tilde{\mathbf{x}}_k^{T-}] =$$

$$= \mathbf{F}_{k-1}E[\tilde{\mathbf{x}}_{k-1}\tilde{\mathbf{x}}_{k-1}^T]\mathbf{F}_{k-1}^T + E[\mathbf{w}_{k-1}\mathbf{w}_{k-1}^T] =$$

$$= \mathbf{F}_{k-1}\mathbf{P}_{k-1}\mathbf{F}_{k-1}^T + \mathbf{Q}_{k-1} \tag{34}$$

which gives the a priori value of the covariance matrix of estimation uncertainty as a function of the previous a posteriori value.

These results obtained with the least-mean-squared estimation error do not depend on what probability distribution is used, as long as it has the required first and second moments.

1.2 Maximum Likelihood approach

Now let's look from the linear Gaussian maximum likelihood estimator prospective. We'll use the meanμ_x of$X$ and the information matrix$Y_{xx} \equiv P_{xx}^{-1}$ as parameters for a Gaussian distribution.

$$p(x, \mu_x, P_{xx}) = \frac{1}{\sqrt{2\pi |P_{xx}|}} e^{-\frac{1}{2}(x-\mu_x)^T P_{xx}^{-1}(x-\mu_x)} \tag{35}$$

whereP_{xx} is the covariance (second central moment).

The Gaussian likelihood function equivalent to(35) would be of the same form

$$\mathcal{L}(x, \mu_x, P_{xx}) = c\, e^{-\frac{1}{2}(x-\mu_x)^T P_{xx}^{-1}(x-\mu_x)} \tag{36}$$

and the log-likelihood

$$\ln \mathcal{L} = \ln\, c - \frac{1}{2}(x-\mu_x)^T Y_{xx}(x-\mu_x) \tag{37}$$

All covariance matrices in Kalman filtering are symmetric and positive definite, because the variances of estimated quantities are never absolutely zero. Consequently, all covariance matricesP_{xx} in Kalman filtering will have a matrix inverse$Y_{xx} = P_{xx}^{-1}$, the corresponding information matrix. AlsoY_{xx} will also be symmetric and positive definite. In fact, they have the same eigenvectors, and the corresponding eigenvalues ofY_{xx} will be the reciprocals of those ofP_{xx}.

In Maximum Likelihood estimation, however, the information matricesY_{xx} are only symmetric and non-negative definite (i.e. with zero eigenvalues possible) and, therefore, not necessarily invertible.

Using the information matrix in place of the covariance matrix in Gaussian likelihood functions allows us to model what estimation theorists would call"flat priors", a condition under which prior assumptions have no influence on the ultimate estimate. This cannot be done using covariance matrices, because it would require that some eigenvalues be infinite. It can be done using information matrices by allowing them to have zero eigenvalues whose eigenvectors represent linear combinations of the state space in which there is zero information. For example, information matrices can be used to represent the information in a measurement, and the dimension of which may be less than the dimension of the state vector.

Using the singular value decomposition we can write

$$Y_{xx} = \sum_i \lambda_i \mathbf{v}_i \mathbf{v}_i^T = V\, diag(\lambda)\, V^T$$

The Moore-Penrose generalized inverse ofY_{xx} can be defined in terms of itssvd as

$$Y_{xx}^{\dagger} = \sum_{\lambda_i \neq 0} \lambda_i^{-1} \mathbf{v}_i \mathbf{v}_i^T \tag{38}$$

which is always symmetric and non-negative definite and of the same rank asY_{xx}.

Two probability distributions are called statistically independent if and only if their joint probability is the product of the individual probabilities. The same is true for likelihoods. Let's denote the joint likelihood function$\mathcal{L}_c(x, \mu_c, Y_c)$ of two independent Gaussian likelihoods$\mathcal{L}_a(x, \mu_a, Y_a)$ and$\mathcal{L}_b(x, \mu_b, Y_b)$.

$$\mathcal{L}_c = c_c e^{-\frac{1}{2}(x-\mu_c)^T Y_c (x-\mu_c)} =$$

$$= \mathcal{L}_a \mathcal{L}_b = c_a e^{-\frac{1}{2}(x-\mu_a)^T Y_a (x-\mu_a)} c_b e^{-\frac{1}{2}(x-\mu_b)^T Y_b (x-\mu_b)} =$$

$$= c_a c_b e^{-\frac{1}{2}(x-\mu_a)^T Y_a (x-\mu_a) - \frac{1}{2}(x-\mu_b)^T Y_b (x-\mu_b)}$$

Taking the logarithm of both sides and differentiating once and twice with respect to$x$ will yield the following sequence of equations:

$$\ln\, c_c - \frac{1}{2}(x-\mu_c)^T Y_c (x-\mu_c) = \ln\, c_a + \ln\, c_b - \frac{1}{2}(x-\mu_a)^T Y_a (x-\mu_a) - \frac{1}{2}(x-\mu_b)^T Y_b (x-\mu_b) \tag{39}$$

$$Y_c (x - \mu_c) = Y_a (x - \mu_a) + Y_b (x - \mu_b) \tag{40}$$

$$Y_c = Y_a + Y_b \tag{41}$$

the last line of which says that information is additive. Setting$x = 0$ in the next-to-last line yields the equation

$$Y_c \mu_c = Y_a \mu_a + Y_b \mu_b \tag{42}$$

The following substitutions will be made in(41) and(42):

$$\left.\begin{aligned} \mu_a &= \hat{\mathbf{x}}_k^- a\, priori\, estimate \\ Y_a &= \mathbf{P}_k^{-1(-)} a\, priori\, information \\ \mu_b &= \mathbf{H}_k^{\dagger} \mathbf{y}_k measurement\, mean \\ Y_b &= \mathbf{H}_k^T \mathbf{R}_k^{-1} \mathbf{H}_k measurement\, information \\ \mu_c &= \hat{\mathbf{x}}_k a\, posteriori\, estimate \\ Y_c &= \mathbf{P}_k^{-1} a\, posteriori\, information \end{aligned}\right\} \tag{43}$$

$$\mathbf{P}_k^{-1} = \mathbf{P}_k^{-1(-)} + \mathbf{H}_k^T \mathbf{R}_k^{-1} \mathbf{H}_k \tag{44}$$

where$\mathbf{P}_k^{-1(-)}$ is the a priori state information and$\mathbf{H}_k^T\mathbf{R}_k^{-1}\mathbf{H}_k$ is the information in the$k$th measurement$\mathbf{y}_k$. The measurement estimations come from(3) by taking the expectation

$$\mathbf{H}^{-1}\mathbf{y} = \mathbf{x} + \mathbf{H}^{-1}\mathbf{v}$$

$$E[\mathbf{H}^{-1}\mathbf{y}] = E[\mathbf{x}] + E[\mathbf{H}^{-1}\mathbf{v}]$$

the last term expands to$\mathbf{H}^{-1}\mathbf{R}\mathbf{H}^{-1(T)}$; reverse it to obtain the information$\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}$. Using an inverse matrix modification formula

$$(A^{-1} + BC^{-1}D)^{-1} = A - AB(C + DAB)^{-1}DA \tag{45}$$

and substituting here

$$\left.\begin{aligned} A^{-1} &= \mathbf{P}_k^{-1(-)} a\ priori\ information\ matrix\ for\ \hat{\mathbf{x}}_k \\ A &= \mathbf{P}_k^{-} a\ priori\ covariance\ matrix\ for\ \hat{\mathbf{x}}_k \\ B &= \mathbf{H}_k^T transpose\ of\ measurement\ sensitivity\ matrix \\ C &= \mathbf{R}_k covariance\ of\ measurement\ noise\ \mathbf{v}_k \\ D &= \mathbf{H}_k measurement\ sensitivity\ matrix \end{aligned}\right\} \tag{46}$$

The equation(45) becomes

$$(\mathbf{P}_k^{-1(-)} + \mathbf{H}_k^T\mathbf{R}_k^{-1}\mathbf{H}_k)^{-1} = \mathbf{P}_k^{-} - \mathbf{P}_k^{-}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_k^{-}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}\mathbf{H}_k\mathbf{P}_k^{-} \tag{47}$$

using(44)

$$\mathbf{P}_k = \mathbf{P}_k^{-} - \mathbf{G}_k\mathbf{H}_k\mathbf{P}_k^{-} \tag{48}$$

where$\mathbf{G}_k = \mathbf{P}_k^{-}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_k^{-}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$ (cf. with(32) and(26)).
Solving(42)

$$\mathbf{P}_k^{-1}\hat{\mathbf{x}}_k = \mathbf{P}_k^{-1(-)}\hat{\mathbf{x}}_k^{-} + \mathbf{H}_k^T\mathbf{R}_k^{-1}\mathbf{H}_k\mathbf{H}_k^{\dagger}\mathbf{y}_k \tag{49}$$

$$\hat{\mathbf{x}}_k = \mathbf{P}_k(\mathbf{P}_k^{-1(-)}\hat{\mathbf{x}}_k^{-} + \mathbf{H}_k^T\mathbf{R}_k^{-1}\mathbf{H}_k\mathbf{H}_k^{\dagger}\mathbf{y}_k) \tag{50}$$

Substituting(48) and expanding$\mathbf{G}_k$

$$\hat{\mathbf{x}}_k = (\mathbf{P}_k^{-} - \mathbf{P}_k^{-}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_k^{-}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}\mathbf{H}_k\mathbf{P}_k^{-})(\mathbf{P}_k^{-1(-)}\hat{\mathbf{x}}_k^{-} + \mathbf{H}_k^T\mathbf{R}_k^{-1}\mathbf{H}_k\mathbf{H}_k^{\dagger}\mathbf{y}_k) \tag{51}$$

Opening the parenthesis and rearranging terms yields

$$\hat{\mathbf{x}}_k = [\mathbf{I} - \mathbf{P}_k^{-}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_k^{-}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}\mathbf{H}_k](\hat{\mathbf{x}}_k^{-} + \mathbf{P}_k^{-}\mathbf{H}_k^T\mathbf{R}_k^{-1}\mathbf{H}_k\mathbf{H}_k^{\dagger}\mathbf{y}_k) \tag{52}$$

and again

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{P}_k^- \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k)^{-1} \{ [(\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k)\mathbf{R}_k^{-1} - \mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T \mathbf{R}_k^{-1}] \mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_k^- \} \tag{53}$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{P}_k^- \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k)^{-1} \{ [\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T \mathbf{R}_k^{-1} + \mathbf{I} - \mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T \mathbf{R}_k^{-1}] \mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_k^- \} \tag{54}$$

and finally

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{P}_k^- \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k)^{-1} (\mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_k^-) \tag{55}$$

Let's define here

$$\mathbf{G}_k \equiv \mathbf{P}_k^- \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k)^{-1} \tag{56}$$

Compare with(26) and(15) (or(28)).

1.3 Maximum A Posteriori Probability approach

There is an alternative class of estimators called maximum a posteriori probability (MAP) estimators which use Bayes' rule to compute the argmax of the a posteriori probability density function to select the value of the variable to be estimated at which its probability density is greatest (maximum mode) (i.e. maximize$\hat{\mathbf{x}}$ in(57)). These estimators are applicable to a more general class of problems (including non-Gaussian and nonlinear) than the Kalman filter, but they tend to have computational complexities that would eliminate them from consideration for real-time practical implementations as filters. They are used for some nonlinear and non-real-time applications, however.

Bayesian estimation combines a priori information with the measurements through a conditional density function of$\mathbf{x}$ given the measurements$\mathbf{y}$. This conditional probability density function is known as the a posteriori distribution of$\mathbf{x}$. Therefore, Bayesian estimation requires the probability density functions of both the measurement noise and unknown parameters. The posterior density function$p(\mathbf{x}|\mathbf{y})$ for$\mathbf{x}$ (taking the measurement sample$\mathbf{y}$ into account) is given by Bayes' rule:

$$p(\mathbf{x}|\mathbf{y}) = \frac{p(\mathbf{y}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{y})} \tag{57}$$

Note since$\mathbf{y}$ is treated as a set of known quantities, then$p(\mathbf{y})$ provides the proper normalization factor to ensure that$p(\mathbf{x}|\mathbf{y})$ is a probability density function. Alternatively,

$$p(\mathbf{y}) = \int_{-\infty}^{\infty} p(\mathbf{y}|\mathbf{x})p(\mathbf{x})d\mathbf{x}$$

If this integral exists then the posterior function$p(\mathbf{x}|\mathbf{y})$ is said to be *proper*; if it does not exist then$p(\mathbf{x}|\mathbf{y})$ is *improper*, in which case we let$p(\mathbf{y}) = 1$.

Since$p(\mathbf{y})$ does not depend on$\mathbf{x}$ we seek to maximize$p(\mathbf{y}|\mathbf{x})p(\mathbf{x})$, or its natural logarithm

$$J_{MAP}(\hat{\mathbf{x}}) = \ln p(\mathbf{y}|\hat{\mathbf{x}}) + \ln p(\hat{\mathbf{x}}) \tag{58}$$

The first term in the sum is actually the log-likelihood function, and the second term gives the a priori information on the to-be-determined parameters.

Therefore, the MAP estimator maximizes

$$J_{MAP}(\hat{\mathbf{x}}) = \ln \mathcal{L}(\mathbf{y}|\hat{\mathbf{x}}) + \ln p(\hat{\mathbf{x}}) \tag{59}$$

Maximum a posteriori estimation has the following properties:

1. if the a priori distribution$p(\hat{\mathbf{x}})$ is uniform, then MAP estimation is equivalent to maximum likelihood estimation;
2. MAP estimation shares the asymptotic consistency and efficiency properties of maximum likelihood estimation;
3. the MAP estimator converges to the maximum likelihood estimator for large samples;
4. the MAP estimator also obeys the invariance principle.

Let's consider a process following a Gaussian distribution. The assumed probability density functions for this case are given by

$$\mathcal{L}(\mathbf{y}|\hat{\mathbf{x}}) = p(\mathbf{y}|\hat{\mathbf{x}}) = \frac{1}{(2\pi)^{m/2}|\mathbf{R}|^{1/2}} e^{-\frac{1}{2}(\mathbf{y}-\mathbf{H}\hat{\mathbf{x}})^T\mathbf{R}^{-1}(\mathbf{y}-\mathbf{H}\hat{\mathbf{x}})} \tag{60}$$

$$p(\hat{\mathbf{x}}) = \frac{1}{(2\pi)^{n/2}|\mathbf{Q}|^{1/2}} e^{-\frac{1}{2}(\hat{\mathbf{x}}^- - \hat{\mathbf{x}})^T\mathbf{Q}^{-1}(\hat{\mathbf{x}}^- - \hat{\mathbf{x}})} \tag{61}$$

where$\mathbf{H}$ has the dimensions of$m \times n$.

Maximizing(59) w.r.t.$\hat{\mathbf{x}}$ leads to the following estimator:

$$\frac{d}{d\hat{\mathbf{x}}}(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}})^T\mathbf{R}^{-1}(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}) + \frac{d}{d\hat{\mathbf{x}}}(\hat{\mathbf{x}}^- - \hat{\mathbf{x}})^T\mathbf{Q}^{-1}(\hat{\mathbf{x}}^- - \hat{\mathbf{x}}) = 0$$

$$\frac{d}{d\hat{\mathbf{x}}}(\mathbf{y}^T\mathbf{R}^{-1}\mathbf{y} - \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{y} - \mathbf{y}^T\mathbf{R}^{-1}\mathbf{H}\hat{\mathbf{x}} + \hat{\mathbf{x}}^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\hat{\mathbf{x}}) +$$

$$+\frac{d}{d\hat{\mathbf{x}}}(\hat{\mathbf{x}}_-^T\mathbf{Q}^{-1}\hat{\mathbf{x}}_- - \hat{\mathbf{x}}^T\mathbf{Q}^{-1}\hat{\mathbf{x}}_- - \hat{\mathbf{x}}_-^T\mathbf{Q}^{-1}\hat{\mathbf{x}} + \hat{\mathbf{x}}^T\mathbf{Q}^{-1}\hat{\mathbf{x}}) = 0$$

$$-\mathbf{H}^T\mathbf{R}^{-1}\mathbf{y} - \mathbf{y}^T\mathbf{R}^{-1}\mathbf{H} + 2\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\hat{\mathbf{x}} - \mathbf{Q}^{-1}\hat{\mathbf{x}}_- - \hat{\mathbf{x}}_-^T\mathbf{Q}^{-1} + 2\hat{\mathbf{x}}^T\mathbf{Q}^{-1} = 0$$

$$-2\mathbf{H}^T\mathbf{R}^{-1}\mathbf{y} + 2\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\hat{\mathbf{x}} - 2\mathbf{Q}^{-1}\hat{\mathbf{x}}_- + 2\mathbf{Q}^{-1}\hat{\mathbf{x}} = 0$$

$$\hat{\mathbf{x}} = (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} + \mathbf{Q}^{-1})^{-1}(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{y} + \mathbf{Q}^{-1}\hat{\mathbf{x}}^-) \tag{62}$$

Let's compare this with our previous results. Substitute(26) into(15) to obtain

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}^- + \mathbf{P}^-\mathbf{H}^T(\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R})^{-1}(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}^-) \tag{63}$$

For our Gaussian(61)$\hat{\mathbf{x}}^- = \mathbf{x} + \mathbf{w}$

$$\mathbf{P}^- = E[(\mathbf{x}-\mathbf{x} - \mathbf{w})(\mathbf{x}-\mathbf{x} - \mathbf{w})^T] = \mathbf{Q}$$

Then(63) becomes

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}^- + \mathbf{Q}\mathbf{H}^T(\mathbf{H}\mathbf{Q}\mathbf{H}^T + \mathbf{R})^{-1}(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}^-) \tag{64}$$

1. Let's consider the coefficient at$\mathbf{y}$:

$$(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} + \mathbf{Q}^{-1})^{-1}\mathbf{H}^T\mathbf{R}^{-1} \stackrel{?}{=} \mathbf{Q}\mathbf{H}^T(\mathbf{H}\mathbf{Q}\mathbf{H}^T + \mathbf{R})^{-1}$$

$$\mathbf{H}^T\mathbf{R}^{-1}(\mathbf{H}\mathbf{Q}\mathbf{H}^T + \mathbf{R}) \stackrel{?}{=} (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} + \mathbf{Q}^{-1})\mathbf{Q}\mathbf{H}^T$$

$$\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{Q}\mathbf{H}^T + \mathbf{H}^T = \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{Q}\mathbf{H}^T + \mathbf{H}^T$$

2. and for$\hat{\mathbf{x}}^-$:

$$(\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} + \mathbf{Q}^{-1})^{-1}\mathbf{Q}^{-1} \stackrel{?}{=} \mathbf{I} - \mathbf{Q}\mathbf{H}^T(\mathbf{H}\mathbf{Q}\mathbf{H}^T + \mathbf{R})^{-1}\mathbf{H}$$

$$\mathbf{Q}^{-1} \stackrel{?}{=} \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} + \mathbf{Q}^{-1} - (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} + \mathbf{Q}^{-1})\mathbf{Q}\mathbf{H}^T(\mathbf{H}\mathbf{Q}\mathbf{H}^T + \mathbf{R})^{-1}\mathbf{H}$$

$$\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} \stackrel{?}{=} (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} + \mathbf{Q}^{-1})\mathbf{Q}\mathbf{H}^T(\mathbf{H}\mathbf{Q}\mathbf{H}^T + \mathbf{R})^{-1}\mathbf{H}$$

$$\mathbf{H}^T\mathbf{R}^{-1}(\mathbf{H}\mathbf{Q}\mathbf{H}^T + \mathbf{R}) \stackrel{?}{=} (\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H} + \mathbf{Q}^{-1})\mathbf{Q}\mathbf{H}^T$$

$$\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{Q}\mathbf{H}^T + \mathbf{H}^T = \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{Q}\mathbf{H}^T + \mathbf{H}^T$$

Q.E.D.

1.4 Example

In this 2-D example I'll simulate a hound chasing a hare (or an air-to-air missile chasing a MiG). The hare is running along an ellipse (dashed red line). The hound is old and his eyes and nose are no so sharp as they used to be. He detects the hare with an error which is normally distributed around the hare position at time$k$ (solid red dots). But his mind is still sharp. He keeps track of his own position(x_1, x_2), speed(v_1, v_2) and acceleration(a_1, a_2), and calculates his projected position at time$k+1$ (green dots):

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}t + \frac{1}{2}\mathbf{a}t^2 + \mathbf{w}(t) = \mathbf{x}_0 + \dot{\mathbf{x}}t + \frac{1}{2}\ddot{\mathbf{x}}t^2 + \mathbf{w}(t)$$

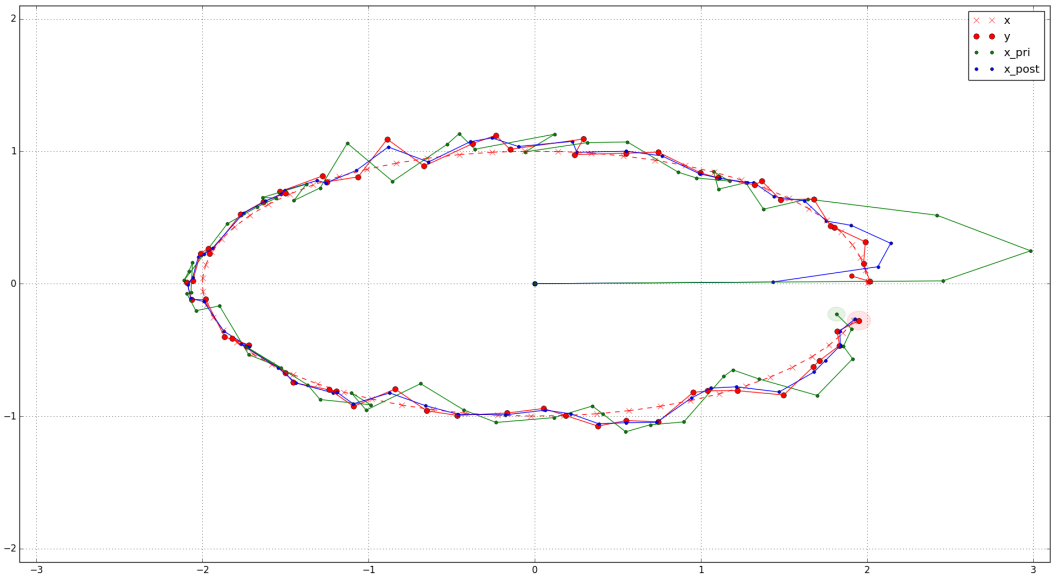
or in discrete form:

$$\mathbf{x}_{k+1} = \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ a_1 \\ a_2 \end{pmatrix}_{k+1} = \begin{pmatrix} 1 & 0 & \triangle t & 0 & \frac{1}{2}\triangle t^2 & 0 \\ 0 & 1 & 0 & \triangle t & 0 & \frac{1}{2}\triangle t^2 \\ 0 & 0 & 1 & 0 & \triangle t & 0 \\ 0 & 0 & 0 & 1 & 0 & \triangle t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ a_1 \\ a_2 \end{pmatrix}_k + \mathcal{N}(0, q)$$

The process noise$\mathbf{w}$ is due to the paws slippage, wind and other factors not included in our model, and assumed to be Gaussian. As an example, the hound observes only his position$\mathbf{x}$ and not speed or acceleration, i.e.

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As mentioned his observation error is also assumed Gaussian:$\mathcal{N}(0, r)$ (solid red dots). The updated positions (the a posteriori approximations) are displayed as blue dots. The last dots also plot circles in light hue with the radius proportional to its respective variance. The initial approximation, arbitrary chosen as$(0, 0)$, is way off but it converges quite fast. As we can see, in most cases the a posteriori position is closer to the truth than both the prediction and observation.


x
y
x_pri
x_post
2
1
0
−1
−2
−3
−2
−1
0
1
2
3

2 Unscented Kalman filter

Let's consider a nonlinear, discrete-time dynamical system with its process equation:

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k, \mathbf{q}_k) \tag{65}$$

where$\mathbf{u}_k$ is the input vector. The process noise$\mathbf{q}_k$ caused by disturbances and modeling errors is assumed to be Gaussian (unlike in the particle filter), with zero mean and with covariance matrix defined by

$$E[\mathbf{q}_n \mathbf{q}_k^T] = \begin{cases} \mathbf{Q}_k & for\ n = k \\ \mathbf{0} & for\ n \neq k \end{cases} \tag{66}$$

In this case it is not additive as it transforms through$f$. The observation equation:

$$\mathbf{y}_k = h(\mathbf{x}_k, \mathbf{u}_k, \mathbf{v}_k) \tag{67}$$

where the measurement noise is Gaussian with zero mean and covariance:

$$E[\mathbf{v}_n \mathbf{v}_k^T] = \begin{cases} \mathbf{R}_k & for\ n = k \\ \mathbf{0} & for\ n \neq k \end{cases} \tag{68}$$

The noises are uncorrelated:$E[\mathbf{q}_n \mathbf{v}_k^T] = 0$.

We seek the minimum-mean-squared error (MMSE) estimate. The MMSE estimate of$\mathbf{x}(k)$ is the conditional mean. Let$\hat{\mathbf{x}}(i|j)$ be the mean of$\mathbf{x}(i)$ conditioned on all of the observations up to and including time$j$

$$\hat{\mathbf{x}}(i|j) = E[\mathbf{x}(i)|\mathbf{Y}^{(j)}] \tag{69}$$

where$\mathbf{Y}^{(j)} = [y(1), \ldots, y(j)]$. The covariance of this estimate is denoted$\mathbf{P}(i|j)$.

The Kalman filter propagates the first two moments of the distribution of$\mathbf{x}(k)$ recursively. Given an estimate$\hat{\mathbf{x}}(k|k)$, the filter first predicts what the future state of the system will be, using the process model.

$$\hat{\mathbf{x}}(k+1|k) = E[f(\mathbf{x}(k), \mathbf{u}(k), \mathbf{q}(k))|\mathbf{Y}^{(k)}]$$

$$\mathbf{P}(k+1|k) = E[\{\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1|k)\}\{\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1|k)\}^T|\mathbf{Y}^{(k)}]$$

The expectations can be calculated only if the distribution of$\mathbf{x}(k)$ conditioned on$\mathbf{Y}^{(k)}$ is known. In general, the distribution cannot be described by a finite number of parameters and most practical systems employ an approximation of some kind. Often the distribution of$\mathbf{x}(k)$ is assumed Gaussian at any time$k$. Two justifications are made. First, only the mean and covariance need to be maintained. Second, given just the first two moments the Gaussian distribution is the entropy maximizing or least informative distribution.

The estimate at time$k+1$ is given through updating the prediction by the linear update rule. See(15),(30) and(26). The EKF exploits the fact that the error in the prediction,$\tilde{\mathbf{x}}(i|j) = \mathbf{x}(i) - \hat{\mathbf{x}}(i|j)$, can be attained by expanding(65),(67) as a Taylor series about the estimate$\hat{\mathbf{x}}(k|k)$. Truncating this series at the first order yields the approximate linear expression for the propagation of state error as

$$\tilde{\mathbf{x}}(k+1|k) \approx \nabla f_x \tilde{\mathbf{x}}(k|k) + \nabla f_q \mathbf{q}(k)$$

Using this approximation, the state prediction equations are

$$\hat{\mathbf{x}}(k+1|k) = f(\hat{\mathbf{x}}(k|k), \mathbf{u}(k), 0) \tag{70}$$

$$\mathbf{P}(k+1|k) = \nabla f_x \mathbf{P}(k|k) \nabla^T f_x + \nabla f_q \mathbf{Q}(k+1) \nabla^T f_q \tag{71}$$

The problems with EKF are that in many practical applications this linearization introduces significant biases or errors. Also calculating Jacobians at every prediction step can be cumbersome and time consuming.

We use the intuition that it is easier to approximate a probability distribution than it is to approximate an arbitrary nonlinear function or transformation. Following this intuition, we generate a set of points whose sample mean and sample covariance are$\hat{\mathbf{x}}(k|k)$ and$\mathbf{P}(k|k)$, respectively. The nonlinear function is applied to each of these points in turn to yield a transformed sample, and the predicted mean and covariance are calculated from the transformed sample. Unlike in a Monte Carlo method, the sample are not drawn at random but rather carefully chosen so they capture specific information about the distribution. In general, this intuition can be applied to capture many kind of information about many types of distribution. Here we consider the special case of (i) capturing the mean and covariance of an (ii) assumed Gaussian distribution.

The$n$-dimensional random variable$\mathbf{x}(k)$ with mean$\hat{\mathbf{x}}(k|k)$ and covariance$\mathbf{P}(k|k)$ is approximated by$2n+1$ weighted samples orσ points selected by the algorithm

$$\begin{aligned}
\mathcal{X}_0(k|k) &= \hat{\mathbf{x}}(k|k) \\
W_0 &= \tfrac{\varkappa}{n+\varkappa} \\
\mathcal{X}_i(k|k) &= \hat{\mathbf{x}}(k|k) + \left(\sqrt{(n+\varkappa)\mathbf{P}(k|k)}\right)_i \\
W_i &= \tfrac{1}{2(n+\varkappa)} \\
\mathcal{X}_{i+n}(k|k) &= \hat{\mathbf{x}}(k|k) - \left(\sqrt{(n+\varkappa)\mathbf{P}(k|k)}\right)_i \\
W_{i+n} &= \tfrac{1}{2(n+\varkappa)}
\end{aligned} \tag{72}$$

where$\varkappa \in \mathbb{R}$,$\left(\sqrt{(n+\varkappa)\mathbf{P}(k|k)}\right)_i$ is the$i$th row (for$\mathbf{P} = \mathbf{A}^T\mathbf{A}$) or column (for$\mathbf{P} = \mathbf{A}\mathbf{A}^T$) of the matrix square root of$(n+\varkappa)\mathbf{P}(k|k)$, and$W_i$ is the weight that is associated with the$i$th point.

Valuable insight into the Unscented Transformation can be gained by relating it to a numerical technique called the Gauss-Hermite quadrature rule in the

context of state estimation. A close similarity also exists between the UT and the central difference interpolation filtering (CDF) techniques.

Theorem 1: The set of samples chosen by(72) have the same sample mean, covariance, and all higher odd-ordered central moments as the distribution of$\mathbf{x}(k)$. The matrix square root and$\varkappa$ affect the 4th and higher order sample moments of the sigma points.

Proof: The matching of the mean, covariance, and all odd-ordered moments can be directly demonstrated. Because the points are symmetrically distributed and chosen with equal weights about$\bar{\mathbf{x}}$, the sample mean is obviously$\bar{\mathbf{x}}$ and all odd-ordered moments are zero. The sample covariance$\mathbf{P}$ is

$$\begin{aligned}\mathbf{P} &= \sum_{i=0}^{2n} W_i \left[\mathcal{X}_i(k|k) - \hat{\mathbf{x}}(k|k)\right]\left[\mathcal{X}_i(k|k) - \hat{\mathbf{x}}(k|k)\right]^T = \\ &= \sum_{i=1}^{n} 2W_i(n+\varkappa)\left(\sqrt{\mathbf{P}(k|k)}\right)_i \left(\sqrt{\mathbf{P}(k|k)}\right)_i^T = \\ &= \sum_{i=1}^{n} \left(\sqrt{\mathbf{P}(k|k)}\right)_i \left(\sqrt{\mathbf{P}(k|k)}\right)_i^T = \\ &= \mathbf{P}(k|k)\end{aligned}$$

■

Remark 1: The above properties hold for any choice of the matrix square root. Efficient and stable methods, such as Cholesky decomposition, should be used.

Given the set of samples generated by(72), the prediction procedure is as follows.

1. The state is augmented with the noises:

$$\mathbf{x}_a(k) = \begin{pmatrix} \mathbf{x}(k) \\ \mathbf{q}(k) \\ \mathbf{v}(k) \end{pmatrix}$$

$$\hat{\mathbf{x}}_a(k) = \begin{pmatrix} \hat{\mathbf{x}}(k) \\ E[\mathbf{q}(k)] \\ E[\mathbf{v}(k)] \end{pmatrix}$$

or in our case

$$\hat{\mathbf{x}}_a(k) = \begin{pmatrix} \hat{\mathbf{x}}(k) \\ 0 \\ 0 \end{pmatrix}$$

n will now be the size of the vector$\mathbf{x}_a(k)$. The covariance matrix is augmented

$$\mathbf{P}_a(k) = \begin{pmatrix} \mathbf{P}(k) & \mathbf{P}_{xq}(k) & \mathbf{P}_{xv}(k) \\ \mathbf{P}_{xq}^T(k) & \mathbf{Q}(k) & \mathbf{P}_{qv}(k) \\ \mathbf{P}_{xv}^T(k) & \mathbf{P}_{qv}^T(k) & \mathbf{R}(k) \end{pmatrix}$$

or in our case

$$\mathbf{P}_a(k) = \begin{pmatrix} \mathbf{P}(k) & 0 & 0 \\ 0 & \mathbf{Q}(k) & 0 \\ 0 & 0 & \mathbf{R}(k) \end{pmatrix}$$

2. The initial values

$$\hat{\mathbf{x}}(0) = E[\mathbf{x}(0)]$$

$$\mathbf{P}(0) = E[(\mathbf{x}(0) - \hat{\mathbf{x}}(0))(\mathbf{x}(0) - \hat{\mathbf{x}}(0))^T]$$

$$\hat{\mathbf{x}}_a(0) = E[\mathbf{x}_a(0)] = \begin{pmatrix} \hat{\mathbf{x}}(0) \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{P}_a(0) = E[(\mathbf{x}_a(0) - \hat{\mathbf{x}}_a(0))(\mathbf{x}_a(0) - \hat{\mathbf{x}}_a(0))^T] = \begin{pmatrix} \mathbf{P}(0) & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{pmatrix}$$

3. Calculate theσ points

$$\mathcal{X}_i^a(k) = \begin{cases} \hat{\mathbf{x}}_a(k) & i = 0 \\ \hat{\mathbf{x}}_a(k) + \left(\sqrt{(n+\varkappa)\mathbf{P}_a(k)}\right)_i & i = 1, \ldots, n \\ \hat{\mathbf{x}}_a(k) - \left(\sqrt{(n+\varkappa)\mathbf{P}_a(k)}\right)_{i-n} & i = n+1, \ldots, 2n \end{cases}$$

for$i$th column.

4. Eachσ point is instantiated through the process model to yield a set of transformed samples

$$\mathcal{X}_i^a(k+1|k) = f(\mathcal{X}_i^a(k|k), \mathbf{u}(k), k) \tag{73}$$

5. The predicted mean is computed as

$$\hat{\mathbf{x}}^{(-)}(k+1|k) = \sum_{i=0}^{2n} W_i \mathcal{X}_i^a(k+1|k) \tag{74}$$

6. The predicted covariance is computed as

$$\mathbf{P}^{(-)}(k+1|k) = \sum_{i=0}^{2n} W_i [\mathcal{X}_i^a(k+1|k) - \hat{\mathbf{x}}^{(-)}(k+1|k)][\mathcal{X}_i^a(k+1|k) - \hat{\mathbf{x}}^{(-)}(k+1|k)]^T \tag{75}$$

7. The predicted observation

$$\mathcal{Y}(k+1|k) = h(\mathcal{X}^a(k+1|k)) \tag{76}$$

8. The mean observation is given by

$$\hat{\mathbf{y}}^{(-)}(k+1|k) = \sum_{i=0}^{2n} W_i \mathcal{Y}_i(k+1|k) \tag{77}$$

9. The update step

$$\mathbf{P}_{yy}(k+1|k+1) = \sum_{i=0}^{2n} W_i \left[\mathcal{Y}_i(k+1|k) - \hat{\mathbf{y}}^{(-)}(k+1|k)\right] \left[\mathcal{Y}_i(k+1|k) - \hat{\mathbf{y}}^{(-)}(k+1|k)\right]^T \tag{78}$$

$$\mathbf{P}_{xy}(k+1|k+1) = \sum_{i=0}^{2n} W_i \left[\mathcal{X}_i^a(k+1|k) - \hat{\mathbf{x}}^{(-)}(k+1|k)\right] \left[\mathcal{Y}_i(k+1|k) - \hat{\mathbf{y}}^{(-)}(k+1|k)\right]^T \tag{79}$$

$$\mathbf{G}(k+1|k+1) = \mathbf{P}_{xy}\mathbf{P}_{yy}^{-1} \tag{80}$$

$$\hat{\mathbf{x}}(k+1|k+1) = \hat{\mathbf{x}}^{(-)}(k+1|k) + \mathbf{G}(k+1|k+1)\left[\mathbf{y}(k+1|k+1) - \hat{\mathbf{y}}^{(-)}(k+1|k)\right] \tag{81}$$

$$\mathbf{P}(k+1|k+1) = \mathbf{P}^{(-)}(k+1|k) - \mathbf{G}(k+1|k+1)\mathbf{P}_{yy}(k+1|k+1)\mathbf{G}^T(k+1|k+1) \tag{82}$$

The innovation vector (or measurement prediction error or residual) points from our predicted measurement to the actual measurement: $\mathbf{y}(k+1|k+1) - \hat{\mathbf{y}}^{(-)}(k+1|k)$, and $\mathbf{G}$ determines how a vector in measurement space maps to a correction in state space.

Theorem 2: The prediction algorithm introduces errors in estimating the mean and covariance at the 4th and higher orders in the Taylor series. These higher order terms are a function of $\varkappa$ and the matrix square root used.

Proof: Let's consider a Gaussian-distributed random variable $\mathbf{x}$ with mean $\bar{\mathbf{x}}$ and covariance $\mathbf{P}_x$. We wish to calculate the mean $\bar{\mathbf{y}}$ and covariance $\mathbf{P}_y$ of the random variable $\mathbf{y}$, which is related to $\mathbf{x}$ through the nonlinear analytic function $\mathbf{y} = f(\mathbf{x})$. Note that $\mathbf{y}$ here is not the observable variable in the KF but rather corresponds to the prior $\mathbf{x}^{(-)}$.

Noting that $\mathbf{x}$ can be written as $\mathbf{x} = \bar{\mathbf{x}} + \delta\mathbf{x}$, where $\delta\mathbf{x}$ is a zero-mean Gaussian random variable with covariance $\mathbf{P}_x$, the nonlinear transformation can be expanded as a Taylor series about $\bar{\mathbf{x}}$

$$\mathbf{y} = f(\bar{\mathbf{x}} + \delta\mathbf{x}) = \sum_{i=0}^{\infty} \left[\frac{(\delta\mathbf{x} \cdot \nabla_x)^i f(\mathbf{x})}{i!}\right]_{\mathbf{x}=\bar{\mathbf{x}}} \tag{83}$$

If we define the operator $\mathbf{D}_{\delta\mathbf{x}}^i f$ as

$$\mathbf{D}_{\delta\mathbf{x}}^i f \equiv \left[(\delta\mathbf{x} \cdot \nabla_x)^i f(\mathbf{x})\right]_{\mathbf{x}=\bar{\mathbf{x}}}$$

then the Taylor series expansion of the nonlinear transformation $\mathbf{y} = f(\mathbf{x})$ can be written as

$$\mathbf{y} = f(\mathbf{x}) = f(\bar{\mathbf{x}}) + \mathbf{D}_{\delta\mathbf{x}} f + \frac{1}{2}\mathbf{D}^2_{\delta\mathbf{x}} f + \frac{1}{3!}\mathbf{D}^3_{\delta\mathbf{x}} f + \frac{1}{4!}\mathbf{D}^4_{\delta\mathbf{x}} f + \dots \tag{84}$$

The true mean of $\mathbf{y}$ is given by

$$\bar{\mathbf{y}} = E[\mathbf{y}] = E[f(\mathbf{x})] =$$

$$= E\left[f(\bar{\mathbf{x}}) + \mathbf{D}_{\delta\mathbf{x}} f + \frac{1}{2}\mathbf{D}^2_{\delta\mathbf{x}} f + \frac{1}{3!}\mathbf{D}^3_{\delta\mathbf{x}} f + \frac{1}{4!}\mathbf{D}^4_{\delta\mathbf{x}} f + \dots \right] \tag{85}$$

If we assume that$\mathbf{x}$ is a symmetrically distributed random variable, then all odd moments will be zero (expectation which is an integral of$\delta\mathbf{x}$ which is$\mathbf{x} - \bar{\mathbf{x}}$ is 0). Also note that

$$E\left[\delta\mathbf{x}\delta\mathbf{x}^T\right] = \mathbf{P}_x \tag{86}$$

Given this, the mean can be reduced further to

$$\bar{\mathbf{y}} = f(\bar{\mathbf{x}}) + \frac{1}{2}\left[\left(\nabla^T \mathbf{P}_x \nabla\right) f(\mathbf{x})\right]_{\mathbf{x}=\bar{\mathbf{x}}} + E\left[\frac{1}{4!}\mathbf{D}^4_{\delta\mathbf{x}} f + \frac{1}{6!}\mathbf{D}^6_{\delta\mathbf{x}} f + \dots\right] \tag{87}$$

The Unscented Transformation calculates the posterior mean from the propagatedσ points using(74). The sigma points are given by (72), i.e.

$$\mathcal{X}_i = \bar{\mathbf{x}} \pm \left(\sqrt{n+\varkappa}\right)\sigma_i = \bar{\mathbf{x}} \pm \tilde{\sigma}_i$$

whereσ_i denotes the$i$th column of the matrix square root of$\mathbf{P}_x$. This implies that

$$\sum_{i=1}^{n} (\sigma_i \sigma_i^T) = \mathbf{P}_x \tag{88}$$

Given this formulation of the sigma points, we can again write the propagation of each point through the nonlinear function as a Taylor series expansion about$\bar{\mathbf{x}}$:

$$f(\mathcal{X}_i) = f(\bar{\mathbf{x}}) + \mathbf{D}_{\tilde{\sigma}_i} f + \frac{1}{2}\mathbf{D}^2_{\tilde{\sigma}_i} f + \frac{1}{3!}\mathbf{D}^3_{\tilde{\sigma}_i} f + \frac{1}{4!}\mathbf{D}^4_{\tilde{\sigma}_i} f + \dots \tag{89}$$

Using(74) and(72), the UT predicted mean is

$$\bar{\mathbf{y}}_{UT} = \frac{\varkappa}{n+\varkappa} f(\bar{\mathbf{x}}) + \frac{1}{2(n+\varkappa)} \sum_{i=1}^{2n} \left(f(\bar{\mathbf{x}}) + \mathbf{D}_{\tilde{\sigma}_i} f + \frac{1}{2}\mathbf{D}^2_{\tilde{\sigma}_i} f + \frac{1}{3!}\mathbf{D}^3_{\tilde{\sigma}_i} f + \frac{1}{4!}\mathbf{D}^4_{\tilde{\sigma}_i} f + \dots \right) =$$

$$= f(\bar{\mathbf{x}}) + \frac{1}{2(n+\varkappa)} \sum_{i=1}^{2n} \left(\mathbf{D}_{\tilde{\sigma}_i} f + \frac{1}{2}\mathbf{D}^2_{\tilde{\sigma}_i} f + \frac{1}{3!}\mathbf{D}^3_{\tilde{\sigma}_i} f + \frac{1}{4!}\mathbf{D}^4_{\tilde{\sigma}_i} f + \dots \right) \tag{90}$$

Since the sigma points are symmetrically distributed around$\bar{\mathbf{x}}$, all the odd moments are zero. This results in the simplification

$$\bar{\mathbf{y}}_{UT} = f(\bar{\mathbf{x}}) + \frac{1}{2(n+\varkappa)} \sum_{i=1}^{2n} \left(\frac{1}{2} \mathbf{D}^2_{\tilde{\sigma}_i} f + \frac{1}{4!} \mathbf{D}^4_{\tilde{\sigma}_i} f + \dots \right) \tag{91}$$

and since

$$\frac{1}{2(n+\varkappa)} \sum_{i=1}^{2n} \frac{1}{2} \mathbf{D}^2_{\tilde{\sigma}_i} f = \frac{1}{2(n+\varkappa)} (\nabla f)^T \left(\frac{1}{2} \sum_{i=1}^{2n} \sqrt{n+\varkappa} \sigma_i \sigma_i^T \sqrt{n+\varkappa} \right) \nabla f =$$

$$= \frac{n+\varkappa}{2(n+\varkappa)} (\nabla f)^T \frac{1}{2} \left(\sum_{i=1}^{2n} \sigma_i \sigma_i^T \right) \nabla f =$$

$$= \frac{1}{2} \left[\left(\nabla^T \mathbf{P}_x \nabla \right) f(\mathbf{x}) \right]_{\mathbf{x}=\bar{\mathbf{x}}}$$

the UT predicted mean can be further simplified to

$$\bar{\mathbf{y}}_{UT} = f(\bar{\mathbf{x}}) + \frac{1}{2} \left[\left(\nabla^T \mathbf{P}_x \nabla \right) f(\mathbf{x}) \right]_{\mathbf{x}=\bar{\mathbf{x}}} + \frac{1}{2(n+\varkappa)} \sum_{i=1}^{2n} \left(\frac{1}{4!} \mathbf{D}^4_{\tilde{\sigma}_i} f + \frac{1}{6!} \mathbf{D}^6_{\tilde{\sigma}_i} f + \dots \right) \tag{92}$$

When we compare(92) and(87), we can clearly see that the true posterior mean and the mean calculated by the UT agrees exactly to the third order and that errors are only introduced in the 4th and higher order terms. The magnitudes of these errors depends on the choice of the composite scaling parameter$\varkappa$ as well as the higher-order derivatives of$f$. The parameter$\varkappa$ provides an extra degree of freedom to"fine tune" the higher order moments of the approximation, and can be used to reduce the overall prediction errors. When$\mathbf{x}(k)$ is assumed Gaussian, a useful heuristic is to select$n + \varkappa = 3$. If a different distribution is assumed for$\mathbf{x}(k)$, then a different choice of$\varkappa$ might be more appropriate.

Now let's look at the *accuracy of the covariance*. The true posterior covariance is given by

$$\mathbf{P}_y = E\left[(\mathbf{y} - \bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}})^T \right] \tag{93}$$

where the expectation is taken over the distribution of$\mathbf{y}$. Substituting(84) and(85) into(93) we get

$$\mathbf{y} - \bar{\mathbf{y}} = f(\bar{\mathbf{x}}) + \mathbf{D}_{\delta\mathbf{x}} f + \frac{1}{2!} \mathbf{D}^2_{\delta\mathbf{x}} f + \frac{1}{3!} \mathbf{D}^3_{\delta\mathbf{x}} f + \dots - f(\bar{\mathbf{x}}) - E\left[\frac{1}{2!} \mathbf{D}^2_{\delta\mathbf{x}} f + \frac{1}{4!} \mathbf{D}^4_{\delta\mathbf{x}} f + \dots \right]$$

post multiplying the state error by the transpose of itself, taking the expectation, and recalling that all odd moments of$\delta\mathbf{x}$ are zero owing to symmetry (e.g.$E[\frac{1}{2!} \mathbf{D}^2_{\delta\mathbf{x}} f \cdot (\mathbf{D}_{\delta\mathbf{x}} f)^T] = 0$)

$$\mathbf{P}_y = E\left[\mathbf{D}_{\delta\mathbf{x}} f \cdot (\mathbf{D}_{\delta\mathbf{x}} f)^T + \frac{1}{3!}\mathbf{D}^3_{\delta\mathbf{x}} f \cdot (\mathbf{D}_{\delta\mathbf{x}} f)^T + \frac{1}{2!}\mathbf{D}^2_{\delta\mathbf{x}} f \cdot (\frac{1}{2!}\mathbf{D}^2_{\delta\mathbf{x}} f)^T + \mathbf{D}_{\delta\mathbf{x}} f \cdot (\frac{1}{3!}\mathbf{D}^3_{\delta\mathbf{x}} f)^T + \ldots\right] -$$

$$-E\left[E\left[\frac{1}{2!}\mathbf{D}^2_{\delta\mathbf{x}} f\right] \cdot (\frac{1}{2!}\mathbf{D}^2_{\delta\mathbf{x}} f)^T + \frac{1}{2!}\mathbf{D}^2_{\delta\mathbf{x}} f \cdot \left(E\left[\frac{1}{2!}\mathbf{D}^2_{\delta\mathbf{x}} f\right]\right)^T - E\left[\frac{1}{2!}\mathbf{D}^2_{\delta\mathbf{x}} f\right] E\left[\frac{1}{2!}\mathbf{D}^2_{\delta\mathbf{x}} f\right]^T + \ldots\right]$$

Limiting to the 3rd order and using(86)

$$\mathbf{P}_y = \left[\left(\nabla \mathbf{P}_x \nabla^T\right) f(\mathbf{x})\right]_{\mathbf{x}=\bar{\mathbf{x}}} + O(\delta\mathbf{x}^4) \tag{94}$$

The unscented KF predicts the covariance using (see(72))

$$\mathcal{X}_0 = \bar{\mathbf{x}}$$

$$\mathcal{X}_i = \bar{\mathbf{x}} \pm \sigma_i$$

which assures that

$$\mathbf{P}_x = \frac{1}{2(n+\varkappa)} \sum_{i=1}^{2n} (\mathcal{X}_i - \bar{\mathbf{x}})(\mathcal{X}_i - \bar{\mathbf{x}})^T$$

The transformed set of sigma points are evaluated by

$$\mathcal{Y}_i = f(\mathcal{X}_i)$$

The predicted mean is computed as

$$\bar{\mathbf{y}} = \frac{1}{n+\varkappa}\left(\varkappa \mathcal{Y}_0 + \frac{1}{2}\sum_{i=1}^{2n} \mathcal{X}_i\right)$$

And the predicted covariance is computed as

$$\mathbf{P}_y = \sum_{i=0}^{2n} W_i (\mathcal{X}_i - \bar{\mathbf{x}})(\mathcal{X}_i - \bar{\mathbf{x}})^T = \frac{1}{n+\varkappa}\left[\varkappa(\mathcal{Y}_0 - \bar{\mathbf{y}})(\mathcal{Y}_0 - \bar{\mathbf{y}})^T + \frac{1}{2}\sum_{i=1}^{2n}(\mathcal{Y}_i - \bar{\mathbf{y}})(\mathcal{Y}_i - \bar{\mathbf{y}})^T\right] \tag{95}$$

Here

$$\mathcal{Y}_0 - \bar{\mathbf{y}} = f(\bar{\mathbf{x}}) - f(\bar{\mathbf{x}}) - \frac{1}{2(n+\varkappa)} \sum_{i=1}^{2n} \left(\frac{1}{2!}\mathbf{D}^2_{\tilde{\sigma}_i} f + \frac{1}{4!}\mathbf{D}^4_{\tilde{\sigma}_i} f + \ldots\right)$$

$$(\mathcal{Y}_0 - \bar{\mathbf{y}})(\mathcal{Y}_0 - \bar{\mathbf{y}})^T = \frac{1}{4(n+\varkappa)^2} \sum_{i=1}^{2n} \left(\frac{1}{2!}\mathbf{D}^2_{\tilde{\sigma}_i} f + \frac{1}{4!}\mathbf{D}^4_{\tilde{\sigma}_i} f + \ldots\right) \sum_{i=1}^{2n} \left(\frac{1}{2!}\mathbf{D}^2_{\tilde{\sigma}_i} f + \frac{1}{4!}\mathbf{D}^4_{\tilde{\sigma}_i} f + \ldots\right)^T = O(\tilde{\sigma}^4)$$

Using(89) and(91)

$$\mathcal{Y}_i - \bar{\mathbf{y}} = \mathbf{D}_{\tilde{\sigma}_i} f + \frac{1}{2!}\mathbf{D}^2_{\tilde{\sigma}_i} f + \frac{1}{3!}\mathbf{D}^3_{\tilde{\sigma}_i} f + \ldots - \frac{1}{2(n+\varkappa)} \sum_{i=1}^{2n} \left(\frac{1}{2!}\mathbf{D}^2_{\tilde{\sigma}_i} f + \frac{1}{4!}\mathbf{D}^4_{\tilde{\sigma}_i} f + \ldots \right)$$

$$(\mathcal{Y}_i - \bar{\mathbf{y}})(\mathcal{Y}_i - \bar{\mathbf{y}})^T = \mathbf{D}_{\tilde{\sigma}_i} f \cdot (\mathbf{D}_{\tilde{\sigma}_i} f)^T + O(\tilde{\sigma}^4)$$

Plugging these back to(95) yields

$$\mathbf{P}_y = \frac{1}{2(n+\varkappa)} \sum_{i=1}^{2n} \mathbf{D}_{\tilde{\sigma}_i} f \cdot (\mathbf{D}_{\tilde{\sigma}_i} f)^T + O(\tilde{\sigma}^4) =$$

$$= \frac{1}{2(n+\varkappa)} \nabla f \left(\sum_{i=1}^{2n} \sqrt{n+\varkappa}\sigma_i \sigma_i^T \sqrt{n+\varkappa} \right) (\nabla f)^T + O(\tilde{\sigma}^4)$$

and substituting(88)

$$\mathbf{P}_y = \nabla f \mathbf{P}_x (\nabla f)^T + O(\tilde{\sigma}^4)$$

Compare it with(94)

■

To reiterate, when a set of carefully chosen sample points (σ points) propagated through the nonlinear system, for non-Gaussian inputs, they capture the posterior mean and covariance accurately to at least the 2nd order (Taylor series expansion) for any nonlinearity, with the accuracy of 3rd and higher order moments being determined by the choice of$\varkappa$. For a symmetrical (e.g. Gaussian) pdf the accuracy is at least the 3rd order.

2.1 Mackey-Glass Example

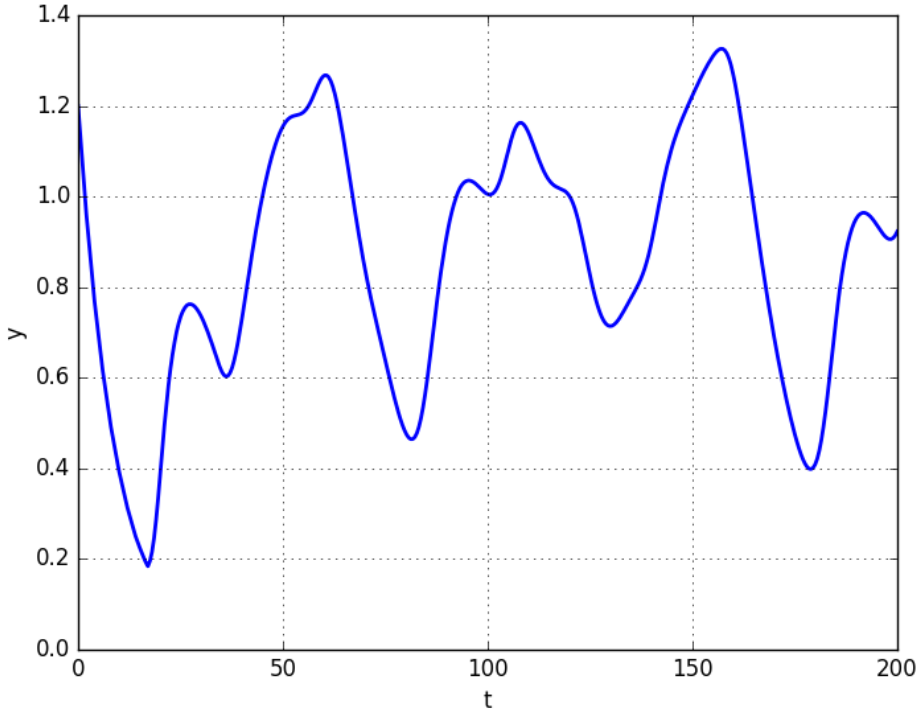
For this example I'll try to predict the Mackey-Glass chaotic series which are defined by

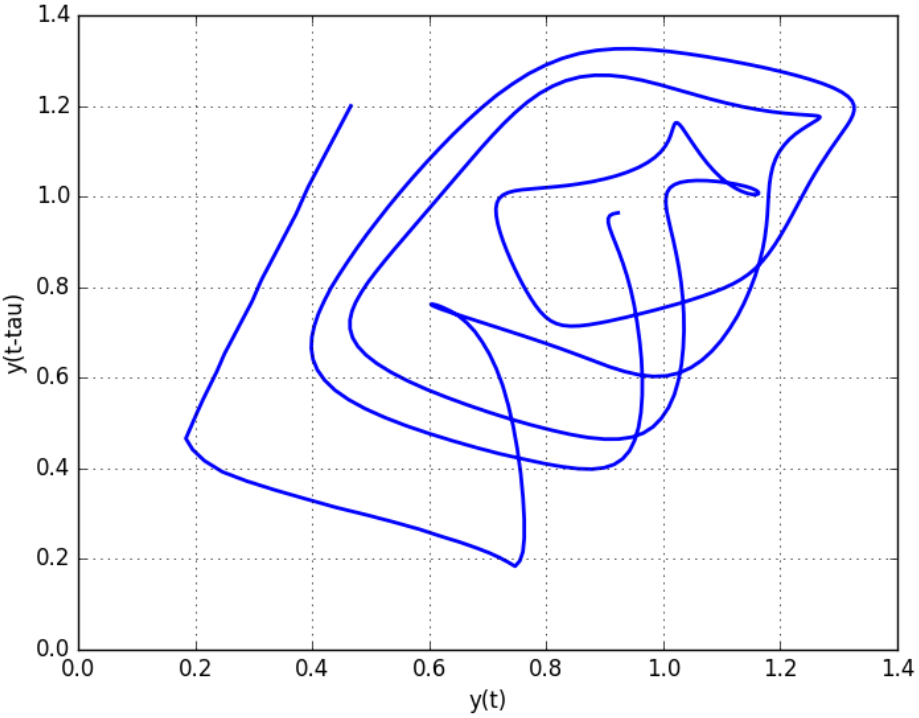
$$\dot{\mathbf{x}} = \beta \frac{\mathbf{x}(t-\tau)}{1+\mathbf{x}(t-\tau)^n} - \gamma \mathbf{x}(t) \tag{96}$$

or for a discrete time t

$$\mathbf{x}(t) = \mathbf{x}(t-1) + \left[\beta \frac{\mathbf{x}(t-\tau)}{1+\mathbf{x}(t-\tau)^n} - \gamma \mathbf{x}(t-1) \right] \triangle t$$

For the delay$\tau = 17$, $\beta = 0.2$, $\gamma = 0.1$, $n = 10$, $\triangle t = 0.1 and x = 1.2 for t \leq 0$ the series look like


1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
y
0
50
100
150
200
t

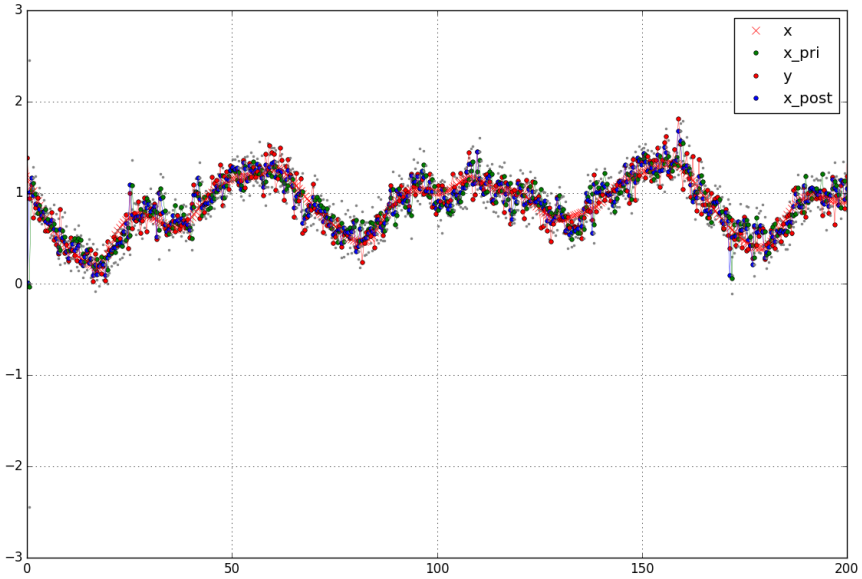


Our augmented state will consist of a scalar function value$x$ and scalar noises:

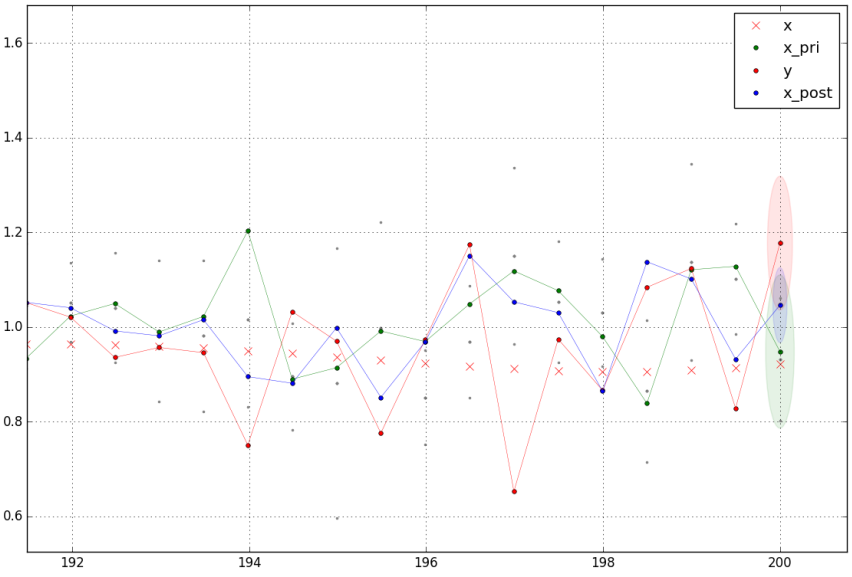
$$\mathbf{x}_a(k) = \begin{pmatrix} x(k) \\ q(k) \\ v(k) \end{pmatrix}_{3\times 1}$$

The initial position is arbitrary chosen at$x = 0$. The initial covariance matrix is equal to

$$\mathbf{P}_a(0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.02 \end{pmatrix}$$



and zoom in at the end



The gray dots indicate the sigma points. The mean a priori values $\hat{\mathbf{x}}^{(-)}$

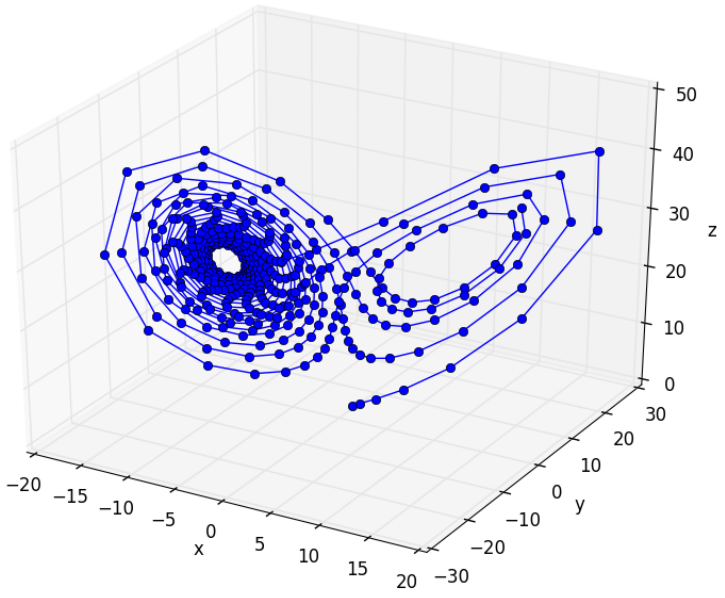
(predictions) are shown in green. The greenish circle of uncertainty has the radius of $\left(\sqrt{\mathbf{P}^{(-)}}\right)_{0,0}$. The observations $\mathbf{y}$ are depicted with red dots. We observe only the first component with some additive Gaussian noise with its standard deviation equal to the square root of r (shown as a reddish circle). The updated a posteriori values $\hat{\mathbf{x}}$ are blue dots with the circle of uncertainty shown in bluish. Its radius is $\left(\sqrt{\mathbf{P}}\right)_{0,0}$.

2.2 Lorenz Attractor Example

The Lorenz equations are

$$\begin{cases} \dot{\mathbf{x}} &= \sigma(\mathbf{y} - \mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{x}(\rho - \mathbf{z}) - \mathbf{y} \\ \dot{\mathbf{z}} &= \mathbf{x}\mathbf{y} - \beta\mathbf{z} \end{cases} \tag{97}$$

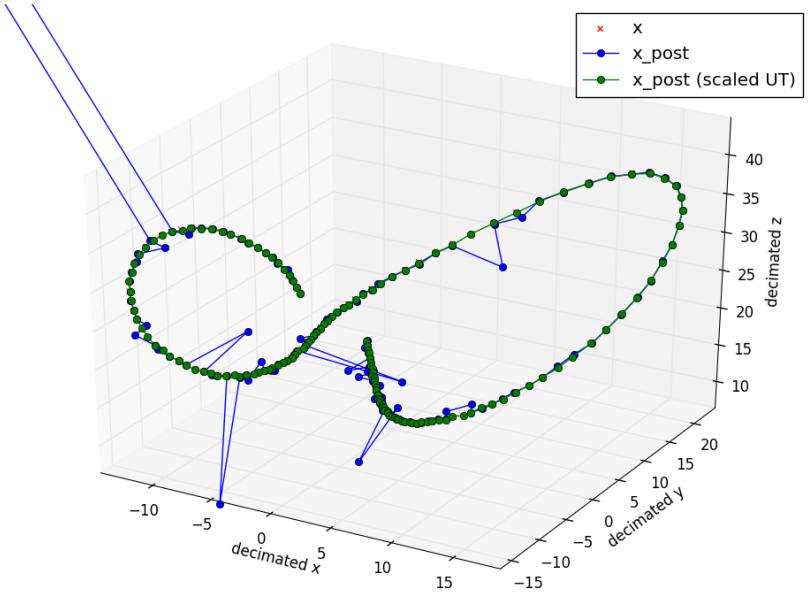
where $\rho = 28,\ \sigma = 10,\ \beta = 8/3$.



The sigma point selection scheme used in the unscented transformation has the property that as the dimension of the state-space increases, the radius of the sphere that bounds all the sigma points increases as well. Even though the mean and covariance of the prior distribution are still captured correctly, it does so at

the cost of sampling non-local effects. If the nonlinearities in question are very severe, this can lead to significant difficulties. In order to address this problem, the sigma points can be scaled towards or away from the mean of the prior distribution by a proper choice of $\varkappa$. The distance of the ith sigma point from $\bar{x}$, $|\mathcal{X}_i - \bar{\mathbf{x}}|$, is proportional to $\sqrt{n+\varkappa}$. When $\varkappa = 0$, the distance is proportional to $\sqrt{n}$. When $\varkappa > 0$ the points are scaled further from $\bar{\mathbf{x}}$ and when $\varkappa < 0$ the points are scaled towards $\bar{\mathbf{x}}$. For the special case of $\varkappa = 3 - n$, the desired dimensional scaling invariance is achieved by canceling the effect of n. However, when $\varkappa = 3 - n < 0$ the weight $W_0 < 0$ and the calculated covariance can be non-positive semidefinite. The scaled unscented transformation was developed to address this problem.

Jumping ahead, here is the comparison of Unscented Kalman filter with the scaled one.



We can see that the scaled UKF (green) is much more stable than the unscaled UKF (blue). Red is the truth.

Our augmented state is:

$$\mathbf{x}_a(k) = \begin{pmatrix} x_1 & x_2 & x_3 & \rho & \sigma & \beta & q_1 & q_2 & q_3 & v_1 & v_2 & v_3 \end{pmatrix}^T$$

The initial position is arbitrary chosen as

$$\mathbf{x}_a(0) = \begin{pmatrix} 0 & 0 & 0 & 28 & 10 & \frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

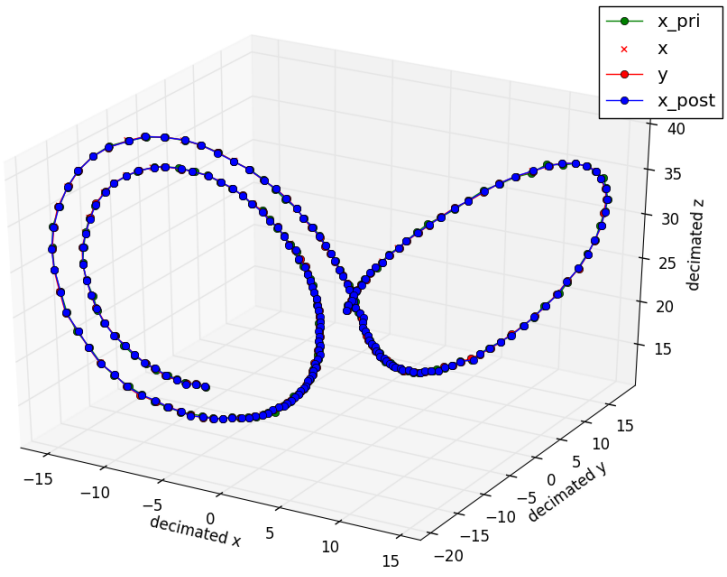
The initial covariance matrix is equal to

$$\mathbf{P}_a(0) = diag\begin{pmatrix} 0.2 & 0.2 & 0.2 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.005 & 0.005 & 0.005 \end{pmatrix}$$

We add a Gaussian noise to the entire state and pass it through f (97) which makes it nonlinear.

We only observe the first 3 components.

For the scaled transformation $\alpha = 0.001,\ \beta = 2$.



As we can see it approximates the function pretty good.