/ Reparametrization Trick d.l 14.12.18

$$\mathbb{E}_{q(z|\lambda)} f(z) \to \max_{\lambda}$$

$$p(x, z|\theta)$$

$$\log p(x|\theta) \geq ELBO(\theta, \lambda) = \mathbb{E}_{q(z|\lambda)} (\log p(x, z|\theta) -$$
$$- \log q(z|\lambda)) \to \max_{\theta, \lambda}$$

VAE, Policy Gradient, Actor-Critic

$$\mathbb{E}_{p(\tau|\theta)} R(\tau) \to \max_{\theta} \quad , \quad \mathbb{E}_{p(s)} \mathbb{E}_{\pi(a|s,\theta)} Q(a, s|w) \to \max_{\theta}$$

RT $\quad z \sim q(z|\lambda) \Leftrightarrow \begin{array}{l} s(z|\lambda) = \varepsilon, \quad \varepsilon \sim q(\varepsilon) \\ z = s^{-1}(\varepsilon|\lambda) \end{array}$

стандартизующее распределение

$$\nabla_\lambda \mathbb{E}_{q(z|\lambda)} f(z) = \nabla_\lambda \mathbb{E}_{q(\varepsilon)} f(s^{-1}(\varepsilon|\lambda)) = \mathbb{E}_{q(\varepsilon)} \underset{\substack{\uparrow \\ 1\times m \\ m = \dim \lambda}}{\nabla_\lambda} f(s^{-1}(\varepsilon|\lambda)) =$$

$$= \mathbb{E}_{q(\varepsilon)} \underbrace{\nabla_z f(z)\Big|_{z = s^{-1}(\varepsilon|\lambda)}}_{1\times d,\ d = \dim z} \underbrace{\nabla_\lambda s^{-1}(\varepsilon|\lambda)}_{d\times m} \approx$$

MC-sampling

$$\approx \frac{1}{M} \sum_{j=1}^{M} \nabla_z f(z_j) \nabla_\lambda s^{-1}(\varepsilon_j|\lambda), \quad \varepsilon_j \sim q(\varepsilon), \ z_j = s^{-1}(\varepsilon_j|\lambda)$$

$z \in \mathbb{R}$, $F(z|\lambda) = \varepsilon$, $\varepsilon \sim R[0,1]$

$z \in \mathbb{R}^d$, $[F(z_1|\lambda), F(z_2|z_1, \lambda), \ldots, F(z_d|z_{d-1} \ldots z_1, \lambda)] =$

$= [\varepsilon_1 \ldots \varepsilon_d]$, $\varepsilon_j \sim R[0,1]$ · стандартизующее распределение F

поиск обратной ф-ии?

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Implicit RT

$$\nabla_\lambda \mathbb{E}_{q(z|\lambda)} f(z) = \mathbb{E}_{q(\varepsilon)} \nabla_z f(z)\Big|_{z=S^{-1}(\varepsilon|\lambda)} \nabla_\lambda S^{-1}(\varepsilon|\lambda) =$$

$$= \mathbb{E}_{q(z|\lambda)} \nabla_z f(z) \nabla_\lambda z \left\{ \nabla_\lambda z = \nabla_\lambda S^{-1}(\varepsilon|\lambda)\Big|_{\varepsilon = S(z|\lambda)} \right\}$$

$$S(z|\lambda) = \varepsilon \quad | \nabla\lambda \qquad \{ z = z(\lambda) \}$$

$$\nabla_z S(z|\lambda) \nabla_\lambda z + \nabla_\lambda S(z|\lambda) = 0$$ Решение СЛАУ

$$\underset{d\times m}{\nabla_\lambda z} = - \underset{d\times d}{(\nabla_z S(z|\lambda))^{-1}} \underset{d\times m}{\nabla_\lambda S(z|\lambda)}$$

Пример I

$$q(z|\lambda) = N(z|\mu, \sigma^2), \; z \in \mathbb{R}$$

$$S(z|\mu,\sigma) = \frac{z-\mu}{\sigma} = \varepsilon \sim N(\varepsilon|0,1)$$

RT : $S^{-1}(\varepsilon|\mu,\sigma) = \mu + \sigma\varepsilon$

$$\nabla_\mu S^{-1}(\varepsilon|\mu,\sigma) = 1, \quad \nabla_\sigma S^{-1}(\varepsilon|\mu,\sigma^2) = \varepsilon = \frac{z-\mu}{\sigma}$$

IRT: $$\nabla_\mu z = - \frac{\nabla_\mu S(z|\mu,\sigma)}{\nabla_z S(z|\mu,\sigma)} = - \frac{-1/\sigma}{1/\sigma} = 1$$

$$\nabla_\sigma z = - \frac{\nabla_\sigma S(z|\mu,\sigma)}{\nabla_z S(z|\mu,\sigma)} = - \frac{-\frac{z-\mu}{\sigma^2}}{1/\sigma} = \frac{z-\mu}{\sigma}$$

Пример II

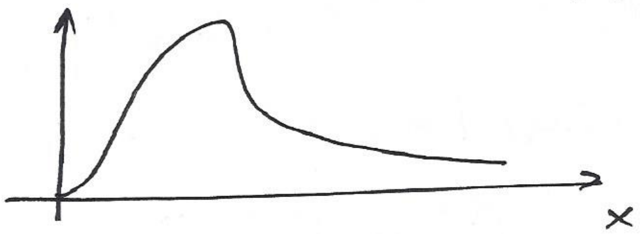
$z \in \mathbb{R}$, $S(z|\lambda) = F(z|\lambda) = \varepsilon \sim R[0,1]$

$$\nabla_\lambda z = - \frac{\nabla_\lambda F(z|\lambda)}{q(z|\lambda)}$$

	Я RT	I RT
Проход вперёд $\mathbb{E}_{q(z\|\lambda)} f(z)$	$\varepsilon \sim q(\varepsilon)$ $z = S^{-1}(\varepsilon\|\lambda)$ $f(z)$	$z \sim q(z\|\lambda)$ $f(z)$
Проход назад $\nabla_\lambda \mathbb{E}_{q(z\|\lambda)} f(z)$	$\nabla_\lambda z = \nabla_\lambda S^{-1}(\varepsilon\|\lambda)$ $\nabla_\lambda f = \nabla_z f(z) \nabla_\lambda z$	$\nabla_\lambda z = -(\nabla_z S(z\|\lambda))^{-1} \nabla_\lambda S(z\|\lambda)$ $\nabla_\lambda f = \nabla_z f(z) \nabla_\lambda z$

Пример: гамма-распределение

$$\mathrm{Gamma}(x|a,b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx), \quad x>0, \; a,b>0$$



$y \sim \mathrm{Gamma}(y|a,1)$

$x = \frac{y}{b} \sim \mathrm{Gamma}(y|a,b)$

$$F(y|a) = \int_0^y \frac{1}{\Gamma(a)} t^{a-1} \exp(-t)\, dt$$

```
def f(x):
    y = np.exp(x)
    z = y + x
    return z
```

$\longrightarrow$

```
def f(x):
    dx = 1
    y = np.exp(x)
    dy = y · dx
    z = y + x
    dz = dy + dx
    return z, dz
```

расчёт производной проходом вперёд

LDA

$$\theta \sim Dir(\theta \mid \alpha) \propto \prod_{j=1}^{k} \theta_j^{\alpha_j - 1}$$

$$\Updownarrow \qquad \sum_{j=1}^{k} \theta_j = 1, \quad \theta_j > 0, \quad \alpha_j > 0$$

$$c_j \sim Gamma(c_j \mid \alpha_j, 1), \quad j = 1 \dots k$$

$$\theta_j = \frac{c_j}{\sum_{i=1}^{k} c_i}$$

$\beta_1 \dots \beta_T$
вер-ти слов
в тематиках

$$\begin{cases} \text{для } d = 1 \dots D \\ \quad \theta_d \sim Dir(\theta \mid \alpha) \\ \quad \text{для } n = 1 \dots N_d \\ \qquad z_{d,n} \sim Cat(z \mid \theta_d) \\ \qquad w_{d,n} \sim Cat(w \mid \beta_{z_{d,n}}) \end{cases}$$

$$p(w, z, \theta \mid \alpha, \beta) = \prod_{d=1}^{D} p(\theta_d \mid \alpha) \cdot \prod_{n=1}^{N_d} p(z_{d,n} \mid \theta_d)\, p(w_{d,n} \mid z_{d,n}, \beta)$$

$$p(w, \theta \mid \alpha, \beta) = \prod_{d=1}^{D} p(\theta_d \mid \alpha) \cdot \prod_{n=1}^{N_d} \left(\sum_{z_{d,n}=1}^{T} p(z_{d,n} \mid \theta_d)\, p(w_{d,n} \mid z_{d,n}, \beta) \right) =$$

$$= \prod_{d=1}^{D} p(\theta_d \mid \alpha) \prod_{n=1}^{N_d} \underbrace{p(w_{d,n} \mid \theta_d, \beta)}_{Cat(w \mid \beta \cdot \theta_d)}$$

$$\log p(w \mid \alpha, \beta) \geq \sum_{d=1}^{D} \left[\mathbb{E}_{q(\theta_d \mid \lambda)} \log p(w_d \mid \theta_d, \beta) - KL(q(\theta_d \mid \lambda) \,\|\, p(\theta_d \mid \alpha)) \right] \to \max_{\alpha, \beta, \lambda}$$

$$q(\theta_d \mid \lambda) = Dir(\theta \mid NN(w_d, \lambda))$$

??

$$p(\theta_d \mid w_d, \beta, \alpha) \quad , \quad q(\theta, z) = q(\theta)\, q(z)\ ?$$

{так делали раньше через VI}

Репараметризация дискретных распр-й

$$z \sim Cat(z \mid \pi) : \quad \begin{matrix} 1 \ldots k \\ \pi_1 \ldots \pi_k \end{matrix} \quad \sum_{i=1}^{k} \pi_i = 1,\ \pi_i \geq 0$$

$$\Updownarrow$$

$$c_i \sim Exp(1) = \exp(-c_i),\ c_i \geq 0$$

$$z = \underset{1 \leq i \leq k}{\operatorname{argmin}} \frac{c_i}{\pi_i} = \underset{1 \leq i \leq k}{\operatorname{argmax}} (\log \pi_i - \log c_i) \ominus$$

$$\underset{Exp(1)}{F(c)} = \int_0^c \exp(-t)\,dt = -\exp(-t)\Big|_0^c = 1 - \exp(-c) = \varepsilon$$

$$\varepsilon \sim R[0,1]$$

$$\Rightarrow \quad c = -\log(1-\varepsilon)$$

$$\Updownarrow$$

$$c = -\log \varepsilon \quad \varepsilon \sim R[0,1]$$

$$\gamma_i = -\log c_i = -\log(-\log \varepsilon_i) \sim Gumbel(\gamma \mid 0, 1)$$

$$\ominus \underset{1 \leq i \leq k}{\operatorname{argmax}} (\log \pi_i + \gamma_i), \quad \text{Gumbel-Max trick}$$

Gumbel-SoftMax:

$$z = \operatorname{Softmax}_\tau(\log \pi + \gamma) = \left\{ \frac{\exp\left(\frac{\log \pi_i + \gamma_i}{\tau}\right)}{\sum_{j=1}^{k} \exp\left(\frac{\log \pi_j + \gamma_j}{\tau}\right)} \right\}_{i=1}^{k}$$

$$\nabla_\pi \mathbb{E}_{Cat(z \mid \pi)} f(z) \approx \mathbb{E}_{Gumbel(\gamma \mid 0,1)^k} \left(\nabla_\pi f(\operatorname{softmax}_\tau(\log \pi + \gamma))\right)$$

$$\tau = const,\ \tau = 1,\ \tau \to 0$$