31.03.17 лк VII

Dual Decomposition

$$\begin{cases} f(x) \to \min \\ g(x) \le 0 \end{cases}$$

прямая задача

$$\mathcal{L}(x,\lambda) = \lambda g(x) + f(x), \quad \lambda \geq 0$$

$$\mathcal{D}(\lambda) = \min_x \mathcal{L}(x,\lambda)$$

$$\mathcal{D}(\lambda) \le f(x^*) \quad \forall \lambda \geq 0$$

$$x^0 = \arg\min_x \mathcal{L}(x,\lambda)$$

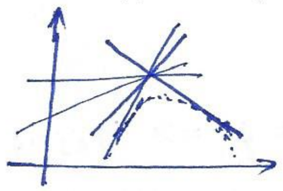
$$\mathcal{D}(\lambda) = f(x^0) + \lambda g(x^0) \le f(x^*) + \lambda \underbrace{g(x^*)}_{\le 0} \le f(x^*)$$

вариационная нижняя оценка

$$\begin{cases} \mathcal{D}(\lambda) \to \max_\lambda \\ \lambda \geq 0 \end{cases}$$

двойственная задача

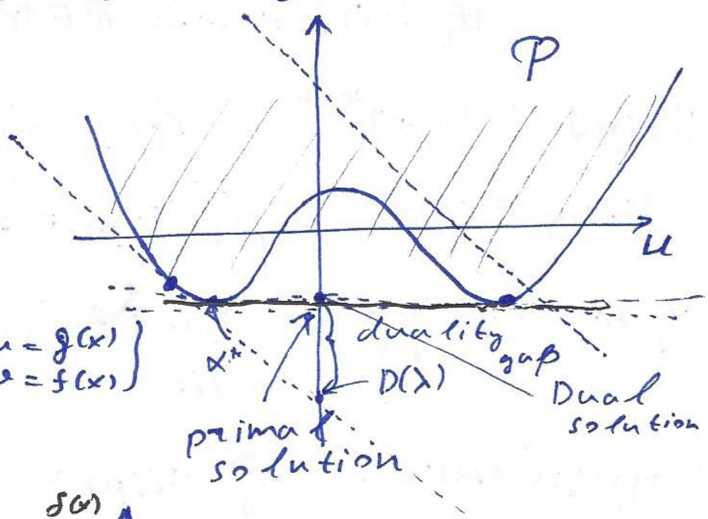
вогнутая ф-ия

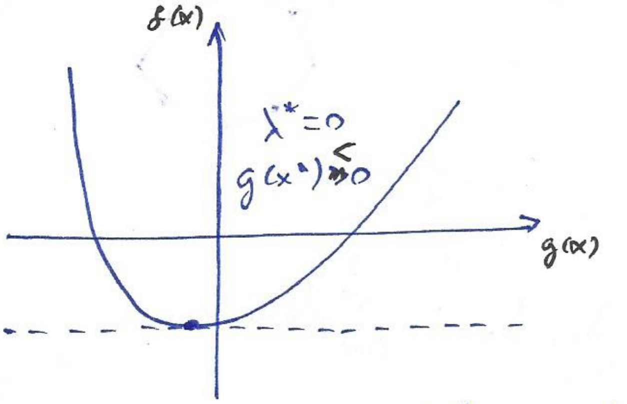


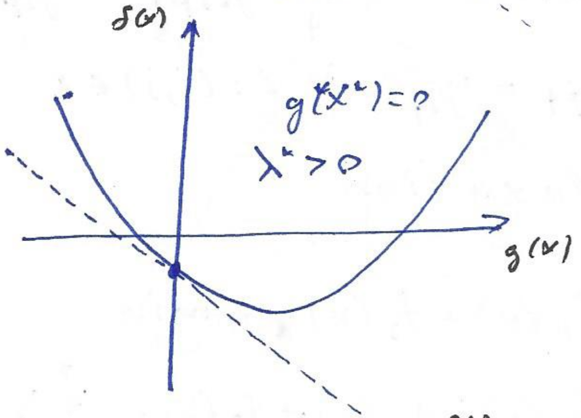
$f(x) + \lambda g(x) = C$

$v = C - \lambda u$

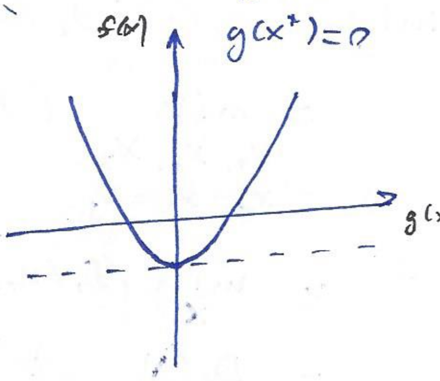
$$\mathcal{P} = \{(u,v) \mid u = g(x),\ v = f(x)\}$$







$$\lambda^* g(x^*) = 0$$



$$T^* = \arg\max_T p(T|X) = \arg\min_T E(T) =$$

$$= \arg\min_T \Big(\sum_{i \in V} \varphi_i(t_i) + \sum_{(i,j) \in \mathcal{E}} \varphi_{ij}(t_i, t_j) \Big)$$

$$\varphi_i(p) = \theta_{ip}, \quad \varphi_{ij}(p,q) = \theta_{ijpq}$$

$$t_i \leftrightarrow y_i \in \{0,1\}^K, \quad t_i = p \Leftrightarrow y_{ip} = 1$$

$$E(Y) = \sum_{i\in V}\sum_{p=1}^{k} \theta_{ip} y_{ip} + \sum_{(i,j)\in E}\sum_{p,q} \theta_{ijpq}\, y_{ip}\, y_{jq} \longrightarrow \min_{Y}$$

s. t. $\quad y_{ip} \in \{0,1\}, \quad \sum_{p} y_{ip} = 1$

примем $\quad y_{ip} \in [0,1] \quad$ QP-relaxation

$$\pi_{ip} = \mathbb{P}\{y_{ip} = 1\}, \quad p(Y) = \prod_{i\in V}\prod_{p} \pi_{ip}^{y_{ip}}$$

$$\mathbb{E}_{\pi} E(Y) = \sum_{i\in V}\sum_{p=1}^{k} \mathbb{E}_{\pi} \theta_{ip} y_{ip} + \ldots = \sum_{i\in V}\sum_{p} \theta_{ip}\pi_{ip} +$$

$$+ \sum_{(i,j)\in E}\sum_{p,q} \theta_{ijpq}\, \mathbb{E}_{\pi} y_{ip} y_{jq} = \sum_{i\in V}\sum_{p} \theta_{ip}\pi_{ip} + \sum_{(i,j)\in E}\sum_{p,q} \theta_{ijpq}\, \pi_{ip}\pi_{jq}$$

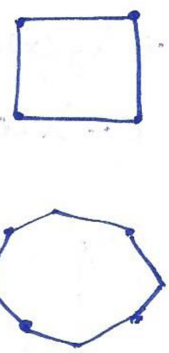
$$\mathbb{E}_{\pi} E(Y) \leq \min \cancel{\mathbb{E}} E(Y)$$

$y_{ij\cancel{pq}} \in \{0,1\}^{k\times k} \qquad t_i = p,\ t_j = q \Rightarrow y_{ijpq} = 1$

$y_{ijpq} \in \{0,1\}, \quad \sum_{p,q} y_{ijpq} = 1$

противоречие: $\quad y_{i2} = 1, \quad y_{ij2q} = 0 \quad \forall q$

$\qquad y_{i2} = 0, \quad y_{ij2q} = 1$



x

разрешение: $\quad \sum_{q} y_{ijpq} = y_{ip} \quad \forall (i,j)\in E$

$\sum_{p} y_{ijpq} = y_{jq} \quad \forall i : (i,j) \in E$

LP-relaxation

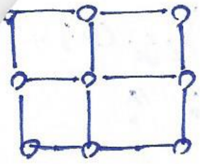
$f_1(x) + f_2(x) + f_3(x) \longrightarrow \min_{x} \qquad \lambda_1, \lambda_2, \lambda_3 : \sum_{i=1}^{3} \lambda_i = 0.$

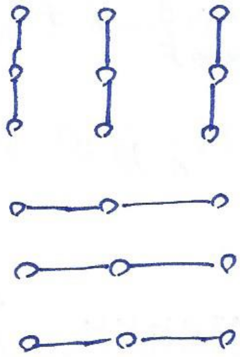
$$\min_{x} \left(f_1(x) + f_2(x) + f_3(x) + \sum_{i=1}^{3} \lambda_i x \right) =$$

$$= \min_{\substack{x_1, x_2, x_3 \\ x_1 = x_2 = x_3}} \left(f_1(x_1) + f_2(x_2) + f_3(x_3) + \sum_{i=1}^{3} \lambda_i x_i \right) \geq$$

$$\geq \min_{x_1} (f_1(x_1) + \lambda_1 x_1) + \min_{x_2} (f_2(x_2) + \lambda_2 x_2) + \min_{x_3} (f_3(x_3) + \lambda_3 x_3)$$

$$= D(\lambda) \quad \forall \lambda : \sum_i \lambda_i = 0$$





$$G \to \{T\} : \cup T = G$$

n_i – число деревьев через вершину i

$n_{ij} = 1$ —"— ребро ij

$$\begin{cases} \theta^T_{ijpq} = \theta_{ijpq} \\ \theta^T_{ip} = \frac{1}{n_i}\theta_{ip} \end{cases}$$

$$E(Y) = \sum_T E_T(Y) = \sum_T \Big(\sum_{i\in T} \theta^T_{ip} y_{ip} + \sum_{(i,j)\in T} \theta^T_{ijpq} y_{ijpq} \Big) =$$

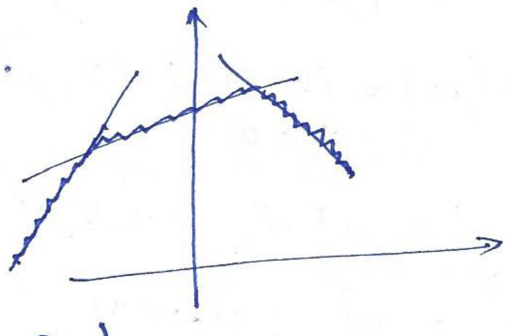
$$= \sum_T (\text{—"—}) + \sum_{i\in V} \sum_p \Big(\sum_T \lambda^T_{ip} \Big) y_{ip}$$

$$\parallel \quad \sum_T \lambda^T_{ip} = 0 \quad \forall i,p$$

$$\min_{\substack{Y \\ Y^T = Y}} \Big(\sum_T \Big(\sum_{i\in T} \sum_p \theta^T_{ip} y^T_{ip} + \sum_{(i,j)\in T} \sum_{p,q} \theta^T_{ijpq} y^T_{ijpq} + \sum_i \sum_p \lambda^T_{ip} y^T_{ip} \Big)\Big) \geq$$

$$\geq \underbrace{\sum_T \min_{y^T} \Big(\sum\sum (\theta^T_{ip} + \lambda^T_{ip}) y^T_{ip} + \sum\sum \theta^T_{ijpq} y^T_{ijpq} \Big)}$$

$$D(\lambda) \to \max_{\lambda : \sum_T \lambda^T_{ip} = 0}$$



$$\frac{\partial D(\lambda)}{\partial \lambda^T_{ip}} = y^{T*}_{ip}$$

$$\lambda^{T,new}_{ip} = \lambda^{T,old}_{ip} + \eta \Big(y^{T*}_{ip} - \frac{1}{n_i} \sum_{T : i\in T} y^{T*}_{ip} \Big)$$

31.03.17 двойственная релаксация

$$\begin{cases} f(x) \to \min_x \\ g_i(x) \le 0, \ i = \overline{1,m} \\ h_j(x) = 0, \ j = \overline{1,p} \end{cases}$$

$$L(x,\lambda,\mu) = f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{j=1}^p \mu_j h_j(x), \quad \lambda_i \ge 0$$

$$D(\lambda,\mu) = \inf_x L(x,\lambda,\mu)$$

двойственная ф-ия

$$D(\lambda,\mu) \le \inf_{x\in F \Leftrightarrow \substack{g_i(x)\le 0 \\ h_j(x)=0}} L(x,\lambda,\mu)$$

$$D(\lambda,\mu) \le \inf_{x\in F} L(x,\lambda,\mu) = \inf_{x\in F} \Big(f(x) + \sum_i \underset{\overset{\vee}{0}}{\lambda_i} \underset{\overset{\wedge}{0}}{g_i(x)} +$$

$$+ \sum_j \mu_j \underset{\overset{\shortparallel}{0}}{h_j(x)}\Big) \le f(x) \quad \forall x \in F$$

двойственная задача оптимизации $\begin{cases} D(\lambda,\mu) \to \max\limits_{\lambda,\mu} \\ \lambda \geqslant 0 \end{cases}$

$x_{opt} \in \operatorname{Arg}\inf\limits_x L(x, \lambda_{opt}, \mu_{opt})$

<u>П1</u> $\begin{cases} c^T x \to \min\limits_x \\ Ax = b \\ x \geqslant 0 \end{cases}$

$$L(x,\lambda,\mu) = c^T x + \mu^T(Ax-b) - \lambda^T x$$

$$D(\lambda,\mu) = \inf_x L(x,\lambda,\mu) = \inf_x \big(x^T(c + A^T\mu - \lambda) - \mu^T b\big) = \begin{cases} -\infty, & c + A^T\mu - \lambda \neq 0 \\ -\mu^T b, & c + A^T\mu - \lambda = 0 \end{cases}$$

$\lambda_i x_i = 0$

$\lambda_{opt,i} > 0 \Rightarrow x_{i,opt} = 0$

$$\begin{cases} \mu^T b \to \min\limits_{\lambda,\mu} \\ \lambda \geqslant 0 \\ c + A^T\mu - \lambda = 0 \end{cases} \Leftrightarrow \begin{cases} \mu^T b \to \min\limits_\mu \\ c + A^T\mu \geqslant 0 \end{cases}$$

<u>П2</u> $L(x,\mu) = c^T x + \mu^T(Ax - b)$

$$D(\mu) = \inf_{x\geqslant 0} L(x,\mu) = \inf_{x \geqslant 0} \big(x^T(c + A^T\mu) - \mu^T b\big) =$$

$$= \begin{cases} -\mu^T b, & c + A^T\mu \geqslant 0 \\ -\infty, & \text{иначе} \end{cases} \qquad \begin{cases} \mu^T b \to \min\limits_\mu \\ c + A^T\mu \geqslant 0 \end{cases}$$

$x_{opt} \in \operatorname{Arg}\inf\limits_x L(x, \mu_{opt})$

если $(c + A^T\mu)_i > 0 \Rightarrow x_{opt,i} = 0$

$$F(x) = f_1(x) + f_2(x) + f_3(x) \to \min_x$$

двойственная декомпозиция — расщепление переменных

$$\begin{cases} f_1(x) + f_2(y) + f_3(z) \to \min\limits_{x,y,z,u} \\ x = u \\ y = u \\ z = u \end{cases} \quad L(x,y,z,u,\mu) = f_1(x) + f_2(y) + f_3(z) + \mu_1(x-u) +$$

$$+ \mu_2(y-u) + \mu_3(z-u)$$

$$D(\mu) = \inf_x (f_1(x) + \mu_1 x) + \inf_y (f_2(y) + \mu_2 y) + \overbrace{\inf_z (f_3(z) + \mu_3 z)}^{D_3(\mu_3)} + \inf_u (-u(\mu_1 + \mu_2 + \mu_3))$$

$$\inf_u (-u(\mu_1+\mu_2+\mu_3)) = \begin{cases} 0, & \mu_1+\mu_2+\mu_3=0 \\ -\infty, & \text{иначе} \end{cases}$$

$$\begin{cases} D_1(\mu_1)+D_2(\mu_2)+D_3(\mu_3) \to \max\limits_{\mu} \\ \mu_1+\mu_2+\mu_3=0 \end{cases}$$

Сильная согласованность – совпадение arginf

$\exists\ \mu_1, \mu_2, \mu_3:$ $\quad \mu_1+\mu_2+\mu_3=0$

$\hat{x}(\mu_1) = \arg\inf (f_1(x)+\mu_1 x)$ $\quad$ и $\hat{x}=\hat{y}=\hat{z}$

$\hat{y}(\mu_2) = \arg\inf (f_2(y)+\mu_2 y)$ $\quad \Downarrow$

$\hat{z}(\mu_3) = \arg\inf (f_3(z)+\mu_3 z)$ $\quad \hat{x} = x_{opt}$

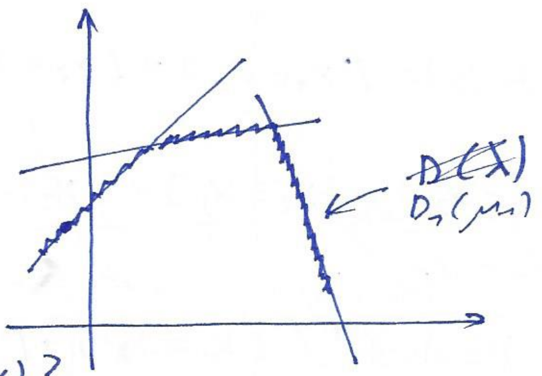
$$D_1(\mu_1) = \inf_x (f_1(x)+\mu_1 x)$$

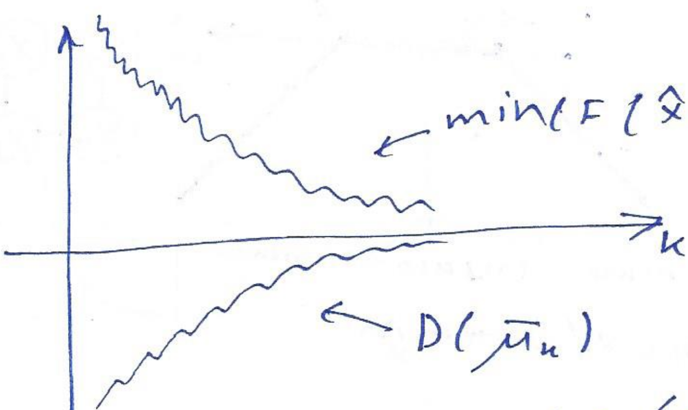
$\partial D_1(\mu_1) = \hat{x}(\mu_2)$?

вне точек излома

$\partial D_1(\mu_1) = \mathrm{conv}\{\hat{x}(\mu_1) \mid$

$\hat{x}(\mu_1) \in \mathrm{Arg}\inf(f_1(x)+\mu_1 x)\}$

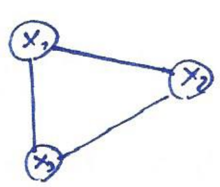




$$\begin{cases} f_1(x)+f_2(y)+f_3(z) \to \min\limits_{x,y,z} \\ y = x \\ z = x \end{cases}$$

$$D(\mu) = \inf_x (f_1(x)-(\mu_1+\mu_2)x) + \inf_y (f_2(y)+\mu_1 y) + \inf_z (f_3(z)+\mu_2 z) \to \max_{\mu_1,\mu_2}$$

\# $E(x) = [x_1=x_2]+[x_2=x_3]+[x_3=x_1] \to \min\limits_{x_1,x_2,x_3 \in \{0,1\}}$

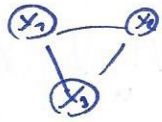


субмодулярность ? нет

$\varphi(0,0)+\varphi(1,1) \le \varphi(0,1)+\varphi(1,0)$

$\quad\ \overset{\shortparallel}{1} \qquad\quad \overset{\shortparallel}{1} \qquad\qquad \overset{\shortparallel}{0} \qquad\quad \overset{\shortparallel}{0}$

$$\begin{cases}[x_1=x_2]+[x_2=x_3]+[x_3=x_4]\to\min\limits_{x_1,x_2,x_3,x_4}\\ x_1=x_4\end{cases}$$



$$D(\mu)=\inf_x\left([x_1=x_2]+[x_2=x_3]+[x_3=x_4]+\mu(x_1-x_4)\right)$$

$\mu=0 \Rightarrow D(\mu)=0$

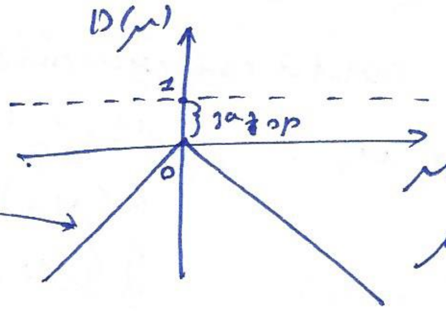
$\mu>0 \Rightarrow D(\mu)=-\mu$

$\mu<0 \Rightarrow D(\mu)=\mu$

$D(\mu)$

$x_1\ x_2\ x_3\ x_4=\begin{cases}0\,1\,0\,1\\1\,0\,1\,0\end{cases}$

сильного согласования нет



$\mu_{opt}=0$

\# $E(x)=[x_1=x_2]+[x_2=x_3]+[x_3=x_4]+[x_4=x_1]\to\min\limits_{x_1,x_2,x_3,x_4\in\{0,1\}}$

$$\begin{cases}[x_1=x_2]+[x_2=x_3]+[x_3=x_4]+[x_4=x_5]\\ x_1=x_5\end{cases}$$

$$D(\mu)=\inf_x\left([x_1=x_2]+[x_2=x_3]+[x_3=x_4]+[x_4=x_5]+\mu(x_1-x_5)\right)$$

$\mu=0 \Rightarrow D(\mu)=0$

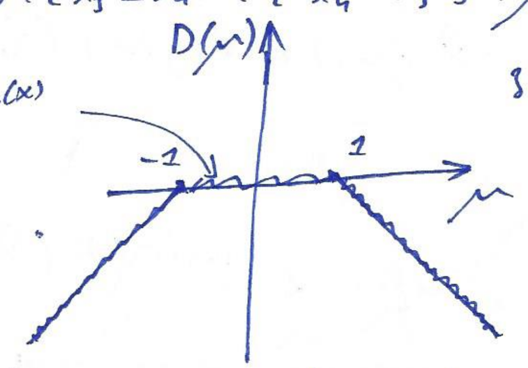
$0<\mu\le 1 \Rightarrow D(\mu)=0$

$\mu>1 \Rightarrow D(\mu)=1-\mu$

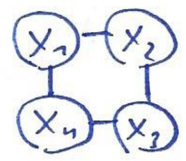
$-1\le\mu<0 \Rightarrow D(\mu)=0$

$\mu<-1 \Rightarrow D(\mu)=1+\mu$

$\min\limits_x E(x)$

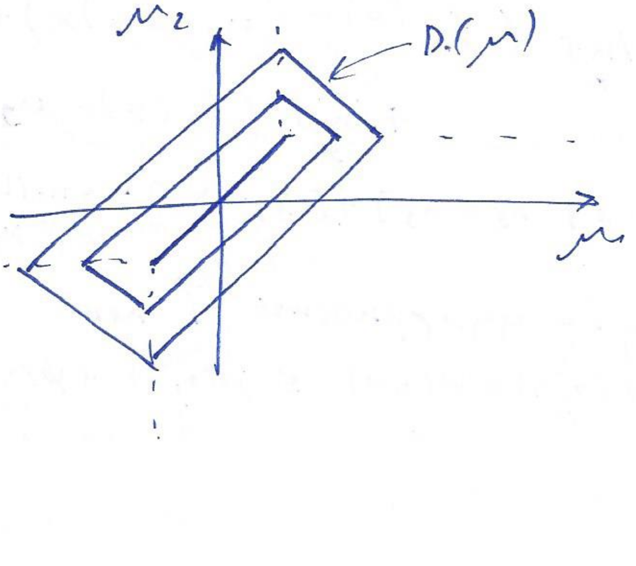


зазора нет



есть согласованность

\# $\begin{cases}[x_1=x_2]+[x_2=x_3]+[x_1'=x_4]+[x_4=x_1']\to\min\limits_x\\ x_1=x_1'\\ x_3=x_3'\end{cases}$



$D(\mu_{opt})=0$