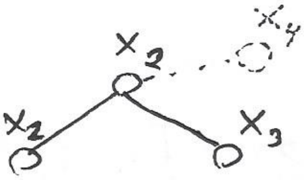
07.04.17 гм VIII

граф вида „дерево": $p(X) = \dfrac{\prod_{(i,j)\in E} p_{ij}(x_i, x_j)}{\prod_{i\in V} p_i(x_i)^{n_i-1}}$

n_i – число рёбер i-й вершины

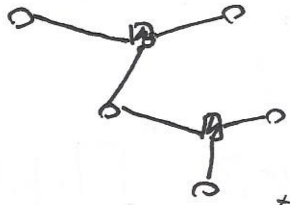


$$p(x_1, x_2, x_3) = \frac{p_{12}(x_1, x_2)\, p_{23}(x_2, x_3)}{p_2(x_2)}$$

$$p(x_1, x_2, x_3, x_4) = \frac{p_{12}(x_1, x_2)\, p_{23}(x_2, x_3)\, p_{24}(x_2, x_4)}{p^2(x_2)}$$

фактор-граф без циклов

$$p(x) = \frac{\prod_f p_f(x_f)}{\prod_{i\in V} p_i^{n_i-1}(x_i)}$$



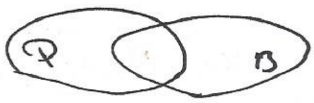
$-\log p(x) - \log Z$

$$p(X) = \frac{1}{Z}\prod_f \psi_f(X_f) = \frac{1}{Z}\exp(-E(X)), \quad E(x) = \sum_f \varphi_f(x_f),$$
$$\varphi_f(x_f) = -\log \psi_f(x_f)$$

ф-л свободной энергии: $F(q) = \mathbb{E}_q E(X) - \mathcal{H}(q) =$
$= \mathbb{E}_q[\log q(x) - \log p(x)] - \log Z = KL(q\|p) - \log Z$

$$q(x) = \frac{\prod_f b_f(x_f)}{\prod_{i\in V} b_i^{n_i-1}(x_i)} \qquad \sum_{x_f} b_f(x_f) = 1, \quad \sum_{x_i} b_i(x_i) = 1$$

$$\sum_{x_f\setminus x_i} b_f(x_f) = b_i(x_i) \quad \forall i\in f \ \forall f$$



$$F(q)\to\min_q \Rightarrow F(b_i(x_i), b_f(x_f))\to\min_{b_i, b_f\in Q}$$

$E(x) = -\sum_f \log\psi_f(x_f)$, $F(b) = -\sum_f\sum_{x_f} b_f(x_f)\log\psi_f(x_f) +$
$+ \sum_f\sum_{x_f} b_f(x_f)\log b_f(x_f) - \sum_{i\in V}\sum_{x_i} b_i(x_i)\log b_i(x_i) \to \min_{b_i, b_f\in Q}$

Bethe Approximation

$$b_i(x_i) = \frac{\prod_f \mu_{f\to i}(x_i)}{\sum_{x_i} —''—}$$

LBP $\mu_{i\to f}(x_i) \propto \prod_{g\neq f}\mu_{g\to i}(x_i)$

$\mu_{f\to i}(x_i) \propto \sum_{x_f\setminus x_i}\psi_f(x_f)\prod_{j\neq i}\mu_{j\to f}(x_j)$

$$b_f(x_f) = \frac{\psi_f(x_f)\prod_i \mu_{i\to f}(x_i)}{\sum_{x_f} —''—}$$

$\min_{b} F(b) \gtrsim -\log Z$

mean-field: $p(x) \simeq q(x) = \prod_{i \in V} q_i(x_i) = \arg\min_q KL(q \| p)$

$\log q_i(x_i) = \mathbb{E}_{x \setminus x_i} \log p(x) + C = \mathbb{E}_{x \setminus x_i} \sum_f \log \psi_f(x_f) + C =$

$// p(x) = \frac{1}{Z} \prod_f \psi_f(x_f)$

$= \sum_f \mathbb{E}_{x_f \setminus x_i} \log \psi_f(x_f) + C = \sum_f \sum_{x_f \setminus x_i} \prod_{\substack{j \neq i \\ j \in f}} q_j(x_j) \log \psi_f(x_f) + C$

$= \sum_{f: i \in f} \sum_{x_f \setminus x_i} \prod_{\substack{j \neq i \\ j \in f}} q_j(x_j) \log \psi_f(x_f) + C''$

$\mu_{j \to f}(x_j)$

$q_i(x_i) \propto \prod_{f: i \in f} \exp \underbrace{\sum_{x_f \setminus x_i} \prod_{j \neq i} q_j(x_j) \log \psi_f(x_f)}_{\mu_{f \to i}(x_i)} = \prod_{f: i \in f} \mu_{f \to i}(x_i)$

Mean-Field

$\mu_{i \to f}(x_i) = q_i(x_i) \propto \prod_{g} \mu_{g \to i}(x_i)$

$\mu_{f \to i}(x_i) = \exp\left(\sum_{x_f \setminus x_i} \log \psi_f(x_f) \prod_{j \neq i} \mu_{j \to f}(x_j) \right)$

$\min_{q \in MF} F(q) \gtrsim -\log Z$

Сэмплирование Гиббса.

$x_i^n \sim p(x_i \mid x_1^n, \dots, x_{i-1}^n, x_{i+1}^{n-1}, \dots, x_m^{n-1}) = \frac{p(x_1^n, \dots, x_{i-1}^n, x_i, x_{i+1}^{n-1}, \dots x_m^{n-1})}{\sum_{x_i} \text{---}''\text{---}} =$

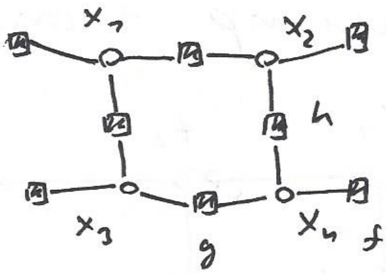
$= \frac{\prod_{f: i \in f} \psi_f(x_i, x_{f \setminus i}^{n/n-1})}{\sum_{x_i} \text{---}''\text{---}}$

Gibbs

$\begin{cases} \mu_{i \to f}(x_i) \sim \text{Sample}\left(\prod_{g} \mu_{g \to i}(x_i)\right) \\ \mu_{f \to i}(x_i) = \psi_f\left(x_i, \bigvee_{j \in f} \mu_{j \to f}(x_j)\right) \end{cases}$

07.04.17 2м сем

$$E(x) = \sum_{i\in V} \varphi_i(x_i) + \sum_{(i,j)\in E} \varphi_{ij}(x_i,x_j)$$



$$\mu_{x_4\to f}(x_4) \propto \mu_{g\to x_4}(x_4)\cdot\mu_{h\to x_4}(x_4)\,\mu_{f\to x_4}(x_4)$$

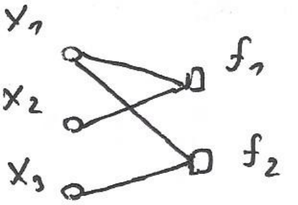
$$\mu_{x_4\to g}(x_4) \propto \text{— " —}$$

$$\mu_{f\to x_4}(x_4)\propto \exp(-\varphi_4(x_4))$$

$$\mu_{g\to x_4}(x_4)\propto \exp\Big(-\sum_{x_3}\varphi_{34}(x_3,x_4)\cdot\mu_{x_3\to g}(x_3)\Big)$$

$$\mu_{i\to f}(x_i) = q_i(x_i) \propto \prod_g \mu_{g\to i}(x_i)$$

$$\mu_{f\to i}(x_i)\propto \exp\Big(\sum_{x_f\setminus x_i} \ln\psi_f(x_f)\prod_{j\ne i}\mu_{j\to f}(x_j)\Big)$$



$$\mu_{f_1\to x_2}(x_2) = \exp\Big(\sum_{x_1}\ln[x_1+x_2=f_1]\,\mu_{x_1\to f}(x_1)\Big) =$$

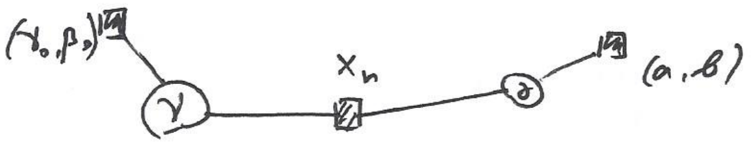
$$=\exp(\ln[x_2+1=f_1]\,\mu_{x_1\to f}(1)+$$

$$+\ln[x_2=f_1]\,\mu_{x_1\to f}(0)) =$$

$$=\exp(\ln[x_2+1=f_1]\,q_1(1)+\ln[x_2=f_1]\,q_1(0)) =$$

$$= [x_2+1=f_1]^{q_1(1)}\cdot[x_2=f_1]^{q_1(0)}$$

$$p(x, x_N, \nu, \tau) = \prod_n \mathcal{N}(x_n|\nu,\tau^{-1})\,\mathcal{N}(\nu|\nu_0,\beta_0^{-1})\,G(\tau|a_0,b_0)$$

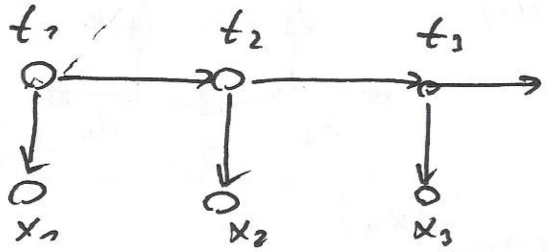


$$\mu_{(a,b)\to\tau}(\tau)\propto\exp(\ln G(\tau|a_0,b_0)) = G(\tau|a_0,b_0)$$

$$\mu_{x\to\tau}(\tau)\propto\exp\Big(\int_\nu \ln\prod_n \mathcal{N}(x_n|\nu,\tau^{-1})\,\mu_{\nu\to x_n}(x)\Big)$$

$$q(\nu,\tau)=q(\nu)q(\tau) = \mathcal{N}(\nu|\nu_0,\beta_0^{-1})\,G(\tau|a_0,b_0)$$

Фильтр частиц



$$p, q \sim \mid \tilde{p}, \tilde{q}$$

$$\mathbb{E}_p f(x) = \mathbb{E}_p \frac{q(x) f(x)}{q(x)} =$$

$$= \mathbb{E}_q \frac{p(x)}{q(x)} f(x) \approx \frac{1}{N} \sum_{x_i \sim q} \frac{p(x_i)}{q(x_i)} f(x_i) \circledast$$

$$Z_q = \int \tilde{q}(x) dx, \quad \int \frac{\tilde{p}(x)}{\tilde{q}(x)} q(x) dx = \frac{Z_p}{Z_q}$$

$$\circledast \sum_{x_i \sim \tilde{q}} w_i f(x_i), \quad w_i = \frac{\tilde{p}(x_i) / \tilde{q}(x_i)}{\sum_j \tilde{p}(x_j) / \tilde{q}(x_j)}$$

$$p(x) = \mathbb{E}_{x_0 \sim p} \delta(x - x_0) \approx \sum_{x_i \sim q} w_i \delta(x - x_i)$$