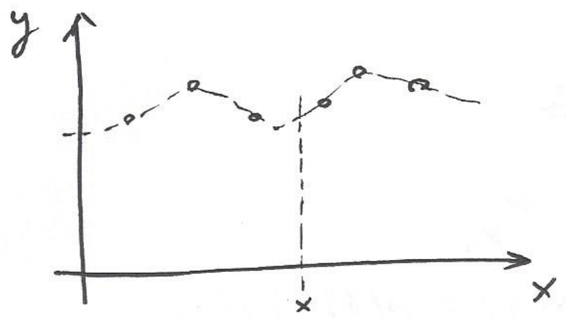
09.12.16 бммо лек XIV

GP $x \in \mathbb{R}^n \to \mathbb{R}$

$\forall T: x_1,\dots,x_T \quad f(x_1),\dots,f(x_T) \sim \mathcal{N}\left(\begin{bmatrix} m(x_1)\\ \vdots \\ m(x_T)\end{bmatrix}, \begin{bmatrix} k(x_1,x_1) & & \\ & \ddots & \\ & & k(x_T,x_T)\end{bmatrix}\right)$



$y = f + \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, \sigma_n^2)$

$p(\cancel{y} f|X) = \mathcal{N}(f|0, k(X,X))$

$X \in \mathbb{R}^{N\times D}$

$p(y|f) = \mathcal{N}(y|f, \sigma_n^2 I)$

$x_* \quad f_* \quad \begin{bmatrix} y\\ f_*\end{bmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} k(X,X)+\sigma^2 I & k(X,x_*) \\ k(x_*,X) & k(x_*,x_*)\end{bmatrix}\right)$

$f_*|y \sim \mathcal{N}(f_* | k(x_*,X)(k(X,X)+\sigma^2 I)^{-1},$
$k(x_*,x_*) - k(x_*,X)(k(X,X)+\sigma^2 I)^{-1}k(X,x_*))$

ковариационная матрица

$k(x_1,x_2) = \sigma_f^2 \exp\left(-\sum_{j=1}^{D} \frac{(x_{1j}-x_{2j})^2}{l^2}\right)$

метод максимизации маргинального правдоподобия.

$\ln p(y|X) = \ln \int p(y|f)p(f|X)\,df =$

$= \ln \mathcal{N}(y|0, k(X,X)+\sigma_n^2 I) \to \max_{\sigma_f, l, \sigma}$ $\qquad O(n^3)$

$p(y=1|f) = \sigma(f)$

идея – inducing inputs

$X \in \mathbb{R}^{N\times D}, \quad y \in \mathbb{R}^N$

$Z \in \mathbb{R}^{M\times D}, \quad u \in \mathbb{R}^M$

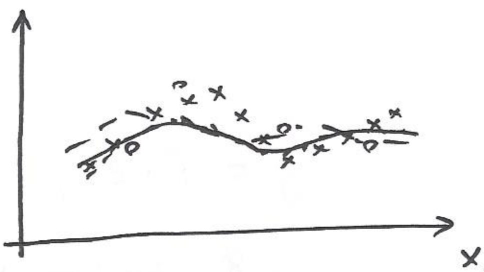
$p(y,f,u|X,Z) = p(y|f)\,\underbrace{p(f|u,X,Z)}_{=}\,\overbrace{p(u|Z)}^{=\ \mathcal{N}(u|0,k(Z,Z))}$

$\mathcal{N}(f|k(X,Z)k^{-1}(Z,Z)u,\ k(X,X) - k(X,Z)k^{-1}(Z,Z)k(Z,X))$

$$q(u,f) = p(f|u,x,z)\,\mathcal{N}(u|\mu,\Sigma) = q(f)q(u)$$

$$\ln p(y) \geq \mathbb{E}_{q(u,f)} \ln \frac{p(y|f)\cancel{p(f|u,x,z)}\,p(u|z)}{\cancel{p(f|u,x,z)}\,q(u)} \; (=)$$

$X \to Z,\ N \to M$



$$q(u,f) \approx p(u,f|y)$$

$$\parallel$$

$$p(f|u,y)p(u|y) = p(f|u)p(u|y) \approx p(f|u)q(u)$$

$$(=)\ \mathbb{E}_{q(f)} \ln p(y|f)\cancel{p(u|z)} - KL(q(u)\,\|\,p(u|z)) \; (=)$$

$$q(f) = \int p(f|u,x,z)\,\mathcal{N}(u|\mu,\Sigma)\,du =$$

$$= \mathcal{N}(f\,|\,k(x,z)k^{-1}(z,z)\mu,\ k(x,x) + k(x,z)k^{-1}(z,z)\cdot$$

$$\cdot(\Sigma - k(z,z))k^{-1}(z,z)k(z,x)) = \mathcal{N}(f|m,s)$$

$$p(y|f) = \prod_{n=1}^{N} p(y_n|f_n)$$

$$(=) \sum_{n=1}^{N} \mathbb{E}_{q(f_n)} \ln p(y_n|f_n) - KL(q(u)\,\|\,p(u))$$

$$\ln p(y) \geq \sum_{n=1}^{N} (\ln \mathcal{N}(y_n\,|\,k(x_n,z)k^{-1}(z,z)\mu,\ \sigma_n^2) -$$

$$- \frac{1}{2\sigma_n^2}\tilde{k}_{nn} - \frac{1}{2} tr(\Sigma\Lambda_n)) - \frac{1}{2}(\ln \frac{|k(z,z)|}{|\Sigma|} - m +$$

$$+ tr(k^{-1}(z,z)\Sigma) + \mu^T k^{-1}(z,z)\mu) \to \max_{\sigma_f, l, \sigma_n, \mu, \Sigma} \quad O(nm^2)$$

$$\Lambda_n = \frac{1}{\sigma_n^2} k^{-1}(z,z)k^T(x_n,z)k(x_n,z)k^{-1}(z,z) \qquad \Sigma = \underset{\Delta}{L}L^T$$

$$\hat{k} = k(x,x) - k(x,z)k^{-1}(z,z)k(z,x)$$

$$\frac{\partial}{\partial\mu}: \ \mu = \frac{\frac{1}{\sigma_n^2}\sum_{i=1}^{N} y_i k^T(x_i,z)}{1 + \frac{1}{\sigma_n^2}\sum_{i=1}^{N} k(x_i,z)k^{-1}(z,z)k(x_i,z)}$$

$$\frac{\partial}{\partial \Sigma}: -\frac{1}{2}\sum_{n=1}^{N} \frac{\partial}{\partial\Sigma} \mathrm{tr}(\Sigma \Lambda_n) + \frac{1}{2} \frac{\partial}{\partial\Sigma}\ln|\Sigma| -$$

$$-\frac{1}{2} \frac{\partial}{\partial\Sigma} \mathrm{tr}(K^{-1}(z,z)\Sigma) = -\frac{1}{2}\sum_{n=1}^{N} \Lambda_n^T + \frac{1}{2} \Sigma^{-1} - \frac{1}{2} K^{-1}(z,z) = 0$$

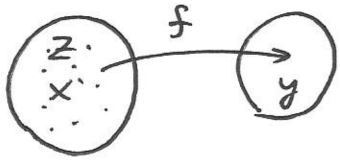
$$\Sigma^{-1} = K^{-1}(z,z) + \sum_{n=1}^{N} \Lambda_n$$

$$\ln p(y) \geq -\frac{1}{2}(\cancel{N}\ln 2\pi + \ln|B| + y^T B^{-1} y + \frac{1}{\sigma_n^2}\mathrm{tr}(\tilde{K})) \to \max_{\sigma_f,\sigma_n,\sigma_d}$$

$$B = \sigma_n^2 I + k(x,z)k^{-1}(z,z)k(z,x)$$

$$\tilde{K} = k(x,x) - k(x,z)k^{-1}(z,z)k(z,x) \qquad O(nm^2)$$

GP - LVM



$y \in \mathbb{R}^{N\times D}$, $X \in \mathbb{R}^{N\times Q}$, $Q << D$

$y_{id} = f_d(x_i) + \varepsilon$, $u_d \in \mathbb{R}^{M}$, $d = 1 \dots D$

$p(x) = \prod_{i=1}^{n} N(x_i|0, I_Q)$, $p(y_d|f_d) = N(y_d|f_d, \sigma_n^2 I)$

~~$p(f_d|x) = N($~~ $p(u_d|z) = N(u_d|0, k(z,z))$

$p(f_d|u_d, x, z) = N(f_d | k(x,z)k^{-1}(z,z)u_d,$
$k(x,x) - k(x,z)k^{-1}(z,z)k(z,x))$

PCA Pr.

$p(y|w,x) = N(y|Xw, \sigma_n^2 I)$, $p(x) = \prod_{i=1}^{n} N(x_i|0, I_Q)$

$k(x_1, x_2) = x_1^T x_2$

$q(\{f_d, u_d\}_{d=1}^{D}, x) = \prod_{d=1}^{D}(p(f_d|u_d, x, z)\,\Phi(u_d))\,q(x)$

$q(x) = \prod_{i=1}^{n} N(x_i|\mu_i, \Sigma_i)$

$$\ln p(y) \geq \mathbb{E}_q \ln \frac{\prod_{d=1}^{D} p(y_d|f_d)\,\cancel{p(f_d|u_d, x, z)}\,p(u_d|z)}{\prod_{d=1}^{D} \cancel{p(f_d|u_d}, x, z)\,\Phi(u_d)} -$$

$$-KL(q(x)\|p(x)) = \sum_{d=1}^{D} \int_{u,f,x} p(f_d|u_d, x, z)\,\Phi(u_d)\,q(x)\cdot$$

$$\ln \frac{p(y_d|f_d)\cdot p(u_d|z)}{\Phi(u_d)} - KL$$