25.10.19 ИО III

Самосогласованные ф-ии.

Self - Concordant.

1. $f(x): \mathbb{R} \to \mathbb{R}$ $\quad \exists f''', \ f''>0$

$\forall x \in \mathbb{R} \quad |f'''(x)| \le 2(f''(x))^{3/2}$

$\{f(x) = -\ln(x)\}$

2. $f(x): \mathbb{R}^n \to \mathbb{R}$ $\quad \left|\frac{d}{dt} g''(t)^{-\frac{1}{2}}\right| \le 1$

$\forall x, h \quad f(x+th)$ - с.с. {экв-е условие}

$\quad = g(t)$ {скалярная ф-ия}

Св-ва:

1. f_1, f_2 с.с. $\Rightarrow f_1 + f_2$ с.с.

$$f_1''' + f_2''' \le 2(f_1''^{3/2} + f_2''^{3/2}) \le 2(f_1'' + f_2'')^{3/2}$$

2. f с.с. $\Rightarrow f(Ax+b)$ с.с.

$$f(A(x+th)+b) = f(\underbrace{Ax+b}_{y}+th) = f(\underbrace{y}_{\text{с.с.}}+th)$$

{по св-ам $f(x,y) = x^2 - y - \ln(x+y)$ с.с.}

$$-1 \le \frac{d}{dt} g''(t)^{-\frac{1}{2}} \le 1 \quad \Big| \int_0^t$$

$$-t \le g''(t)^{-\frac{1}{2}} - g''(0)^{-\frac{1}{2}} \le t$$

$$t + g''(0)^{-\frac{1}{2}} = \frac{1 + t g''(0)^{1/2}}{g''(0)^{1/2}} ; \quad \frac{1 - t g''(0)^{1/2}}{g''(0)^{1/2}}$$

$$\boxed{\frac{g''(0)}{(1 + t g''(0)^{1/2})^2} \leq g''(t) \leq \frac{g''(0)}{(1 - t g''(0)^{1/2})^2} \quad (*)}$$

Ньютоновское напр-е

$$\Delta x_N = - \nabla^2 f(x)^{-1} \nabla f(x)$$

$$\nabla f(x)^T \Delta x_N = -\lambda^2$$

$$\Delta x_N^T \nabla^2 f(x) \Delta x_N = \lambda^2$$

$$x^+ = x + T \Delta x_N$$

$$(*) \int_0^t : \quad g'(t) - g'(0) \leq \int_0^t \frac{g''(0)}{(1 - s g''(0)^{1/2})^2} ds =$$

$$= \frac{g''(0)^{1/2}}{1 - s g''(0)^{1/2}} \Bigg|_0^t = \frac{g''(0)^{1/2}}{1 - t g''(0)^{1/2}} - g''(0)^{1/2}$$

$$g'(t) \leq g'(0) - g''(0)^{1/2} + \frac{g''(0)^{1/2}}{1 - t g''(0)^{1/2}} \quad \Big| \int_0^t$$

$(**)$

$$\boxed{g(t) - g(0) \leq (g'(0) - g''(0)^{1/2}) t - \ln(1 - t g''(0)^{1/2})}$$

$$\int_0^{t} \frac{g''(0)^{1/2}}{1 - s g''(0)^{1/2}} ds = - \ln(1 - s g''(0)^{1/2}) \Big|_0^t =$$

$$= - \ln(1 - t g''(0)^{1/2})$$

$$g(t) = f(x+th)$$

$$g'(0) = \nabla f(x)^T \Delta x_N \, , \quad g''(0) = \Delta x_N^T \nabla^2 f(x) \Delta x_N$$

$$g'(0) = -\lambda^2 \, , \quad g''(0) = \lambda^2 \quad \longrightarrow (**)$$

$$g(t) - g(0) \le -\lambda^2 t - \lambda t - \ln(1 - t\lambda) = \left\{ \overset{t}{\ddot{t}} = \frac{1}{1+\lambda} \right\} =$$

$$= -\frac{\lambda^2}{1+\lambda} - \frac{\lambda}{1+\lambda} - \ln\Big(\underbrace{1 - \frac{\lambda}{1+\lambda}}_{= \frac{1}{1+\lambda}}\Big) =$$

$$= -\frac{\lambda(1+\lambda)}{\cancel{1+\lambda}} + \ln(1+\lambda) = -\lambda + \ln(1+\lambda) \le$$

$$\le -\frac{1}{2} \frac{\lambda^2}{1+\lambda} = -\frac{1}{2} t \lambda^2$$

$$\left(-\lambda + \ln(1+\lambda)\right)' \le \left(-\frac{1}{2} \frac{\lambda^2}{1+\lambda}\right)' \quad \forall \lambda \ge 0$$

{док-во нер-ва через производную} $\qquad \lambda = 0 : \ \equiv$

$$-1 + \frac{1}{1+\lambda} \ \vee \ -\frac{1}{2} \frac{2\lambda + 2\lambda^2 - \lambda^2}{(1+\lambda)^2}$$

$$\frac{1}{1+\lambda} \ \vee \ 1 - \frac{1}{2} \frac{(2+\lambda)\lambda}{(1+\lambda)^2}$$

$$1 \ \vee \ (1+\lambda) - \frac{1}{2} \frac{(2+\lambda)\lambda}{1+\lambda}$$

$$0 \ \vee \ \lambda - \frac{1}{2} \frac{(1+\lambda)\lambda + \lambda}{1+\lambda}$$

$$0 \ \vee \ \lambda - \frac{1}{2}\lambda - \frac{1}{2} \frac{\lambda}{1+\lambda}$$

$$0 \ \vee \ \frac{1}{2}\lambda - \frac{1}{2} \frac{\lambda}{1+\lambda}$$

$$\frac{1}{1+\lambda} \ \vee \ 1 \, , \quad 1 \ \vee \ 1+\lambda \, , \quad 0 \ \overset{\le}{\vee} \ \lambda \quad \blacksquare$$

$$g(t) - g(0) \leq -\frac{1}{2} \frac{\lambda^2}{1+\lambda} = -\frac{\tau}{2}\lambda^2$$

Оценка скорости сх-ти метода Ньютона, не зависящая от геометрии ф-ии (констант Липшица L, сильной вып. μ).

Дальше необходимо получить оценку опт-ти: $f^* - f(x) \geq \dots$

$$g'(t) - g'(0) \geq \int_0^t \frac{g''(0)}{1+s\, g''(0)^{1/2}} ds =$$

$$= -\frac{g''(0)^{1/2}}{1+s f''(0)^{1/2}} \Big|_0^t = -\frac{g''(0)^{1/2}}{1+t g''(0)^{1/2}} + g''(0)^{1/2}$$

$$g(t) - g(0) \geq g'(0)t + g''(0)^{1/2} t - \int_0^t \frac{g''(0)^{1/2}}{1+s\, g''(0)^{1/2}} ds =$$

$$= \boxed{g'(0)t + g''(0)^{1/2} t - \ln(1 + g''(0)^{1/2} t)} \qquad \overset{\shortparallel}{\ln(1+s g''(0)^{1/2})\Big|_0^t}$$

$$f(x+th) - f(x) \geq t \nabla f(x)^T h + t (h^T \nabla^2 f(x) h)^{1/2} -$$
$$- \ln(1 + t(h^T \nabla^2 f(x) h)^{1/2}) \quad \Big| \min_{th=v}$$

$$\{ u(x) \geq v(x) \ \forall x \Leftarrow\Rightarrow \min u(x) \geq \min v(x) \}$$

$$f^* - f(x) \geq \cancel{\min t \nabla f(x)^T h + t(h^T \nabla^2 f(x) h)}$$
$$\min_v \{ \nabla f(x)^T v + (v^T \nabla^2 f(x) v)^{1/2} - \ln(1 + (v^T \nabla^2 f(x) v)^{1/2})$$

$$\max_{v} \frac{\nabla f^T v}{(v^T \nabla^2 f v)^{\frac{1}{2}}} = \max_{z} \frac{\nabla f^T A^{1/2} z}{\| z \|} =$$

$\left\{ \begin{array}{l} \nabla^2 f = A \\ v = A^{1/2} z \end{array} \right\}$ максимум достигается на коллинеарных векторах

$$\{ z = A^{1/2} \nabla f \}$$

$$= \frac{\nabla f^T A \nabla f}{\| A^{1/2} \nabla f \|} = \{ A^{1/2} \nabla f = (\nabla f^T A \nabla f)^{1/2} \} =$$

$$= (\nabla f^T A \nabla f)^{1/2} = (\nabla f^T \nabla^2 f \nabla f)^{1/2} =$$

$$= (\Delta x_N^T \nabla^2 f(x) \Delta x_N)^{1/2} = \lambda$$

Тогда $\max\limits_{v} \dfrac{\nabla f^T v}{(v^T \nabla^2 f v)^{\frac{1}{2}}} = \lambda < 1$

и $|(v^T \nabla^2 f(x) v)^{\frac{1}{2}}| \geq |\nabla f(x)^T v|$

Бесконечно возрастающая ф-ия,

компакт

$\exists$ компакт с конечными значениями ф-ии,

по т. Вейерштрасса $\exists$ локальный минимум

и он достижим: $\boxed{f(x) \to +\infty \text{ при } \|x\| \to +\infty}$

$f(x^+) - f(x) \leq -\frac{\tau}{2} \lambda^2$

{условие на сх-ть}

λ^2 обязано $\to 0$, тогда и $\lambda \to 0$

и $\exists$ итер. $\lambda < 1$