22. 02. 19 нейробайес II

Стохастические градиенты

$f(x)$	stoch grad
$\sum\limits_{i=1}^{n} f_i(x)$	$n \dfrac{\partial f_j(x)}{\partial x}, \quad j = \mathcal{U}\{1 \dots n\}$
$\int p(y) g(x,y) dy$	$\dfrac{\partial}{\partial x} \int p(y) g(x,y) dy = \int p(y) \dfrac{\partial}{\partial x} g(x,y) dy \approx \dfrac{\partial}{\partial x} g(x, \hat{y})$, $\hat{y} \sim p(y)$ $\dfrac{\partial}{\partial x} g(x, \hat{y})$
$\int p(y) \sum\limits_{i=1}^{n} g_i(x,y) dy$	$n \dfrac{\partial g_j(x, \hat{y})}{\partial x}, \quad \hat{y} \sim p(y), \; j \sim \mathcal{U}\{1 \dots n\}$

$\int p(y\|x) g(x,y) dy$
 // log-der
 // trick
	$\left\{ \dfrac{\partial}{\partial x} g(x, \hat{y}) \right.$? $\hat{y} \sim p(y\|x)$
	 $\dfrac{\partial}{\partial x} \int p(y\|x) g(x,y) dy = \int \dfrac{\partial}{\partial x} p(y\|x) g(x,y) dy =$
	 $= \int \dfrac{\partial}{\partial x} p(y\|x) \, g(x,y) dy + \int p(y\|x) \dfrac{\partial}{\partial x} g(x,y) dy =$
	 $= \int p(y\|x) \dfrac{\partial \log p(y\|x)}{\partial x} g(x,y) dy + \int p(y\|x) \dfrac{\partial}{\partial x} g(x,y) dy \approx$
	 $\approx \dfrac{\partial \log p(\hat{y}\|x)}{\partial x} g(x, \hat{y}) + \dfrac{\partial g(x, \hat{y})}{\partial x}$
	 $\hat{y} \sim p(y\|x)$

$$p(x, \theta) = p(x|\theta) p(\theta) = \prod_{i=1}^{n} p(x_i|\theta) p(\theta), \quad \theta \in \mathbb{R}^d, \quad \begin{matrix} n \gg 1 \\ d \gg 1 \end{matrix}$$

$$X = (x_1 \dots x_n), \quad p(\theta|X) \approx q(\theta|\varphi) = \arg\min_{\varphi} KL(q(\theta|\varphi) \| p(\theta|X))$$

сопряжений нет $\qquad = \arg\max_{\varphi} \mathcal{L}(\varphi)$

$$\mathcal{L}(\varphi) = \int q(\theta|\varphi) \log \frac{p(x,\theta)}{q(\theta|\varphi)} \, d\theta$$

$$\frac{\partial}{\partial\varphi} \mathcal{L}(\varphi) = \frac{\partial}{\partial\varphi} \int q(\theta|\varphi) [\log p(x|\theta) + \log p(\theta) -$$

$$- \log q(\theta|\varphi)] \, d\theta = \int q(\theta|\varphi) \frac{\partial \log q(\theta|\varphi)}{\partial\varphi} \cdot$$

$$\left[\sum_{i=1}^{n} \log p(x_i|\theta) + \log \frac{p(\theta)}{q(\theta|\varphi)}\right] d\theta - \underbrace{\int \cancel{q(\theta|\varphi)} \frac{\partial \log q(\theta|\varphi)}{\partial\varphi} \, d\theta}_{\to 0} =$$

$$\simeq \frac{\partial \log q(\hat\theta|\varphi)}{\partial\varphi} \left[\sum_{i=1}^{n} \log p(x_i|\hat\theta) + \log \frac{p(\hat\theta)}{q(\hat\theta|\varphi)}\right] \simeq$$

$$\simeq \frac{\partial \log q(\hat\theta|\varphi)}{\partial\varphi} \left[n \log p(x_j|\hat\theta) + \log \frac{p(\hat\theta)}{q(\hat\theta|\varphi)}\right]$$

$$\hat\theta \sim q(\theta|\varphi), \quad j \sim U\{1..n\}$$

$\mathbb{E}_{p(x)} f(x)$ - ? Методы уменьшения дисперсии

$\mathbb{D} f(x)$, $\mathbb{E} g(x)$

$$\mathbb{E} f(x) = \mathbb{E}(f(x) + \lambda g(x)) \quad \forall \lambda$$

$$\mathbb{D}(f(x) + \lambda g(x)) = \mathbb{D} f(x) + \lambda^2 \mathbb{D} g(x) + 2\lambda \operatorname{Cov}(f,g) \to \min_\lambda$$

$$\lambda \mathbb{D} g = -\operatorname{Cov}(f,g), \quad \lambda^* = -\frac{\operatorname{Cov}(f,g)}{\mathbb{D} g}$$

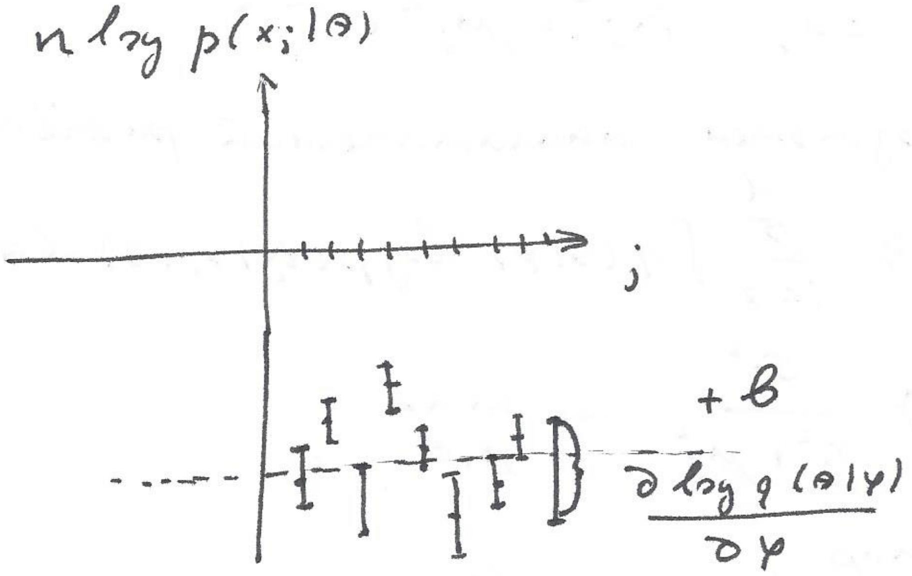
$$\mathbb{D}(f(x) + \lambda^* g(x)) = \mathbb{D} f - \frac{\operatorname{Cov}^2(f,g)}{\mathbb{D} g} = \boxed{\mathbb{D} f(1 - \rho^2)}$$

$$\operatorname{Cov}(f,g) = \sqrt{\mathbb{D} f}\sqrt{\mathbb{D} g} \, \underbrace{\operatorname{Corr}(f,g)}_{=\rho}$$

$$\mathbb{E} f(x) \approx f(\hat x) + \lambda^* g(\hat x)$$

$$\frac{\partial \log q(\theta|\varphi)}{\partial\varphi} \cdot \left[n \log p(x_j|\theta) + \log \frac{p(\theta)}{q(\theta|\varphi)} + \underbrace{b}_{\lambda^*}\right]$$

$$b \, \mathbb{E}_{q(\theta|\varphi)} \frac{\partial \log q(\theta|\varphi)}{\partial\varphi} = 0$$



Пусть $b = b(x_i)$: $\frac{\partial \log q(\theta|\varphi)}{\partial \varphi}\left[n \log p(x_i|\hat\theta) + \log \frac{p(\hat\theta)}{q(\hat\theta|\varphi)} + b(x_i,\varphi)\right]$

$$p(t,\theta|x) = p(t|x,\theta)\,p(\theta) = \frac{1}{1+\exp(-t\theta^T x)} \prod_{i=1}^{d} \mathcal{N}(\theta_i|0,\lambda_i^2)$$

$$\Lambda^* = \arg\max_{\Lambda} (T|X,\Lambda) = \arg\max_{\Lambda} \int \prod_{i=1}^{n} p(t_i|x_i,\theta)\,p(\theta|\Lambda)\,d\theta$$

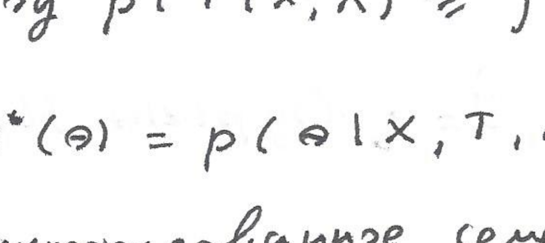
$$\log p(T|X,\Lambda) \geq \int q(\theta|\varphi) \log \frac{p(T|X,\theta)\,p(\theta|\Lambda)}{p(\theta|\varphi)}\,d\theta \to \max_{\Lambda,\varphi}$$

$$q^*(\theta) = p(\theta|X,T,\Lambda) \qquad q(\theta|\varphi) = \prod_{j=1}^{d} \mathcal{N}(\theta_j|\mu_j,\sigma_j^2)$$

факторизованные семейства: $\varphi = \{\mu_j, \sigma_j\}$

$$\log p(T|X,\Lambda) \geq \int q(\theta|\varphi) \log p(T|X,\theta)\,d\theta - \underbrace{\int q(\theta|\varphi) \log \frac{q(\theta|\varphi)}{p(\theta|\Lambda)}\,d\theta}_{KL(q(\theta|\varphi)\,\|\,p(\theta|\lambda))} =$$

$$= \mathbb{E}_{q(\theta|\varphi)} \log p(T|X,\theta) - KL(q\|p) =$$



$$= \sum_{i=1}^{n} \int q(\theta|\varphi) \log p(t_i|x_i,\theta)\,d\theta - \sum_{j=1}^{d} KL(\mathcal{N}(\theta_j|\mu_j,\sigma_j^2)\,\|\,\mathcal{N}(\theta_j|0,\lambda_j^2)) \to \max_{\mu,\sigma,\lambda}$$

$$KL(q\|p) = \log \frac{\lambda_j}{\sigma_j} + \frac{\sigma_j^2 + \mu_j^2}{2\lambda_j^2} \;\Big|\; \frac{\partial}{\partial \lambda_j}$$

$$\frac{1}{\lambda_j} - \frac{\sigma_j^2 + \mu_j^2}{\lambda_j^3} = 0, \quad \tilde{\lambda}_j^2 = \mu_j^2 + \sigma_j^2$$

возможен аналитический расчёт λ

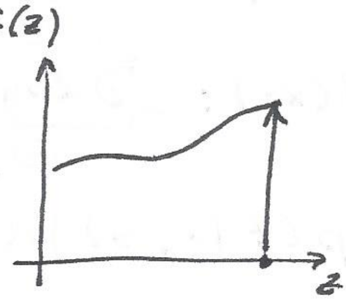
$$\log p(T \mid X, \Lambda) \geq \sum_{i=1}^{d} \int q(\theta \mid \varphi) \log p(t_i \mid x_i, \theta)\, d\theta +$$

$$+ \frac{1}{2} \sum_{j=1}^{d} \log \frac{\sigma_j^2}{\sigma_j^2 + \mu_j^2} \to \max_{\mu, \sigma},$$

семинар

алгоритм REINFORCE

$$\max_\theta \mathbb{E}_{p(z \mid \theta)} f(z)$$



$$\nabla_\theta \mathbb{E}_{p(z \mid \theta)} f(z) = \nabla_\theta \int_z p(z \mid \theta) f(z)\, dz =$$

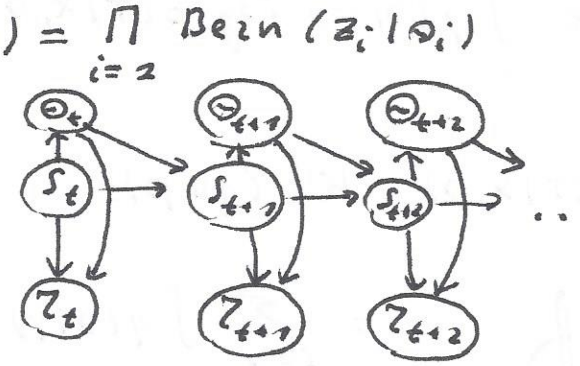
$$= \int_z \nabla_\theta p(z \mid \theta) f(z)\, dz = \int_z p(z \mid \theta) \nabla_\theta \log p(z \mid \theta) f(z)\, dz =$$

$$= \mathbb{E}_{p(z \mid \theta)} \nabla_\theta \log p(z \mid \theta) f(z), \quad \hat{g} = \nabla_\theta \log p(z \mid \theta) f(z)$$

1) $\max_{p(z) \in h(Z)} \mathbb{E}_{p(z)} f(z) \geq \max_\theta \mathbb{E}_{p(z \mid \theta)} f(z)$

$Z \in \{0, 1\}^d, \quad p(z \mid \theta) = \prod_{i=1}^{d} \mathrm{Bern}(z_i \mid \theta_i)$

2) 



$p(s_{t+1} \mid s_t, a_t), \; p(s_0)$

$\pi(a_t \mid s_t, \theta), \; r(a_t, s_t) \quad : \mathbb{E}_{\{s_t, a_t\}} \sum_{t=1}^{\infty} \gamma^t r(a_t, s_t) \to \max_\theta$

$$\hat{g} = \sum_{t=1}^{\infty} \nabla_\theta \log \pi(a_t \mid s_t, \theta)\, \gamma^t r(a_t, s_t)$$