Literature Review

Eeo Jun

1 Introduction & Definitions

Our project would be to implement an algorithm for edge colouring, and analyse how the output changes for different kinds of graphs - in particular random graphs. An extended goal would be to analyse colouring for graphs which have a small number of vertices with maximum degree. Some of the many applications of edge colouring are round-robin tournament scheduling[7], as well as link scheduling for TDMA network protocols.

An example of the link scheduling problem is as follows; we have 4 devices which are linked to some number of other devices. They want to communicate with one another, but if two devices communicate with the same one at the same time, their signals will interfere with one another. One solution to this is to allow communication between different pairs of devices every time period t, in such a way so two devices won't communicate with the same device in the same time period; after k of these periods, all devices would have communicated to one another.

If we represent this as a graph, with the vertices being the devices and the edges being $\{uv : \text{u can communicate with v}\}$, then the problem of minimising k is the same as that of computing the chromatic index of this graph.

We take definitions from Diestel[1] - a **graph** is a pair $G = (V, E)$ of vertices and edges, where $E \subseteq V^2$. In our study we only consider cases where $|V|$ and $|E|$ are finite. The **degree** of a vertex v, $d(v)$ is the number of edges at v. The **maximum degree** of a graph G, $\Delta(G)$ (or Δ where there is no ambiguity) is the maximum value of $d(v)$ over all vertices of G. An **edge colouring** of some graph $G = (V, E)$ is a mapping $\varphi : E \to \mathcal{C}$ where $\mathcal{C}$ is a set of colours, such that for all pairs of distinct edges $e, f \in E$, if e and f share a common vertex then $\varphi(e) \neq \varphi(f)$. The **chromatic index** of a graph, $\chi'(G)$ is the minimum number of colours required for a valid edge colouring of G.

A **Class 1** graph has a chromatic index of $\chi' = \Delta$, and a **Class 2** graph has a chromatic index of $\chi' = \Delta + 1$. If $Z \subseteq V$ is a set of nodes, then the **induced subgraph** $(Z, \{uv \in E(G) \,|\, u, v \in Z\})$ is denoted by $G[Z]$. The set of vertices of G which have degree Δ is denoted by Λ_G. The **core** and **semi-core**[2] of a graph G is denoted by $G[\Lambda_G]$, and $G[N_G(\Lambda_G))]$, respectively, where $N_G(Z) = \{v : uv \in E(G),\, u \in Z\} \cup Z$; i.e. the subgraph induced by the verticles of maximum degree and their neighbours.

A graph is **complete** if $\forall u, v \in V$, $u \neq v$, $uv \in E$. A complete graph with n nodes is denoted K_n. A **bipartite graph** is a graph $G = (V, E)$ such that V

can be partitioned into two sets, A and B such that $A \cap B = \emptyset$, and there is no edge uv in E such that $u, v \in A$ or $u, v \in B$.

2 Theorems

The foundation of edge colouring is Vizing's Theorem[4], which establishes a tight lower and upper bound on the chromatic index of any (simple) graph. Vizing's Theorem states that $\forall G$, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. As mentioned previously, graphs which only need Δ colours are in Class 1, and those that need $\Delta + 1$ in Class 2. It was shown that determining whether a graph is in Class 1 or Class 2 is NP-Complete[5].

Some graph classes do have polynomial time algorithms to compute a proper and optimal edge colouring (and by extension, determine if the graph is in Class 1/2). For instance, Behzad et al[6] used a simple construction with addition of edges modulo n to show that the chromatic index for K_n is $n - 1$ if n is even, and n if n is odd and $n \geq 3$ (the case where $n = 1$ is trivial). For bipartite graphs, there is Kőnig's Theorem[3]: *The chromatic index of any bipartite graph is equal to its maximum degree;* i.e. every bipartite graph is Class 1.

In proofs of both Konig's and Vizing's theorem, a common technique used is the path switching method. It is used when an edge cannot be coloured using one of the free colours of its endpoints. (The *free* colours at some node u refer to the set $\mathcal{C} \setminus \{\gamma : uv \in E(G), \varphi(uv) = \gamma\}$).

Given two colours $\alpha, \beta \in \mathcal{C}$ ($\mathcal{C}$ being the set of colours), we use $G_\varphi[\alpha, \beta]$ to denote the subgraph of G induced by only the edges which have been coloured α and β by φ. Every component of $G_\varphi[\alpha, \beta]$ is either an even-cycle or a path. By properly selecting α and β and exchanging the colours along some component (usually a path) in $G_\varphi[\alpha, \beta]$, we free up one of previously forbidden α or β to be used at a vertex while maintaining validity of the colouring.

The algorithm to be implemented as part of our project will be Misra and Gries's algorithm[9] that uses colouring fans. An alternative algorithm exists that isn't based on colouring fans, but rather counting arguments was discovered by Ehrenfeucht et al[10], which might be implemented as part of our project.

3 Results for other Graph Classes

Even though Misra and Gries's algorithm will yield *good enough* results - since the upper bound on colours used is at worst, $\Delta + 1$; there exists specialised algorithms to decide whether graphs are in Class 1/2 for certain graph classes.

The problem of deciding whether a graph is in Class 1 or 2- known as the Graph Classification Problem still has unknown complexity for many well-studied graph classes[12]- the most surprising being co-graphs (graphs which are P_4-free). Among a series of papers to investigate this problem, Vizing[16] proved that every graph in Class 2 has at least 3 vertices of maximum degree.

Erdős and Wilson[11] proved that *almost every simple random graph is Class 1*. If $\mathcal{G}(\mathrm{n}, \mathrm{p})$ is the probability space consisting of graphs of n vertices, and each possible edge is included independently with probability p, then almost every graph in $\mathcal{G}(\mathrm{n}, \frac{1}{2})$ has 1 unique vertex of degree Δ and is in Class 1. In particular, as $n \to \infty$, then $P(n) = U_n/V_n \to 1$, where U_n is the number of class 1 graphs and V_n is the total number of graphs with n vertices.

It is worth noting that a method to determine whether a graph is Class 1/2 is by looking at structural properties of the core and semi-core of the graph. For example, Fournier[8] proved that if the core $G[\Lambda_G]$ of a graph has no cycles, then G is Class 1 - this is consistent with Vizing's findings, since two vertices cannot produce a cycle.

Another result from Machado and de Figueiredo[2] builds on the previous result and shows that the chromatic index for a graph is equal to the chromatic index of it's semi-core. These results would benefit our project when we investigate graphs where $|\Lambda_G| \leq k$ - we should bound $k \geq 3$, and we know that if there is a cycle induced by Λ_G then G may be in Class 2. Another nice consequence of this is that we only have to investigate connected graphs.

A related result on the core of the graph was proven recently by Cranston and Rabern[20]. The Hilton-Zhao conjecture is as follows: if G is connected with Λ_G having maximum degree 2, then $\chi'(G) > \Delta$ iff G is overfull. A graph G is *overfull* when $|E(G)| > \Delta\lfloor|V(G)|/2\rfloor$. They proved that it holds true for $\Delta = 4$. Furthermore, any graph that is overfull is necessarily in Class 2; thus it suffices for our research to look at graphs which are not overfull. An interesting branch of research in this direction is to consider the case for when $\Delta = 5$ and we still have the same constraints on the core of the graph.

For planar graphs with $\Delta \geq 8$, it is shown by Chrobrak et al[13, 14] that the colours needed is Δ, and we can compute such a colouring in polynomial time - $O(n \log n)$ if $\Delta \geq 9$, and $O(n)$ if $\Delta \geq 19$.

For the well known family of partial k-trees, which are subgraphs of k-trees, there is a polynomial-time algorithm by Zhou et al[15] to compute the minimal edge colouring for fixed k. A k-tree is defined recursively as follows[15]:

1. K_k is a k-tree.
2. A k-tree with $n+1$ vertices ($n \geq k$) can be constructed from a k-tree H with n vertices by adding a vertex adjacent to exactly k vertices that form a k-clique in H.
3. No other graphs are k-trees.

A partial k-tree is a subgraph of a k-tree.

4 Random Graph Generation

There are a variety of random graph models - the most commonly studied model being the Erdős–Rényi model[17], also known as the binomial graphs model –

$\mathcal{G}(\mathrm{n},\mathrm{p})$ is the probability space consisting of graphs of n vertices, and each possible edge is included independently with probability p.

Another model is the Barabási–Albert Preferential Attachment model[18], which has 3 parameters: n, m, m_0 where $n \geq m_0 \geq m$. A graph is generated as follows: we start with K_{m_0}, and then we add $n - m_0$ nodes one by one as follows: each new node being added is attached to m random existing nodes. The probability that the new node is connected to node i is $p_i = d(i)/\sum_j d(j)$. This accurately simulates the growth of many real-life networks, such as citation networks and the graph of hyperlinks on the internet.

There are also some algorithms[19] for generating k-regular graphs - graphs which vertices all have degree k, but there seems to be very few if not no results on generating graphs with certain forbidden subgraphs or graphs that have certain induced subgraph(s).

5 Conclusions

As a start, we would implement the algorithm by Misra and Gries[9]. Then we would test the colours needed on a number of random graphs generated using the binomial graph model $\mathcal{G}(\mathrm{n},\mathrm{p})$ - maybe parameterized over different values of both n and p. When computing an edge colouring for graphs, as mentioned before Machado et al's results[2] help us greatly since we need only colour the semi-core of the graph instead of the full graph – this would speed up the time needed for those tests. In particular, recreating Erdős and Wilson's result[11] empirally would be interesting.

One area of interest is in graphs where $|\Lambda_G| \leq k$, where k is small compared to $|V|$. Another is where $\Delta = 5$ and $G[\Lambda_G]$ have maximum degree 2. We would randomly generate such graphs and check if any of them need $\Delta + 1$ colours by the algorithm; we would investigate their structure further should any such instances appear. Note that since we are computing the colouring using an algorithm that makes no guarantees about optimality, we cannot state any rigorous results (e.g. that such graphs are definitely in Class 1/2), however it would maybe serve as a good exploratory study in this field.

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