(Actual) Shortest Paths

Consider the following:

$$\mathrm{SP} \equiv \mathrm{AddZero}(\infty, (\mathbb{R}^+, \min, +) \overrightarrow{\times} (2^{2^E}, \cup, \hat{\cup}))$$

we combine paths by adding their weights and unioning their edge sets (thus, second component can be seen as set of paths). We choose by using min; if both have the same weight, we combine the path sets.

ref	S	$\oplus$	$\otimes$	$\bar{0}$	$\bar{1}$
(1)	$\mathbb{R}^+$	min	$+$		0
(2)	2^{2^E}	$\cup$	$\hat{\cup}$	$\varnothing$	$\{\varnothing\}$
(3)	$\mathbb{R}^+ \times 2^{2^E}$	$\min \overrightarrow{\times} \cup$	$+ \times \hat{\cup}$		$(0, \{\varnothing\})$
SP	$\{\infty\} \uplus (\mathbb{R}^+ \times 2^{2^E})$	$(\min \overrightarrow{\times} \cup)^{\mathrm{id}}_{\infty}$	$(+ \overrightarrow{\times} \hat{\cup})^{\mathrm{an}}_{\infty}$	$\mathrm{inl}(\infty)$	$\mathrm{inr}((0, \{\varnothing\}))$

We show that SP is a semiring.

First, We show that (2) is $\mathbb{LD}$ ($\mathbb{RD}$ is similar):

$$\begin{aligned}(A \,\hat{\cup}\, B) \cup (A \,\hat{\cup}\, C) &= \{a \cup b : a \in A, b \in B\} \cup \{a \cup c : a \in A, c \in C\} \\ &= \{a \cup x : a \in A, x \in B \cup C\} \\ &= A \,\hat{\cup}\, (B \cup C)\end{aligned}$$

$\mathbb{LD}$ for (3) requires:

$$\mathbb{LD}((1) \overrightarrow{\times} (2)) \equiv \underbrace{\mathbb{LD}((1))}_{\text{yes}} \wedge \underbrace{\mathbb{LD}((2))}_{\text{yes}} \wedge (\underbrace{\mathbb{LC}(\mathbb{R}^+, +)}_{\text{yes!}} \vee \mathbb{LK}(2^{2^E}, \hat{\cup}))$$

$\mathbb{LD}$ for SP follows, since AddZero preserves $\mathbb{LD}$.

$\mathbb{RD}$ is similar. Thus, SP is both $\mathbb{LD}$ and $\mathbb{RD}$.

We need $\oplus$ to be commutative:

$$\mathbb{CM}((\mathbb{R}^+, \min) \overrightarrow{\times} (2^{2^E}, \cup)) \equiv \underbrace{\mathbb{CM}(2^{2^E}, \cup)}_{\text{yes!}}$$

We have a zero by construction, annihilating $\otimes$. We show that $\mathrm{inr}((0, \{\varnothing\}))$ is a multiplicative ID. Consider (left operand inl case, as well as the case with operands switched around is trivial):

$$\mathrm{inr}((a, B)) \otimes \mathrm{inr}((0, \{\varnothing\})) = \mathrm{inr}((a + 0, B \,\hat{\cup}\, \{\varnothing\})) = \mathrm{inr}((a, B)) \;\Box$$

It follows that SP is a semiring.

SP adjacency matrix, A for graph $G = (V, E)$ with weights $w : E \mapsto \mathbb{R}^+$:

$$A_{ij} = \begin{cases} \mathrm{inr}((w(ij), \{\{ij\}\})) & \text{if } ij \in E \\ \mathrm{inl}(\infty) & \text{otherwise} \end{cases}$$