Relativistic Vortices

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July 2021

1 Abstract

In Lam's paper in 2021, vortices within non-relativistic quantum fluid have been investigated. In this short report, we try to expand it into relativistic case. Our result shows that relativistic vortices do exist. Relativistic quantum fluid has to been represented in Klein-Gordon equation and boosted by Lorentz transformation instead of Schrodinger-Poisson equation and Galilean transformation.

2 Basics

2.1 Classical Quantum Fluid

Consider Klein-Gordon equation:

$$(\partial^{\nu}\partial_{\nu} + m^2)\phi = (\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2)\phi = 0 \tag{1}$$

$$\phi(\vec{x}, t) = \frac{1}{2m}(\psi e^{-imt} + \psi^* e^{imt}) \tag{2}$$

$$\tag{3}$$

with non-relativistic limit: $|\ddot{\psi}| << m|\psi|$, we can get schrodinger-Poisson equation:

$$i\partial_t = \frac{-1}{2m}\nabla^2\psi + m\Phi\psi \tag{4}$$

$$\nabla^2\Phi = \frac{\rho}{2M_{PI}^2} \tag{5}$$

$$\tag{6}$$

By Madelung's transformation, we can represent the schrodinger equation into classical quantum fluid:

$$\rho = m|\psi|^2 \tag{7}$$

$$\vec{v} = \frac{1}{m}\vec{\nabla}\theta = \frac{1}{m|\psi|^2} Im[\psi^*\nabla\psi] \tag{8}$$

$$\tag{9}$$

Applying this transformation into Schrodinger-Poisson equation, we can get the continuity equation and Euler equation, which correspond to imaginary and real part respectively.

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \tag{10}$$

$$\partial_t \vec{v} + (\vec{v} \cdot) \vec{v} = -\vec{\nabla} \Phi + \frac{1}{2m^2} \vec{\nabla} (\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}) \tag{11}$$

$$\tag{12}$$

With the equations, we can predict the evolution of the classical quantum fluid. In the fluid, there are some places with velocity ill-defined, which are called vortices. At those places the circulation is not equal to zero, however, the density is equal to zero. From Lam's paper, we know that the density around vortices $|\psi|^2\ x^2$, and the velocity of vortices $\frac{1}{mR}$.

2.2 Klein-Gordon equation and Relativistic Quantum Fluid

In relativistic case, since $|\ddot{\psi}|$ no longer $<< m|\psi|$, we cannot apply Schrodinger-Poisson equation but Klein-Gordon equation. Similar to non-relativistic case, we can represent the Klein-Gordon equation into relativistic quantum fluid:

$$Im[\phi^* (\partial_t^2 - \nabla^2 + m^2) \phi] = 0 \tag{13}$$

$$\rightarrow \phi^* \partial_t^2 \phi - \phi \partial_t^2 \phi^* - \phi^* \nabla^2 \phi + \phi \nabla^2 \phi^* \tag{14}$$

$$= \partial_t (\phi^* \partial_t \phi - \phi \partial_t \phi^*) + \nabla \cdot (\phi \nabla \phi^* - \phi^* \nabla \phi) = 0 \tag{15}$$

$$\tag{16}$$

If we define:

$$\rho_{KG} = \frac{i}{2m} (\phi^* \partial_t \phi - \phi \partial_t \phi^*) \tag{17}$$

$$\vec{j_{KG}} = \frac{1}{2mi} (\phi^* \nabla \phi - \phi \nabla \phi^*) \tag{18}$$

$$\tag{19}$$

We can get continuity equation:

$$\partial_t \rho_{KG} + \nabla \cdot \vec{j_{KG}} = 0 \tag{20}$$

Thus we can also represent and evolve relativistic quantum fluid by the continuity and Euler equations.

2.3 Lorentz transformation

Lorentz transformation are 4x4 matrices, which can be written as $L(\vec{theta}, \vec{w}) = exp(\vec{v} \cdot \vec{J} + \vec{w} \cdot \vec{K})$. $\vec{J}$ are rotational matrices and $\vec{K}$ are boosting matrices.

	Classical Quantum Fluid	Relativistic Quantum Fluid
Eqaution	Scrodinger Poisson Equation	Klein-Gordon Equation
Definition of density/velocity	$\rho = \lvert\psi\rvert^2$. $\vec{v} = \frac{i}{m} Im[\psi^* \nabla^2 \psi]$	$\rho = \frac{E}{m}\lvert\psi\rvert^2, E = \sqrt{m^2 + p^2}$ $, \vec{v} = \frac{i}{m} Im[\psi^* \nabla^2 \psi]$
Coordinate transformation	Galilean Transformation	Lorentz transformation

Figure 1: This graph summarize the equations used in relativistic and non-relativistic quantum fluid.

We can write Lorentz transformation of $K_i, i = 1, 2, 3$ into one-dimensional representation, take i=1 for example:

$$L(w^1) = exp(w^1 K_1), K_1 = x^0 \partial_1 - x^1 \partial_0 \tag{21}$$

$$R = +\sqrt{(x^0)^2 - (x^1)^2} (x^0 \geq 0) \tag{22}$$

$$R = -\sqrt{(x^0)^2 - (x^1)^2} (x^0 < 0) \tag{23}$$

$$tanh\Omega = \frac{x^1}{x^0} \tag{24}$$

$$\rightarrow L(w^1) f(\Omega) = exp(w^1 \frac{\partial}{\partial \Omega}) f(\Omega) = f(\Omega + w^1) \tag{25}$$

$$\tag{26}$$

Since both $\partial^\nu \partial_\nu$ and m^2 are scalar, which are invariant under Lorentz transformation. A solution to Klein-Gordon equation will remain to be a solution after Lorentz transformation. This means that different observers will see different relativistic scalar fields, however, all scalar fields follows the evolution predicted by Klein-Gordon equation. The scalar fields can be transformed to each other by Lorentz transformation.

3 Relativistic Vortices

Since quantum fluid can also be represented in relativistic scenarios, we can ask whether there are relativistic vortices? Recall that the definition of vortices are places where circulation around them are not equal to zero.

3.1 Example

We consider a simple wave function: $\phi = (x + iy)e^{imt}$. Circulation at t=0 around (x,y)=(0,0) is $\frac{2\pi}{m} \neq 0$, which is a vortex. We find that it is a solution to both Klein-Gordon equation and schrodinger equation, which means that it is a vortex in both relativistic and non-relativistic case. For different observers in non-relativistic case, we can do Galilean transformation. The wave function after the transformation would satisfy schrodinger equation. On the other hand, for observers in relativistic case, we have to do Lorentz transformation to get a new wave function, which satisfies Klein-Gordon equation but isn't a solution to schrodinger equation. The wave functions are different between Galilean and Lorentz transformation. It is easy to understand that non-relativistic case is an approximation to relativistic case. They will be the same to the observers, which are static relative to the wave function ($\phi = (x + iy)e^{imt}$).

3.2 Lorentz Transformation

3.2.1 Lorentz boost to z axis

$$L(w^3)\phi = e^{w^3 K_3}(x + iy)e^{-imt} = e^{(w^3 \frac{\partial}{\partial\Omega})}(x + iy)e^{-imRcosh(\Omega)} \tag{27}$$

$$= (x + iy)e^{-imRcosh(\Omega+w^3)} = (x + iy)e^{i(pz-Et)} \tag{28}$$

$$t = Rcosh(\Omega), z = Rsinh(\Omega), E = \sqrt{p^2 + m^2}, p = \gamma mv, v = \frac{sinh(w^3)}{cosh(w^3)} \tag{29}$$

$$\tag{30}$$

3.2.2 Lorentz boost to x axis

$$L(w^1)\phi = e^{w^1 K_1}(x + iy)e^{-imt} \tag{31}$$

$$= e^{(w^1 \frac{\partial}{\partial\Omega})}(Rsinh(\Omega) + iy)e^{-imRcosh(\Omega)} \tag{32}$$

$$= (Rsinh(\Omega + w^1) + iy)e^{-imRcosh(\Omega+w^1)} \tag{33}$$

$$= [Rsinh(\Omega)cosh(w^1) + Rcosh(\Omega)sinh(w^1) + iy] \tag{34}$$

$$e^{-im[Rcosh(\Omega)cosh(w^1)+Rsinh(\Omega)sinh(w^1)]} \tag{35}$$

$$= (\frac{x}{\sqrt{1 - v^2}} + \frac{vt}{\sqrt{1 - v^2}} + iy)e^{i(px-Et)} \tag{36}$$

$$t = Rcosh(\Omega), x = Rsinh(\Omega), E = \sqrt{p^2 + m^2}, p = \gamma mv, v = \frac{sinh(w^1)}{cosh(w^1)} \tag{37}$$

$$\tag{38}$$

We can see that both the wave function after the two Lorentz transformations satisfy Klein-Gordon equation, but not solutions to schrodinger equation.

3.2.3 Circulation

Consider static case: $\phi = (x + iy)e^{imt}$.

$$\vec{v} = \frac{i}{m} Im[\phi^* \nabla\phi] \tag{39}$$

$$= \frac{1}{m}(\frac{-y}{x^2 + y^2}\hat{e_x} + \frac{x}{x^2 + y^2}\hat{e_y}) \tag{40}$$

$$\tag{41}$$

Integrate throw the contour: $r(\theta) = cos(\theta)\hat{e_x} + sin(\theta)\hat{e_y}$

$$\rightarrow \int_0^{2\pi} \vec{v} \cdot d\vec{l} = \int_0^{2\pi} \vec{v}(\theta) \cdot \vec{r'}(\theta) d\theta \tag{42}$$

$$= \int_0^{2\pi} \frac{1}{m}(-sin(\theta)\hat{e_x} + cos(\theta)\hat{e_y}) \cdot (-sin(\theta)\hat{e_x} + cos(\theta)\hat{e_y}) d\theta = \frac{2\pi}{m} \tag{43}$$

$$\tag{44}$$

The circulation around (x,y)=(0,0) is not equal to zero, which means that there is a vortex. Since the circulation is a scalar, it remain the same after Lorentz transformation. Observers at different coordinates would see the relativistic vortex. Although it is a really simple vortex, we can do Lorentz transformation (include rotation and boost) on it to make complicate vortices