Scattering Kernels in 3D

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

Cornette-Shanks

[Cornette and Shanks 1992] - *Physically reasonable analytic expression for the single-scattering phase function.*
Independently proposed [Liu and Weng 2006]

In[∘]:=
```
pCornetteShanks[u_, g_] := 3/(8 Pi) ((1 - g^2) (1 + u^2))/((2 + g^2) (1 + g^2 - 2 g u)^(3/2))
```

$$\text{pCornetteShanks}[u\_, g\_] := \frac{3}{8\,\text{Pi}} \frac{\left(1-g^2\right)\left(1+u^2\right)}{\left(2+g^2\right)\left(1+g^2-2\,g\,u\right)^{3/2}}$$

Normalization condition

In[∘]:=
```
Integrate[2 Pi pCornetteShanks[u, g], {u, -1, 1}, Assumptions → -1 < g < 1]
```

Out[∘]= 1

Mean-cosine

In[∘]:=
```
Integrate[2 Pi pCornetteShanks[u, g] u, {u, -1, 1}, Assumptions → -1 < g < 1]
```

Out[∘]=
$$\frac{3\,g\left(4+g^2\right)}{5\left(2+g^2\right)}$$

Legendre expansion coefficients

In[∘]:=
```
Integrate[
 2 Pi (2 k + 1) pCornetteShanks[Cos[y], g] LegendreP[k, Cos[y]] Sin[y] /. k → 0,
 {y, 0, Pi}, Assumptions → -1 < g < 1]
```

Out[∘]= 1

In[∘]:=
```
Integrate[
 2 Pi (2 k + 1) pCornetteShanks[Cos[y], g] LegendreP[k, Cos[y]] Sin[y] /. k → 1,
 {y, 0, Pi}, Assumptions → -1 < g < 1]
```

Out[∘]=
$$\frac{9\,g\left(4+g^2\right)}{5\left(2+g^2\right)}$$

In[∘]:=
```
Integrate[
 2 Pi (2 k + 1) pCornetteShanks[Cos[y], g] LegendreP[k, Cos[y]] Sin[y] /. k → 2,
 {y, 0, Pi}, Assumptions → -1 < g < 1]
```

Out[∘]=
$$\frac{7 + 80\, g^2 + 18\, g^4}{14 + 7\, g^2}$$

In[∘]:=
```
Integrate[
 2 Pi (2 k + 1) pCornetteShanks[Cos[y], g] LegendreP[k, Cos[y]] Sin[y] /. k → 3,
 {y, 0, Pi}, Assumptions → -1 < g < 1]
```

Out[∘]=
$$\frac{g\,\left(27 + 238\, g^2 + 50\, g^4\right)}{15\,\left(2 + g^2\right)}$$

sampling

In[∘]:=
```
cdf = Integrate[2 Pi pCornetteShanks[u, g],
  {u, -1, x}, Assumptions → -1 < g < 1 && 0 < x < 1]
```

Out[∘]=
$$\frac{1}{4\, g^3\,\left(2 + g^2\right)\,\sqrt{1 + g^2 - 2\, g\, x}}\left(2 - 2\, g^6 - 2\, g\, x - 2\,\sqrt{1 + g^2 - 2\, g\, x} + 4\, g^3\,\sqrt{1 + g^2 - 2\, g\, x} + g^4\,\left(-5 + x^2\right) + 2\, g^5\,\left(x + \sqrt{1 + g^2 - 2\, g\, x}\right) - g^2\,\left(-5 + x^2 + 4\,\sqrt{1 + g^2 - 2\, g\, x}\right)\right)$$

This CDF can be inverted by solving a quartic equation and simplifying:

In[∘]:=
```
sampleCornetteShanksSimplified[xi_, g_] :=
 Module[{T1, T1a, T2, T3, T4, T5, T6, T7},
```

$$T1a = 1 + 2\, g^2 - 2\, g^3 - g^5 + 4\, g^3\, xi + 2\, g^5\, xi;$$

$$T1 = (T1a)^2;$$

$$T2 = -4\,\left(-144\, g^2 + 288\, g^4 - 144\, g^6\right)^3;$$

$$T4 = 432\,\left(-1 + g^4\right)^3 - 1296\,\left(1 - g^2\right)\,\left(-1 + g^4\right)\,\left(-1 - 5\, g^2 + 5\, g^4 + g^6\right);$$

$$T3 = \left(T4 + 1728\,\left(1 - g^2\right)\, T1\right)^2;$$

$$T7 = \left(1728\,\left(1 - g^2\right)\, T1 + \sqrt{T2 + T3} + T4\right)^{1/3};$$

$$T5 = \frac{48 \times 2^{1/3}\,\left(-g^2 + 2\, g^4 - g^6\right)}{\left(1 - g^2\right)\, T7};$$

$$T6 = \frac{2\,\left(-1 + g^4\right)}{1 - g^2} + T5 + \frac{T7}{3 \times 2^{1/3}\,\left(1 - g^2\right)};$$

$$\frac{1 + g^2 - \frac{1}{4}\left(\sqrt{6 + 6\, g^2 + T6} - \sqrt{\frac{-6 + 6\, g^4 - \frac{16\, T1a}{\sqrt{4 + \frac{4\, g^2\,(-4 + T5)}{T5} + T5}} + T6 - g^2\, T6}{-1 + g^2}}\right)^2}{2\, g}$$

```
 ]
```

```
With[{g = .6},
 Show[
  Histogram[Table[sampleCornetteShanksSimplified[RandomReal[], g],
    {i, 1, 100 000}], 100, "PDF", ScalingFunctions → "Log"],
  LogPlot[2 Pi pCornetteShanks[u, g], {u, -1, 1}, PlotRange → All]
 ]
]
```

Out[11060]=