Bessel K NDF

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

notation

$u = \mathbf{m} . \mathbf{n} = \mathrm{Cos}[\theta_m]$

α = roughness

Definitions and derivations

In[2745]:=
```
BesselK`D[u_, α_, a_] :=
```
$$\frac{2\left(1-u^2\right)^{-\frac{1}{2}+\frac{a}{2}}(u\,\alpha)^{-a}\,\mathrm{BesselK}\left[1-a,\frac{2\sqrt{1-u^2}}{u\,\alpha}\right]}{\pi\,u^3\,\alpha\,\mathrm{Gamma}[a]}\ \mathrm{HeavisideTheta}[u]$$

derivation

```
Beckmann`D[u_, α_] :=
```
$$\frac{e^{-\frac{-1+\frac{1}{u^2}}{\alpha^2}}}{\alpha^2\,\pi\,u^4}\ \mathrm{HeavisideTheta}[u]$$

In[1489]:=
```
Integrate[Beckmann`D[u, α √m] × PDF[GammaDistribution[a, 1]][m],
 {m, 0, Infinity}, Assumptions → 0 < u < 1 && α > 0 && a > 0]
```

Out[1489]=
$$\frac{2\left(1-u^2\right)^{-\frac{1}{2}+\frac{a}{2}}(u\,\alpha)^{-a}\,\mathrm{BesselK}\left[1-a,\frac{2\sqrt{1-u^2}}{u\,\alpha}\right]}{\pi\,u^3\,\alpha\,\mathrm{Gamma}[a]}$$

In[2527]:=
```
FullSimplify[
 Integrate[Beckmann`D[u, α/√2 m] × PDF[ChiDistribution[2 a]][m], {m, 0, Infinity},
  Assumptions → 0 < u < 1 && α > 0 && a > 0], Assumptions → a > 0 && α > 0]
```

Out[2527]=
$$\frac{2\left(1-u^2\right)^{\frac{1}{2}(-1+a)}(u\,\alpha)^{-1-a}\,\mathrm{BesselK}\left[-1+a,\frac{2\sqrt{1-u^2}}{u\,\alpha}\right]}{\pi\,u^2\,\mathrm{Gamma}[a]}$$

shape invariant f(x)

In[1493]:=
$$\texttt{FullSimplify}\Big[\texttt{BesselK`D}[u, \alpha, a]\, u^4 \alpha^2 \;/.\; u \to \frac{1}{\sqrt{1 + x^2 \alpha^2}},\;\texttt{Assumptions} \to 1 - \frac{1}{\sqrt{1 + x^2 \alpha^2}} > 0 \;\&\&\; x > 0 \;\&\&\; \alpha > 0 \;\&\&\; a > 0\Big]$$

Out[1493]=
$$\frac{2\, x^{-1+a}\, \texttt{BesselK}[-1 + a, 2\, x]}{\pi\, \texttt{Gamma}[a]}$$

height field normalization

In[2549]:=
```
NIntegrate[2 Pi u BesselK`D[u, .6, 1.6], {u, 0, 1}]
```

Out[2549]= 1.

distribution of slopes

In[1494]:=
$$\texttt{FullSimplify}\Big[\texttt{BesselK`D}\Big[\frac{1}{\sqrt{p^2 + q^2 + 1}}, \alpha, a\Big] \left(\frac{1}{\sqrt{p^2 + q^2 + 1}}\right)^4,\;\texttt{Assumptions} \to 0 < \alpha < 1 \;\&\&\; p > 0 \;\&\&\; q > 0\Big]$$

Out[1494]=
$$\frac{2\, (p^2 + q^2)^{\frac{1}{2}(-1+a)}\, \alpha^{-1-a}\, \texttt{BesselK}\Big[-1 + a, \frac{2\sqrt{p^2+q^2}}{\alpha}\Big]}{\pi\, \texttt{Gamma}[a]}$$

In[1495]:=
$$\texttt{BesselK`P22}[p\_, q\_, \alpha\_, a\_] := \frac{2\, (p^2 + q^2)^{\frac{1}{2}(-1+a)}\, \alpha^{-1-a}\, \texttt{BesselK}\Big[-1 + a, \frac{2\sqrt{p^2+q^2}}{\alpha}\Big]}{\pi\, \texttt{Gamma}[a]}$$

In[2550]:=
$$\texttt{Integrate}[\texttt{BesselK`P22}[p, q, \alpha, a], \{p, -\texttt{Infinity}, \texttt{Infinity}\}, \{q, -\texttt{Infinity}, \texttt{Infinity}\}, \texttt{Assumptions} \to 0 < \alpha < 1 \;\&\&\; a > 0]$$

Out[2550]= 1