# Infinite 3D medium, Isotropic Point Source, Rayleigh Scattering

Chi-3 Random Flight

This is code to accompany the book:

A Hitchhiker’s Guide to Multiple Scattering



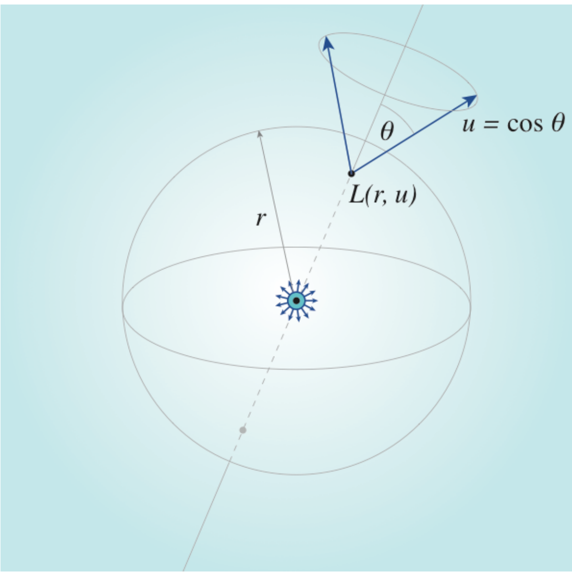
www.eugenedeon.com/hitchhikers

## Path Setup

Put a file at ~/.hitchhikerpath with the path to your hitchhiker repo so that these worksheets can find the MC data from the C++ simulations for verification

In[∘]:=
```
SetDirectory[Import["~/.hitchhikerpath"]]
```

## Notation




c - single-scattering albedo
r - radial position coordinate in medium (distance from point source at origin)
u = cos $\theta$ - direction cosine

### Namespace

In[3414]:=
```
Begin["inf3DisopointRayleighscatterChi3`"]
```
Out[3414]=
```
inf3DisopointRayleighscatterChi3`
```

# Analytical results

## Collision rate density

collision rate density Cc due to correlated emission:

### derivation

In[•]:=
```
Clear[cpc, c];
```
$$\text{cpc}[s\_] := c\,\frac{e^{-\frac{s^2}{4}}\,s^2}{2\sqrt{\pi}}$$

In[•]:=
```
f00 = Fpc[0, 0, cpc, u];
f10 = Fpc[1, 0, cpc, u];
f11 = Fpc[1, 1, cpc, u];
f20 = Fpc[2, 0, cpc, u];
f22 = Fpc[2, 2, cpc, u];
```

In[•]:=
```
o = 3;
Clear[A, b, c, r, h, F];
A[n_] := 0;
A[0] := 1;
A[1] := 0;
A[2] := 1/2;
hsystem =
  Table[h[k] == 2/Pi u F[k, 0] + Sum[A[m] × h[m] × F[k, m], {m, 0, o - 1}], {k, 0, o - 1}];
hsystemsolve = Simplify[
  Solve[hsystem, Table[h[i], {i, 0, o - 1}]] /. F[0, 0] → f00 /. F[0, 1] → -f10 /.
       F[1, 1] → f11 /. F[1, 0] → f10 /.
     F[2, 0] → f20 /. F[0, 2] → f20 /. F[2, 2] → f22]
```

Out[•]=

$$\Big\{\Big\{h[0] \to \Big(2\,c\,u\,\big(8\,u^2\,(9\,c - 4\,e^{u^2}\,u^4) + 18\,c\,e^{u^2}\sqrt{\pi}\,u\,(-1 + 2\,u^2)\,\mathrm{Erf}[u] - 9\,c\,e^{2u^2}\,\pi\,\mathrm{Erf}[u]^2\big)\Big) \Big/ \Big(\pi\,\Big(-4\,u^2\,\big(18\,c^2 + 8\,e^{2u^2}\,u^4 - 3\,c\,e^{u^2}\,(9 + 4\,u^2 + 4\,u^4)\big) + 6\,c\,e^{u^2}\sqrt{\pi}\,u\,\big(c\,(3 - 6\,u^2) + e^{u^2}\,(-9 + 2\,u^2)\big)\,\mathrm{Erf}[u] + 9\,c^2\,e^{2u^2}\,\pi\,\mathrm{Erf}[u]^2\Big)\Big),$$

$$h[1] \to \Big(4\,c\,e^{u^2}\,\Big(\big(1 + 2\,u^2\big)\,\mathrm{DawsonF}[u]\,\big(-16\,e^{u^2}\,u^5 + c\,(54\,u + 24\,u^3 + 8\,u^5) + 3\,c\,e^{u^2}\sqrt{\pi}\,(-9 + 2\,u^2)\,\mathrm{Erf}[u]\big) + u\,\Big(-2\,c\,u\,(27 + 12\,u^2 + 4\,u^4) - 3\,e^{u^2}\sqrt{\pi}\,\mathrm{Erf}[u]\,\big(c\,(-9 + 2\,u^2) + 2\sqrt{\pi}\,u^3\,F[1, 2]\big) + 4\,u^4\,\big(4\,e^{u^2}\,u + \sqrt{\pi}\,(3 + 2\,u^2)\,F[1, 2]\big)\Big)\Big)\Big) \Big/ \Big(\pi^{3/2}\,\Big(-4\,u^2\,\big(18\,c^2 + 8\,e^{2u^2}\,u^4 - 3\,c\,e^{u^2}\,(9 + 4\,u^2 + 4\,u^4)\big) + 6\,c\,e^{u^2}\sqrt{\pi}\,u\,\big(c\,(3 - 6\,u^2) + e^{u^2}\,(-9 + 2\,u^2)\big)\,\mathrm{Erf}[u] + 9\,c^2\,e^{2u^2}\,\pi\,\mathrm{Erf}[u]^2\Big)\Big),$$

$$h[2] \to -\Big(\Big(16\,c\,e^{u^2}\,u^4\,\big(-6\,u - 4\,u^3 + 3\,e^{u^2}\sqrt{\pi}\,\mathrm{Erf}[u]\big)\Big) \Big/ \Big(\pi\,\Big(-4\,u^2\,\big(18\,c^2 + 8\,e^{2u^2}\,u^4 - 3\,c\,e^{u^2}\,(9 + 4\,u^2 + 4\,u^4)\big) + 6\,c\,e^{u^2}\sqrt{\pi}\,u\,\big(c\,(3 - 6\,u^2) + e^{u^2}\,(-9 + 2\,u^2)\big)\,\mathrm{Erf}[u] + 9\,c^2\,e^{2u^2}\,\pi\,\mathrm{Erf}[u]^2\Big)\Big)\Big)\Big\}\Big\}$$

In[•]:=
```
Clear[r, c];
First[(2 k + 1) 1/(4 Pi c) (h[k]) u SphericalBesselJ[k, r u] /. k → 0 /. hsystemsolve //
  FullSimplify]
```

Out[•]=

$$\Big(u\,\Big(72\,c\,u^2 - 9\,c\,e^{2u^2}\,\pi\,\mathrm{Erf}[u]^2 - 2\,e^{u^2}\,u\,\big(16\,u^5 + 9\,c\sqrt{\pi}\,(1 - 2\,u^2)\,\mathrm{Erf}[u]\big)\Big)\,\mathrm{Sin}[r\,u]\Big) \Big/ \Big(2\,\pi^2\,r\,\Big(-72\,c^2\,u^2 - 6\,c\,e^{u^2}\,u\,\big(-2\,u\,(9 + 4\,(u^2 + u^4)) + 3\,c\sqrt{\pi}\,(-1 + 2\,u^2)\,\mathrm{Erf}[u]\big) + e^{2u^2}\,\big(-32\,u^6 + 6\,c\sqrt{\pi}\,u\,(-9 + 2\,u^2)\,\mathrm{Erf}[u] + 9\,c^2\,\pi\,\mathrm{Erf}[u]^2\big)\Big)\Big)$$

## result

In[3415]:=
```
Ccexact[r_, c_] := NIntegrate[
  (u (72 c u^2 - 9 c E^(2 u^2) π Erf[u]^2 - 2 E^(u^2) u (16 u^5 + 9 c Sqrt[π] (1 - 2 u^2) Erf[u])) Sin[r u]) /
   (2 π^2 r (-72 c^2 u^2 - 6 c E^(u^2) u (-2 u (9 + 4 (u^2 + u^4)) + 3 c Sqrt[π] (-1 + 2 u^2) Erf[u]) +
       E^(2 u^2) (-32 u^6 + 6 c Sqrt[π] u (-9 + 2 u^2) Erf[u] + 9 c^2 π Erf[u]^2))), {u, 0, Infinity}]
```

# load MC data

In[3482]:=
```
ppoints[xs_, dr_, maxx_] :=
  Table[{dr (i) - 0.5 dr, xs[[i]]}, {i, 1, Length[xs]}][[1 ;; -2]]
```

In[3483]:=
```
ppointsu[xs_, du_, Σt_] :=
  Table[{-1.0 + du (i) - 0.5 du, xs[[i]] / (2 Σt)}, {i, 1, Length[xs]}][[1 ;; -1]]
```

In[3484]:=
```
fs = FileNames["code/3D_medium/infinite3Dmedium/Isotropicpointsource/MCdata/
     inf3D_isotropicpoint_rayleighscatter_chi3_*"];
```

In[3485]:=
```
index[x_] := Module[{data, c},
    data = Import[x, "Table"];
    c = data[[2, 3]];
    {c, data}];
simulations = index /@ fs;
cs = Union[#[[1]] & /@ simulations]
```

Out[3487]=
```
{0.01, 0.1, 0.3, 0.5, 0.7, 0.8, 0.9, 0.95, 0.99, 0.999}
```

In[3488]:=
```
numcollorders = simulations[[1]][[-1]][[2, 13]];
```

# Compare analytic and MC

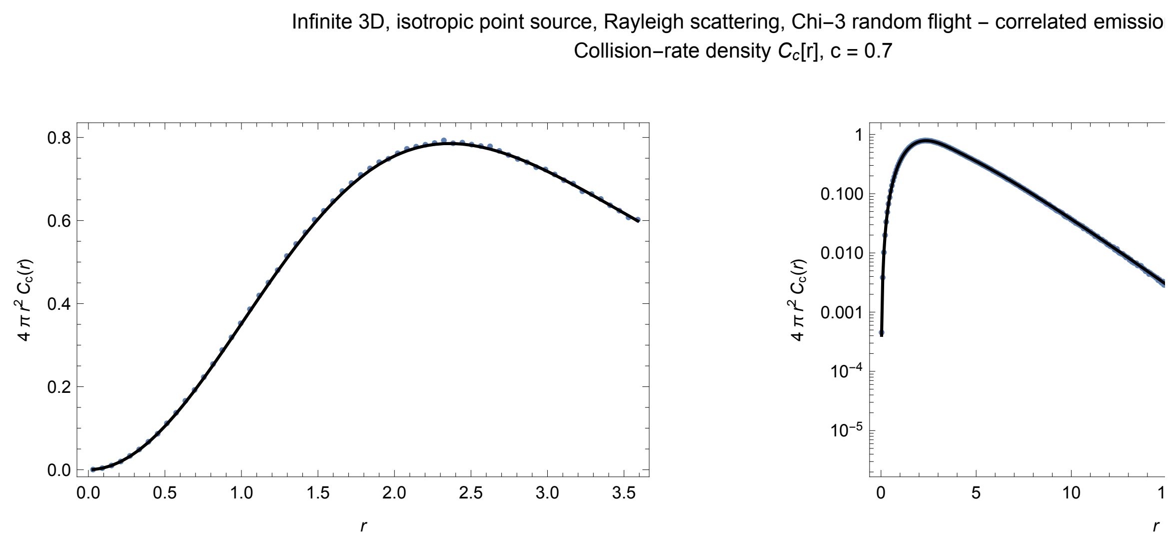
## Collision-rate density - Exact solution - comparison to MC

In[3423]:=
```
{ActionMenu["Set c", "c = " <> ToString[#] :> (c = #;) & /@ cs], Dynamic[c]}
```

Out[3423]=
```
{ Set c , 0.99}
```

In[3489]:=
```
data = SelectFirst[simulations, #[[1]] == c &][[2]];
maxr = data[[2, 5]];
dr = data[[2, 7]];
MCCollisionRate = ppoints[data[[4]], dr, maxr];
exact1CRShallow =
  Quiet[{#[[1]], 4 Pi #[[1]]^2 Ccexact[#[[1]], c]} & /@ MCCollisionRate[[1 ;; 60]];
exact1CR = Quiet[{#[[1]], 4 Pi #[[1]]^2 Ccexact[#[[1]], c]}] & /@
   MCCollisionRate[[61 ;; -1 ;; 10]];
plotϕshallow = Quiet[Show[
    ListPlot[MCCollisionRate[[1 ;; 60]],
     PlotRange → All, PlotStyle → PointSize[.01]],
    ListPlot[exact1CRShallow, PlotRange → All, Joined → True, PlotStyle → Black],
    Frame → True,
    FrameLabel -> {{4 π r^2 C"c"[r],}, {r,}}
   ]];
logplotϕ = Quiet[Show[
    ListLogPlot[MCCollisionRate, PlotRange → All, PlotStyle → PointSize[.01]],
    ListLogPlot[exact1CR, PlotRange → All, Joined → True, PlotStyle → Black],
    ListLogPlot[exact1CRShallow,
     PlotRange → All, Joined → True, PlotStyle → Black],
    Frame → True,
    FrameLabel -> {{4 π r^2 C"c"[r],}, {r,}}
   ]];
Show[GraphicsGrid[{{plotϕshallow, logplotϕ}}, ImageSize → 800],
 PlotLabel -> "Infinite 3D, isotropic point source,
    Rayleigh scattering, Chi-3 random flight - correlated
    emission\nCollision-rate density Cc[r], c = " <> ToString[c]]
```

Out[3497]=



# Moments

In[3498]:= $\text{pc}[s\_] := \frac{e^{-\frac{s^2}{4}} s^2}{2\sqrt{\pi}}$

## correlation emission

### collision rate

In[3499]:= $\text{m0Cc}[c\_] := \frac{1}{1-c}$

In[3500]:= $\text{m2Cc}[c\_, s\_, s2\_, g\_] := \frac{s2}{(1-c)^2}\left(1 + c\, g\, \frac{2\, s^2}{s2\,(1 - c\, g)}\right)$

### fluence

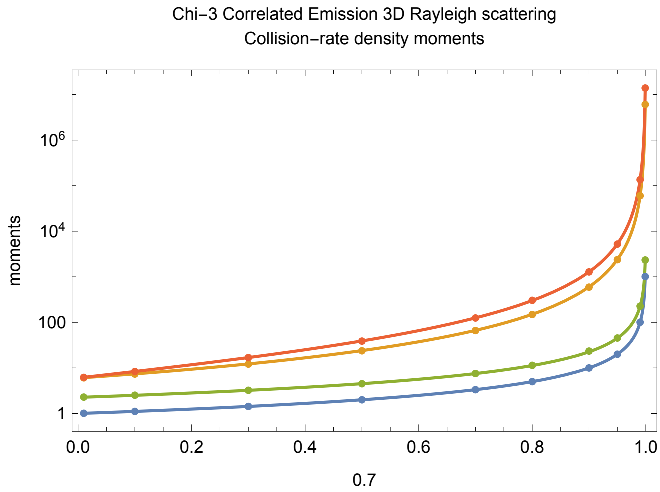
In[3511]:= $\text{m0}\phi\text{c}[c\_, s\_] := \frac{s}{1-c}$

In[3512]:= $\text{m2}\phi\text{c}[c\_, s\_, s2\_, s3\_, g\_] := \frac{\left(s3\,(1-c)\,(1-g\,c) + 3\,c\,s\left(2\,g\,c\,s^2 + s2\left(-2\,g\,c + g + 1\right)\right)\right)}{3\,(1-c)^2\,(1-c\,g)}$

```
In[3513]:= simsC = simulations;
```

In[3523]:=
```
m0Ccs = {#[[1]], #[[-1, 8, 1]]} & /@ simsC;
m2Ccs = {#[[1]], #[[-1, 8, 3]]} & /@ simsC;
m0ϕcs = {#[[1]], #[[-1, 10, 1]]} & /@ simsC;
m2ϕcs = {#[[1]], #[[-1, 10, 3]]} & /@ simsC;
mfp = Integrate[pc[s] s, {s, 0, Infinity}];
mfp2 = Integrate[pc[s] s s, {s, 0, Infinity}];
mfp3 = Integrate[pc[s] s s s, {s, 0, Infinity}];
g = 0;
Show[
 LogPlot[{m0Cc[c], m2Cc[c, mfp, mfp2, g], m0ϕc[c, mfp],
   m2ϕc[c, mfp, mfp2, mfp3, g]}, {c, 0.01, 0.999}, PlotRange → All],
 ListLogPlot[{m0Ccs, m2Ccs, m0ϕcs, m2ϕcs}, PlotRange → All],
 PlotRange → All, Frame → True,
 FrameLabel → {{"moments",}, {c, "Chi-3 Correlated Emission 3D Rayleigh
      scattering\nCollision-rate density moments"}}
]
```

Out[3531]=




# Namespace

In[3532]:=
```
End[]
```

Out[3532]=
```
inf3DisopointRayleighscatterChi3`
```