Beckmann NDF

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

notation

u = **m** . **n** = Cos[θ_m]

α = roughness

Definitions and derivations

$$\text{Beckmann`D}[u\_, \alpha\_] := \frac{e^{-\frac{-1+\frac{1}{u^2}}{\alpha^2}}}{\alpha^2\,\pi\,u^4}\,\text{HeavisideTheta}[u]$$

In[1149]:= $$\text{Beckmann`}\sigma[u\_, \alpha\_] := \frac{1}{2}\left(u\left(1+\text{Erf}\left[\frac{u}{\alpha\sqrt{1-u^2}}\right]\right)+\alpha\sqrt{1-u^2}\,\frac{E^{\frac{u^2}{\alpha^2\,(u^2-1)}}}{\sqrt{\text{Pi}}}\right)$$

In[2558]:= $$\text{Beckmann`}\Lambda[u\_, \alpha\_] := \frac{1}{2}\left(-1+\frac{e^{\frac{u^2}{(-1+u^2)\,\alpha^2}}\sqrt{1-u^2}\,\alpha}{\sqrt{\pi}\,u}+\text{Erf}\left[\frac{u}{\sqrt{1-u^2}\,\alpha}\right]\right)$$

In[604]:= $$\left(1+\text{Beckmann`}\Lambda[u, \alpha]\right)u == \text{Beckmann`}\sigma[u, \alpha]\ //\ \text{FullSimplify}$$

Out[604]= True

In[605]:= $$\left(\text{Beckmann`}\Lambda[u, \alpha]\right)u == \text{Beckmann`}\sigma[-u, \alpha]\ //\ \text{FullSimplify}$$

Out[605]= True

In[2643]:= $$\text{FullSimplify}\left[\text{Beckmann`}\Lambda\left[u, \frac{u}{\sqrt{1-u^2}\,x}\right], \text{Assumptions} \to 0 < u < 1\ \&\&\ x > 0\right]$$

Out[2643]= $$\frac{1}{2}\left(-1+\frac{e^{-x^2}}{\sqrt{\pi}\,x}+\text{Erf}[x]\right)$$

shape invariant f(x)

In[1231]:= $\text{FullSimplify}\left[\text{Beckmann`D}[u,\alpha]\, u^4\, \alpha^2 \text{ /. } u \text{ -> } \frac{1}{\sqrt{1+x^2\alpha^2}},\ \text{Assumptions} \to 1-\frac{1}{\sqrt{1+x^2\alpha^2}} > 0\right]$

Out[1231]= $\frac{e^{-x^2}}{\pi}$

height field normalization

In[606]:= $\text{Integrate}[2\, \text{Pi}\, u\, \text{Beckmann`D}[u,\alpha], \{u, 0, 1\}, \text{Assumptions} \to 0<\alpha<1]$

Out[606]= 1

distribution of slopes

In[607]:= $\text{FullSimplify}\left[\text{Beckmann`D}\left[\frac{1}{\sqrt{p^2+q^2+1}},\alpha\right]\left(\frac{1}{\sqrt{p^2+q^2+1}}\right)^4,\ \text{Assumptions} \to 0<\alpha<1\ \&\&\ p>0\ \&\&\ q>0\right]$

Out[607]= $\frac{e^{-\frac{p^2+q^2}{\alpha^2}}}{\pi\,\alpha^2}$

In[608]:= $\text{Beckmann`P22}[p\_, q\_, \alpha\_] := \frac{e^{-\frac{p^2+q^2}{\alpha^2}}}{\pi\,\alpha^2}$

In[609]:= $\text{Integrate}[\text{Beckmann`P22}[p, q, \alpha], \{p, -\text{Infinity}, \text{Infinity}\}, \{q, -\text{Infinity}, \text{Infinity}\}, \text{Assumptions} \to 0<\alpha<1]$

Out[609]= 1

In[610]:= $\text{Integrate}[\text{Beckmann`P22}[p, q, \alpha], \{q, -\text{Infinity}, \text{Infinity}\}, \text{Assumptions} \to \alpha>0\ \&\&\ \text{Im}[p] == 0]$

Out[610]= $\frac{e^{-\frac{p^2}{\alpha^2}}}{\sqrt{\pi}\,\alpha}$

In[613]:= $\text{Beckmann`P2}[p\_, \alpha\_] := \frac{e^{-\frac{p^2}{\alpha^2}}}{\sqrt{\pi}\,\alpha}$

derivation of Λ(u)

In[614]:=

$$\text{FullSimplify}\Big[\frac{\sqrt{1-u^2}}{u}\,\text{Integrate}\Big[\Big(q-\frac{u}{\sqrt{1-u^2}}\Big)\,\text{Beckmann`P2}[q,\alpha],\ \Big\{q,\ \frac{u}{\sqrt{1-u^2}},\ \text{Infinity}\Big\},\ \text{Assumptions}\to 0<u<1\ \&\&\ 0<\alpha<1\Big],\ \text{Assumptions}\to 0<u<1\ \&\&\ 0<\alpha<1\Big]$$

Out[614]=

$$\frac{1}{2}\left(-1+\frac{e^{\frac{u^2}{(-1+u^2)\,\alpha^2}}\sqrt{1-u^2}\;\alpha}{\sqrt{\pi}\;u}+\text{Erf}\Big[\frac{u}{\sqrt{1-u^2}\;\alpha}\Big]\right)$$

compare σ to delta integral:

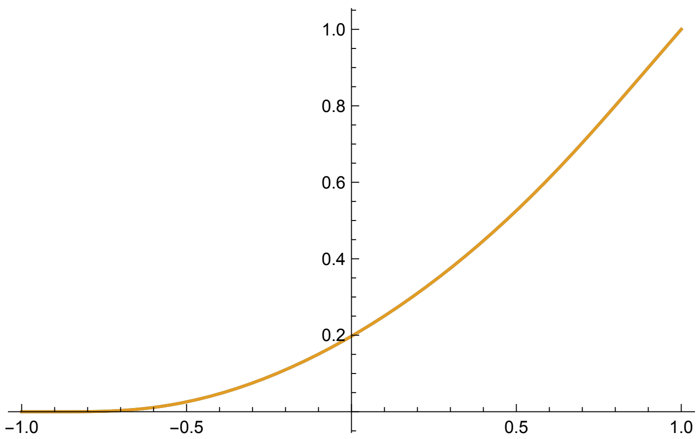
In[615]:=

$$\text{Delta`}\sigma[\text{u\_},\text{ui\_}] := \text{Re}\Big[2\left(\sqrt{1-u^2-ui^2}+u\,ui\,\text{ArcCos}\Big[-\frac{u\,ui}{\sqrt{1-u^2}\,\sqrt{1-ui^2}}\Big]\right)\Big]$$

In[616]:=

```
With[{α = .7},
 Plot[{
   Quiet[NIntegrate[Beckmann`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
   Quiet[Beckmann`σ[u, α]]
  }, {u, -1, 1}]
]
```

Out[616]=



importance sampling

In[4437]:=
```
With[{α = 1.3},
 Show[
  Histogram[
   Table[1/(√(1 - α² (Log[RandomReal[]]))), {i, Range[10 000]}], 200, "PDF"],
  Plot[Beckmann`D[u, α] 2 Pi u, {u, 0, 1}, PlotRange → All]
 ]
]
```

General: Exp[-1.41787 × 10^9] is too small to represent as a normalized machine number; precision may be lost.

Out[4437]=