Scattering Kernels in 3D

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com

Isotropic Scattering

```
pIsotropic[u_] := 1/(4 Pi)
```

Normalization condition

```
Integrate[2 Pi pIsotropic[u], {u, -1, 1}]
1
```

Mean-cosine

```
Integrate[2 Pi pIsotropic[u] u, {u, -1, 1}]
0
```

Legendre expansion coefficients

```
Integrate[
 2 Pi (2 k + 1) pIsotropic[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 0, {y, 0, Pi}]
1
```

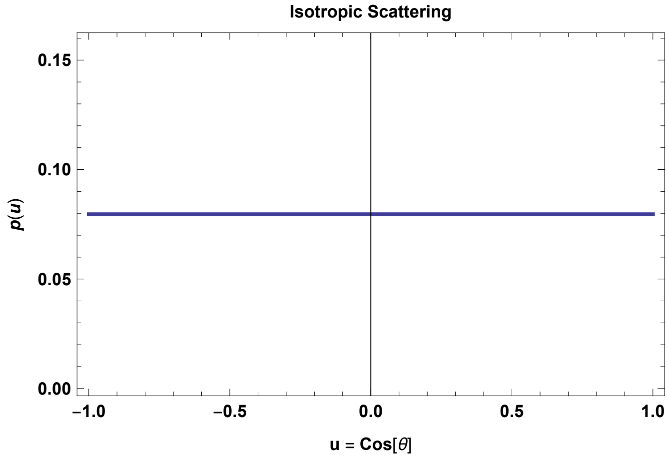
```
Integrate[
 2 Pi (2 k + 1) pIsotropic[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 1, {y, 0, Pi}]
0
```

sampling

```
cdf = Integrate[2 Pi pIsotropic[u], {u, -1, x}]
(1 + x)/2
```

```
Solve[cdf == e, x]
{{x → -1 + 2 e}}
```

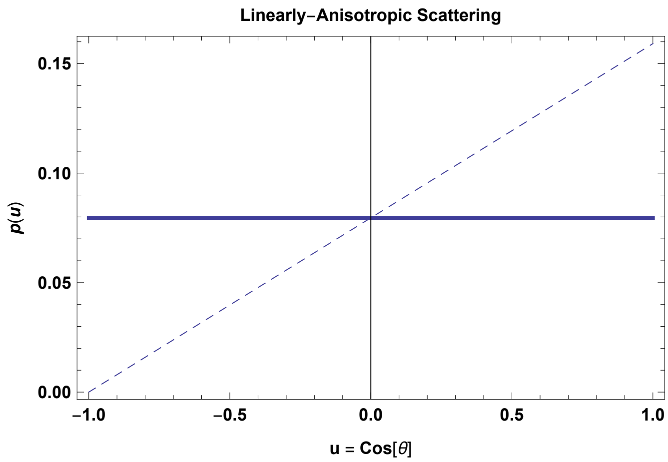
```
Clear[u]; Show[
 Plot[pIsotropic[u], {u, -1, 1}, PlotStyle → Thick]
 , Frame → True,
 FrameLabel → {{p[u],}, {"u = Cos[θ]", "Isotropic Scattering"}}]
```



Linearly-Anisotropic Scattering (Eddington)

$$\text{pLinaniso[u\_, b\_] := } \frac{1}{4\,\text{Pi}}\,(1 + b\,u)$$

```
Clear[u];
Show[
 Plot[pIsotropic[u], {u, -1, 1}, PlotStyle → Thick],
 Plot[pLinaniso[u, 1], {u, -1, 1}, PlotStyle → Dashed]
 , Frame → True,
 FrameLabel → {{p[u],}, {"u = Cos[θ]", "Linearly-Anisotropic Scattering"}}]
```



Normalization condition

```
Integrate[2 Pi pLinaniso[u, b], {u, -1, 1}, Assumptions → b > -1 && b < 1]
1
```

Mean cosine (g)

```
Integrate[2 Pi pLinaniso[u, b] u, {u, -1, 1}, Assumptions → b > -1 && b < 1]
```

$$\frac{b}{3}$$

Legendre expansion coefficients

```
Integrate[
 2 Pi (2 k + 1) pLinaniso[Cos[y], b] LegendreP[k, Cos[y]] Sin[y] /. k → 0, {y, 0, Pi}]
```

1

```
Integrate[
 2 Pi (2 k + 1) pLinaniso[Cos[y], b] LegendreP[k, Cos[y]] Sin[y] /. k → 1, {y, 0, Pi}]
```

b

sampling

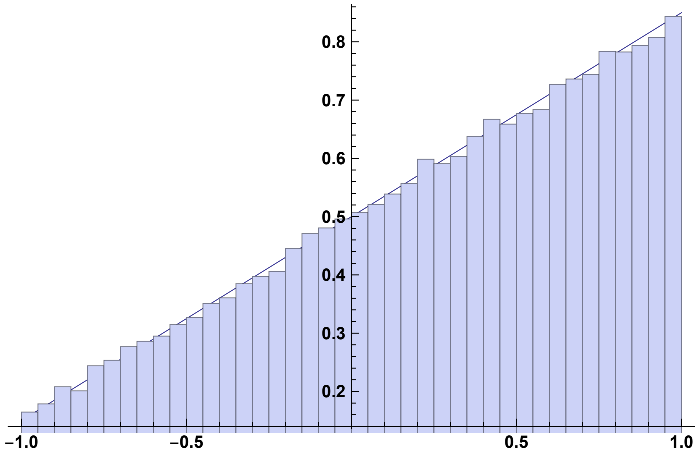
```
cdf = Integrate[2 Pi pLinaniso[u, b], {u, -1, x}]
```

$$\frac{1}{2} - \frac{b}{4} + \frac{x}{2} + \frac{b\,x^2}{4}$$

```
Solve[cdf == e, x]
```

$$\left\{\left\{x \to \frac{-1 - \sqrt{1 - 2\,b + b^2 + 4\,b\,e}}{b}\right\}, \left\{x \to \frac{-1 + \sqrt{1 - 2\,b + b^2 + 4\,b\,e}}{b}\right\}\right\}$$

```
b = 0.7;
Show[
 Plot[2 Pi pLinaniso[u, b], {u, -1, 1}],
 Histogram[
  Map[(-1 + Sqrt[1 - 2 b + b^2 + 4 b #])/b &, Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"]
]
Clear[b];
```



Rayleigh Scattering

General form:

```
pRayleigh[u_, γ_] := 1/(4 Pi) 3/(4 (1 + 2 γ)) ((1 + 3 γ) + (1 - γ) u^2)
```

$$\text{pRayleigh}[u\_, \gamma\_] := \frac{1}{4\,\text{Pi}} \frac{3}{4\,(1 + 2\,\gamma)} \left((1 + 3\,\gamma) + (1 - \gamma)\,u^2\right)$$

Common special case ($\gamma = 0$):

$$\text{pRayleigh}[u\_] := \left(1 + u^2\right) \frac{3}{16\,\text{Pi}}$$

Normalization condition

```
Integrate[2 Pi pRayleigh[u], {u, -1, 1}]
1

Integrate[2 Pi pRayleigh[u, y], {u, -1, 1}, Assumptions → y > 0] // Simplify
1
```

Mean cosine (g)

```
Integrate[2 Pi pRayleigh[u] u, {u, -1, 1}]
0

Integrate[2 Pi pRayleigh[u, y] u, {u, -1, 1}, Assumptions → y > 0] // Simplify
0
```

Legendre expansion coefficients

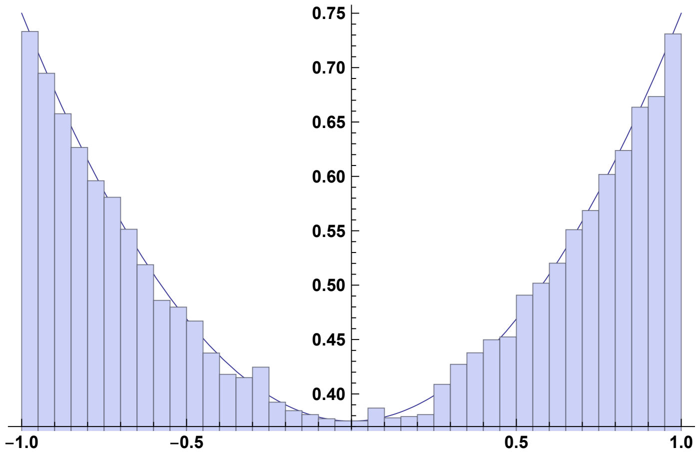
```
Integrate[
 2 Pi (2 k + 1) pRayleigh[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 0, {y, 0, Pi}]
1

Integrate[
 2 Pi (2 k + 1) pRayleigh[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 1, {y, 0, Pi}]
0

Integrate[
 2 Pi (2 k + 1) pRayleigh[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 2, {y, 0, Pi}]
```

$$\frac{1}{2}$$

sampling

```
Show[
 Plot[2 Pi pRayleigh[u], {u, -1, 1}],
 Histogram[Map[(1 - (2 - 4 # + Sqrt[5 + 16 (-1 + #) #])^(2/3)) / (2 - 4 # + Sqrt[5 + 16 (-1 + #) #])^(1/3) &,
   Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"]
]
Clear[b];
```

$$\text{Histogram}\left[\text{Map}\left[\frac{1-\left(2-4\,\#+\sqrt{5+16\,(-1+\#)\,\#}\right)^{2/3}}{\left(2-4\,\#+\sqrt{5+16\,(-1+\#)\,\#}\right)^{1/3}}\,\&,\right.\right.$$



Lambertian Sphere

geometrical optics far-field phase function of a white Lambertian sphere in 3D: [Esposito and Lumme 1977, Blinn 1982, Porco et al. 2008]

In[224]:= pLambertSphere[u_] := $\dfrac{2\left(\sqrt{1-u^2}-u\,\text{ArcCos}[u]\right)}{3\,\pi^2}$

MC testing

Normalization condition

In[ ]:= `Integrate[2 Pi pLambertSphere[u], {u, -1, 1}]`

Out[ ]= 1

forward scattering probability

In[226]:= `Clear[u]; Integrate[2 Pi pLambertSphere[u], {u, 0, 1}]`

Out[226]= $\frac{1}{6}$

Mean cosine (g)

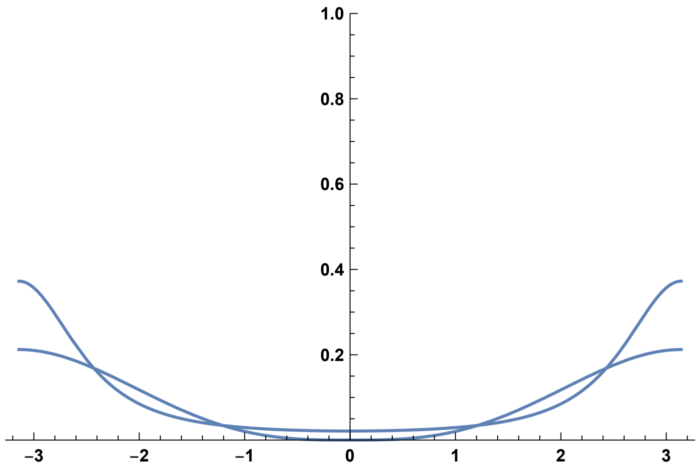
In[•]:=
```
Integrate[2 Pi pLambertSphere[u] u, {u, -1, 1}]
```
Out[•]= $-\frac{4}{9}$

Mean square cosine

In[227]:=
```
Integrate[2 Pi pLambertSphere[u] u^2, {u, -1, 1}]
```
Out[227]= $\frac{3}{8}$

This phase function is not particularly well approximated by Henyey Greenstein:

In[•]:=
```
Show[
 Plot[pHG[Cos[t], -4/9], {t, -Pi, Pi}, PlotRange → {0, 1}],
 Plot[pLambertSphere[Cos[t]], {t, -Pi, Pi}, PlotRange → All]
]
```
Out[•]=



Legendre expansion coefficients

In[•]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 0,
 {y, 0, Pi}]
```
Out[•]= 1

In[•]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 1,
 {y, 0, Pi}]
```
Out[•]= $-\frac{4}{3}$

In[•]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 2,
 {y, 0, Pi}]
```
Out[•]= $\frac{5}{16}$

In[ ]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 3,
  {y, 0, Pi}]
```
Out[ ]= 0

In[ ]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 4,
  {y, 0, Pi}]
```
Out[ ]= $\frac{1}{64}$

In[ ]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 6,
  {y, 0, Pi}]
```
Out[ ]= $\frac{13}{4096}$

In[ ]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 8,
  {y, 0, Pi}]
```
Out[ ]= $\frac{17}{16\,384}$

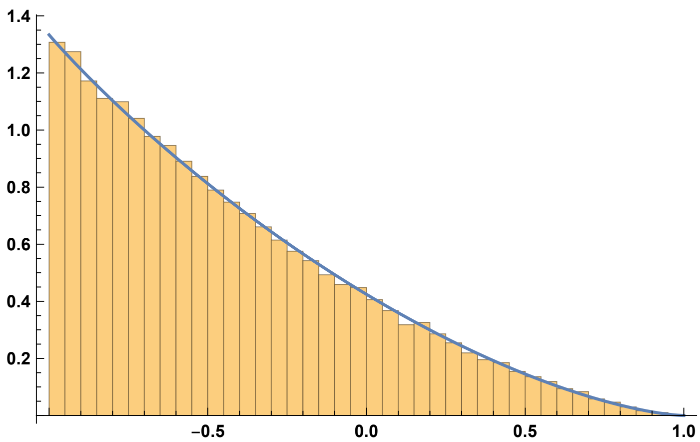
In[ ]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 10,
  {y, 0, Pi}]
```
Out[ ]= $\frac{343}{786\,432}$

Importance sampling:

The cosine of deflection can be sampled from:

In[3531]:=
```
Show[
 Histogram[Table[
   Sin[2 Pi RandomReal[]] Sqrt[(1 - #1) (1 - #2)] - Sqrt[#1 #2] &[RandomReal[], RandomReal[]]
   , {i, Range[100 000]}], 50, "PDF"],
 Plot[2 Pi pLambertSphere[u], {u, -1, 1}]
]
```

Out[3531]=



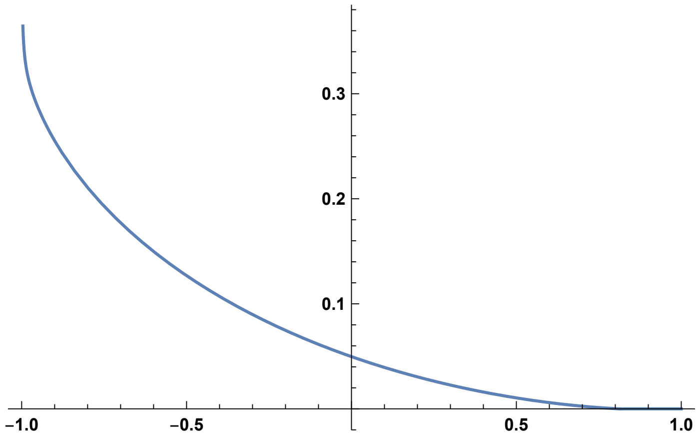
Callisto

[Porco et al. 2008] - doi:10.1088/0004-6256/136/5/2172

In[∘]:=
```
pCallisto[u_] := HeavisideTheta[2.521 - ArcCos[-u]]
  2.2/(4 Pi (1.0004369822233856`)) (2 - 0.79333 ArcCos[-u] + Exp[-21.2 ArcCos[-u]])
  (1 + Sin[ArcCos[-u]/2] Tan[ArcCos[-u]/2] Log[Tan[ArcCos[-u]/4]])
```

In[∘]:=
```
Plot[pCallisto[u], {u, -1, 1}]
```

Out[∘]=



Normalization condition

In[∘]:=
```
NIntegrate[ 2 Pi pCallisto[u], {u, -1, 1}]
```

Out[∘]= 1.

Mean cosine (g)

In[∘]:=
```
NIntegrate[2 Pi pCallisto[u] u, {u, -1, 1}]
```

Out[∘]= -0.560001

Legendre expansion coefficients

In[∘]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 0, {y, 0, Pi}]
```

Out[∘]= 1.

In[∘]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 1, {y, 0, Pi}]
```

Out[∘]= -1.68

In[∘]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 2, {y, 0, Pi}]
```

Out[∘]= 0.851712

In[ ]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 3, {y, 0, Pi}]
```
Out[ ]= -0.285211

In[ ]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 4, {y, 0, Pi}]
```
Out[ ]= 0.182995

In[ ]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 6, {y, 0, Pi}]
```
Out[ ]= 0.0908047

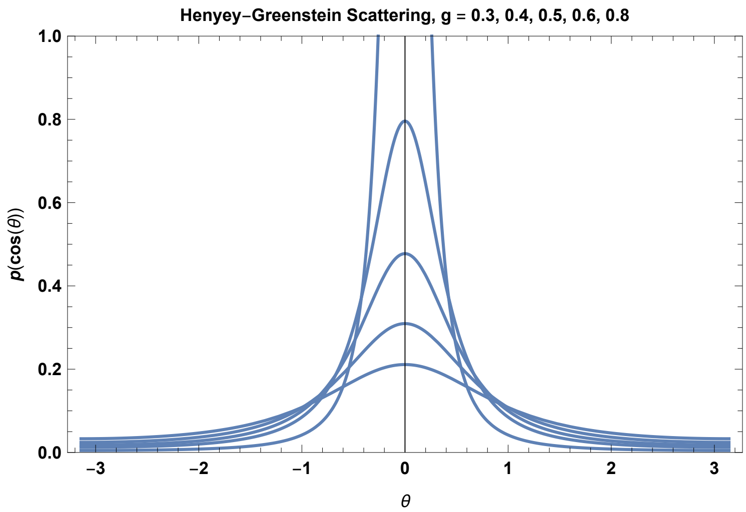
In[ ]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 8, {y, 0, Pi}]
```
Out[ ]= 0.064234

In[ ]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 10, {y, 0, Pi}]
```
Out[ ]= 0.0552028

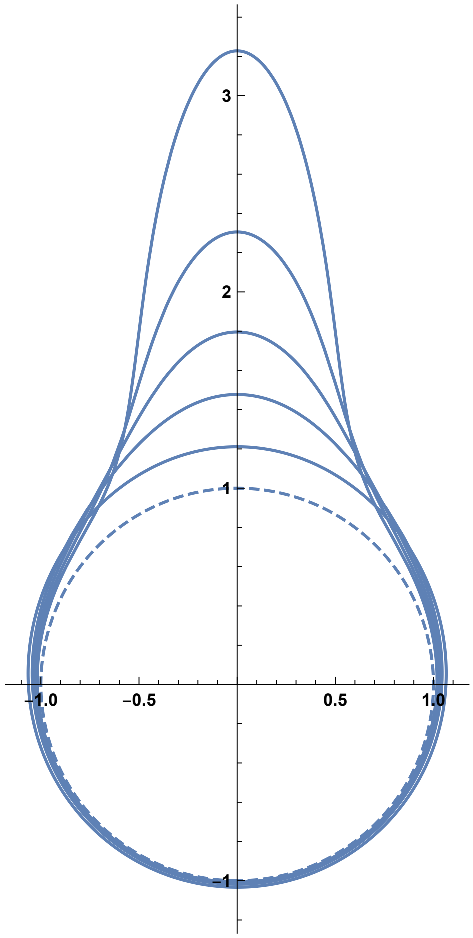
Henyey-greenstein Scattering

In[ ]:= Clear[pHG]; pHG[dot_, g_] := $\frac{1}{4\,\mathrm{Pi}} \frac{\left(1 - g^2\right)}{\left(1 + g^2 - 2\,g\,\mathrm{dot}\right)^{\frac{3}{2}}}$

```
pHGplot = Show[
  Plot[pHG[Cos[t], .8], {t, -Pi, Pi}, PlotRange → {0, 1}],
  Plot[pHG[Cos[t], .6], {t, -Pi, Pi}, PlotRange → All],
  Plot[pHG[Cos[t], .5], {t, -Pi, Pi}, PlotRange → All],
  Plot[pHG[Cos[t], .4], {t, -Pi, Pi}, PlotRange → All],
  Plot[pHG[Cos[t], .3], {t, -Pi, Pi}, PlotRange → All],
  Frame → True,
  ImageSize → 400,
  FrameLabel → {{p[Cos[θ]],},
    {θ, "Henyey-Greenstein Scattering, g = 0.3, 0.4, 0.5, 0.6, 0.8"}}]
```



```
Show[
 ParametricPlot[{Sin[t], Cos[t]} (1),
  {t, -Pi, Pi}, PlotRange → All, PlotStyle → Dashed],
 ParametricPlot[{Sin[t], Cos[t]} (1 + pHG[Cos[t], 0.75]),
  {t, -Pi, Pi}, PlotRange → All],
 ParametricPlot[{Sin[t], Cos[t]} (1 + pHG[Cos[t], 0.68]),
  {t, -Pi, Pi}, PlotRange → All],
 ParametricPlot[{Sin[t], Cos[t]} (1 + pHG[Cos[t], 0.6]),
  {t, -Pi, Pi}, PlotRange → All],
 ParametricPlot[{Sin[t], Cos[t]} (1 + pHG[Cos[t], 0.5]),
  {t, -Pi, Pi}, PlotRange → All],
 ParametricPlot[{Sin[t], Cos[t]} (1 + pHG[Cos[t], 0.3]),
  {t, -Pi, Pi}, PlotRange → All]
]
```



Normalization condition

```
Integrate[2 Pi pHG[u, g], {u, -1, 1}, Assumptions → g > -1 && g < 1]
1
```

Legendre expansion coefficients

```
Integrate[2 Pi (2 k + 1) pHG[u, g] LegendreP[k, u] /. k → 0,
 {u, -1, 1}, Assumptions → g > -1 && g < 1]
```

1

```
Integrate[2 Pi (2 k + 1) pHG[u, g] LegendreP[k, u] /. k → 1,
 {u, -1, 1}, Assumptions → g > -1 && g < 1]
```

3 g

sampling

```
cdf = Integrate[2 Pi pHG[u, g], {u, -1, x}, Assumptions → g > -1 && g < 1 && x < 1]
```

$$\frac{(-1+g)\left(-1-g+\sqrt{1+g^2-2\,g\,x}\right)}{2\,g\,\sqrt{1+g^2-2\,g\,x}}$$

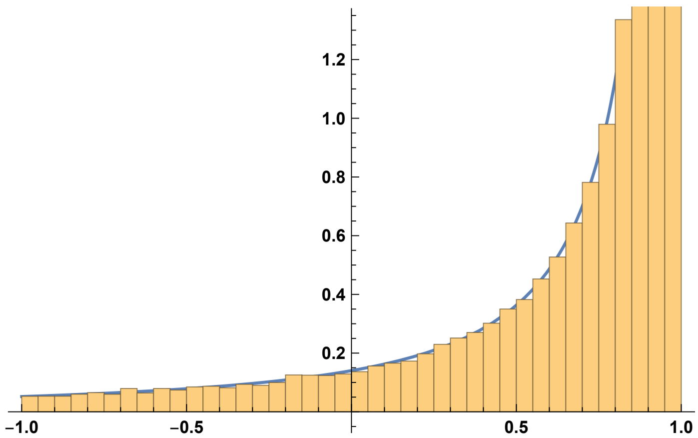
```
Solve[cdf == e, x]
```

$$\left\{\left\{x\to\frac{-1+2\,e+2\,g-2\,e\,g+2\,e^2\,g-g^2+2\,e\,g^2-2\,e\,g^3+2\,e^2\,g^3}{(1-g+2\,e\,g)^2}\right\}\right\}$$

```
FullSimplify[%]
```

$$\left\{\left\{x\to-\frac{(-1+g)^2+2\,e\,(-1+g)\,(1+g^2)-2\,e^2\,(g+g^3)}{(1+(-1+2\,e)\,g)^2}\right\}\right\}$$

```
g = 0.7;
Show[
 Plot[2 Pi pHG[u, g], {u, -1, 1}],
 Histogram[Map[- ((-1 + g)^2 + 2 # (-1 + g) (1 + g^2) - 2 #^2 (g + g^3)) / (1 + (-1 + 2 #) g)^2 &,
   Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"]
]
Clear[b, g];
```



Henyey-greenstein Scattering (Flatland)

Definition:

$$\text{pH2}[\theta\_, g\_] := \frac{1}{2\,\text{Pi}} \frac{1 - g^2}{1 + g^2 - 2\,g\,\text{Cos}[\theta]};$$

Moments

```
Integrate[pH2[t, g] Cos[t], {t, -Pi, Pi}, Assumptions → g > -1 && g < 1 && g ≠ 0 && n ≥ 0]
```

g

```
Integrate[pH2[t, g] Cos[2 t], {t, -Pi, Pi},
 Assumptions → g > -1 && g < 1 && g ≠ 0 && n ≥ 0]
```

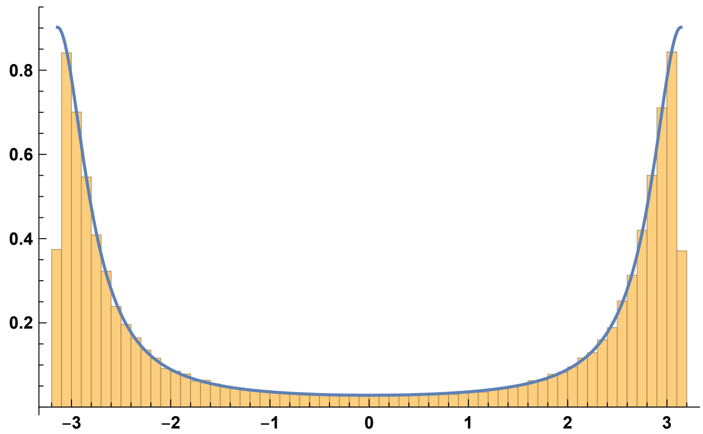
g^2

```
Integrate[pH2[t, g] Cos[7 t], {t, -Pi, Pi},
 Assumptions → g > -1 && g < 1 && g ≠ 0 && n ≥ 0]
```

g^7

Sampling:

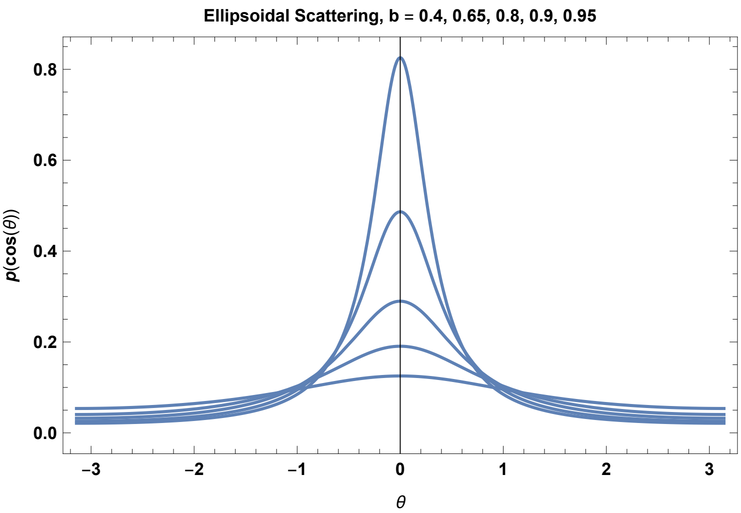
```
g = -0.7;
Show[
 Histogram[Map[2 ArcTan[(1 - g)/(1 + g) Tan[Pi/2 (1 - 2 #)]] &,
   Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"],
 Plot[ pH2[θ, g], {θ, -Pi, Pi}, PlotRange → All]
]
Clear[g];
```



Kagiwada-Kalaba (Ellipsoidal) Scattering

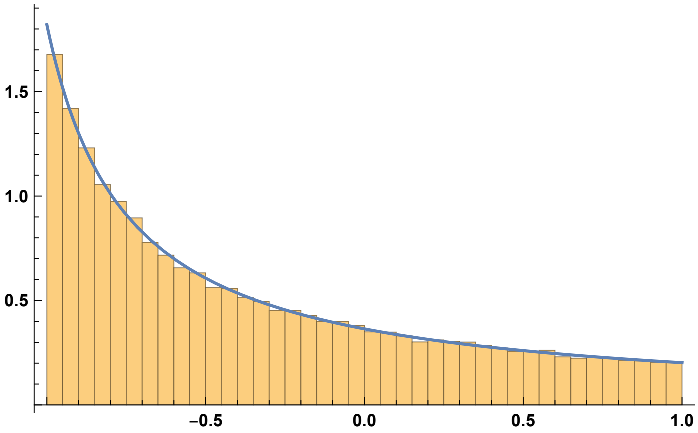
```
pEllipsoidal[u_, b_] := b (2 Pi Log[(1 + b) / (1 - b)] (1 - b u))^-1
```

```
pEllplot = Show[
  Plot[pEllipsoidal[Cos[t], .9], {t, -Pi, Pi}, PlotRange → All],
  Plot[pEllipsoidal[Cos[t], .8], {t, -Pi, Pi}, PlotRange → All],
  Plot[pEllipsoidal[Cos[t], .65], {t, -Pi, Pi}, PlotRange → All],
  Plot[pEllipsoidal[Cos[t], .4], {t, -Pi, Pi}, PlotRange → All],
  Plot[pEllipsoidal[Cos[t], .95], {t, -Pi, Pi}, PlotRange → All],
  Frame → True,
  ImageSize → 400,
  FrameLabel → {{p[Cos[θ]],},
    {θ, "Ellipsoidal Scattering, b = 0.4, 0.65, 0.8, 0.9, 0.95"}}]
```



sampling

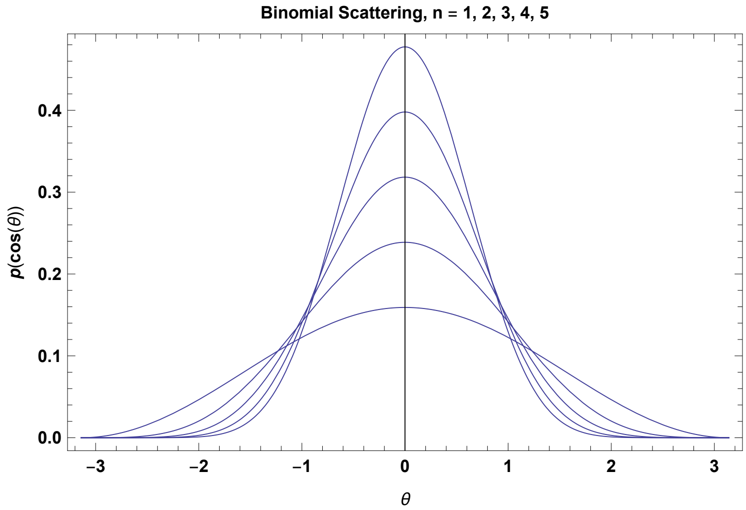
```
b = -0.8;
Show[Histogram[
   Map[(1 - (1 + b) ((1+b)/(1-b))^-#)/b &, Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"],
  Plot[2 Pi pEllipsoidal[u, b], {u, -1, 1}]

]
Clear[b];
```



Binomial Scattering

```
pBinomial[u_, n_] := Pi^-1 ((n + 1) / 2^(n+2)) (1 + u)^n
```

```
pBinplot = Show[
  Plot[pBinomial[Cos[t], 1], {t, -Pi, Pi}, PlotRange → All],
  Plot[pBinomial[Cos[t], 2], {t, -Pi, Pi}, PlotRange → All],
  Plot[pBinomial[Cos[t], 3], {t, -Pi, Pi}, PlotRange → All],
  Plot[pBinomial[Cos[t], 4], {t, -Pi, Pi}, PlotRange → All],
  Plot[pBinomial[Cos[t], 5], {t, -Pi, Pi}, PlotRange → All],
  Frame → True,
  ImageSize → 400,
  FrameLabel → {{p[Cos[θ]],}, {θ, "Binomial Scattering, n = 1, 2, 3, 4, 5"}}]
```



Normalization condition

```
Integrate[2 Pi pBinomial[u, n], {u, -1, 1}, Assumptions → n ≥ 0]
```

1

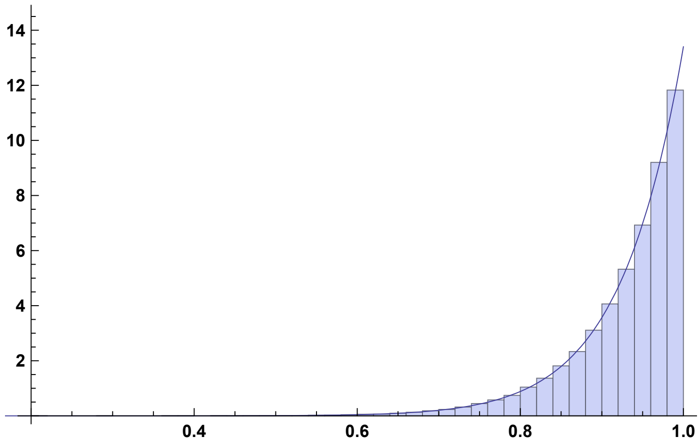
Mean cosine (g)

```
Integrate[2 Pi pBinomial[u, n] u, {u, -1, 1}, Assumptions → n ≥ 0]
```

$$\frac{n}{2+n}$$

sampling

```
n = 25.8;
Show[
 Histogram[Map[-1 + (2^(1+n) #)^(1/(1+n)) &, Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"],
 Plot[2 Pi pBinomial[u, n], {u, -1, 1}, PlotRange → All]

]
Clear[b];
```

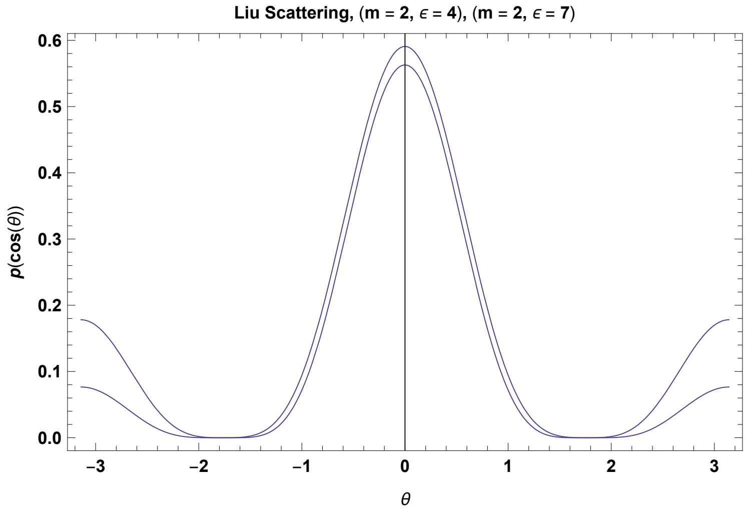


Liu Scattering

$$\text{pLiu[u\_, e\_, m\_] :=} \frac{e\,(2\,m+1)\,(1+e\,u)^{2\,m}}{2\,\text{Pi}\,\left((1+e)^{2\,m+1}-(1-e)^{2\,m+1}\right)}$$

```
Clear[m]
```

```
pLiuplot = Show[
  Plot[pLiu[Cos[t], 4, 2], {t, -Pi, Pi}, PlotRange → All],
  Plot[pLiu[Cos[t], 7, 2], {t, -Pi, Pi}, PlotRange → All],
  Frame → True,
  ImageSize → 400,
  FrameLabel →
   {{p[Cos[θ]],}, {θ, "Liu Scattering, (m = 2, ϵ = 4), (m = 2, ϵ = 7)"}}]
```



Normalization condition

```
Integrate[2 Pi pLiu[u, e, m], {u, -1, 1}, Assumptions → e > 0 && m > 0 && m ∈ Integers]
```

1

Mean cosine (g)

```
Integrate[2 Pi u pLiu[u, e, m], {u, -1, 1},
 Assumptions → e > 0 && m > 0 && m ∈ Integers && e < 1]
```

$$\frac{(1+e)^{1+2m}\left(-1+e+2em\right)+(1-e)^{1+2m}\left(1+e+2em\right)}{2e\left(-(1-e)^{1+2m}+(1+e)^{1+2m}\right)(1+m)}$$

Legendre expansion coefficients

```
Integrate[2 Pi (2 k + 1) pLiu[u, e, m] LegendreP[k, u] /. k → 0, {u, -1, 1},
 Assumptions → m > 0 && m ∈ Integers && e ∈ Reals && e ≠ 0 && Abs[e] < 1]
```

1

```
Integrate[2 Pi (2 k + 1) pLiu[u, e, m] LegendreP[k, u] /. k → 2, {u, -1, 1},
 Assumptions → m > 0 && m ∈ Integers && e ∈ Reals && e ≠ 0 && Abs[e] < 1]
```

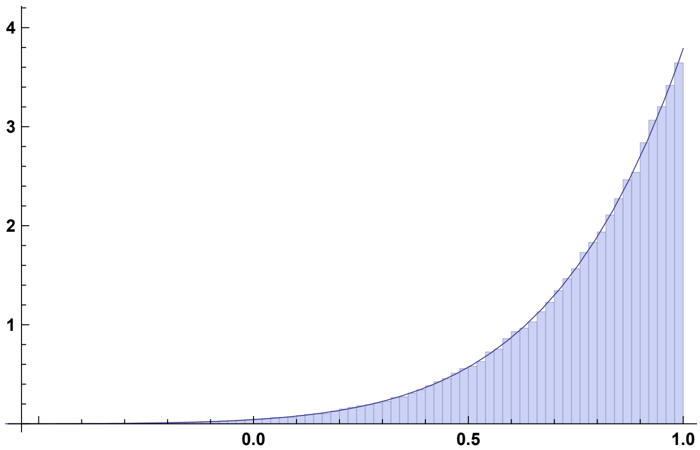
$$\left(5\left((1+e)^{1+2m}\left(3+e\left(-3+2m\left(-3+2e(1+m)\right)\right)\right)+(1-e)^{2m}(-1+e)\left(3+e\left(3+2m\left(3+2e(1+m)\right)\right)\right)\right)\right)\Big/\left(2e^2\left(-(1-e)^{1+2m}+(1+e)^{1+2m}\right)(1+m)(3+2m)\right)$$

sampling

```
m = 3.5;
ϵ = 0.9;
```

$$\text{Show}\left[\text{Histogram}\left[\text{Map}\left[\frac{-1+\left((-1+\#)\,(1-\epsilon)^{2\,m}\,(-1+\epsilon)+\#\,(1+\epsilon)^{1+2\,m}\right)^{\frac{1}{1+2\,m}}}{\epsilon}\,\&,\right.\right.\right.$$

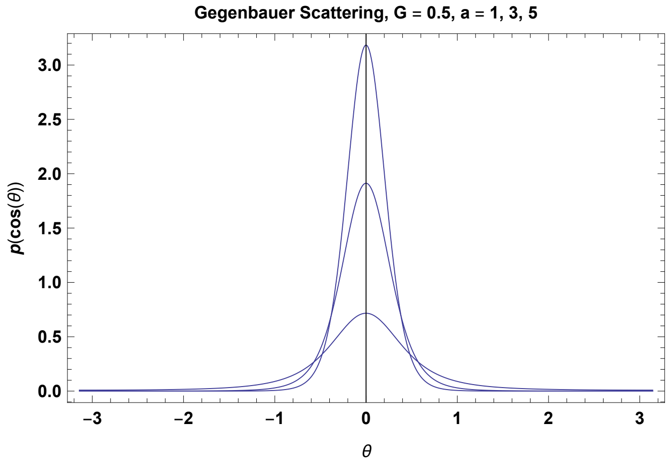
```
    Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"],
  Plot[2 Pi pLiu[u, ϵ, m], {u, -1, 1}, PlotRange → All]

]
Clear[m, ϵ];
```



Gegenbauer Scattering

$$\text{pGegenbauer}[u\_,\,g\_,\,a\_] := \frac{\left(1+g^2-2\,g\,u\right)^{-(a+1)}}{\frac{\left((1-g)^{-2\,a}-(1+g)^{-2\,a}\right)\,\pi}{a\,g}}$$

```
Show[
 Plot[pGegenbauer[Cos[t], 0.5, 1], {t, -Pi, Pi}, PlotRange → All],
 Plot[pGegenbauer[Cos[t], 0.5, 3], {t, -Pi, Pi}, PlotRange → All],
 Plot[pGegenbauer[Cos[t], 0.5, 5], {t, -Pi, Pi}, PlotRange → All],

 Frame → True,
 FrameLabel →
  {{p[Cos[θ]],}, {θ, "Gegenbauer Scattering, G = 0.5, a = 1, 3, 5"}}]
```



Normalization condition

```
Integrate[2 Pi pGegenbauer[u, g, a], {u, -1, 1}, Assumptions → -1 ≤ g ≤ 1 && a > 0]
```

1

Mean cosine (g)

```
Integrate[2 Pi u pGegenbauer[u, g, a], {u, -1, 1}, Assumptions → -1 ≤ g ≤ 1 && a > 0]
```

$$\frac{(1+g)^{2a}\left(1-2ag+g^2\right)-(1-g)^{2a}\left(1+2ag+g^2\right)}{2(-1+a)g\left((1-g)^{2a}-(1+g)^{2a}\right)}$$

Legendre expansion coefficients

```
Integrate[2 Pi (2 k + 1) pGegenbauer[u, g, a] LegendreP[k, u] /. k → 0,
 {u, -1, 1}, Assumptions → -1 ≤ g ≤ 1 && a > 0]
```

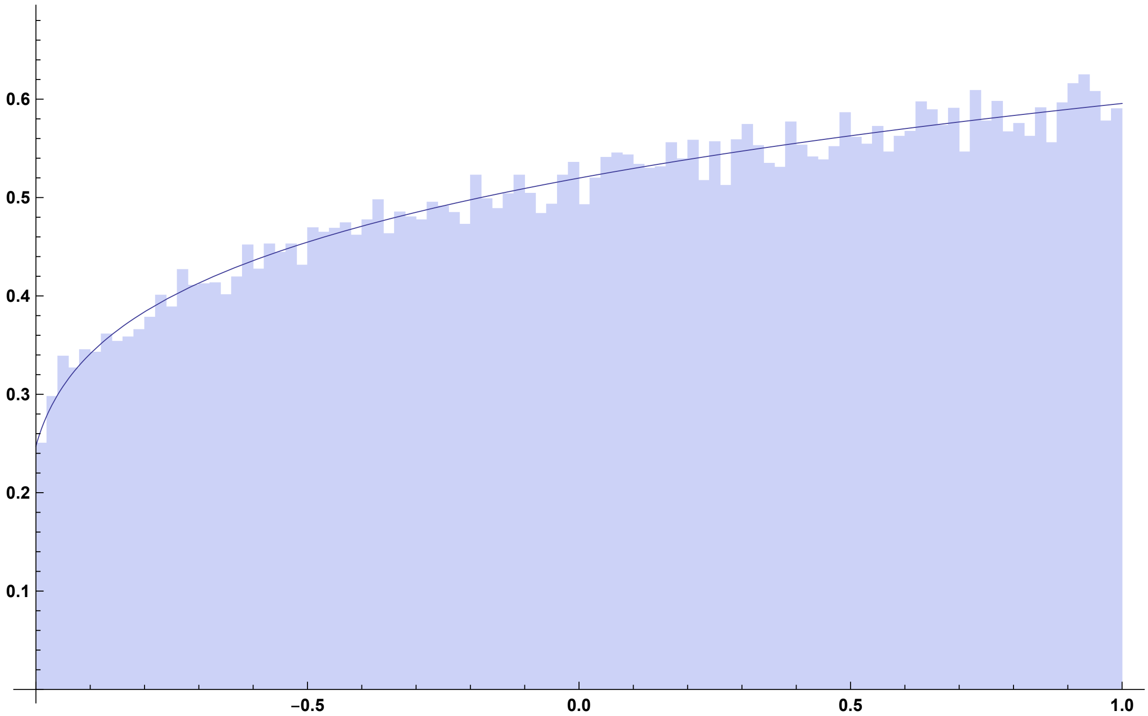
1

```
FullSimplify[Integrate[2 Pi (2 k + 1) pGegenbauer[u, g, a] LegendreP[k, u] /. k → 3,
  {u, -1, 1}, Assumptions → -1 ≤ g ≤ 1 && a > 0]]
```

$$-\left(7\left(24a^2g^2\left(1+g^2\right)\left((1-g)^{2a}-(1+g)^{2a}\right)+3\left(5+3g^2+3g^4+5g^6\right)\left((1-g)^{2a}-(1+g)^{2a}\right)+8a^3g^3\left((1-g)^{2a}+(1+g)^{2a}\right)+2ag\left(15+14g^2+15g^4\right)\left((1-g)^{2a}+(1+g)^{2a}\right)\right)\right)\Big/\left(8(-3+a)(-2+a)(-1+a)g^3\left((1-g)^{2a}-(1+g)^{2a}\right)\right)$$

sampling

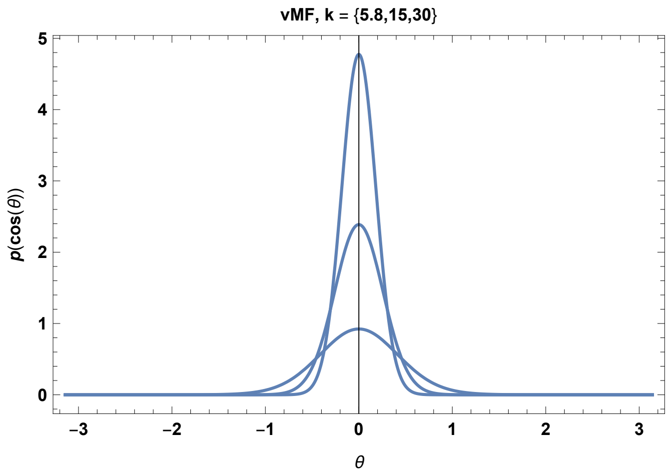
```
g = -0.8;
a = -1.2;
Show[Histogram[Map[(1 + g^2 - (# (1 - g)^(-2 a) - (-1 + #) (1 + g)^(-2 a))^(-1/a))/(2 g) &,
    Table[RandomReal[], {i, 1, 100 000}]], 100, "PDF"],
 Plot[2 Pi pGegenbauer[u, g, a], {u, -1, 1}, PlotRange → All]

]
Clear[g, a];
```



vMF (spherical Gaussian) Scattering

```
pVMF[u_, k_] := k/(4 Pi Sinh[k]) Exp[k u]
```

```
Show[
 Plot[pVMF[Cos[t], 5.8], {t, -Pi, Pi}, PlotRange → All],
 Plot[pVMF[Cos[t], 15], {t, -Pi, Pi}, PlotRange → All],
 Plot[pVMF[Cos[t], 30], {t, -Pi, Pi}, PlotRange → All],

 Frame → True,
 FrameLabel → {{p[Cos[θ]],}, {θ, "vMF, k = {5.8,15,30}"}}]
```



Normalization condition

```
Integrate[2 Pi pVMF[u, k], {u, -1, 1}, Assumptions → k > 0]
```

1

Mean cosine (g)

```
Integrate[2 Pi u pVMF[u, k], {u, -1, 1}, Assumptions → k > 0]
```

$$-\frac{1}{k} + \mathrm{Coth}[k]$$

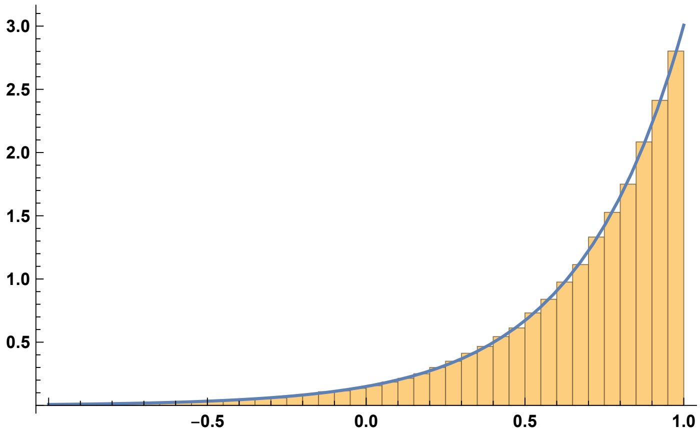
Legendre expansion coefficients

```
Integrate[2 Pi (2 o + 1) pVMF[u, k] LegendreP[o, u] /. o → 4,
 {u, -1, 1}, Assumptions → k > 0]
```

$$\frac{9\left(105 + 45 k^2 + k^4 - 5 k \left(21 + 2 k^2\right) \mathrm{Coth}[k]\right)}{k^4}$$

sampling

```
k = 3;
Show[Histogram[
  Map[Log[E^-k (1 - #) + E^k #]/k &, Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"],
 Plot[2 Pi pVMF[u, k], {u, -1, 1}, PlotRange → All]

]
Clear[k];
```



Klein-Nishina

Normalized variant of Klein-Nishina - energy parameter “e” = $\frac{E_\gamma}{m_e c^2}$

$$\text{pKleinNishina}[u\_, e\_] := \frac{1}{1 + e\,(1 - u)}\;\frac{1}{\frac{2\,\pi\,\text{Log}[1+2\,e]}{e}}$$

Normalization condition

In[∘]:= `Integrate[2 Pi pKleinNishina[u, e], {u, -1, 1}, Assumptions → e > 0]`

Out[∘]= 1

Mean-cosine

In[∘]:= `Integrate[2 Pi pKleinNishina[u, e] u, {u, -1, 1}, Assumptions → e > 0]`

Out[∘]= $1 + \frac{1}{e} - \frac{2}{\text{Log}[1 + 2\,e]}$

Legendre expansion coefficients

In[•]:=
```
Integrate[
 2 Pi (2 k + 1) pKleinNishina[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 0,
 {y, 0, Pi}, Assumptions → e > 0]
```

Out[•]= 1

In[•]:=
```
Integrate[
 2 Pi (2 k + 1) pKleinNishina[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 1,
 {y, 0, Pi}, Assumptions → e > 0]
```

Out[•]= $3 + \frac{3}{e} - \frac{6}{\mathrm{Log}[1 + 2 e]}$

In[•]:=
```
Integrate[
 2 Pi (2 k + 1) pKleinNishina[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 2,
 {y, 0, Pi}, Assumptions → e > 0]
```

Out[•]= $\frac{5}{4}\left(1 + \frac{3\left(2 + 4 e + e^2 - \frac{4 e (1+e)}{\mathrm{Log}[1+2 e]}\right)}{e^2}\right)$

In[•]:=
```
Integrate[
 2 Pi (2 k + 1) pKleinNishina[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 3,
 {y, 0, Pi}, Assumptions → e > 0]
```

Out[•]= $\frac{7\left(15 + 45 e + 36 e^2 + 6 e^3 - \frac{2 e (15+30 e+11 e^2)}{\mathrm{Log}[1+2 e]}\right)}{6 e^3}$

sampling

In[•]:=
```
cdf = Integrate[2 Pi pKleinNishina[u, e], {u, -1, x}, Assumptions → e > 0 && 0 < x < 1]
```

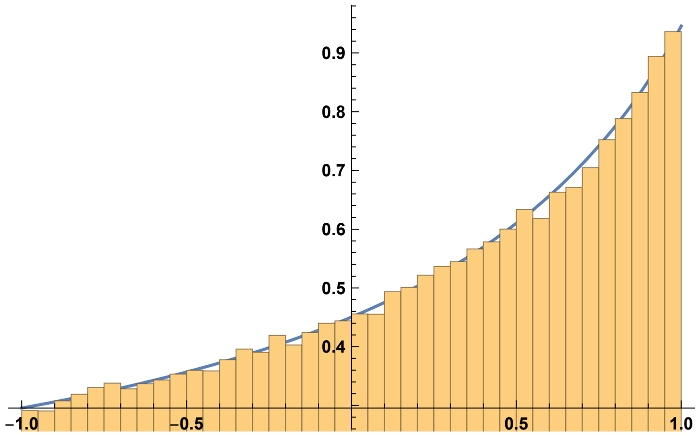
Out[•]= $1 - \frac{\mathrm{Log}[1 + e - e x]}{\mathrm{Log}[1 + 2 e]}$

In[•]:=
```
Solve[cdf == k, x]
```

Out[•]= $\left\{\left\{x \to \mathrm{ConditionalExpression}\left[\frac{1 + e - (1 + 2 e)^{1-k}}{e}, -\pi \le \mathrm{Im}\left[(-1 + k)\, \mathrm{Log}[1 + 2 e]\right] < \pi\right]\right\}\right\}$

In[•]:=
```
With[{e = 1.1},
 Show[
  Plot[2 Pi pKleinNishina[u, e], {u, -1, 1}],
  Histogram[
   Map[(1 + e - (1 + 2 e)^(1-#))/e &, Table[RandomReal[], {i, 1, 100000}]], 50, "PDF"]
  ]
 ]
```

Out[•]=



Cornette-Shanks

[Cornette and Shanks 1992] - *Physically reasonable analytic expression for the single-scattering phase function.*
Independently proposed [Liu and Weng 2006]

In[•]:= pCornetteShanks[u_, g_] := $\frac{3}{8\,\text{Pi}}\frac{(1-g^2)(1+u^2)}{(2+g^2)(1+g^2-2\,g\,u)^{3/2}}$

Normalization condition

In[•]:= Integrate[2 Pi pCornetteShanks[u, g], {u, -1, 1}, Assumptions → -1 < g < 1]

Out[•]= 1

Mean-cosine

In[•]:= Integrate[2 Pi pCornetteShanks[u, g] u, {u, -1, 1}, Assumptions → -1 < g < 1]

Out[•]= $\frac{3\,g\,(4+g^2)}{5\,(2+g^2)}$

Legendre expansion coefficients

In[•]:=
```
Integrate[
 2 Pi (2 k + 1) pCornetteShanks[Cos[y], g] LegendreP[k, Cos[y]] Sin[y] /. k → 0,
 {y, 0, Pi}, Assumptions → -1 < g < 1]
```

Out[•]= 1

In[•]:=
```
Integrate[
 2 Pi (2 k + 1) pCornetteShanks[Cos[y], g] LegendreP[k, Cos[y]] Sin[y] /. k → 1,
 {y, 0, Pi}, Assumptions → -1 < g < 1]
```

Out[•]= $$\frac{9\,g\,(4+g^2)}{5\,(2+g^2)}$$

In[•]:=
```
Integrate[
 2 Pi (2 k + 1) pCornetteShanks[Cos[y], g] LegendreP[k, Cos[y]] Sin[y] /. k → 2,
 {y, 0, Pi}, Assumptions → -1 < g < 1]
```

Out[•]= $$\frac{7+80\,g^2+18\,g^4}{14+7\,g^2}$$

In[•]:=
```
Integrate[
 2 Pi (2 k + 1) pCornetteShanks[Cos[y], g] LegendreP[k, Cos[y]] Sin[y] /. k → 3,
 {y, 0, Pi}, Assumptions → -1 < g < 1]
```

Out[•]= $$\frac{g\,(27+238\,g^2+50\,g^4)}{15\,(2+g^2)}$$

sampling

In[•]:=
```
cdf = Integrate[2 Pi pCornetteShanks[u, g],
  {u, -1, x}, Assumptions → -1 < g < 1 && 0 < x < 1]
```

Out[•]=
$$\frac{1}{4\,g^3\,(2+g^2)\,\sqrt{1+g^2-2\,g\,x}}\left(2-2\,g^6-2\,g\,x-2\,\sqrt{1+g^2-2\,g\,x}+4\,g^3\,\sqrt{1+g^2-2\,g\,x}+g^4\,(-5+x^2)+2\,g^5\left(x+\sqrt{1+g^2-2\,g\,x}\right)-g^2\left(-5+x^2+4\,\sqrt{1+g^2-2\,g\,x}\right)\right)$$

Draine

Draine, B.T. (2003) '*Scattering by interstellar dust grains. 1: Optical and ultraviolet*', ApJ., 598, 1017–25.

In[•]:=
```
pDraine[u_, g_, α_] := 1/(4 Pi) ((1 - g^2)/(1 + g^2 - 2 g u)^(3/2) (1 + α u^2)/(1 + α (1 + 2 g^2)/3))
```
$$\text{pDraine}[u\_,g\_,\alpha\_] := \frac{1}{4\,\text{Pi}}\left(\frac{1-g^2}{(1+g^2-2\,g\,u)^{3/2}}\,\frac{1+\alpha\,u^2}{1+\alpha\,(1+2\,g^2)/3}\right)$$

Normalization condition

In[∘]:= Integrate[2 Pi pDraine[u, g, a], {u, -1, 1}, Assumptions → 0 < a < 1 && -1 < g < 1]

Out[∘]= 1

Mean-cosine

In[∘]:= Integrate[2 Pi pDraine[u, g, a] u, {u, -1, 1}, Assumptions → 0 < a < 1 && -1 < g < 1]

Out[∘]= $\frac{3}{5}\left(g+\frac{2\,(1+a)\,g}{3+a+2\,a\,g^2}\right)$

In[∘]:= $\frac{3}{5}\left(g+\frac{2\,(1+a)\,g}{3+a+2\,a\,g^2}\right)$ /. a → 0

Out[∘]= g

Legendre expansion coefficients

In[∘]:= Integrate[2 Pi (2 k + 1) pDraine[Cos[y], g, a] LegendreP[k, Cos[y]] Sin[y] /. k → 0,
{y, 0, Pi}, Assumptions → 0 < a < 1 && -1 < g < 1]

Out[∘]= 1

In[∘]:= Integrate[2 Pi (2 k + 1) pDraine[Cos[y], g, a] LegendreP[k, Cos[y]] Sin[y] /. k → 1,
{y, 0, Pi}, Assumptions → 0 < a < 1 && -1 < g < 1]

Out[∘]= $\frac{9\,g\,\left(5+a\,\left(3+2\,g^2\right)\right)}{5\,\left(3+a+2\,a\,g^2\right)}$

In[∘]:= Integrate[2 Pi (2 k + 1) pDraine[Cos[y], g, a] LegendreP[k, Cos[y]] Sin[y] /. k → 2,
{y, 0, Pi}, Assumptions → 0 < a < 1 && -1 < g < 1]

Out[∘]= $\frac{14\,a+5\,\left(21+11\,a\right)\,g^2+36\,a\,g^4}{7\,\left(3+a+2\,a\,g^2\right)}$

In[∘]:= Integrate[2 Pi (2 k + 1) pDraine[Cos[y], g, a] LegendreP[k, Cos[y]] Sin[y] /. k → 3,
{y, 0, Pi}, Assumptions → 0 < a < 1 && -1 < g < 1]

Out[∘]= $\frac{g\,\left(54\,a+7\,\left(45+23\,a\right)\,g^2+100\,a\,g^4\right)}{15\,\left(3+a+2\,a\,g^2\right)}$

sampling

In[∘]:=
```
cdf = Integrate[2 Pi pDraine[u, g, a],
  {u, -1, x}, Assumptions → 0 < a < 1 && -1 < g < 1 && -1 < x < 1]
```

Out[∘]=
$$\left(3\,(-1+g)\,g^2\left(-1-g+\sqrt{1+g^2-2\,g\,x}\right)+a\left(2-2\,g^6-2\,g\,x-2\sqrt{1+g^2-2\,g\,x}+g^3\sqrt{1+g^2-2\,g\,x}+g^4\left(-2+x^2\right)+2\,g^5\left(x+\sqrt{1+g^2-2\,g\,x}\right)-g^2\left(-2+x^2+\sqrt{1+g^2-2\,g\,x}\right)\right)\right)\Big/\left(2\,g^3\left(3+a+2\,a\,g^2\right)\sqrt{1+g^2-2\,g\,x}\right)$$

Schlick

In[∘]:=
$$\text{pSchlick[u\_, k\_]} := \frac{1}{4\,\text{Pi}}\left(\frac{1-k^2}{(1+k\,u)^2}\right)$$

Normalization condition

In[∘]:=
```
Integrate[2 Pi pSchlick[u, k], {u, -1, 1}, Assumptions → -1 < k < 1]
```

Out[∘]= 1

Mean-cosine

In[∘]:=
```
Integrate[2 Pi pSchlick[u, k] u, {u, -1, 1}, Assumptions → -1 < k < 1]
```

Out[∘]=
$$-\frac{k-\text{ArcTanh}[k]+k^2\,\text{ArcTanh}[k]}{k^2}$$

Legendre expansion coefficients

In[∘]:=
```
Integrate[2 Pi (2 k + 1) pSchlick[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 0,
  {y, 0, Pi}, Assumptions → -1 < e < 1]
```

Out[∘]= ConditionalExpression[1, e ≠ 0]

In[∘]:=
```
Integrate[2 Pi (2 k + 1) pSchlick[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 1,
  {y, 0, Pi}, Assumptions → -1 < e < 1]
```

Out[∘]=
$$\text{ConditionalExpression}\left[-\frac{3\left(e+\left(-1+e^2\right)\text{ArcTanh}[e]\right)}{e^2},\ e\neq 0\right]$$

In[∘]:=
```
Integrate[2 Pi (2 k + 1) pSchlick[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 2,
  {y, 0, Pi}, Assumptions → -1 < e < 1]
```

Out[∘]=
$$\text{ConditionalExpression}\left[-\frac{5\left(-6\,e+4\,e^3-6\left(-1+e^2\right)\text{ArcTanh}[e]\right)}{2\,e^3},\ e\neq 0\right]$$

In[•]:=
```
Integrate[2 Pi (2 k + 1) pSchlick[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 3,
  {y, 0, Pi}, Assumptions → -1 < e < 1]
```

Out[•]= $\text{ConditionalExpression}\left[-\frac{7\left(30\,e - 26\,e^3 - 6\left(5 - 6\,e^2 + e^4\right)\text{ArcTanh}[e]\right)}{4\,e^4},\ e \neq 0\right]$

sampling

In[•]:=
```
cdf = Integrate[2 Pi pSchlick[u, e], {u, -1, x}, Assumptions → -1 < e < 1 && 0 < x < 1]
```

Out[•]= $\frac{(1 + e)\ (1 + x)}{2 + 2\,e\,x}$

In[•]:=
```
Solve[cdf == k, x]
```

Out[•]= $\left\{\left\{x \to \frac{1 + e - 2\,k}{-1 - e + 2\,e\,k}\right\}\right\}$

In[•]:=
```
With[{e = -.7},
 Show[
  Plot[2 Pi pSchlick[u, e], {u, -1, 1}],
  Histogram[Map[(1 + e - 2 #)/(-1 - e + 2 e #) &, Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"]
 ]
]
```

Out[•]=