# Beckmann NDF

This is code to accompany the book:

## A Hitchhiker's Guide to Multiple Scattering



www.eugenedeon.com/hitchhikers

### notation

u = **m** . **n** = Cos[$\theta_m$]

$\alpha$ = roughness

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## Definitions and derivations

$$\text{Beckmann`D}[u\_, \alpha\_] := \frac{e^{-\frac{-1+\frac{1}{u^2}}{\alpha^2}}}{\alpha^2\, \pi\, u^4}\ \text{HeavisideTheta}[u]$$

In[1149]:= $$\text{Beckmann`}\sigma[u\_, \alpha\_] := \frac{1}{2}\left(u\left(1+\text{Erf}\left[\frac{u}{\alpha\sqrt{1-u^2}}\right]\right)+\alpha\sqrt{1-u^2}\,\frac{E^{\frac{u^2}{\alpha^2\,(u^2-1)}}}{\sqrt{\text{Pi}}}\right)$$

In[2558]:= $$\text{Beckmann`}\Lambda[u\_, \alpha\_] := \frac{1}{2}\left(-1+\frac{e^{\frac{u^2}{(-1+u^2)\,\alpha^2}}\sqrt{1-u^2}\,\alpha}{\sqrt{\pi}\,u}+\text{Erf}\left[\frac{u}{\sqrt{1-u^2}\,\alpha}\right]\right)$$

In[604]:= $$(1+\text{Beckmann`}\Lambda[u, \alpha])\, u == \text{Beckmann`}\sigma[u, \alpha] \ \text{// FullSimplify}$$

Out[604]= True

In[605]:= $$(\text{Beckmann`}\Lambda[u, \alpha])\, u == \text{Beckmann`}\sigma[-u, \alpha] \ \text{// FullSimplify}$$

Out[605]= True

In[2643]:= $$\text{FullSimplify}\left[\text{Beckmann`}\Lambda\left[u, \frac{u}{\sqrt{1-u^2}\,x}\right], \text{Assumptions} \to 0 < u < 1 \,\&\&\, x > 0\right]$$

Out[2643]= $$\frac{1}{2}\left(-1+\frac{e^{-x^2}}{\sqrt{\pi}\,x}+\text{Erf}[x]\right)$$

## shape invariant f(x)

In[1231]:= $\text{FullSimplify}\left[\text{Beckmann`D}[u, \alpha]\, u^4\, \alpha^2 \text{ /. } u \to \frac{1}{\sqrt{1 + x^2\, \alpha^2}},\ \text{Assumptions} \to 1 - \frac{1}{\sqrt{1 + x^2\, \alpha^2}} > 0\right]$

Out[1231]= $\frac{e^{-x^2}}{\pi}$

## height field normalization

In[606]:= `Integrate[2 Pi u Beckmann`D[u, α], {u, 0, 1}, Assumptions → 0 < α < 1]`

Out[606]= 1

## distribution of slopes

In[607]:= $\text{FullSimplify}\left[\text{Beckmann`D}\left[\frac{1}{\sqrt{p^2 + q^2 + 1}}, \alpha\right] \left(\frac{1}{\sqrt{p^2 + q^2 + 1}}\right)^4,\ \text{Assumptions} \to 0 < \alpha < 1 \,\&\&\, p > 0 \,\&\&\, q > 0\right]$

Out[607]= $\frac{e^{-\frac{p^2+q^2}{\alpha^2}}}{\pi\, \alpha^2}$

In[608]:= $\text{Beckmann`P22}[p\_, q\_, \alpha\_] := \frac{e^{-\frac{p^2+q^2}{\alpha^2}}}{\pi\, \alpha^2}$

In[609]:= `Integrate[Beckmann`P22[p, q, α], {p, -Infinity, Infinity}, {q, -Infinity, Infinity}, Assumptions → 0 < α < 1]`

Out[609]= 1

In[610]:= `Integrate[Beckmann`P22[p, q, α], {q, -Infinity, Infinity}, Assumptions → α > 0 && Im[p] == 0]`

Out[610]= $\frac{e^{-\frac{p^2}{\alpha^2}}}{\sqrt{\pi}\, \alpha}$

In[613]:= $\text{Beckmann`P2}[p\_, \alpha\_] := \frac{e^{-\frac{p^2}{\alpha^2}}}{\sqrt{\pi}\, \alpha}$

## derivation of Λ(u)

In[614]:=
```
FullSimplify[
  Sqrt[1 - u^2]/u Integrate[(q - u/Sqrt[1 - u^2]) Beckmann`P2[q, α], {q, u/Sqrt[1 - u^2], Infinity},
   Assumptions → 0 < u < 1 && 0 < α < 1], Assumptions → 0 < u < 1 && 0 < α < 1]
```

Out[614]=

$$\frac{1}{2}\left(-1+\frac{e^{\frac{u^2}{(-1+u^2)\,\alpha^2}}\sqrt{1-u^2}\;\alpha}{\sqrt{\pi}\;u}+\mathrm{Erf}\left[\frac{u}{\sqrt{1-u^2}\;\alpha}\right]\right)$$

## compare σ to delta integral:

In[615]:=
```
Delta`σ[u_, ui_] := Re[2 (Sqrt[1 - u^2 - ui^2] + u ui ArcCos[-(u ui)/(Sqrt[1 - u^2] Sqrt[1 - ui^2])])]
```

In[616]:=
```
With[{α = .7},
 Plot[{
   Quiet[NIntegrate[Beckmann`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
   Quiet[Beckmann`σ[u, α]]
  }, {u, -1, 1}]
]
```

Out[616]=