Picard/Lalesco kernel

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

Integral equation

$f(x) = S_0(x) + c \int_0^\infty \frac{1}{2} e^{-|x-t|} f(t)\, dt$

In[ ]:=
```
Picard`K[x_] := 1/2 Exp[-Abs[x]]
```

This kernel has a known explicit H-function [d'Eon and McCormick 2019]

References

- Fock, V. 1944. Some integral equations of mathematical physics. In: Doklady AN SSSR, vol. 26, 147–51, http://mi.mathnet.ru/eng/msb6183.
- Case, K. M. 1957. On Wiener-Hopf equations. Ann. Phys. (USA) 2(4): 384–405. doi:10.1016/0003-4916(57)90027-1
- Krein, M. G. 1962. Integral equations on a half-line with kernel depending upon the difference of the arguments. Amer. Math. Soc. Transl. 22: 163–288.
- Atkinson, K. 1969. The Numerical Solution of Integral Equations on the Half-Line. *SIAM J. Numer. Anal.*, 6(3), 375–397. doi: 10.1137/0706035
- Eugene d'Eon & Norman J. McCormick (2019) Radiative Transfer in Half Spaces of Arbitrary Dimension, *Journal of Computational and Theoretical Transport*, 48:7, 280-337, DOI: 10.1080/23324309.2019.1696365

Applications

The Picard kernel arises for isotropic scattering problems including:

- classical exponential random flights in a 1D rod
- BesselK0 random flights in the 1D rod
- Gamma/Erlang-2 random flights in 3D
- $\frac{1}{2} e^{-s} (1 + s)$ random flights in 4D
- $\frac{2^{\frac{1}{2}-\frac{d}{2}}\, d\, s^{\frac{1}{2}(-1+d)}\, \mathrm{BesselK}\left[\frac{1}{2}(-1+d), s\right]}{\sqrt{\pi}\, \mathrm{Gamma}\left[1+\frac{d}{2}\right]}$ random flights in dD

Normalization

In[∘]:=
```
Integrate[Picard`K[x], {x, -Infinity, Infinity}]
```
Out[∘]= 1

Fourier transform

Plane-parallel

In[∘]:=
```
√2 Pi FourierTransform[Picard`K[x], x, z]
```
Out[∘]= $\frac{1}{1+z^2}$

Radial symmetry

In[∘]:= Picard\`pc[r_, d_] := $\frac{2^{-d/2}\, d\, r^{d/2}\, \mathrm{BesselK}\left[\frac{1}{2}(-2+d), r\right]}{\mathrm{Gamma}\left[1+\frac{d}{2}\right]}$

In[∘]:=
```
TableForm[Table[{"d = " <> ToString[d], Picard`pc[r, d]}, {d, Range[10]}]]
```
Out[∘]//TableForm=

d = 1	e^{-r}
d = 2	r BesselK[0, r]
d = 3	$e^{-r}\, r$
d = 4	$\frac{1}{2} r^2$ BesselK[1, r]
d = 5	$\frac{1}{3} e^{-r}\left(1+\frac{1}{r}\right) r^2$
d = 6	$\frac{1}{8} r^3$ BesselK[2, r]
d = 7	$\frac{1}{15} e^{-r}\left(1+\frac{3}{r^2}+\frac{3}{r}\right) r^3$
d = 8	$\frac{1}{48} r^4$ BesselK[3, r]
d = 9	$\frac{1}{105} e^{-r}\left(1+\frac{15}{r^3}+\frac{15}{r^2}+\frac{6}{r}\right) r^4$
d = 10	$\frac{1}{384} r^5$ BesselK[4, r]

In[∘]:=
```
FullSimplify[πd[d, Picard`pc[r, d]], Assumptions → z > 0]
```
Out[∘]= $\frac{1}{1+z^2}$

Laplace transform

In[∘]:=
```
LaplaceTransform[Picard`K[x], x, s]
```
Out[∘]= $\frac{1}{2\,(1+s)}$

Laplace expression of the kernel

In[∘]:= $\texttt{Picard`k[s\_]} := \dfrac{\texttt{DiracDelta}[s-1]}{2}$

In[∘]:=
```
LaplaceTransform[Picard`k[s], s, Abs[x]]
```

Out[∘]= $\frac{1}{2}\,e^{-\text{Abs}[x]}$

Resolvent / solution

[**Atkinson 1969**], p.382:

Resolvent

In[∘]:= $\texttt{R[x0\_, x\_, c\_]} := \dfrac{\left(-2+2\sqrt{1-c}+c\right) e^{-\sqrt{1-c}\,(x+x0)} + c\, e^{-\sqrt{1-c}\,\text{Abs}[-x+x0]}}{2\sqrt{1-c}}$

Solution

Test with a gamma forcing function:

In[∘]:=
```
S0[t_] := Exp[-t] t
```

In[∘]:=
```
S0[x] + Integrate[R[x, t, c] × S0[t], {t, 0, Infinity}, Assumptions → x > 0 && 0 < c < 1]
```

Out[∘]=

$$e^{-x}x + \frac{1}{\left(1+\sqrt{1-c}\right)^2\sqrt{1-c}\,c}\,e^{-\left(1+2\sqrt{1-c}\right)x+\sqrt{1-c}\,x}\Big(4e^x + 4\sqrt{1-c}\,e^x - 5c\,e^x - \sqrt{1-c}\,c\,e^x + c^2e^x - 4e^{\sqrt{1-c}\,x} - 4\sqrt{1-c}\,e^{\sqrt{1-c}\,x} + 4c\,e^{\sqrt{1-c}\,x} + 2\sqrt{1-c}\,c\,e^{\sqrt{1-c}\,x} - 2c\,e^{\sqrt{1-c}\,x}x - 2\sqrt{1-c}\,c\,e^{\sqrt{1-c}\,x}x + 2c^2e^{\sqrt{1-c}\,x}x + \sqrt{1-c}\,c^2e^{\sqrt{1-c}\,x}x\Big)$$

In[∘]:=
```
FullSimplify[%, Assumptions → x > 0 && 0 < c < 1]
```

Out[∘]=

$$\frac{e^{-\left(1+\sqrt{1-c}\right)x}\left(\left(-1+\sqrt{1-c}\right)(-4+c)\,e^x - 2c\,e^{\sqrt{1-c}\,x}\right)}{c^2}$$

$$\texttt{S0[x]} + c\,\texttt{Integrate}\left[\left(\left(\frac{e^{-\left(1+\sqrt{1-c}\right)x}\left(\left(-1+\sqrt{1-c}\right)(-4+c)\,e^x - 2c\,e^{\sqrt{1-c}\,x}\right)}{c^2}\right)\texttt{/. x} \to \texttt{t}\right)\texttt{Picard`K[t - x], \{t, 0, Infinity\}, Assumptions} \to x>0\ \&\&\ 0<c<1\right] - \left(\frac{e^{-\left(1+\sqrt{1-c}\right)x}\left(\left(-1+\sqrt{1-c}\right)(-4+c)\,e^x - 2c\,e^{\sqrt{1-c}\,x}\right)}{c^2}\right)$$

Out[∘]=

$$-\frac{e^{\left(-1-\sqrt{1-c}\right)x}\left(\left(-1+\sqrt{1-c}\right)(-4+c)\,e^x - 2c\,e^{\sqrt{1-c}\,x}\right)}{c^2} + e^{-x}x + \frac{e^{-x-\sqrt{1-c}\,x}\left(4e^x - 4\sqrt{1-c}\,e^x - c\,e^x + \sqrt{1-c}\,c\,e^x - 2c\,e^{\sqrt{1-c}\,x} - c^2e^{\sqrt{1-c}\,x}x\right)}{c^2}$$

In[∘]:=
```
FullSimplify[%, Assumptions → 0 < c < 1 && x > 0] // Expand
```

Out[∘]= 0