Exponential NDF

This is code to accompany the book:

A Hitchhiker’s Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

notation

u = **m** . **n** = Cos[θ_m]
α = roughness

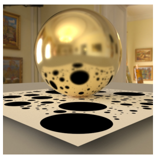
Definitions and derivations

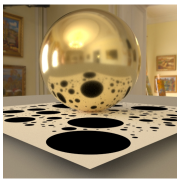
In[2787]:=
```
Exponential`D[u_, α_] := (2 E^(-(2 Sqrt[1 - u^2])/(u α)))/(π u^4 α^2) HeavisideTheta[u]
```

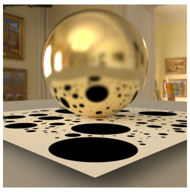
$$\text{Exponential`D}[u\_, \alpha\_] := \frac{2\, e^{-\frac{2\sqrt{1-u^2}}{u\,\alpha}}}{\pi\, u^4\, \alpha^2}\ \text{HeavisideTheta}[u]$$

comparison to Beckmann and GGX

In[4189]:=
```
Clear[im];
```

im[exp] := ;

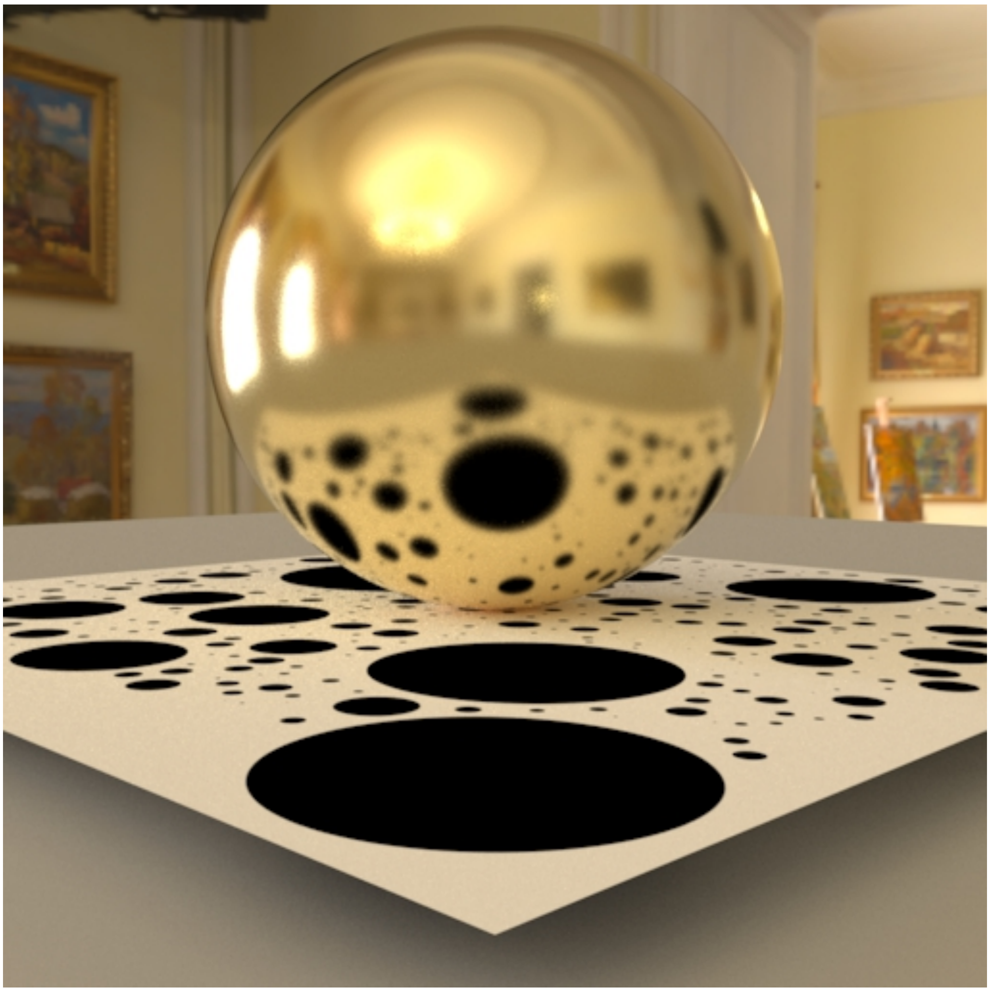
im[ggx] := ;

im[beckmann] := ;

In[4192]:=
```
Manipulate[Show[im[s], ImageSize → 512], {s, {exp, ggx, beckmann}}]
```

s exp ggx beckmann

Out[4192]=



relationship to Beckmann

The Exponential NDF can be written as a Gamma superposition of Beckmanns:

In[1900]:=
```
Integrate[Beckmann`D[u, α √m] × PDF[GammaDistribution[3/2, 1]][m],
  {m, 0, Infinity}, Assumptions → 0 < u < 1 && α > 0 && a > 0]
```

Out[1900]=

$$\frac{2\, e^{-\frac{2\sqrt{1-u^2}}{u\,\alpha}}}{\pi\, u^4\, \alpha^2}$$

or as a Chi-3 superposition:

In[2306]:=
```
Integrate[Beckmann`D[u, α/√2 m] × PDF[ChiDistribution[3]][m],
 {m, 0, Infinity}, Assumptions → 0 < u < 1 && α > 0 && a > 0]
```

Out[2306]=

$$\frac{2\, e^{-\frac{2\sqrt{1-u^2}}{u\,\alpha}}}{\pi\, u^4\, \alpha^2}$$

importance sampling (for Walter's method)

Use the Chi-3 expansion of the NDF we can sample the full NDF using 3 Gaussian random variates and 1 uniform random variate:

$$1\Big/\left(\sqrt{\left(1-\left(\frac{\alpha}{\sqrt{2}}\,\mathrm{Norm}[\{g_1, g_2, g_3\}]\right)^2 (\mathrm{Log}[u_1])\right)}\right)$$

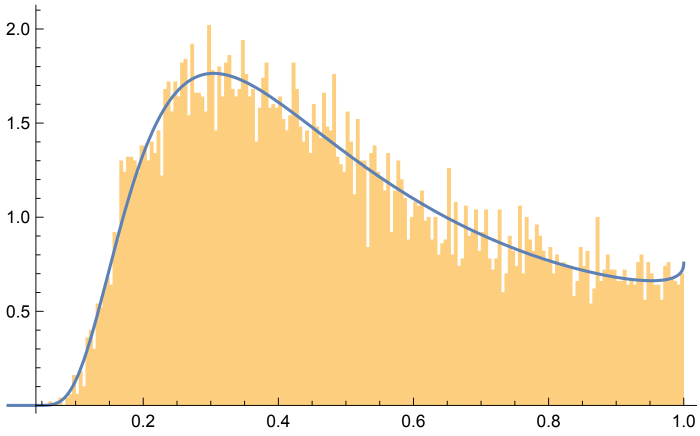
Out[2939]=

$$\frac{1}{\sqrt{1-\frac{1}{2}\alpha^2\left(\mathrm{Abs}[g_1]^2+\mathrm{Abs}[g_2]^2+\mathrm{Abs}[g_3]^2\right)\mathrm{Log}[u_1]}}$$

In[2932]:=
```
With[{α = 2.3},
 Show[
  Histogram[Table[
    1/(√(1 - (α/√2 Norm[{RandomVariate[NormalDistribution[]], RandomVariate[
                NormalDistribution[]],
               RandomVariate[NormalDistribution[]]}])^2
          (Log[RandomReal[]]))), {i, Range[10 000]}], 200, "PDF"],
  Plot[Exponential`D[u, α] 2 Pi u, {u, 0, 1}, PlotRange → All]
 ]
]
```

... General: Exp[-42566.1] is too small to represent as a normalized machine number; precision may be lost.

Out[2932]=



shape invariant f(x)

In[1885]:=
$$\text{FullSimplify}\left[\text{Exponential`D}[u,\alpha]\,u^4\,\alpha^2\ /.\ u \to \frac{1}{\sqrt{1+x^2\,\alpha^2}},\ \text{Assumptions}\to 1-\frac{1}{\sqrt{1+x^2\,\alpha^2}}>0\,\&\&\,x>0\,\&\&\,\alpha>0\,\&\&\,a>0\right]$$

Out[1885]=
$$\frac{2\,e^{-2x}}{\pi}$$

distribution of slopes

In[1887]:=
$$\text{FullSimplify}\left[\text{Exponential`D}\left[\frac{1}{\sqrt{p^2+q^2+1}},\alpha\right]\left(\frac{1}{\sqrt{p^2+q^2+1}}\right)^4,\ \text{Assumptions}\to 0<\alpha<1\,\&\&\,p>0\,\&\&\,q>0\right]$$

Out[1887]=
$$\frac{2\,e^{-\frac{2\sqrt{p^2+q^2}}{\alpha}}}{\pi\,\alpha^2}$$

In[1889]:=
$$\text{Exponential`P22}[p\_,q\_,\alpha\_]:=\frac{2\,e^{-\frac{2\sqrt{p^2+q^2}}{\alpha}}}{\pi\,\alpha^2}$$

In[1890]:=
```
Integrate[Exponential`P22[p, q, α], {p, -Infinity, Infinity},
  {q, -Infinity, Infinity}, Assumptions → 0 < α < 1]
```

Out[1890]= 1

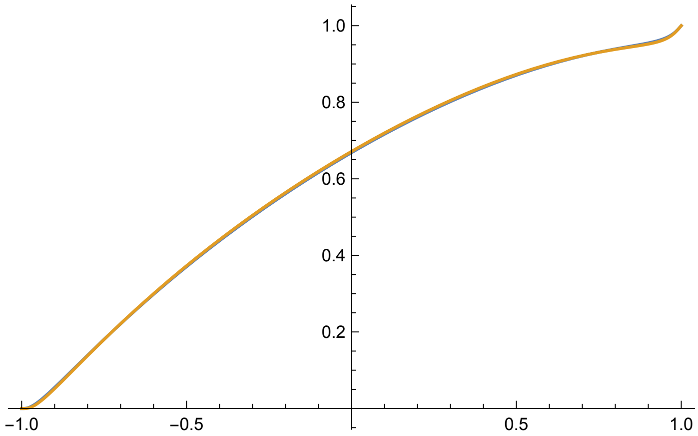
sigma - approximate with Ei sigma:

The cross section and Lambda() function are not known analytically, but we noted the following close approximation:

In[•]:=
$$\text{Ei`}\sigma[u\_,\alpha\_]:=\frac{1}{6\sqrt{\pi}\left(-1+u^2\right)\alpha^2}\left(\alpha\left(3\sqrt{\pi}\,u\left(-1+u^2\right)\alpha+2\,e^{\frac{u^2}{\left(-1+u^2\right)\alpha^2}}\sqrt{1-u^2}\left(-\alpha^2+u^2\left(-1+\alpha^2\right)\right)\right)+3\sqrt{\pi}\,u\left(-\alpha^2+u^2\left(-2+\alpha^2\right)\right)\text{Erf}\left[\frac{u}{\sqrt{1-u^2}\,\alpha}\right]+2\sqrt{\pi}\,u^2\,\text{Abs}[u]\left(1+2\,\text{Erf}\left[\frac{u^2\sqrt{1-u^2}}{\alpha\,\text{Abs}[u]-u^2\,\alpha\,\text{Abs}[u]}\right]\right)\right)$$

In[1948]:=
```
With[{α = 2.1},
 Plot[{
   Quiet[NIntegrate[Exponential`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
   Quiet[Ei`σ[u, 1.7 α]]
  }, {u, -1, 1}]
]
```

Out[1948]=



G1 shadow

In[∘]:=

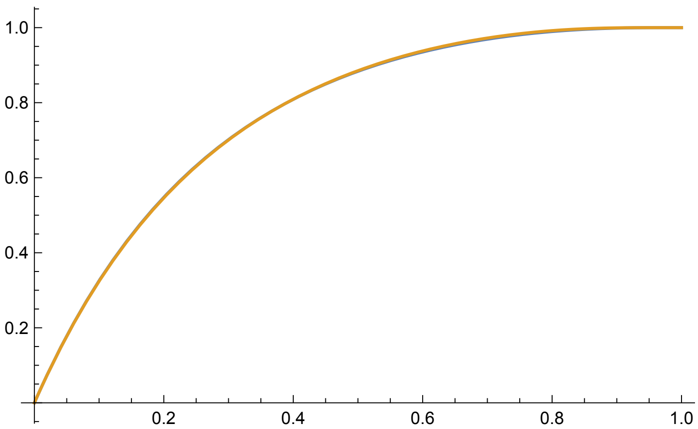
$$\text{Ei`}\Lambda[\text{u\_}, \alpha\_] := \frac{2\, e^{\frac{u^2}{(-1+u^2)\,\alpha^2}} \sqrt{1-u^2}\, \alpha \left(-\alpha^2 + u^2 \left(-1+\alpha^2\right)\right) + \sqrt{\pi}\, u \left(3\, \alpha^2 + u^2 \left(2 - 3\, \alpha^2\right)\right) \text{Erfc}\left[\frac{u}{\sqrt{1-u^2}\ \alpha}\right]}{6 \sqrt{\pi}\, u \left(-1+u^2\right) \alpha^2}$$

In[2628]:=

$$\text{Ei`G1}[\text{u\_}, \alpha\_] := \frac{1}{1 + \text{Ei`}\Lambda[u, \alpha]}$$

In[2634]:=
```
With[{α = .8},
 Plot[{
   Quiet[u / NIntegrate[Exponential`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
   Quiet[Ei`G1[u, 1.7 α]]
  }, {u, 0, 1}, PlotRange → All]
]
```

Out[2634]=



Rational approximation for G1:

Similar to Walter's approximation for Beckmann G1 we find:

In[3048]:=
```
Exponential`G1approx[u_, α_] := If[#1 < 1.5,
    (4.15428848706139`*^-8 + 5.3171271839868055` #1 + 1.5604787398328208` #1^2) /
     (1 + 2.9490717540313374` #1 + 3.812617848664264` #1^2 -
       1.0148658798039354` #1^3 + 0.18113230976734568` #1^4)
    ,
    1] &[1/1.7 α^-1 (Sqrt[1 - u^2]/u)^-1]
```

In[2938]:=
```
With[{α = 0.8},
 Plot[{
   Exponential`G1approx[u, α],
   u / Quiet[NIntegrate[Exponential`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]]
  }, {u, 0, 1}, PlotRange → All]
]
```

Out[2938]=