Student-T NDF

This is code to accompany the book:

A Hitchhiker’s Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

notation

u = **m** . **n** = Cos[θ_m]

α = roughness

---

definitions and derivations

In[2785]:= $$\text{ST`D}[u\_, \alpha\_, \gamma\_] := \frac{\left(1+\frac{1-u^2}{u^2\,\alpha^2\,(-1+\gamma)}\right)^{-\gamma}}{\pi\, u^4\, \alpha^2}\ \text{HeavisideTheta}[u]$$

In[1183]:= $$\text{ST`}\sigma[u\_, \alpha\_, \gamma\_] := \frac{u}{2} + \frac{\alpha\left(1-\frac{u^2}{(-1+u^2)\,\alpha^2\,(-1+\gamma)}\right)^{\frac{3}{2}-\gamma}\sqrt{(1-u^2)\,(-1+\gamma)}\ \text{Gamma}\left[-\frac{3}{2}+\gamma\right]}{2\sqrt{\pi}\ \text{Gamma}[-1+\gamma]} + \frac{u^2\ \text{Gamma}\left[-\frac{1}{2}+\gamma\right]\ \text{Hypergeometric2F1}\left[\frac{1}{2}, -\frac{1}{2}+\gamma, \frac{3}{2}, \frac{u^2}{(-1+u^2)\,\alpha^2\,(-1+\gamma)}\right]}{\sqrt{\pi}\ \alpha\sqrt{(1-u^2)\,(-1+\gamma)}\ \text{Gamma}[-1+\gamma]}$$

In[1184]:= $$\text{ST`}\Lambda[u\_, \alpha\_, \gamma\_] := \frac{1}{4\sqrt{\pi}\ u\ \text{Gamma}[-1+\gamma]}\left(2\,\alpha^{-2+2\gamma}\left(u^2-(-1+u^2)\,\alpha^2\,(-1+\gamma)\right)^{\frac{3}{2}-\gamma}\left((1-u^2)\,(-1+\gamma)\right)^{-1+\gamma} + (-1)^{-\gamma}\,u\,(-3+2\gamma)\ \text{Beta}\left[\frac{(-1+u^2)\,\alpha^2\,(-1+\gamma)}{u^2}, -1+\gamma, \frac{3}{2}-\gamma\right]\right)\text{Gamma}\left[-\frac{3}{2}+\gamma\right]$$

In[1185]:= $(1+\text{ST`}\Lambda[u,\alpha,\gamma])\ u == \text{ST`}\sigma[u,\alpha,\gamma]$ /. $\gamma\to 3$ /. $u\to 1/2$ /. $\alpha\to 1/3$ // FullSimplify

Out[1185]= True

In[1187]:= $\text{ST`}\Lambda[u,\alpha,\gamma]\ u == \text{ST`}\sigma[-u,\alpha,\gamma]$ /. $\gamma\to 3$ /. $u\to 1/2$ /. $\alpha\to 1/3$ // FullSimplify

Out[1187]= True

In[2705]:=
```
FullSimplify[ST`Λ[u, u/(Sqrt[1 - u^2] x), γ], Assumptions → 0 < u < 1 && x > 0 && γ > 3/2]
```

Out[2705]=
$$\frac{1}{4\sqrt{\pi}\,x}(-1+\gamma)^{\gamma}\,\mathrm{Gamma}\left[-\frac{3}{2}+\gamma\right]\left(\frac{2\left(-1+x^2+\gamma\right)^{\frac{3}{2}-\gamma}}{\mathrm{Gamma}[\gamma]}-\frac{x^{3-2\gamma}\,(-3+2\gamma)\,\mathrm{Hypergeometric2F1Regularized}\left[-1+\gamma,-\frac{1}{2}+\gamma,\gamma,\frac{1-\gamma}{x^2}\right]}{-1+\gamma}\right)$$

shape invariant f(x)

In[1240]:=
```
FullSimplify[ST`D[u, α, γ] u^4 α^2 /. u -> 1/Sqrt[1 + x^2 α^2],
  Assumptions → 1 - 1/Sqrt[1 + x^2 α^2] > 0 && α > 2]
```

Out[1240]=
$$\frac{\left(\frac{-1+\gamma}{-1+x^2+\gamma}\right)^{\gamma}}{\pi}$$

Relationship to Beckmann NDF:

In[2766]:=
```
Integrate[PDF[GammaDistribution[γ - 1, 1]][αB] × Beckmann`D[u, α Sqrt[(γ - 1)/αB]],
  {αB, 0, Infinity}, Assumptions → α > 0 && 0 < u < 1 && γ > 3/2]
```

Out[2766]=
$$\frac{\left(1+\frac{1-u^2}{u^2\,\alpha^2\,(-1+\gamma)}\right)^{-\gamma}}{\pi\,u^4\,\alpha^2}$$

height field normalization

In[2783]:=
```
Integrate[2 Pi u ST`D[u, α, γ], {u, 0, 1}, Assumptions → 0 < α && γ > 3/2]
```

Out[2783]= 1

distribution of slopes

In[1206]:=
```
FullSimplify[ST`D[1/Sqrt[p^2 + q^2 + 1], α, γ] (1/Sqrt[p^2 + q^2 + 1])^4,
  Assumptions → 0 < α < 1 && p > 0 && q > 0]
```

Out[1206]=
$$\frac{\alpha^{-2+2\gamma}\left(p^2+q^2+\alpha^2\,(-1+\gamma)\right)^{-\gamma}(-1+\gamma)^{\gamma}}{\pi}$$

In[1208]:=
```
ST`P22[p_, q_, α_, γ_] := (α^(-2+2 γ) (p^2 + q^2 + α^2 (-1 + γ))^-γ (-1 + γ)^γ)/π
```

In[1209]:=
```
Integrate[ST`P22[p, q, α, γ], {p, -Infinity, Infinity},
  {q, -Infinity, Infinity}, Assumptions → 0 < α < 1 && γ > 2]
```

Out[1209]= 1

In[1210]:=
```
Integrate[ST`P22[p, q, α, γ],
  {q, -Infinity, Infinity}, Assumptions → 0 < α < 1 && γ > 2]
```

Out[1210]= ConditionalExpression$\left[\dfrac{\alpha^{-2+2\gamma}\left(p^2+\alpha^2(-1+\gamma)\right)^{\frac{1}{2}-\gamma}(-1+\gamma)^{\gamma}\,\mathrm{Gamma}\left[-\frac{1}{2}+\gamma\right]}{\sqrt{\pi}\,\mathrm{Gamma}[\gamma]},\ \alpha^2\gamma+\mathrm{Re}\left[p^2\right]>\alpha^2\right]$

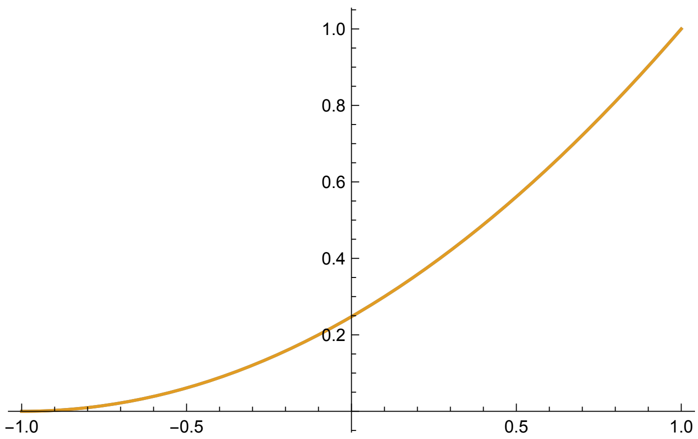
In[1211]:= ST`P2[p_, α_, γ_] := $\dfrac{\alpha^{-2+2\gamma}\left(p^2+\alpha^2(-1+\gamma)\right)^{\frac{1}{2}-\gamma}(-1+\gamma)^{\gamma}\,\mathrm{Gamma}\left[-\frac{1}{2}+\gamma\right]}{\sqrt{\pi}\,\mathrm{Gamma}[\gamma]}$

compare σ to delta integral:

In[1157]:= Delta`σ[u_, ui_] := $\mathrm{Re}\left[2\left(\sqrt{1-u^2-ui^2}+u\,ui\,\mathrm{ArcCos}\left[-\dfrac{u\,ui}{\sqrt{1-u^2}\sqrt{1-ui^2}}\right]\right)\right]$

In[1166]:=
```
With[{α = .7, γ = 3},
  Plot[{
     Quiet[NIntegrate[ST`D[ui, α, γ] × Delta`σ[u, ui], {ui, 0, 1}]],
    ST`σ[u, α, γ]
   }, {u, -1, 1}]
]
```

Out[1166]=



importance sampling

[Ribardiere et al. 2017]:

In[4521]:=

```
With[{α = 1.2, γ = 2.3},
 Show[
  Histogram[Table[
```

$$\frac{1}{\sqrt{1+\left(\alpha\sqrt{(\gamma-1)\ \left(\left(1-\text{RandomReal}[\,]\right)^{1/(1-\gamma)}-1\right)}\right)^{2}}},$$

```
    {i, Range[100 000]}], 200, "PDF"],
  Plot[ST`D[u, α, γ] 2 Pi u, {u, 0, .999}, PlotRange → All]
 ]
]
```

Out[4521]=