Fresnel Boundaries

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com

Non-polarized

Reflection and Refraction

Helper functions to reflect and refract vectors:

In[2300]:=
```
refract[w_, n_, eta1_, eta2_] := -eta1/eta2 (w - (w.n) n) - (Sqrt[1 - (eta1/eta2)^2 (1 - (w.n)^2)]) n;

reflect[v_, n_] := -v + 2 n n.v;
```

$$\text{refract}[w\_, n\_, \text{eta1}\_, \text{eta2}\_] := \frac{-\text{eta1}}{\text{eta2}}(w - (w.n)\,n) - \left(\sqrt{1 - \left(\frac{\text{eta1}}{\text{eta2}}\right)^2 \left(1 - (w.n)^2\right)}\right) n;$$

Dielectric Fresnel

Dielectric reflectance for incoming light with incoming cosine *costhetai* and ratio of internal to external indices of *etaratio*:

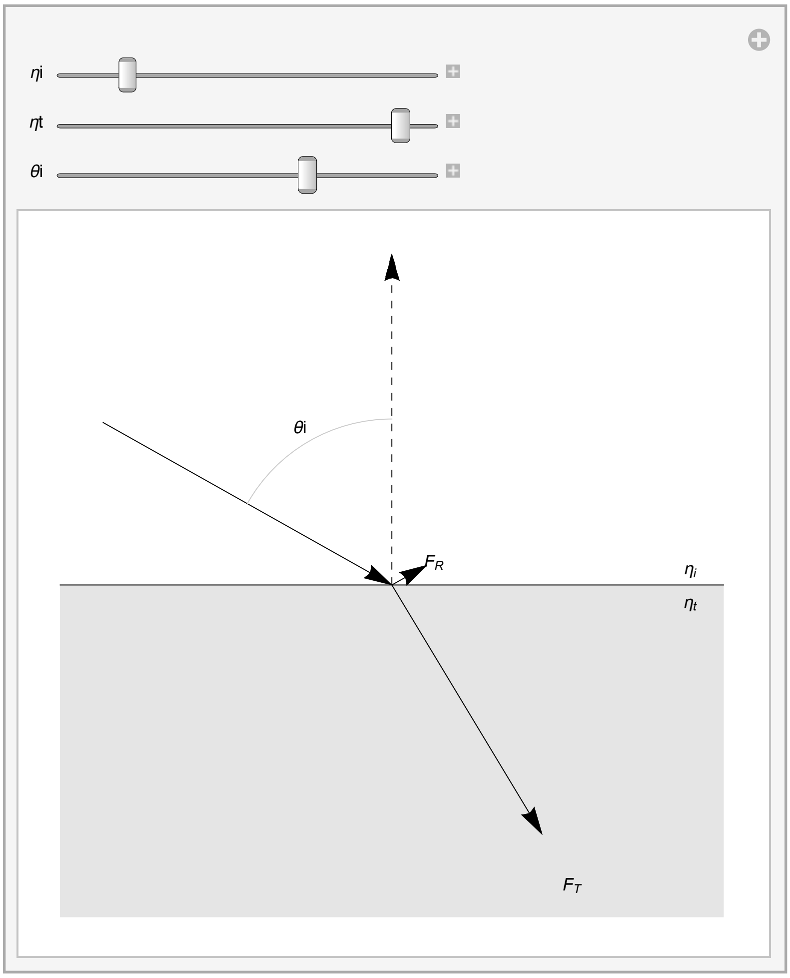
In[2302]:=
```
evalF[g_, c_] := 1/2 (g - c)^2/(g + c)^2 (1 + (c (g + c) - 1)^2/(c (g - c) + 1)^2);
FR[etaratio_, costhetai_] := If[
  etaratio^2 - 1 + costhetai^2 ≥ 0,
  evalF[Sqrt[etaratio^2 - 1 + costhetai^2], costhetai],
  1]
```

$$\text{evalF}[g\_, c\_] := \frac{1}{2}\frac{(g-c)^2}{(g+c)^2}\left(1 + \frac{(c\,(g+c)-1)^2}{(c\,(g-c)+1)^2}\right);$$

interactive Snell's law explorer

```
In[•]:= Manipulate[
  plotFresnel = Graphics[
    {
     Gray,
     GrayLevel[.9],
     Rectangle[{-1, -1}, {1, 0}],
     Black,
     Line[{{-1, 0}, {1, 0}}],
     {Dashed,
      Arrow[{{0, 0}, {0, 1}}]},
     Arrow[{{-Sin[θi], Cos[θi]}, {0, 0}}],
     Arrow[{{0, 0}, FR[ηt/ηi, Cos[θi]] {Sin[θi], Cos[θi]}}],
     Arrow[{{0, 0},
        (1 - FR[ηt/ηi, Cos[θi]]) refract[{-Sin[θi], Cos[θi]}, {0, 1}, ηi, ηt]}],
     Text["ηᵢ", {.9, 0.05}],
     Text["ηₜ", {.9, -0.05}],
     Text["F_R", 1.2 FR[ηt/ηi, Cos[θi]] {Sin[θi], Cos[θi]}],
     Text["F_T",
      1.2 (1 - FR[ηt/ηi, Cos[θi]]) refract[{-Sin[θi], Cos[θi]}, {0, 1}, ηi, ηt]],
     GrayLevel[.8],
     Circle[{0, 0}, 0.5, {Pi/2, Pi/2 + θi}],
     Black,
     Text["θi", .55 {-Sin[θi/2], Cos[θi/2]}]
    }
   ], {ηi, 1, 2}, {ηt, 1, 2}, {θi, 0, Pi/2}]
```

Out[ ]=



Benchmark data

In[2335]:=
```
ns = {5/10, 7/10, 9/10, 99/100, 101/100, 11/10, 14/10, 2};
FRdata = Table[NumberForm[FR[n, Cos[t]], {8, 8}],
   {n, ns}, {t, {0., 0.2, 0.5, 1., 1.2, 1.5}}];
Transpose[Join[{Table[N[n], {n, ns}]}, Transpose[FRdata]]] // Grid
```

Out[2337]=

0.5	0.11111111	0.11175198	0.28326783	1	1	1
0.7	0.03114187	0.03124674	0.03870036	1	1	1
0.9	0.00277008	0.00277588	0.00309559	0.04529667	1	1
0.99	0.00002525	0.00002530	0.00002756	0.00018462	0.00132454	1
1.01	0.00002475	0.00002479	0.00002690	0.00016095	0.00096253	0.14322539
1.1	0.00226757	0.00227070	0.00242550	0.00987357	0.03762339	0.50722589
1.4	0.02777778	0.02779989	0.02881388	0.06118057	0.13573578	0.65791346
2.	0.11111111	0.11114523	0.11262121	0.15018951	0.22278595	0.68340009

Hemispherical Albedo - Smooth Dielectric

[Dunkle 1963, Ozisik 1973 p. 60]

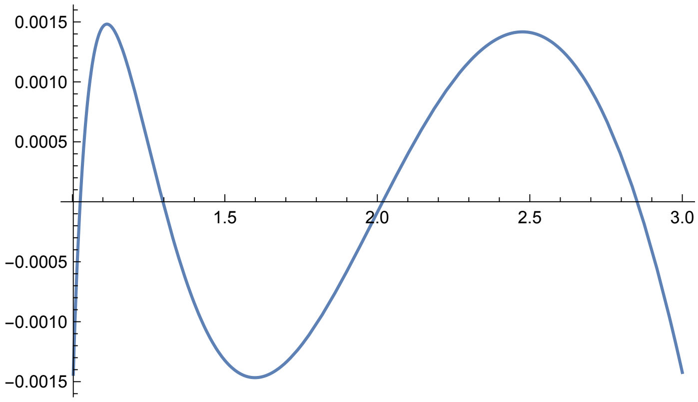
In[∘]:= $$\text{DielectricHemisphericalAlbedo}[n\_] := \frac{1}{2} + \frac{(n-1)\,(3\,n+1)}{6\,(n+1)^2} - \frac{2\,n^3\,(n^2+2\,n-1)}{(n^2+1)\,(n^4-1)} + \frac{8\,n^4\,(n^4+1)}{(n^2+1)\,(n^4-1)^2}\,\text{Log}[n] + \frac{n^2\,(n^2-1)^2}{(n^2+1)^3}\,\text{Log}\left[\frac{n-1}{n+1}\right]$$

Our approximation [Aug 2019] for $1 < \eta < 3$:

In[∘]:= $$\text{DielectricHemisphericalAlbedoApprox}[n\_] := \text{Log}\left[\frac{10\,893\,n - 1438.2}{1 + 10\,212\,n - 774.4\,n^2}\right]$$

Error plot:

In[∘]:=
```
Plot[DielectricHemisphericalAlbedo[n] -
  DielectricHemisphericalAlbedoApprox[n], {n, 1.003, 3}]
```

Out[∘]=


Hemispherical Albedo - Smooth Dielectric - Flatland

[Allen 1973]

In[2295]:= $$\text{DielectricHemisphericalAlbedoFlatland}[n\_] := \frac{1}{15}\left(-\frac{2\,(1+31\,n^2+15\,n^4+31\,n^6+n^8)\,\text{EllipticE}\left[\frac{1}{1-n^2}\right]}{(-1+n^2)^{3/2}\,(1+n^2)^2} + \frac{1}{(-1+n^4)^3}\left((-1+n)\,(1+n)\,\left((1+n^2)\,(7+24\,n^2+102\,n^4+10\,n^6+15\,n^8) + 30\,n^3\,\sqrt{1+n^2}\,(2+n^2+2\,n^4)\,\text{ArcSinh}[n]\right) + 2\,n^2\,\sqrt{-1+n^2}\,\left((-22-15\,n^2-47\,n^4-n^6+9\,n^8+n^{10})\,\text{EllipticK}\left[\frac{1}{1-n^2}\right] + 15\,n^4\,(2+n^2+2\,n^4)\,\text{EllipticPi}\left[1+n^2,\frac{1}{1-n^2}\right]\right)\right)\right)$$

Conductor Fresnel

Exact

In[2308]:=
```
Clear[p, q, rhoPerp, rhoPar, Rs, Rp];
p[ni_, n_, k_, theta_] :=
```
$$\sqrt{\left(\frac{1}{2}\left(\sqrt{\left(n^2-k^2-ni^2\,\mathrm{Sin}[theta]^2\right)^2+4\,n^2\,k^2}+\left(n^2-k^2-ni^2\,\mathrm{Sin}[theta]^2\right)\right)\right)};$$
```
q[ni_, n_, k_, theta_] :=
```
$$\sqrt{\left(\frac{1}{2}\left(\sqrt{\left(n^2-k^2-ni^2\,\mathrm{Sin}[theta]^2\right)^2+4\,n^2\,k^2}-\left(n^2-k^2-ni^2\,\mathrm{Sin}[theta]^2\right)\right)\right)};$$

In[2311]:=
$$\mathrm{rhoPerp}[p\_,\,q\_,\,n1\_,\,t\_] := \frac{\left(n1\,\mathrm{Cos}[t]-p\right)^2+q^2}{\left(n1\,\mathrm{Cos}[t]+p\right)^2+q^2};$$
$$\mathrm{rhoPar}[p\_,\,q\_,\,n1\_,\,t\_] := \frac{\left(p-n1\,\mathrm{Sin}[t]\,\mathrm{Tan}[t]\right)^2+q^2}{\left(p+n1\,\mathrm{Sin}[t]\,\mathrm{Tan}[t]\right)^2+q^2}\,\mathrm{rhoPerp}[p,\,q,\,n1,\,t]$$

In[2313]:=
$$\mathrm{FR}[ni\_,\,n\_,\,k\_,\,t\_] := \frac{1}{2}\left(\mathrm{rhoPerp}[p[ni,n,k,t],\,q[ni,n,k,t],\,ni,\,t]+\mathrm{rhoPar}[p[ni,n,k,t],\,q[ni,n,k,t],\,ni,\,t]\right)$$

Reflectance at normal incidence:

In[∘]:=
$$\mathrm{ConductorNormalReflectance}[ni\_,\,n\_,\,k\_] := 1-\frac{4\,ni\,n}{\left(ni+n\right)^2+k^2}$$

modified Gulbrandsen mapping for ni = 1:

These two functions take F0 (**r**) and **G** tint parameter and map to n + i k:

In[∘]:=
$$\mathrm{nmaprG}[r\_,\,G\_] := \frac{-1+\left(2-4\left(1-\mathrm{Sin}\left[\frac{G\,\pi}{2}\right]^2\right)\right)\sqrt{r}-r}{-1+r}$$

In[∘]:=
$$\mathrm{kmaprG}[r\_,\,G\_] := \sqrt{\frac{-\left(-1+\frac{-1-r+\sqrt{r}\,\left(2-4\left(1-\mathrm{Sin}\left[\frac{G\pi}{2}\right]^2\right)\right)}{-1+r}\right)^2+r\left(1+\frac{-1-r+\sqrt{r}\,\left(2-4\left(1-\mathrm{Sin}\left[\frac{G\pi}{2}\right]^2\right)\right)}{-1+r}\right)^2}{1-r}}$$

These two functions find **G** from n + i k.   (**r** is ConductorNormalReflectance[1,n,k])

In[∘]:=
$$\mathrm{gmap}[n\_,\,k\_] := 1-\frac{-1-k^2+n^2+\sqrt{\left(1+k^2\right)^2+2\left(-1+k^2\right)n^2+n^4}}{2\sqrt{\left(1+k^2\right)^2+2\left(-1+k^2\right)n^2+n^4}}$$

In[∘]:=
$$\mathrm{Gmap}[n\_,\,k\_] := \frac{2\,\mathrm{ArcSin}\left[\sqrt{\mathrm{gmap}[n,\,k]}\right]}{\pi}$$

Schlick's Approximation

In[∘]:=
```
mix[a_, b_, t_] := b t + (1 - t) a;
FRSchlickFresnel[ni_, n_, k_, theta_] := mix[
  ConductorReflectance[ni, n, k],
  1,
  (1 - Cos[theta])^5
 ]
```

Additional approximate form

Mentioned in [Pharr and Humphreys - Physically Based Rendering], first edition, [9.1], [9.2] (more accurate for larger k). See also [Dunkle 1963]

In[2307]:=
```
FRPharrHumphreys[n_, k_, t_] :=
```
$$\frac{1}{2}\left(\frac{(n^2+k^2)\,\text{Cos}[t]^2-2\,n\,\text{Cos}[t]+1}{(n^2+k^2)\,\text{Cos}[t]^2+2\,n\,\text{Cos}[t]+1}+\frac{(n^2+k^2)-2\,n\,\text{Cos}[t]+\text{Cos}[t]^2}{(n^2+k^2)+2\,n\,\text{Cos}[t]+\text{Cos}[t]^2}\right)$$

Hemispherical Albedo - Smooth Conductor

approximation 1

[Dunkle 1963, Ozisik 1973 p. 60] - approximation, exact for PharrHumphreys approximate form:

(parallel and perpendicular integrals):

In[∘]:=
```
ConductorAlbedo1[n_, k_] :=
```
$$1-\left(8\,n-8\,n^2\,\text{Log}\left[\frac{1+2\,n+n^2+k^2}{n^2+k^2}\right]+\frac{8\,n\,(n^2-k^2)}{k}\,\text{ArcTan}\left[\frac{k}{n+n^2+k^2}\right]\right)$$

In[2306]:=
```
ConductorAlbedo2[n_, k_] :=
```
$$1-\left(\frac{8\,n}{n^2+k^2}-\frac{8\,n^2}{(n^2+k^2)^2}\,\text{Log}\left[1+2\,n+n^2+k^2\right]+\frac{8\,n\,(n^2-k^2)}{k\,(n^2+k^2)^2}\,\text{ArcTan}\left[\frac{k}{1+n}\right]\right)$$

unpolarized average:

In[2305]:=
```
ConductorHemisphericalAlbedoApprox1[n_, k_] :=
```
$$\frac{1}{2}\,(\text{ConductorAlbedo1}[n,k]+\text{ConductorAlbedo2}[n,k])$$

approximation 2

Our 1st approximation [Aug 2019] for $0.1 < \eta < 4$, $1.7 < k < 8$ - accurate to within about 0.001

In[∘]:=
```
ConductorHemisphericalAlbedoApprox2[n_, k_] :=
 (-8.214737476609672` + (133.7359744765862` - 98.9832978804228` n) n +
    k (-182.3702003078492` + (59.56171376011442` - 3.9828814738544107` n) n) +
    k^2 (-62.591907101534154` + (-13.109290817253394` + 0.3081804514395202` n) n)) /
  ((-395.2681486337069` - 78.47602559071363` n) n +
    k (-187.16625841046798` + (94.65173464738277` - 15.855805815946251` n) n) +
    k^2 (-62.07520207202666` + n (-15.438703954274438` + 1.` n)))
```

approximation 2

Our 2nd approximation [Aug 2019] for $0.1 < \eta < 4$, $1.7 < k < 8$ - accurate to within about 0.00006

In[∘]:=
```
rGfit[r_, G_] :=
 (-2.0451558353360357` + 14.9839211083631` G + 1.3890774118488598` G^2 +
    60.88405996107553` r + 97.32802959940146` G r - 145.19694441443767` G^2 r +
    322.2806535116558` r^2 - 687.9935126644131` G r^2 + 338.6486829760271` G^2 r^2) /
  (230.27759868799` G - 198.15004297964407` G^2 + 512.5359759247783` r -
    1005.5551398186406` G r + 470.70682665699746` G^2 r - 127.45924551986035` r^2 +
    191.70842828834142` G r^2 - 73.78542717613104` G^2 r^2)
```

In[∘]:=
```
ConductorHemisphericalAlbedoApprox3[n_, k_] :=
 rGfit[ConductorReflectance[1, n, k], Gmap[n, k]]
```

Schlick approximation

$$\text{ConductorHemisphericalAlbedoSchlick[n\_, k\_] := } 1 - \frac{80\,n}{21\,(k^2 + (1+n)^2)}$$

Asymptotic behaviour (large k)

In[∘]:=
$$\text{ConductorHemisphericalAlbedoLargek[n\_, k\_] := } 1 - \frac{16\,n}{3\,k^2}$$

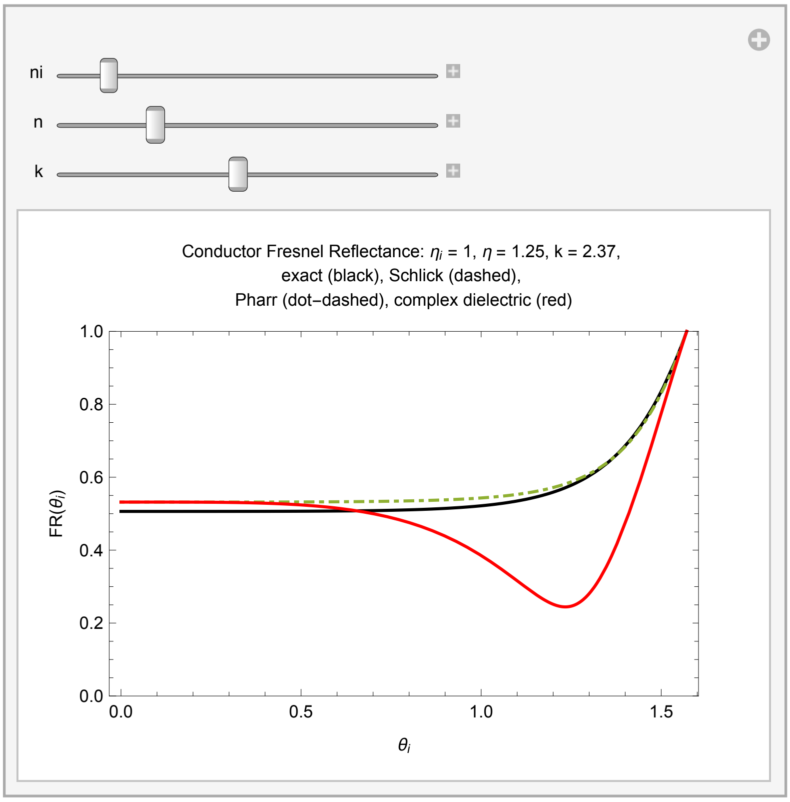
Misunderstood form: Substitute complex index (n+ik) into the *dielectric* formula and take the magnitude of the result, Abs[FR]

Remove the conditional so that this works with complex numbers:

In[2304]:=
$$\text{FR2[etaratio\_, costhetai\_] := evalF}\left[\sqrt{\text{etaratio}^2 - 1 + \text{costhetai}^2},\ \text{costhetai}\right]$$

Conductor Fresnel Reflectance Comparison

```
Manipulate[
 condFRplot = Show[
   Plot[{
     FR[ni, n, k, t],
     FRSchlickFresnel[ni, n, k, t],
     FRPharrHumphreys[n, k, t],
     Abs[FR2[(n + I k), Cos[t]]]
    }, {t, 0, Pi/2}, PlotRange → {0, 1},
    PlotStyle → {Black, Dashed, DotDashed, Red}], Frame → True,
   FrameLabel -> {{FR[θi],},
     {θi, "Conductor Fresnel Reflectance: ηi = 1, η = "<>
       ToString[n] <> ", k = " <> ToString[k] <>
       ", \nexact (black), Schlick (dashed), \nPharr (dot-dashed), complex
         dielectric (red)"}}], {ni, 1, 2}, {n, 0.1, 5}, {k, 0, 5}]
```



Benchmark data

In[2328]:=
```
ns = {1.01, 1.1, 1.4, 2, 10};
ni = 1;
k = 0.5;
FRdata1 = Table[NumberForm[FR[ni, n, k, t], {8, 8}],
   {n, ns}, {t, {0., 0.2, 0.5, 1., 1.2, 1.5}}];
k = 5;
FRdata2 = Table[NumberForm[FR[ni, n, k, t], {8, 8}],
   {n, ns}, {t, {0., 0.2, 0.5, 1., 1.2, 1.5}}];
Join[Transpose[Join[{Table[n, {n, ns}]}, Transpose[FRdata1]]],
  Transpose[Join[{Table[n, {n, ns}]}, Transpose[FRdata2]]]] // Grid
```

Out[2334]=

1.01	0.05829701	0.05837392	0.06178289	0.14339752	0.27067729	0.77121604
1.1	0.05579399	0.05585623	0.05861520	0.12773108	0.24394224	0.75157831
1.4	0.06821963	0.06826528	0.07026499	0.12163143	0.21644127	0.71680339
2	0.13513514	0.13517164	0.13673298	0.17490309	0.24661001	0.69400413
10	0.67010309	0.67006997	0.66870975	0.64017903	0.59473323	0.50049776
1.01	0.86088202	0.86086219	0.86006999	0.84542467	0.82700656	0.88043233
1.1	0.85039102	0.85037015	0.84953608	0.83411201	0.81472471	0.87118142
1.4	0.81794538	0.81792166	0.81697257	0.79937929	0.77724082	0.84183787
2	0.76470588	0.76467863	0.76358448	0.74310011	0.71697927	0.78954494
10	0.72602740	0.72599530	0.72467855	0.69690083	0.65143060	0.52292443