MacDonald kernel: **H-function**

Definition and application

This H function arises for isotropic scattering problems including:

- classical exponential random flights in Flatland
- BesselK0 random flights in the 1D rod
- $\frac{2\,s\,\mathrm{BesselK}[1,s]}{\pi}$ random flights in 3D
- $\frac{1}{2}\,e^{-s}\,(1+s)$ random flights in 4D
- $\frac{2^{\frac{1}{2}-\frac{d}{2}}\,d\,s^{\frac{1}{2}(-1+d)}\,\mathrm{BesselK}\left[\frac{1}{2}(-1+d),s\right]}{\sqrt{\pi}\,\mathrm{Gamma}\left[1+\frac{d}{2}\right]}$ random flights in dD

References

- Fock, V. 1944. Some integral equations of mathematical physics. In: Doklady AN SSSR, vol. 26, 147–51, http://mi.mathnet.ru/eng/msb6183.
- Case, K. M. 1957. On Wiener-Hopf equations. Ann. Phys. (USA) 2(4): 384–405. doi:10. 1016/0003-4916(57)90027-1
- Krein, M. G. 1962. Integral equations on a half-line with kernel depending upon the difference of the arguments. Amer. Math. Soc. Transl. 22: 163–288.
- Eugene d'Eon & M. M. R. Williams (2018): Isotropic Scattering in a Flatland Half-Space, *Journal of Computational and Theoretical Transport*, DOI: 10.1080/23324309.2018.1544566
- Eugene d’Eon & Norman J. McCormick (2019) Radiative Transfer in Half Spaces of Arbitrary Dimension, *Journal of Computational and Theoretical Transport*, 48:7, 280-337, DOI: 10.1080/23324309.2019.1696365

Explicit general solution

The H-function is known explicitly by adapting a derivation of V.A. Fock 1944 [d’Eon and McCormick 2019, Eq.(B.8)].

In[1206]:= $\mathrm{IFock}[x\_] := -2\,(x)\,\mathrm{ArcTanh}\left[e^{i\,(x)}\right] - 2\,i\,\mathrm{PolyLog}\left[2,\,e^{i\,(x)}\right] + \frac{1}{2}\,i\,\mathrm{PolyLog}\left[2,\,e^{2\,i\,(x)}\right]$

In[1207]:= $\mathrm{MacDonald`H}[u\_,\,c\_] := \sqrt{\frac{1+u}{1+u\,\sqrt{1-c^2}}}\,\mathrm{Abs}\left[\mathrm{Exp}\left[\frac{1}{\mathrm{Pi}}\,\mathrm{IFock}[\mathrm{ArcSec}[u]+\mathrm{ArcSin}[c]]\right]\right]$

Identities

A unique identify for the MacDonald kernel is [d’Eon and Williams 2018, Eq.(A.7)]

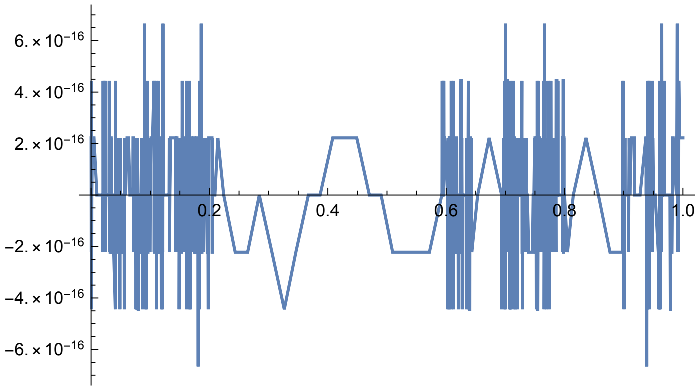
Out[•]//TraditionalForm=

$$H(u,\,c)\,H(u,\,-c) = \frac{1+u}{1+\sqrt{1-c^2}\;u}$$

In[•]:=

$$\texttt{With}\Big[\{\texttt{c = 0.9}\},\ \texttt{Plot}\Big[\Big\{\texttt{MacDonald`H[u, c]} \times \texttt{MacDonald`H[u, -c]} - \left(\frac{(1+u)}{1+\sqrt{1-c^2}\;u}\right)\Big\},\ \{\texttt{u, 0, 1}\}\Big]\Big]$$

Out[•]=



special case c = 1:

[d’Eon and Williams 2018]

In[•]:=

$$\texttt{MacDonald`Hc1[u\_, 1] :=}\ \sqrt{1+u}\ \texttt{Exp}\left[\texttt{Re}\left[\frac{\texttt{HypergeometricPFQ}\left[\left\{\frac{1}{2},1,1\right\},\left\{\frac{3}{2},\frac{3}{2}\right\},\frac{1}{u^2}\right]}{\pi\,u}\right]\right]$$

In[•]:=

$$\texttt{MacDonald`Hc2[u\_, 1] :=}\ \sqrt{1+u}\ \texttt{Exp}\left[\frac{\texttt{HypergeometricPFQ}\left[\left\{\frac{1}{2},1,1\right\},\left\{\frac{3}{2},\frac{3}{2}\right\},\frac{1}{u^2}\right]}{\pi\,u}+\frac{\texttt{ArcCosh[u]}}{2}\right]$$

In[•]:=

$$\texttt{MacDonald`Hc3[u\_, 1] :=}\ \texttt{Chop}\left[\sqrt{(1+u)}\ e^{\frac{i\left(\texttt{PolyLog}\left[2,-\frac{i\left(1+\sqrt{1-u^2}\right)}{u}\right]-\texttt{PolyLog}\left[2,\frac{i\left(1+\sqrt{1-u^2}\right)}{u}\right]\right)}{\pi}}\left(-i\,u-\sqrt{1-u^2}\right)^{-\frac{\texttt{ArcSec[u]}}{\pi}}\right]$$

In[•]:=

$$\texttt{N}\left[\{\texttt{MacDonald`Hc1[u, 1], MacDonald`Hc1[u, 1], MacDonald`Hc1[u, 1]}\}\ \texttt{/.}\ u \to \frac{1}{3}\right]$$

Out[•]= {1.55799, 1.55799, 1.55799}

special case $\mu = 1$:

[d’Eon and Williams 2018]

In[•]:=
```
MacDonald`Hu1[1, c_] :=
 Sqrt[2/(1 + Sqrt[1 - c^2])] Exp[c/Pi HypergeometricPFQ[{1/2, 1, 1}, {3/2, 3/2}, c^2]]
```

Benchmark values

Validated against independent implementation by Barry Ganapol, Dec 2019.

In[1212]:=
```
Style[TableForm[
  Table[NumberForm[Chop[N[MacDonald`H[u, c], 12]], 12],
   {u, 1/10, 1, 1/10}, {c, 1/10, 1, 1/10}]
  , TableHeadings → {{"μ=0.1", "μ=0.2", "μ=0.3", "μ=0.4", "μ=0.5", "μ=0.6",
     "μ=0.7", "μ=0.8", "μ=0.9", "μ=1.0"}, {"c=0.1", "c=0.2", "c=0.3",
     "c=0.4", "c=0.5", "c=0.6", "c=0.7", "c=0.8", "c=0.9", "c=1.0"}}], Small]
```

Out[1212]=

	c=0.1	c=0.2	c=0.3	c=0.4	c=0.5	c=0.6	c=0.7
μ=0.1	1.00986237220	1.02035969089	1.03160591899	1.04375534182	1.05702663988	1.07174978069	1.08846
μ=0.2	1.01545117571	1.03214541676	1.05032602070	1.07032520184	1.09261788807	1.11792683741	1.14745
μ=0.3	1.01955400755	1.04090070058	1.06441474026	1.09061229185	1.12023814909	1.15443681754	1.19513
μ=0.4	1.02277520414	1.04783494385	1.07568183575	1.10701407777	1.14284807666	1.18476064184	1.23543
μ=0.5	1.02539994920	1.05352430029	1.08499779016	1.12069475183	1.16189837189	1.21061713441	1.27030
μ=0.6	1.02759251403	1.05830376879	1.09287380232	1.13234524502	1.17825957533	1.23304942534	1.30093
μ=0.7	1.02945790873	1.06238935388	1.09964260728	1.14241993772	1.19251069370	1.25275993701	1.32814
μ=0.8	1.03106780654	1.06592963874	1.10553506623	1.15123714379	1.20506177355	1.27025228749	1.35252
μ=0.9	1.03247341547	1.06903152309	1.11071857647	1.15902972215	1.21621583758	1.28590306999	1.37452
μ=1.0	1.03371259147	1.07177452765	1.11531854174	1.16597352752	1.22620388089	1.30000256645	1.39450

special values

[d’Eon and Williams 2018, Eq.(A.18)]

In[•]:=
```
MacDonald`H[1, 1]
```

Out[•]= $\sqrt{2}\, e^{\frac{2\,\text{Catalan}}{\pi}}$

[d’Eon and McCormick 2019, Eq.(B.13)]

$$H_{2D}(1, c = 1/2) = \frac{2e^{\frac{4C}{3\pi}}}{(2+\sqrt{3})^{2/3}}.$$

In[•]:=
```
FullSimplify[
 Log[MacDonald`H[1, 1/2]] == Log[2 Exp[4 Catalan / (3 Pi)] / (2 + Sqrt[3])^(2/3)]]
```

Out[•]= True

Additional representations

Form 2

[d’Eon and McCormick 2019 - Eq.(B.8)]

In[1214]:=
```
MacDonald`H2[u_, c_] := Sqrt[(1 + u)/(1 + u Sqrt[1 - c^2])]
  Exp[1/(2 Pi) (IFock[ArcSec[u] + ArcSin[c]] - IFock[ArcSec[u] - ArcSin[c]])]
```

In[1215]:=
```
Style[TableForm[
  Table[NumberForm[Chop[N[MacDonald`H2[u, c], 12]], 12],
   {u, 1/10, 1, 1/10}, {c, 1/10, 1, 1/10}]
  , TableHeadings -> {{"μ=0.1", "μ=0.2", "μ=0.3", "μ=0.4", "μ=0.5", "μ=0.6",
     "μ=0.7", "μ=0.8", "μ=0.9", "μ=1.0"}, {"c=0.1", "c=0.2", "c=0.3",
     "c=0.4", "c=0.5", "c=0.6", "c=0.7", "c=0.8", "c=0.9", "c=1.0"}}], Small]
```

Out[1215]=

	c=0.1	c=0.2	c=0.3	c=0.4	c=0.5	c=0.6	c=0.7
μ=0.1	1.00986237220	1.02035969089	1.03160591899	1.04375534182	1.05702663988	1.07174978069	1.08846
μ=0.2	1.01545117571	1.03214541676	1.05032602070	1.07032520184	1.09261788807	1.11792683741	1.14745
μ=0.3	1.01955400755	1.04090070058	1.06441474026	1.09061229185	1.12023814909	1.15443681754	1.19513
μ=0.4	1.02277520414	1.04783494385	1.07568183575	1.10701407777	1.14284807666	1.18476064184	1.23543
μ=0.5	1.02539994920	1.05352430029	1.08499779016	1.12069475183	1.16189837189	1.21061713441	1.27030
μ=0.6	1.02759251403	1.05830376879	1.09287380232	1.13234524502	1.17825957533	1.23304942534	1.30093
μ=0.7	1.02945790873	1.06238935388	1.09964260728	1.14241993772	1.19251069370	1.25275993701	1.32814
μ=0.8	1.03106780654	1.06592963874	1.10553506623	1.15123714379	1.20506177355	1.27025228749	1.35252
μ=0.9	1.03247341547	1.06903152309	1.11071857647	1.15902972215	1.21621583758	1.28590306999	1.37452
μ=1.0	1.03371259147	1.07177452765	1.11531854174	1.16597352752	1.22620388089	1.30000256645	1.39450

Form 3

In[•]:=
```
Sum[(Sqrt[π] Gamma[(1 + i)/2])/(j^2 Gamma[i/2]) Sin[y]^j, {j, 1, Infinity, 2}]
```

Out[•]=
```
HypergeometricPFQ[{1/2, 1, 1}, {3/2, 3/2}, Sin[y]^2] Sin[y]
```

In[•]:=
```
D[HypergeometricPFQ[{1/2, 1, 1}, {3/2, 3/2}, x^2] x, x] // FullSimplify
```

Out[•]=
```
ArcSin[x]/(x Sqrt[1 - x^2])
```

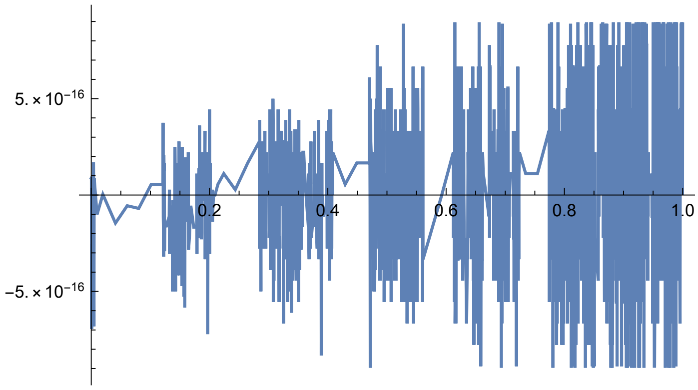
In[•]:=
```
Sum[(Sqrt[π] Gamma[(1 + i)/2])/(j^1 Gamma[i/2]) x^(j - 1), {j, 1, Infinity, 2}]
```

Out[•]=
```
ArcSin[x]/(x Sqrt[1 - x^2])
```

In[1218]:=
```
MacDonald`X[y_] := Re[HypergeometricPFQ[{1/2, 1, 1}, {3/2, 3/2}, Sin[y]^2] Sin[y]]
```

In[•]:=
```
Plot[Re[IFock[x]] - MacDonald`X[x], {x, 0, 1}]
```

Out[•]=



In[1216]:=

$$\text{MacDonald`H3}[u\_,\ c\_] := \sqrt{\frac{1+u}{1+u\sqrt{1-c^2}}}\ \text{Exp}\left[\frac{1}{2\,\text{Pi}}\left(\text{MacDonald`X}[\text{ArcSec}[u]+\text{ArcSin}[c]] - \text{MacDonald`X}[\text{ArcSec}[u]-\text{ArcSin}[c]]\right)\right]$$

In[•]:=
```
MacDonald`H3[1, 1/2] // FullSimplify
```

Out[•]=

$$\frac{2\,e^{\frac{4\,\text{Catalan}}{3\pi}}}{\left(2+\sqrt{3}\right)^{2/3}}$$

In[1219]:=
```
Style[TableForm[
  Table[NumberForm[Chop[N[MacDonald`H3[u, c], 12]], 12],
   {u, 1/10, 1, 1/10}, {c, 1/10, 1, 1/10}]
  , TableHeadings → {{"μ=0.1", "μ=0.2", "μ=0.3", "μ=0.4", "μ=0.5", "μ=0.6",
     "μ=0.7", "μ=0.8", "μ=0.9", "μ=1.0"}, {"c=0.1", "c=0.2", "c=0.3",
     "c=0.4", "c=0.5", "c=0.6", "c=0.7", "c=0.8", "c=0.9", "c=1.0"}}], Small]
```

… N: Internal precision limit $MaxExtraPrecision = 50.` reached while evaluating

$$\text{HypergeometricPFQ}\left[\left\{\frac{1}{2}, 1, 1\right\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, \text{Sin}\left[\text{ArcSec}\left[\frac{4}{5}\right]+\text{ArcSin}\left[\frac{7}{10}\right]\right]^2\right].$$

… N: Internal precision limit $MaxExtraPrecision = 50.` reached while evaluating

$$\text{HypergeometricPFQ}\left[\left\{\frac{1}{2}, 1, 1\right\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, \text{Sin}\left[\text{ArcSec}\left[\frac{4}{5}\right]-\text{ArcSin}\left[\frac{7}{10}\right]\right]^2\right].$$

… N: Internal precision limit $MaxExtraPrecision = 50.` reached while evaluating

$$\text{HypergeometricPFQ}\left[\left\{\frac{1}{2}, 1, 1\right\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, \text{Sin}\left[\text{ArcSec}\left[\frac{9}{10}\right]+\text{ArcSin}\left[\frac{9}{10}\right]\right]^2\right].$$

… General: Further output of N::meprec will be suppressed during this calculation.

Out[1219]=

	c=0.1	c=0.2	c=0.3	c=0.4	c=0.5	c=0.6	c=0.7
μ=0.1	1.00986237220	1.02035969089	1.03160591899	1.04375534182	1.05702663988	1.07174978069	1.08846
μ=0.2	1.01545117571	1.03214541676	1.05032602070	1.07032520184	1.09261788807	1.11792683741	1.14745
μ=0.3	1.01955400755	1.04090070058	1.06441474026	1.09061229185	1.12023814909	1.15443681754	1.19513
μ=0.4	1.02277520414	1.04783494385	1.07568183575	1.10701407777	1.14284807666	1.18476064184	1.23543
μ=0.5	1.02539994920	1.05352430029	1.08499779016	1.12069475183	1.16189837189	1.21061713441	1.27030
μ=0.6	1.02759251403	1.05830376879	1.09287380232	1.13234524502	1.17825957533	1.23304942534	1.30093
μ=0.7	1.02945790873	1.06238935388	1.09964260728	1.14241993772	1.19251069370	1.25275993701	1.32814
μ=0.8	1.03106780654	1.06592963874	1.10553506623	1.15123714379	1.20506177355	1.27025228749	1.35252
μ=0.9	1.03247341547	1.06903152309	1.11071857647	1.15902972215	1.21621583758	1.28590306999	1.37452
μ=1.0	1.03371259147	1.07177452765	1.11531854174	1.16597352752	1.22620388089	1.30000256645	1.39450

Form 4

$$\text{MacDonald`H4}[u\_,c\_] := \sqrt{\frac{1+u}{1+u\sqrt{1-c^2}}}\ \text{Exp}\left[\frac{1}{\text{Pi}}\ \text{Re}\left[\text{HypergeometricPFQ}\left[\left\{\frac{1}{2},1,1\right\},\left\{\frac{3}{2},\frac{3}{2}\right\},\text{Sin}[\#]^2\right]\text{Sin}[\#]\ \&[\text{ArcSec}[u]+\text{ArcSin}[c]]\right]\right]$$

Form 5

In[•]:= $\text{Leftover}[x\_] := +i\left(\text{PolyLog}\left[2,-e^{i x}\right]-\text{PolyLog}\left[2,e^{i x}\right]\right)$

In[•]:= $$\text{MacDonald`H5}[u\_,c\_] := \sqrt{\frac{1+u}{1+u\sqrt{1-c^2}}}\ \text{Abs}\Big[\left(\left(1-e^{i(\text{ArcSec}[u]+\text{ArcSin}[c])}\right)^{\text{ArcSec}[u]+\text{ArcSin}[c]}\left(1+e^{i(\text{ArcSec}[u]+\text{ArcSin}[c])}\right)^{-\text{ArcSec}[u]-\text{ArcSin}[c]}\right)^{1/\text{Pi}} \times \text{Exp}\left[\frac{1}{\text{Pi}}\ \text{Leftover}[\text{ArcSec}[u]+\text{ArcSin}[c]]\right]\Big]$$

In[•]:= $\text{MacDonald`H5}\left[1,\frac{1}{2}\right] == \text{MacDonald`H}\left[1,\frac{1}{2}\right]$ // FullSimplify

Out[•]= True

Expansion of Fock's integral

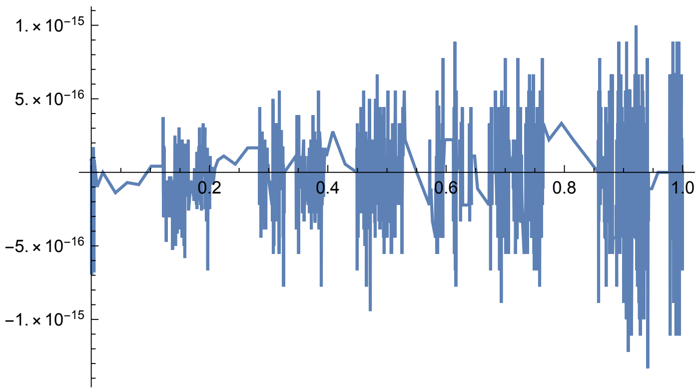
In[•]:= $\text{Integrate}\left[\frac{x}{\text{Sin}[x]},x\right]$

Out[•]= $x\left(\text{Log}\left[1-e^{i x}\right]-\text{Log}\left[1+e^{i x}\right]\right)+i\left(\text{PolyLog}\left[2,-e^{i x}\right]-\text{PolyLog}\left[2,e^{i x}\right]\right)$

In[•]:= $\text{IFocksum}[x\_,J\_] := \text{Sum}\left[-\frac{\left(i^j\left(-2+2^j\right)\text{BernoulliB}[j]\right)x^{j+1}}{j!\,(j+1)},\{j,0,J,2\}\right]$

In[•]:= Plot[{Re[IFock[x]] - Re[IFocksum[x, 50]]}, {x, 0, 1}, PlotRange → All]

Out[•]=



In[•]:=
```
IFocksum[x, 15]
```

Out[•]=
$$x+\frac{x^3}{18}+\frac{7\,x^5}{1800}+\frac{31\,x^7}{105\,840}+\frac{127\,x^9}{5\,443\,200}+\frac{73\,x^{11}}{37\,635\,840}+\frac{1\,414\,477\,x^{13}}{8\,499\,883\,392\,000}+\frac{8191\,x^{15}}{560\,431\,872\,000}$$

In[•]:=
```
Series[x (Log[1 - ⅇ^(ⅈ x)] - Log[1 + ⅇ^(ⅈ x)]) + ⅈ (PolyLog[2, -ⅇ^(ⅈ x)] - PolyLog[2, ⅇ^(ⅈ x)]),
  {x, 0, 15}, Assumptions → 0 < x < 1]
```

Out[•]=
$$-\frac{i\,\pi^2}{4}+x+\frac{x^3}{18}+\frac{7\,x^5}{1800}+\frac{31\,x^7}{105\,840}+\frac{127\,x^9}{5\,443\,200}+\frac{73\,x^{11}}{37\,635\,840}+\frac{1\,414\,477\,x^{13}}{8\,499\,883\,392\,000}+\frac{8191\,x^{15}}{560\,431\,872\,000}+O[x]^{16}$$

Numerical Integration - Form 1

In[•]:=
```
MacDonald`NH1[u_, c_] :=
  (1 + u)/(1 + Sqrt[1 - c^2] u) Exp[NIntegrate[c/Pi (t ArcTan[u t])/((t^2 + 1) (c + Sqrt[t^2 + 1])), {t, 0, Infinity}]]
```

In[•]:=
```
MacDonald`NH1b[u_, c_] :=
  Exp[NIntegrate[c/Pi (t ArcTan[u t])/((t^2 + 1) (-c + Sqrt[t^2 + 1])), {t, 0, Infinity}]]
```

In[•]:=
```
MacDonald`NH1c[u_, c_] :=
  Sqrt[(1 + u)/(1 + Sqrt[1 - c^2] u)] Exp[NIntegrate[c/Pi (t ArcTan[t u])/(Sqrt[1 + t^2] (1 - c^2 + t^2)), {t, 0, Infinity}]]
```

In[•]:=
```
MacDonald`NH1c[u_, c_, J_] := Sqrt[(1 + u)/(1 + Sqrt[1 - c^2] u)]
  Exp[Sum[(c^j Gamma[(1+j)/2] Hypergeometric2F1Regularized[1, (1+j)/2, (2+j)/2, 1 - 1/u^2])/(2 j Sqrt[π] u),
    {j, 1, J - 1, 2}] + NIntegrate[(c^J t ArcTan[t u])/((1 + t^2)^(J/2) (π - c^2 π + π t^2)), {t, 0, Infinity}]]
```

In[•]:=
```
MacDonald`NH1c[u_, c_, 3] := Sqrt[(1 + u)/(1 + Sqrt[1 - c^2] u)] Exp[Chop[(c ArcSin[Sqrt[#1]])/(π u Sqrt[-(-1 + #1) #1])] &[1 - 1/u^2] +
  NIntegrate[(c^3 t ArcTan[t u])/((1 + t^2)^(3/2) (π - c^2 π + π t^2)), {t, 0, Infinity}]]
```

In[•]:=
```
MacDonald`NH1c[u_, c_, 5] :=
  Sqrt[(1 + u)/(1 + Sqrt[1 - c^2] u)] Exp[Chop[(-c^3 Sqrt[-(-1 + #1) #1] + c ArcSin[Sqrt[#1]] (c^2 + 3 #1))/(3 π u Sqrt[1 - #1] #1^(3/2))] &[1 - 1/u^2] +
  NIntegrate[(c^5 t ArcTan[t u])/((1 + t^2)^(5/2) (π - c^2 π + π t^2)), {t, 0, Infinity}]]
```

In[•]:=
$$\texttt{MacDonald`NH1c}[u\_, c\_, 7] := \sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\ \texttt{Exp}\Big[\texttt{Chop}\Big[\frac{1}{15\,\pi\,u\,\sqrt{1-\#1}\ \#1^{5/2}}\Big(c\,\texttt{ArcSin}\big[\sqrt{\#1}\big]\left(3\,c^4+5\,c^2\,\#1+15\,\#1^2\right) - c^3\left(\left(5+2\,c^2\right)\sqrt{1-\#1}\ \#1^{3/2}+3\,c^2\sqrt{-(-1+\#1)\,\#1}\right)\Big)\Big]\ \&\Big[1-\frac{1}{u^2}\Big] + \texttt{NIntegrate}\Big[\frac{c^7\,t\,\texttt{ArcTan}[t\,u]}{\left(1+t^2\right)^{7/2}\left(\pi-c^2\,\pi+\pi\,t^2\right)}, \{t, 0, \texttt{Infinity}\}\Big]\Big]$$

In[•]:=
$$\texttt{MacDonald`NH1d}[u\_, c\_, J\_] := \sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\ \texttt{Exp}\Big[\texttt{Sum}\Big[\frac{c^j\,\texttt{Gamma}\big[\frac{1+j}{2}\big]\,\texttt{Hypergeometric2F1Regularized}\big[1, \frac{1+j}{2}, \frac{2+j}{2}, 1-\frac{1}{u^2}\big]}{2\,j\,\sqrt{\pi}\,u}, \{j, 1, J-1, 2\}\Big] + \texttt{NIntegrate}\Big[\frac{c^J\,t\,\texttt{ArcTan}[t\,u]}{\left(1+t^2\right)^{J/2}\left(\pi-c^2\,\pi+\pi\,t^2\right)}, \{t, 0, \texttt{Infinity}\}\Big]\Big]$$

In[•]:=
$$\texttt{N}\Big[\Big\{\texttt{MacDonald`H}\Big[\frac{7}{10}, \frac{9}{10}\Big], \texttt{MacDonald`NH1}\Big[\frac{7}{10}, \frac{9}{10}\Big], \texttt{MacDonald`NH1b}\Big[\frac{7}{10}, \frac{9}{10}\Big], \texttt{MacDonald`NH1c}\Big[\frac{7}{10}, \frac{9}{10}\Big], \texttt{MacDonald`NH1c}\Big[\frac{7}{10}, \frac{9}{10}, 10\Big]\Big\}\Big]$$

Out[•]= {1.58199, 1.58199, 1.58199, 1.58199, 1.5848}

Numerical Integration - Form 2

In[•]:=
$$\texttt{MacDonald`NH2f}[u\_, c\_] := \frac{(1+u)}{1+\sqrt{1-c^2}\,u}\ \texttt{Exp}\Big[\frac{-u}{\texttt{Pi}}\ \texttt{NIntegrate}\Big[\frac{\texttt{Log}\Big[\Big(1-\frac{c}{\sqrt{1+t^2}}\Big)\frac{t^2+1}{t^2+1-c^2}\Big]}{1+t^2\,u^2}, \{t, 0, \texttt{Infinity}\}\Big]\Big]$$

In[•]:=
$$\texttt{MacDonald`NH2}[u\_, c\_] := \frac{(1+u)}{1+\sqrt{1-c^2}\,u}\ \texttt{Exp}\Big[\texttt{NIntegrate}\Big[\frac{u}{\texttt{Pi}}\ \frac{\texttt{Log}\Big[1+\frac{c}{\sqrt{1+t^2}}\Big]}{1+t^2\,u^2}, \{t, 0, \texttt{Infinity}\}\Big]\Big]$$

In[•]:=
$$\texttt{MacDonald`NH2b}[u\_, c\_] := \texttt{Exp}\Big[\texttt{NIntegrate}\Big[\frac{-u}{\texttt{Pi}}\ \frac{\texttt{Log}\Big[1-\frac{c}{\sqrt{1+t^2}}\Big]}{1+t^2\,u^2}, \{t, 0, \texttt{Infinity}\}\Big]\Big]$$

In[•]:=
$$\texttt{MacDonald`NH2c}[u\_, c\_, J\_] := \sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\ \texttt{Exp}\Big[\texttt{Sum}\Big[\frac{c^j\,\texttt{Gamma}\big[\frac{1+j}{2}\big]\,\texttt{Hypergeometric2F1Regularized}\big[1, \frac{1+j}{2}, \frac{2+j}{2}, 1-\frac{1}{u^2}\big]}{2\,j\,\sqrt{\pi}\,u}, \{j, 1, J-1, 2\}\Big] + \texttt{NIntegrate}\Big[\frac{u}{\texttt{Pi}}\ \frac{\frac{(-1)^{1+J}\,c^J\left(1+t^2\right)^{-J/2}\,\texttt{Hypergeometric2F1}\big[1, \frac{J}{2}, 1+\frac{J}{2}, \frac{c^2}{1+t^2}\big]}{J}}{1+t^2\,u^2}, \{t, 0, \texttt{Infinity}\}\Big]\Big]$$

In[•]:= $$\text{N}\left[\left\{\text{MacDonald`H}\left[\frac{7}{10},\frac{9}{10}\right],\ \text{MacDonald`NH2f}\left[\frac{7}{10},\frac{9}{10}\right],\ \text{MacDonald`NH2}\left[\frac{7}{10},\frac{9}{10}\right],\ \text{MacDonald`NH2b}\left[\frac{7}{10},\frac{9}{10}\right],\ \text{MacDonald`NH2c}\left[\frac{7}{10},\frac{9}{10},100\right]\right\}\right]$$

Out[•]= {1.58199, 1.58199, 1.58199, 1.58199, 1.58199}

Numerical Integration - Form 3

In[•]:= MacDonald`NH3[u_, c_] :=

$$\frac{(1+u)}{1+\sqrt{1-c^2}\ u}\ \text{Exp}\left[\text{NIntegrate}\left[\frac{1}{\text{Pi}}\ \frac{u\ y\ \text{Log}\left[1+\frac{c}{y}\right]}{\sqrt{-1+y^2}\ \left(1+u^2\ (-1+y^2)\right)},\ \{y,1,\text{Infinity}\}\right]\right]$$

In[•]:= MacDonald`NH3b[u_, c_] :=

$$\text{Exp}\left[\text{NIntegrate}\left[\frac{-u}{\text{Pi}}\ \frac{y\ \text{Log}\left[1-\frac{c}{y}\right]}{\sqrt{-1+y^2}\ \left(1+u^2\ (-1+y^2)\right)},\ \{y,1,\text{Infinity}\}\right]\right]$$

In[•]:= MacDonald`NH3c[u_, c_] :=

$$\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\ u}}\ \text{Exp}\left[\text{NIntegrate}\left[\frac{u\ y\ \text{ArcTanh}\left[\frac{c}{y}\right]}{\pi\sqrt{-1+y^2}\ \left(1-u^2+u^2\ y^2\right)},\ \{y,1,\text{Infinity}\}\right]\right]$$

In[•]:= $$\text{N}\left[\left\{\text{MacDonald`H}\left[\frac{7}{10},\frac{9}{10}\right],\ \text{MacDonald`NH3}\left[\frac{7}{10},\frac{9}{10}\right],\ \text{MacDonald`NH3b}\left[\frac{7}{10},\frac{9}{10}\right],\ \text{MacDonald`NH3c}\left[\frac{7}{10},\frac{9}{10}\right]\right\}\right]$$

Out[•]= {1.58199, 1.58199, 1.58199, 1.58199}

Numerical Integration - Form 4

In[•]:= MacDonald`NH4[u_, c_] :=

$$\frac{(1+u)}{1+\sqrt{1-c^2}\ u}\ \text{Exp}\left[\text{NIntegrate}\left[\frac{u\ \text{Csc}[x]^2\ \text{Log}[1+c\ \text{Sin}[x]]}{\pi+\pi\ u^2\ \text{Cot}[x]^2},\ \{x,0,\text{Pi}/2\}\right]\right]$$

In[•]:= MacDonald`NH4o[u_, c_] :=

$$\frac{(1+u)}{1+\sqrt{1-c^2}\ u}\ \text{Exp}\left[\text{NIntegrate}\left[\frac{u}{\text{Pi}}\ \frac{\text{Log}[1+c\ \text{Sin}[x]]}{u^2\ \text{Cos}[x]^2+\text{Sin}[x]^2},\ \{x,0,\text{Pi}/2\}\right]\right]$$

In[•]:= MacDonald`NH4oo[u_, c_] :=

$$\frac{(1+u)}{1+\sqrt{1-c^2}\ u}\ \text{Exp}\left[\frac{u}{\text{Pi}}\left(\frac{1}{2}\pi\sqrt{\frac{1}{u^2}}\ \text{Log}[1+c]-\text{NIntegrate}\left[\frac{c\ \text{ArcCot}[u\ \text{Cot}[x]]\ \text{Cos}[x]}{u+c\ u\ \text{Sin}[x]},\ \{x,0,\text{Pi}/2\}\right]\right)\right];$$

In[•]:= MacDonald`NH4oo2[u_, c_] :=

$$\frac{(1+u)\ \sqrt{1+c}}{1+\sqrt{1-c^2}\ u}\ \text{Exp}\left[\frac{-u}{\text{Pi}}\left(\text{NIntegrate}\left[\frac{c\ \text{ArcCot}[u\ \text{Cot}[x]]\ \text{Cos}[x]}{u+c\ u\ \text{Sin}[x]},\ \{x,0,\text{Pi}/2\}\right]\right)\right];$$

In[∘]:= MacDonald`NH4oo3[u_, c_] := $\text{Exp}\left[\frac{-u}{\text{Pi}}\left(\frac{1}{2}\pi\sqrt{\frac{1}{u^2}}\,\text{Log}[1-c] - \text{NIntegrate}\left[\frac{-c\,\text{ArcCot}[u\,\text{Cot}[x]]\,\text{Cos}[x]}{u - c\,u\,\text{Sin}[x]}, \{x, 0, \text{Pi}/2\}\right]\right)\right];$

In[∘]:= MacDonald`NH4oo4[u_, c_] := $\left(\sqrt{1-c}\right)^{-1}\text{Exp}\left[\frac{u}{\text{Pi}}\left(\text{NIntegrate}\left[\frac{-c\,\text{ArcCot}[u\,\text{Cot}[x]]\,\text{Cos}[x]}{u - c\,u\,\text{Sin}[x]}, \{x, 0, \text{Pi}/2\}\right]\right)\right];$

In[∘]:= MacDonald`NH4oo5[u_, c_] := $\text{Chop}\left[\frac{(1+u)\sqrt{1+c}}{1+\sqrt{1-c^2}\,u}\text{Exp}\left[\frac{-u}{\text{Pi}}\left(\text{NIntegrate}\left[-\frac{c^2\,\text{ArcCot}[u\,\text{Cot}[x]]\,\text{Cos}[x]}{c\,u + u\,\text{Csc}[x]}, \{x, 0, \text{Pi}/2\}\right]\right) - \frac{c\left(\pi + \frac{2\,i\,u\,\text{ArcSec}[u]}{\sqrt{1-u^2}}\right)}{2\pi}\right]\right];$

In[∘]:= MacDonald`NH4b[u_, c_] := $\text{Exp}\left[\text{NIntegrate}\left[\frac{-u\,\text{Csc}[x]^2\,\text{Log}[1 - c\,\text{Sin}[x]]}{\pi + \pi u^2\,\text{Cot}[x]^2}, \{x, 0, \text{Pi}/2\}\right]\right]$

In[∘]:= MacDonald`NH4c[u_, c_] := $\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\,\text{Exp}\left[\text{NIntegrate}\left[\frac{u\,\text{ArcTanh}[c\,\text{Sin}[x]]\,\text{Csc}[x]^2}{\pi\left(1+u^2\,\text{Cot}[x]^2\right)}, \{x, 0, \text{Pi}/2\}\right]\right]$

In[∘]:= MacDonald`NH4c[u_, c_, J_] := $\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\,\text{Exp}\left[\text{Sum}\left[\frac{c^j\,\text{Gamma}\left[\frac{1+j}{2}\right]\text{Hypergeometric2F1Regularized}\left[1, \frac{1+j}{2}, \frac{2+j}{2}, 1-\frac{1}{u^2}\right]}{2\,j\sqrt{\pi}\,u}, \{j, 1, J-1, 2\}\right] + \text{NIntegrate}\left[\left((-1)^J c^J\,u\,\text{Csc}[x]^2\,\text{Hypergeometric2F1}\left[1, \frac{J}{2}, 1+\frac{J}{2}, c^2\,\text{Sin}[x]^2\right]\left(-\text{Sin}[x]\right)^J\right)\Big/\left(J\pi\left(1+u^2\,\text{Cot}[x]^2\right)\right), \{x, 0, \text{Pi}/2\}\right]\right]$

In[∘]:= $\text{N}\left[\left\{\text{MacDonald`H}\left[\frac{7}{10}, \frac{9}{10}\right], \text{MacDonald`NH4}\left[\frac{7}{10}, \frac{9}{10}\right], \text{MacDonald`NH4o}\left[\frac{7}{10}, \frac{9}{10}\right], \text{MacDonald`NH4oo}\left[\frac{7}{10}, \frac{9}{10}\right], \text{MacDonald`NH4oo2}\left[\frac{7}{10}, \frac{9}{10}\right], \text{MacDonald`NH4oo2}\left[\frac{7}{10}, \frac{9}{10}\right], \text{MacDonald`NH4oo3}\left[\frac{7}{10}, \frac{9}{10}\right], \text{MacDonald`NH4oo4}\left[\frac{7}{10}, \frac{9}{10}\right], \text{MacDonald`NH4oo5}\left[\frac{7}{10}, \frac{9}{10}\right], \text{MacDonald`NH4b}\left[\frac{7}{10}, \frac{9}{10}\right], \text{MacDonald`NH4c}\left[\frac{7}{10}, \frac{9}{10}\right], \text{MacDonald`NH4c}\left[\frac{7}{10}, \frac{9}{10}, 11\right]\right\}\right]$

Out[∘]= {1.58199, 1.58199, 1.58199, 1.58199, 1.58199, 1.58199, 1.58199, 1.58199, 1.58199, 1.58199, 1.58199, 1.58199}

Numerical Integration - Form 5

In[∘]:= MacDonald`NH5c[u_, c_] :=

$$\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\ \mathrm{Exp}\left[\mathrm{NIntegrate}\left[\frac{c^2\,u\,\mathrm{ArcTanh}[y]}{\pi\sqrt{(c-y)\,(c+y)}\,\left(y^2+u^2\,(c-y)\,(c+y)\right)},\{y,0,c\}\right]\right]$$

In[∘]:= $N\left[\left\{\mathrm{MacDonald`H}\left[\frac{7}{10},\frac{9}{10}\right],\mathrm{MacDonald`NH5c}\left[\frac{7}{10},\frac{9}{10}\right]\right\}\right]$

Out[∘]= {1.58199, 1.58199}

Numerical Integration - Form 6

In[∘]:= MacDonald`NH6c[u_, c_] :=

$$\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\ \mathrm{Exp}\left[\mathrm{NIntegrate}\left[\frac{c^2\,u\,\mathrm{ArcTanh}\left[\sqrt{c^2-Y^2}\right]}{\pi\,Y\,\left(c^2+\left(-1+u^2\right)Y^2\right)}\left(\frac{Y}{\sqrt{c^2-Y^2}}\right),\{Y,0,c\}\right]\right]$$

In[∘]:= $N\left[\left\{\mathrm{MacDonald`H}\left[\frac{7}{10},\frac{9}{10}\right],\mathrm{MacDonald`NH6c}\left[\frac{7}{10},\frac{9}{10}\right]\right\}\right]$

Out[∘]= {1.58199, 1.58199}

Numerical Integration - Fox / Mullikin / Case & Zweifel forms

In[∘]:= MacDonald`HFox[u_, c_] :=

$$\frac{\sqrt{1+c}\,(1+u)}{1+\sqrt{1-c^2}\,u}\ \mathrm{Exp}\left[\frac{-1}{\mathrm{Pi}}\,\mathrm{NIntegrate}\left[\frac{\mathrm{ArcTan}\left[\frac{c\,t}{\sqrt{1-t^2}}\right]}{t+u},\{t,0,1\}\right]\right]$$

In[∘]:= MacDonald`HMullikin54[u_, c_] :=

$$\frac{1+u}{1+\sqrt{1-c^2}\,u}\ \mathrm{Exp}\left[\frac{u}{\mathrm{Pi}}\,\mathrm{NIntegrate}\left[\frac{\mathrm{ArcTan}\left[\frac{c\,t}{\sqrt{1-t^2}}\right]}{t\,(t+u)},\{t,0,1\}\right]\right]$$

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In[∘]:= MacDonald`HCZ1[u_, c_] := $\frac{\sqrt{1+c}\,(1+u)}{1+\sqrt{1-c^2}\,u}$

$$\mathrm{Exp}\left[\frac{-1}{\mathrm{Pi}}\left(\mathrm{Log}[1+u]\,\frac{\mathrm{Pi}}{2}-\mathrm{NIntegrate}\left[\mathrm{Log}[t+u]\,\frac{c}{\sqrt{1-t^2}\,\left(1+\left(-1+c^2\right)t^2\right)},\{t,0,1\}\right]\right)\right]$$

In[∘]:= MacDonald`HCZ2[u_, c_] := $\frac{\sqrt{1+c}\,\sqrt{(1+u)}}{1+\sqrt{1-c^2}\,u}$

$$\mathrm{Exp}\left[\frac{1}{\mathrm{Pi}}\left(\mathrm{NIntegrate}\left[\mathrm{Log}[t+u]\,\frac{c}{\sqrt{1-t^2}\,\left(1+\left(-1+c^2\right)t^2\right)},\{t,0,1\}\right]\right)\right]$$

In[◦]:=
```
N[{MacDonald`H[7/10, 9/10], MacDonald`HFox[7/10, 9/10], MacDonald`HMullikin54[7/10, 9/10],
  MacDonald`HCZ1[7/10, 9/10], MacDonald`HCZ2[7/10, 9/10]}]
```

Out[◦]=
```
{1.58199, 1.58199, 1.58199, 1.58199, 1.58199}
```