Exponential NDF

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

notation

$u = \mathbf{m} . \mathbf{n} = \text{Cos}[\theta_m]$

α = roughness

Definitions and derivations

In[2787]:=
```
Exponential`D[u_, α_] := (2 e^(-2 Sqrt[1-u^2]/(u α)))/(π u^4 α^2) HeavisideTheta[u]
```

$$\text{Exponential`D}[u\_, \alpha\_] := \frac{2\, e^{-\frac{2\sqrt{1-u^2}}{u\,\alpha}}}{\pi\, u^4\, \alpha^2} \text{HeavisideTheta}[u]$$

relationship to Beckmann

In[1900]:=
```
Integrate[Beckmann`D[u, α Sqrt[m]] × PDF[GammaDistribution[3/2, 1]][m],
 {m, 0, Infinity}, Assumptions → 0 < u < 1 && α > 0 && a > 0]
```

Out[1900]= $$\frac{2\, e^{-\frac{2\sqrt{1-u^2}}{u\,\alpha}}}{\pi\, u^4\, \alpha^2}$$

In[2306]:=
```
Integrate[Beckmann`D[u, α/Sqrt[2] m] × PDF[ChiDistribution[3]][m],
 {m, 0, Infinity}, Assumptions → 0 < u < 1 && α > 0 && a > 0]
```

Out[2306]= $$\frac{2\, e^{-\frac{2\sqrt{1-u^2}}{u\,\alpha}}}{\pi\, u^4\, \alpha^2}$$

shape invariant f(x)

In[1885]:=
```
FullSimplify[Exponential`D[u, α] u^4 α^2 /. u -> 1/Sqrt[1 + x^2 α^2],
 Assumptions → 1 - 1/Sqrt[1 + x^2 α^2] > 0 && x > 0 && α > 0 && a > 0]
```

Out[1885]= $$\frac{2\, e^{-2\,x}}{\pi}$$

distribution of slopes

In[1887]:=
```
FullSimplify[Exponential`D[1/Sqrt[p^2 + q^2 + 1], α] (1/Sqrt[p^2 + q^2 + 1])^4,
  Assumptions → 0 < α < 1 && p > 0 && q > 0]
```

Out[1887]=
$$\frac{2\, e^{-\frac{2\sqrt{p^2+q^2}}{\alpha}}}{\pi\,\alpha^2}$$

In[1889]:=
$$\text{Exponential`P22[p\_, q\_, }\alpha\text{\_] := }\frac{2\, e^{-\frac{2\sqrt{p^2+q^2}}{\alpha}}}{\pi\,\alpha^2}$$

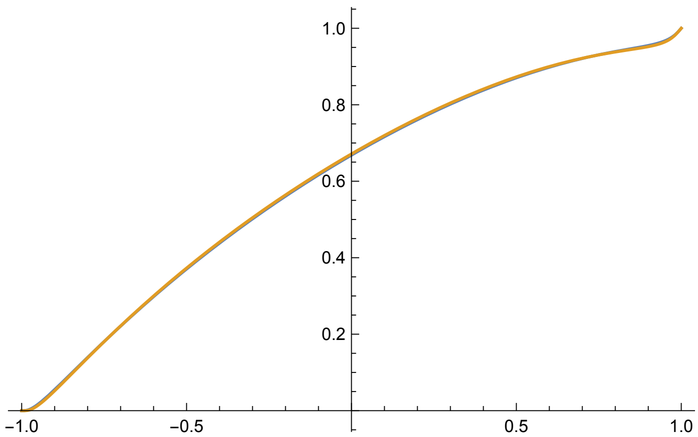
In[1890]:=
```
Integrate[Exponential`P22[p, q, α], {p, -Infinity, Infinity},
  {q, -Infinity, Infinity}, Assumptions → 0 < α < 1]
```

Out[1890]= 1

sigma - approximate with Ei sigma:

In[∘]:=
$$\text{Ei`}\sigma\text{[u\_, }\alpha\text{\_] :=}$$
$$\frac{1}{6\sqrt{\pi}\,(-1+u^2)\,\alpha^2}\left(\alpha\left(3\sqrt{\pi}\,u\,(-1+u^2)\,\alpha + 2\,e^{\frac{u^2}{(-1+u^2)\,\alpha^2}}\sqrt{1-u^2}\,\left(-\alpha^2+u^2\,(-1+\alpha^2)\right)\right) + 3\sqrt{\pi}\,u\,\left(-\alpha^2+u^2\,(-2+\alpha^2)\right)\,\text{Erf}\left[\frac{u}{\sqrt{1-u^2}\,\alpha}\right] + 2\sqrt{\pi}\,u^2\,\text{Abs[u]}\left(1+2\,\text{Erf}\left[\frac{u^2\sqrt{1-u^2}}{\alpha\,\text{Abs[u]} - u^2\,\alpha\,\text{Abs[u]}}\right]\right)\right)$$

In[1948]:=
```
With[{α = 2.1},
 Plot[{
   Quiet[NIntegrate[Exponential`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
   Quiet[Ei`σ[u, 1.7 α]]
  }, {u, -1, 1}]
]
```

Out[1948]=


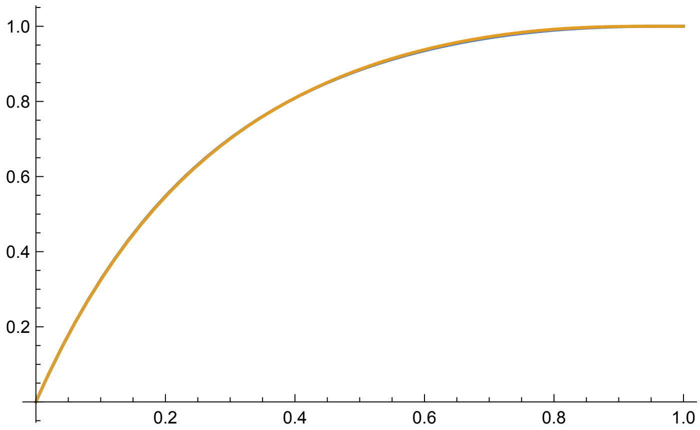
G1 shadow

In[•]:=
```
Ei`Λ[u_, α_] :=
```
$$\frac{2\, e^{\frac{u^2}{(-1+u^2)\,\alpha^2}} \sqrt{1-u^2}\, \alpha \left(-\alpha^2 + u^2 \left(-1+\alpha^2\right)\right) + \sqrt{\pi}\, u \left(3\alpha^2 + u^2\left(2 - 3\alpha^2\right)\right) \text{Erfc}\left[\frac{u}{\sqrt{1-u^2}\ \alpha}\right]}{6\sqrt{\pi}\, u \left(-1+u^2\right) \alpha^2}$$

In[2628]:=
```
Ei`G1[u_, α_] := 1/(1 + Ei`Λ[u, α])
```

In[2634]:=
```
With[{α = .8},
 Plot[{
   Quiet[u / NIntegrate[Exponential`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
   Quiet[Ei`G1[u, 1.7 α]]
  }, {u, 0, 1}, PlotRange → All]
]
```

Out[2634]=



In[2872]:=
```
Exponential`G1approx[u_, α_] := If[#1 < 1.5,
    (4.15428848706139`*^-8 + 5.3171271839868055` #1 + 1.5604787398328208` #1^2) /
     (1 + 2.9490717540313374` #1 + 3.812617848664264` #1^2 -
       1.0148658798039354` #1^3 + 0.18113230976734568` #1^4)
    ,
    1] &[α^-1 (Sqrt[1 - u^2]/u)^-1]
```

In[2886]:=
```
Plot[Exponential`G1approx[u, .8], {u, 0, 1}, PlotRange → All]
```

Out[2886]=