Infinite 3D medium, Isotropic Point Source, Isotropic Scattering

Exponential Random Flight

This is code to accompany the book:

A Hitchhiker’s Guide to Multiple Scattering

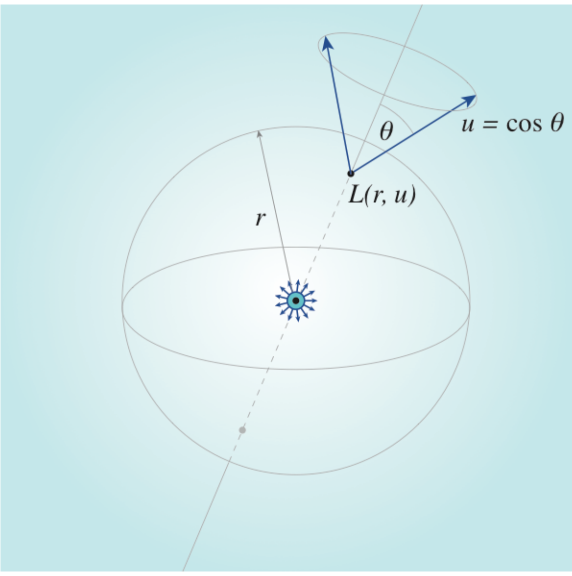
www.eugenedeon.com/hitchhikers

Path Setup

Put a file at ~/.hitchhikerpath with the path to your hitchhiker repo so that these worksheets can find the MC data from the C++ simulations for verification

In[725]:=
```
SetDirectory[Import["~/.hitchhikerpath"]]
```

Notation



c - single-scattering albedo

Σt - extinction coefficient

r - radial position coordinate in medium (distance from point source at origin)

u = cos θ - direction cosine

Namespace

In[977]:= `Begin["inf3Disopointisoscatter`"]`

Out[977]= `inf3Disopointisoscatter``

Util

In[988]:=
$$\text{SA}[d\_, r\_] := d \frac{\text{Pi}^{d/2}}{\text{Gamma}\left[\frac{d}{2}+1\right]} r^{d-1}$$

Diffusion modes

In[989]:=
$$\text{diffusionMode}[v\_, d\_, r\_] := (2\pi)^{-d/2} r^{1-\frac{d}{2}} v^{-1-\frac{d}{2}} \text{BesselK}\left[\frac{1}{2}(-2+d), \frac{r}{v}\right]$$

Analytic solutions

Caseology quantities

In[862]:=
$$\text{CaseN0}[c\_, v0\_] := \frac{1}{2} c\, v0^3 \left(\frac{c}{v0^2-1} - \frac{1}{v0^2}\right)$$

In[863]:=
$$\text{Casev0}[c\_?\text{NumericQ}] := \text{FindRoot}\left[c\, v\, \text{ArcTanh}\left[\frac{1}{v}\right] - 1 == 0, \{v, 1.00000000001, 10^{10}\}, \text{Method} \to \text{"Brent"}\right][[1]][[2]]$$

In[1051]:=
$$\text{Casev0approx}[c\_] := 1 \Big/ \sqrt{1 - c^{2.4429445001914587` + \frac{0.5786368322364553`}{c} - 0.021581332427913873`\, c}}$$

In[864]:=
$$\text{CaseN}[c\_, v\_] := v \left(\text{Case}\lambda[v, c]^2 + \left(\frac{\pi c v}{2}\right)^2\right)$$

In[865]:=
$$\text{Case}\lambda[v\_, c\_] := 1 - c\, v\, \text{ArcTanh}[v]$$

Fluence: exact solution (1)

[Bothe 1942]

In[990]:=
$$\phi\text{exact1a}[r\_, \Sigma t\_, c\_] := \frac{1}{2\,\text{Pi}^2 r} \text{NIntegrate}\left[\frac{z\, \text{ArcTan}[z/\Sigma t]}{z - c\,\Sigma t\, \text{ArcTan}[z/\Sigma t]} \text{Sin}[r z], \{z, 0, \text{Infinity}\}, \text{Method} \to \text{"ExtrapolatingOscillatory"}\right]$$

[Case et al. 1953]

In[992]:=
$$\phi\text{exact1b}[r\_, \Sigma t\_, c\_] := \frac{\text{Exp}[-\Sigma t\, r]}{4\,\text{Pi}\, r^2} + c \frac{\Sigma t}{2\,\text{Pi}^2 r} \text{NIntegrate}\left[\frac{\text{ArcTan}[z]^2}{z - c\,\text{ArcTan}[z]} \text{Sin}[r\,\Sigma t\, z], \{z, 0, \text{Infinity}\}, \text{Method} \to \text{"LevinRule"}\right]$$

Rigorous diffusion approximation

In[994]:= $\phi\text{rigourousDiffusion}[r\_, \Sigma t\_, c\_] := \frac{\Sigma t}{4\,\text{Pi}\,r}\,\frac{E^{-r\,\Sigma t/\#}}{\#\,\text{CaseN0}[c, \#]}\,\&[\text{Casev0}[c]]$

In[993]:= $\phi\text{transient}[r\_, \Sigma t\_, c\_] := \frac{\Sigma t}{4\,\text{Pi}\,r}\,\text{NIntegrate}\left[\frac{e^{-\Sigma t\,r/v}}{v\,\text{CaseN}[c, v]}, \{v, 0, 1\}\right]$

Expansion of transient term [Case et al. 1953]

In[995]:= $\phi\text{transient2}[r\_, \Sigma t\_, c\_, M\_] := \frac{\text{Exp}[-r\,\Sigma t]}{4\,\text{Pi}\,r^2} + \frac{1}{4\,\text{Pi}\,r}\,\text{Sum}\left[\text{ExpIntegralE}[2\,n, r\,\Sigma t]\right.$
$\left.\text{SeriesCoefficient}\left[v / \text{CaseN}[c, v], \{v, 0, 2\,n\}\right], \{n, 1, M\}\right]$

Fluence: exact solution (2)

[Davison 1947]

In[1761]:= $\phi\text{exact2a}[r\_, \Sigma t\_, c\_] := \phi\text{rigourousDiffusion}[r, \Sigma t, c] +$
$\frac{\Sigma t}{4\,\text{Pi}\,r}\,\text{NIntegrate}\left[\frac{e^{-\Sigma t\,r\,y}}{\frac{c^2\,\pi^2}{4\,y^2} + \left(1 - \frac{c}{2\,y}\,\text{Log}\left[\frac{y+1}{y-1}\right]\right)^2}, \{y, 1, \text{Infinity}\}\right]$

[Case and Zwiefel 1967]

In[1762]:= $\phi\text{exact2b}[r\_, \Sigma t\_, c\_] :=$
$\phi\text{rigourousDiffusion}[r, \Sigma t, c] + \frac{\Sigma t}{4\,\text{Pi}\,r}\,\text{NIntegrate}\left[\frac{e^{-\Sigma t\,r/v}}{v\,\text{CaseN}[c, v]}, \{v, 0, 1\}\right]$

n-th scattered fluence

In[998]:= $\phi\text{exact1}[r\_, \Sigma t\_, c\_, n\_] := \frac{(c\,\Sigma t)^n}{2\,\pi^2\,r}\,\text{NIntegrate}\left[\frac{\text{ArcTan}\left[\frac{z}{\Sigma t}\right]^{1+n}\,\text{Sin}[r\,z]}{z^n},\right.$
$\left.\{z, 0, \text{Infinity}\}, \text{Method} \to \text{"ExtrapolatingOscillatory"}\right]$

In[999]:= $\phi\text{exact2}[r\_, \Sigma t\_, c\_, n\_] := \frac{c^n\,\Sigma t}{2^{n+3}\,\text{Pi}^2\,\text{I}\,r}\,\text{Chop}\left[\text{NIntegrate}\left[\right.\right.$
$\left.\left.\frac{\text{Exp}[-r\,z\,\Sigma t]}{z^n}\left(\left(\text{Log}\left[\frac{z+1}{z-1}\right] + \text{I}\,\text{Pi}\right)^{n+1} - \left(\text{Log}\left[\frac{z+1}{z-1}\right] - \text{I}\,\text{Pi}\right)^{n+1}\right), \{z, 1, \text{Infinity}\}\right]\right]$

In[1000]:= $\phi\text{Gaussian}[r\_, \Sigma t\_, c\_, n\_] := \frac{3\,\sqrt{3}\,e^{-\frac{3\,r^2\,\Sigma t^2}{4\,(1+n)}}\,c^n\,\Sigma t^2}{8\,\sqrt{(1+n)^3}\,\pi^{3/2}}$

Moments

In[1001]:= $\phi\text{m}[c\_, \Sigma t\_, m\_?\text{IntegerQ}, n\_] :=$

$$\text{Limit}\left[\text{Simplify}\left[(-1)^{m/2}\left(\frac{2\,\text{Gamma}\left[\frac{3+m}{2}\right]}{\text{Gamma}\left[\frac{1+m}{2}\right]}\,\text{D}\left[\frac{\left(\frac{c\,\Sigma t\,\text{ArcTan}\left[\frac{z}{\Sigma t}\right]}{z}\right)^{1+n}}{c\,\Sigma t}, \{z, m\}\right]\right)\right], z \to 0\right]$$

In[1004]:= TableForm[Table[ϕm[c, Σt, m, n], {m, 0, 6, 2}]]

Out[1004]//TableForm=

$$\frac{c^n}{\Sigma t}$$
$$\frac{2\, c^n\, (1+n)}{\Sigma t^3}$$
$$\frac{4\, c^n\, (1+n)\, (18+5\, n)}{3\, \Sigma t^5}$$
$$\frac{8\, c^n\, (1+n)\, \left(810+343\, n+35\, n^2\right)}{9\, \Sigma t^7}$$

In[1005]:= ϕm[c_, Σt_, m_?IntegerQ] :=

$$\text{Limit}\left[\text{Simplify}\left[(-1)^{m/2}\left(\frac{2\,\text{Gamma}\left[\frac{3+m}{2}\right]}{\text{Gamma}\left[\frac{1+m}{2}\right]}\,\text{D}\left[\frac{\text{ArcTan}\left[\frac{z}{\Sigma t}\right]}{z - c\,\Sigma t\,\text{ArcTan}\left[\frac{z}{\Sigma t}\right]}, \{z, m\}\right]\right)\right], z \to 0\right]$$

TableForm[Table[ϕm[c, Σt, m], {m, 0, 6, 2}]]

Out[1007]//TableForm=

$$\frac{1}{\Sigma t - c\, \Sigma t}$$
$$\frac{2}{(-1+c)^2\, \Sigma t^3}$$
$$\frac{8\, (-9+4\, c)}{3\, (-1+c)^3\, \Sigma t^5}$$
$$\frac{16\, \left(135-144\, c+44\, c^2\right)}{3\, (-1+c)^4\, \Sigma t^7}$$

Recurrence derivation [Case et al. 1953]

In[1024]:= $\text{CaseB}[0, c\_] := \frac{1}{1-c};$

$$\text{CaseB}[m\_, c\_] := \frac{1}{(1-c)^2}\,\text{Sum}\left[\text{Caseb}[m, s]\left(\frac{c}{1-c}\right)^{s-1}, \{s, 1, m\}\right];$$

$$\text{Caseb}[m\_, 1] := \frac{1}{2\,m+1};$$

$$\text{Caseb}[m\_, s\_] := \text{Sum}\left[\frac{\text{Caseb}[n, s-1]}{1+2\,(m-n)}, \{n, s-1, m-1\}\right]$$

In[1028]:= $\phi\text{mCase}[c\_, \Sigma t\_, m\_?\text{IntegerQ}] := \frac{1}{\Sigma t^{m+1}}\,\text{CaseB}\left[m/2, \alpha\right]\,\text{Factorial}[m+1]$

In[1030]:= TableForm[Table[FullSimplify[ϕmCase[c, Σt, m]], {m, 0, 6, 2}]]

Out[1030]//TableForm=

$$\frac{1}{\Sigma t - \alpha\, \Sigma t}$$
$$\frac{2}{(-1+\alpha)^2\, \Sigma t^3}$$
$$\frac{8\, (-9+4\, \alpha)}{3\, (-1+\alpha)^3\, \Sigma t^5}$$
$$\frac{16\, (135+4\, \alpha\, (-36+11\, \alpha))}{3\, (-1+\alpha)^4\, \Sigma t^7}$$

Classical diffusion approximation

In[1033]:= $\phi\text{Diffusion}[r\_, \Sigma t\_, c\_] := \frac{1}{\Sigma t\,(1-c)}\,\text{diffusionMode}\left[\frac{1}{\sqrt{3\,(1-c)}\,\Sigma t}, 3, r\right]$

In[1034]:= FullSimplify[ϕDiffusion[r, Σt, c], Assumptions → 0 < c < 1 && Σt > 0]

Out[1034]= $$\frac{3\, e^{-\sqrt{3-3\,c}\; r\, \Sigma t}\, \Sigma t}{4\, \pi\, r}$$

Grosjean-style diffusion approximation

In[1036]:=
$$\phi\text{Grosjean}[r\_, \Sigma t\_, c\_] := \frac{\text{Exp}[-r\,\Sigma t]}{4\,\text{Pi}\,r^2} + \frac{c}{\Sigma t\,(1-c)}\,\text{diffusionMode}\left[\frac{\sqrt{2-c}}{\sqrt{3\,(1-c)}\;\Sigma t}, 3, r\right]$$

In[1037]:=
```
FullSimplify[ϕGrosjean[r, Σt, c], Assumptions → 0 < c < 1 && Σt > 0]
```

Out[1037]=
$$\frac{e^{-r\,\Sigma t} - \frac{3\,c\,e^{-\sqrt{3+\frac{3}{-2+c}}\,r\,\Sigma t}\,r\,\Sigma t}{-2+c}}{4\,\pi\,r^2}$$

Angular ϕ Integral

Note: this form leaves out the singular term $\frac{e^{-r\Sigma_t}}{4\pi r^2}\delta(u-1)$, because it doesn't plot:

In[1039]:=
$$\text{Lintegral}[r\_, u\_, \Sigma t\_, c\_, \phi\_] := \frac{c\,\Sigma t}{4\,\text{Pi}}\,\text{NIntegrate}\left[\phi\left[\sqrt{r^2+t^2-2\,r\,t\,u}, \Sigma t, c\right]\,\text{Exp}[-\Sigma t\,t], \{t, 0, \text{Infinity}\}\right]$$

Angular Classical diffusion approximation

In[1040]:=
$$\text{Ldiffusion}[r\_, u\_, \Sigma t\_, c\_] := \frac{1}{4\,\text{Pi}}\,\phi\text{Diffusion}[r, \Sigma t, c] + \frac{1}{4\,\text{Pi}}\,u\,\frac{3\,e^{-r\,\sqrt{3-3\,c}\,\Sigma t}\left(1+r\,\sqrt{3-3\,c}\,\Sigma t\right)}{4\,\pi\,r^2}$$

---

load MC data

In[1992]:=
```
ppoints[xs_, dr_, maxx_] :=
  Table[{dr (i) - 0.5 dr, xs[[i]]}, {i, 1, Length[xs]}][[1 ;; -2]]
```

In[1993]:=
```
ppointsu[xs_, du_, Σt_] :=
  Table[{-1.0 + du (i) - 0.5 du, xs[[i]] / (2 Σt)}, {i, 1, Length[xs]}][[1 ;; -1]]
```

In[1994]:=
```
fs = FileNames["code/3D_medium/infinite3Dmedium/Isotropicpointsource/MCdata/
     inf3D_isotropicpoint_isotropicscatter*"];
```

In[1995]:=
```
index[x_] := Module[{data, α, Σt},
    data = Import[x, "Table"];
    Σt = data[[1, 13]];
    α = data[[2, 3]];
    {α, Σt, data}];
simulations = index /@ fs;
cs = Union[#[[1]] & /@ simulations]
```

Out[1997]=
```
{0.01, 0.1, 0.3, 0.5, 0.7, 0.8, 0.9, 0.95, 0.99, 0.999}
```

In[1998]:=
```
mfps = Union[#[[2]] & /@ simulations]
```

Out[1998]=
```
{0.3, 1}
```

In[1999]:=
```
numcollorders = simulations[[1]][[3]][[2, 13]];
maxr = simulations[[1]][[3]][[2, 5]];
dr = simulations[[1]][[3]][[2, 7]];
numr = Floor[maxr / dr];
```

Select simulation

In[1650]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :> (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :> (mfp = #;) & /@ mfps],
  Dynamic[mfp]}}
```

Out[1650]= {{ Set c , 0.9}, { Set mfp , 0.3}}

Compare Deterministic and MC

Mean Track Length

In[1963]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :> (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :> (mfp = #;) & /@ mfps],
  Dynamic[mfp]}}
```

Out[1963]= {{ Set c , 0.9}, { Set mfp , 0.3}}

In[1964]:=
```
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
meanTL = data[[-1]]
```

$$\frac{mfp}{1 - c}$$

Out[1965]= {Mean, track, length:, 1.11109}

Out[1966]= 1.11111

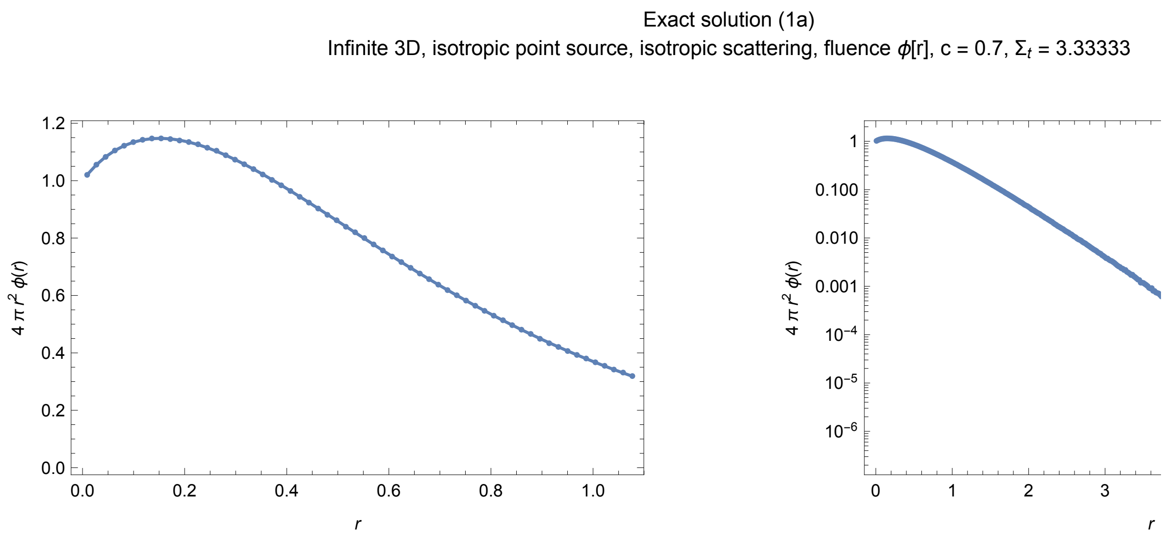
Fluence - Exact solution (1a) comparison to MC

In[ ]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :> (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :> (mfp = #;) & /@ mfps],
  Dynamic[mfp]}}
```

Out[ ]= {{ Set c , 0.9}, { Set mfp , 0.3}}

In[1720]:=
```
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
maxr = data[[2, 5]];
dr = data[[2, 7]];
pointsCR = ppoints[data[[4]], dr, maxr];
pointsFluence = ppoints[data[[6]], dr, maxr];
exact1FluenceShallow =
  Quiet[{#[[1]], 4 Pi #[[1]]^2 ϕexact1a[#[[1]], 1/mfp, c]}] & /@
   pointsCR[[1 ;; 60]];
exact1Fluence = Quiet[{#[[1]], 4 Pi #[[1]]^2 ϕexact1a[#[[1]], 1/mfp, c]}] & /@
   pointsCR[[1 ;; -1 ;; 10]];
plotϕshallow = Quiet[Show[
    ListPlot[pointsFluence[[1 ;; 60]],
     PlotRange → All, PlotStyle → PointSize[.01]],
    ListPlot[exact1FluenceShallow, PlotRange → All, Joined → True],
    Frame → True,
    FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
   ]];
logplotϕ = Quiet[Show[
    ListLogPlot[pointsFluence, PlotRange → All, PlotStyle → PointSize[.01]],
    ListLogPlot[exact1Fluence, PlotRange → All, Joined → True],
    Frame → True,
    FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
   ]];
Show[GraphicsGrid[{{plotϕshallow, logplotϕ}}, ImageSize → 800],
 PlotLabel -> "Exact solution (1a)\nInfinite 3D, isotropic
     point source, isotropic scattering, fluence ϕ[r], c = " <>
   ToString[c] <> ", Σt = " <> ToString[1/mfp]]
```

Out[1729]=



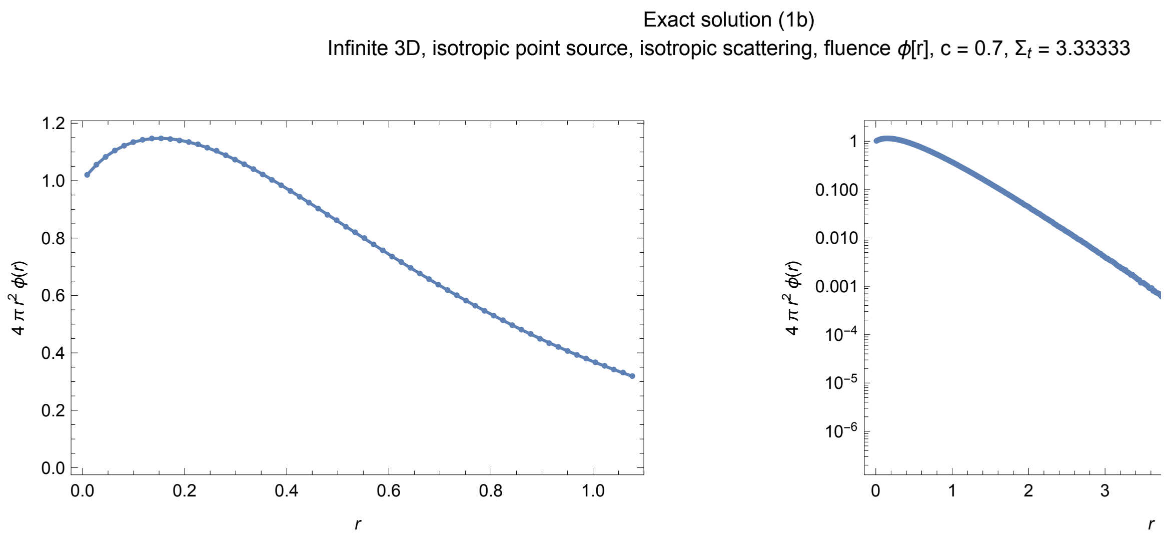
Fluence - Exact solution (1b) comparison to MC

In[•]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :→ (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :→ (mfp = #;) & /@ mfps],
  Dynamic[mfp]}}
```

Out[•]=
```
{{ Set c , 0.9}, { Set mfp , 0.3}}
```

In[1730]:=
```
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
maxr = data[[2, 5]];
dr = data[[2, 7]];
pointsCR = ppoints[data[[4]], dr, maxr];
pointsFluence = ppoints[data[[6]], dr, maxr];
exact1FluenceShallow =
  Quiet[{#[[1]], 4 Pi #[[1]]^2 ϕexact1b[#[[1]], 1/mfp, c]}] & /@
   pointsCR[[1 ;; 60]];
exact1Fluence = Quiet[{#[[1]], 4 Pi #[[1]]^2 ϕexact1b[#[[1]], 1/mfp, c]}] & /@
   pointsCR[[1 ;; -1 ;; 10]];
plotϕshallow = Quiet[Show[
    ListPlot[pointsFluence[[1 ;; 60]],
     PlotRange → All, PlotStyle → PointSize[.01]],
    ListPlot[exact1FluenceShallow, PlotRange → All, Joined → True],
    Frame → True,
    FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
   ]];
logplotϕ = Quiet[Show[
    ListLogPlot[pointsFluence, PlotRange → All, PlotStyle → PointSize[.01]],
    ListLogPlot[exact1Fluence, PlotRange → All, Joined → True],
    Frame → True,
    FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
   ]];
Show[GraphicsGrid[{{plotϕshallow, logplotϕ}}, ImageSize → 800],
 PlotLabel -> "Exact solution (1b)\nInfinite 3D, isotropic
    point source, isotropic scattering, fluence ϕ[r], c = " <>
   ToString[c] <> ", Σt = " <> ToString[1/mfp]]
```

Out[1739]=



Fluence - Exact solution (2a) comparison to MC

In[•]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :→ (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :→ (mfp = #;) & /@ mfps],
  Dynamic[mfp]}}
```

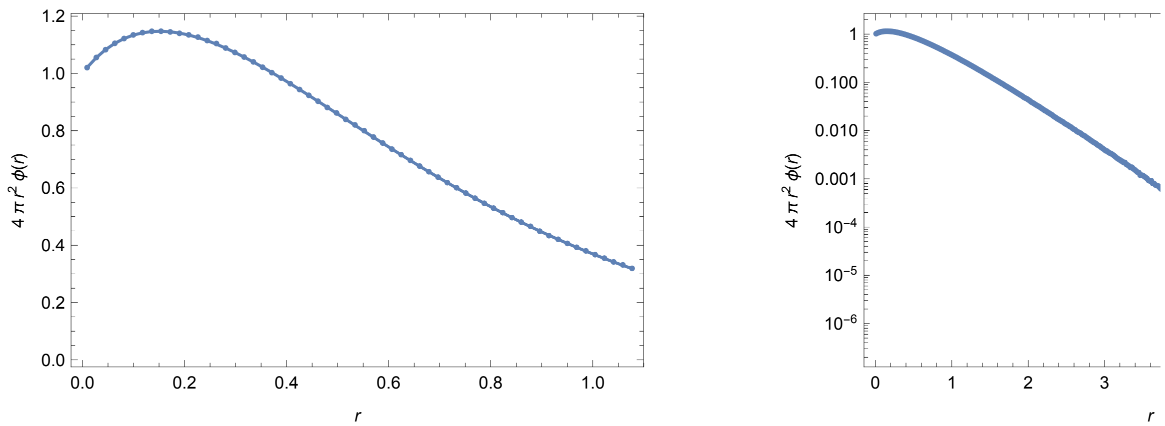
Out[•]=
```
{{ Set c , 0.9}, { Set mfp , 0.3}}
```

In[1763]:=
```
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
maxr = data[[2, 5]];
dr = data[[2, 7]];
pointsCR = ppoints[data[[4]], dr, maxr];
pointsFluence = ppoints[data[[6]], dr, maxr];
exact1FluenceShallow =
  Quiet[{#[[1]], 4 Pi #[[1]]^2 ϕexact2a[#[[1]], 1/mfp, c]}] & /@
   pointsCR[[1 ;; 60]];
exact1Fluence = Quiet[{#[[1]], 4 Pi #[[1]]^2 ϕexact2a[#[[1]], 1/mfp, c]}] & /@
   pointsCR[[1 ;; -1 ;; 10]];
plotϕshallow = Quiet[Show[
    ListPlot[pointsFluence[[1 ;; 60]],
     PlotRange → All, PlotStyle → PointSize[.01]],
    ListPlot[exact1FluenceShallow, PlotRange → All, Joined → True],
    Frame → True,
    FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
   ]];
logplotϕ = Quiet[Show[
    ListLogPlot[pointsFluence, PlotRange → All, PlotStyle → PointSize[.01]],
    ListLogPlot[exact1Fluence, PlotRange → All, Joined → True],
    Frame → True,
    FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
   ]];
Show[GraphicsGrid[{{plotϕshallow, logplotϕ}}, ImageSize → 800],
 PlotLabel -> "Exact solution (2a)\nInfinite 3D, isotropic
     point source, isotropic scattering, fluence ϕ[r], c = " <>
   ToString[c] <> ", Σt = " <> ToString[1/mfp]]
```

Out[1772]=

Exact solution (2a)

Infinite 3D, isotropic point source, isotropic scattering, fluence ϕ[r], c = 0.7, Σ_t = 3.33333



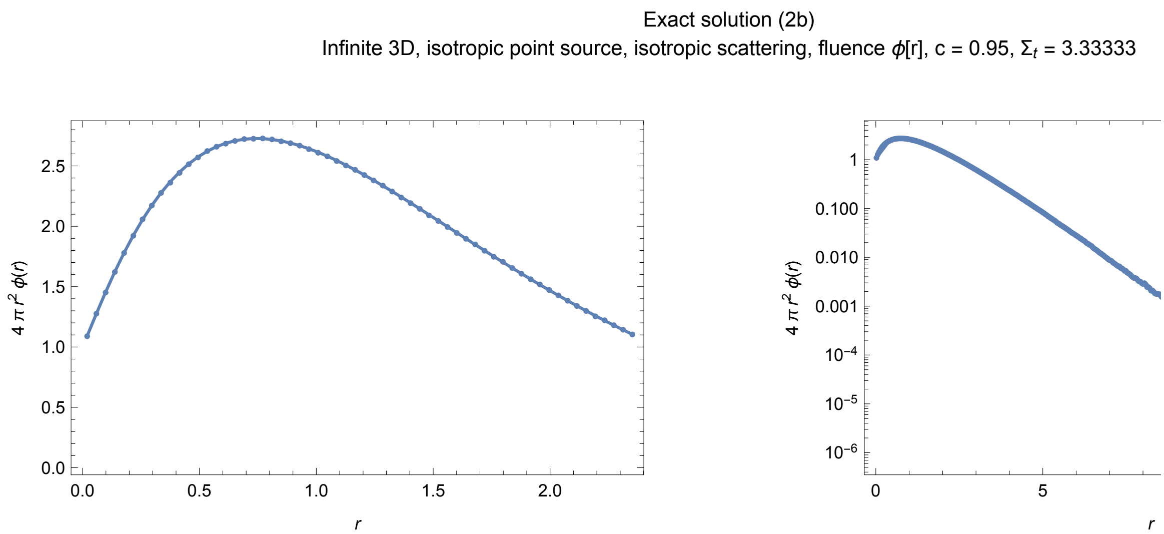
Fluence - Exact solution (2b) comparison to MC

In[1783]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :> (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :> (mfp = #;) & /@ mfps],
  Dynamic[mfp]}}
```

Out[1783]=
```
{{Set c, 0.9}, {Set mfp, 0.3}}
```

In[1784]:=
```
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
maxr = data[[2, 5]];
dr = data[[2, 7]];
pointsCR = ppoints[data[[4]], dr, maxr];
pointsFluence = ppoints[data[[6]], dr, maxr];
exact1FluenceShallow =
  Quiet[{#[[1]], 4 Pi #[[1]]^2 ϕexact2b[#[[1]], 1/mfp, c]}] & /@
   pointsCR[[1 ;; 60]];
exact1Fluence = Quiet[{#[[1]], 4 Pi #[[1]]^2 ϕexact2b[#[[1]], 1/mfp, c]}] & /@
   pointsCR[[1 ;; -1 ;; 10]];
plotϕshallow = Quiet[Show[
     ListPlot[pointsFluence[[1 ;; 60]],
      PlotRange → All, PlotStyle → PointSize[.01]],
     ListPlot[exact1FluenceShallow, PlotRange → All, Joined → True],
     Frame → True,
     FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
    ]];
logplotϕ = Quiet[Show[
     ListLogPlot[pointsFluence, PlotRange → All, PlotStyle → PointSize[.01]],
     ListLogPlot[exact1Fluence, PlotRange → All, Joined → True],
     Frame → True,
     FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
    ]];
Show[GraphicsGrid[{{plotϕshallow, logplotϕ}}, ImageSize → 800],
 PlotLabel -> "Exact solution (2b)\nInfinite 3D, isotropic
    point source, isotropic scattering, fluence ϕ[r], c = " <>
   ToString[c] <> ", Σt = " <> ToString[1/mfp]]
```

Out[1793]=



Fluence - Diffusion approximations (Classical and Grosjean) comparison to MC

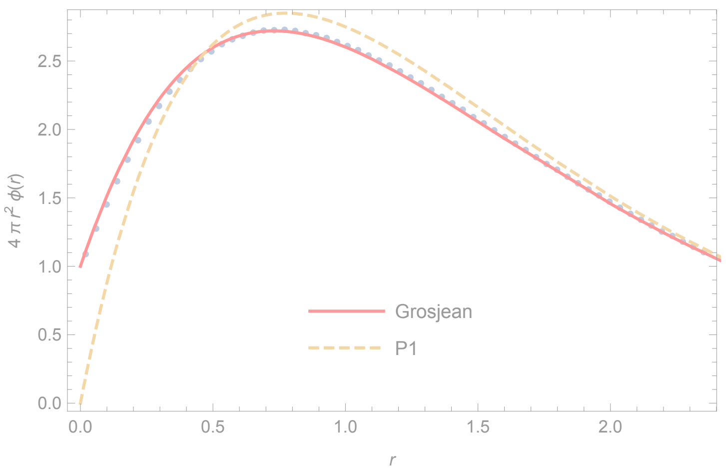
In[∘]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :→ (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :→ (mfp = #;) & /@ mfps],
  Dynamic[mfp]}}
```

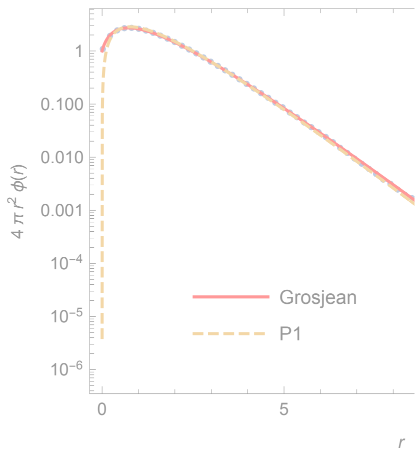
Out[∘]=
```
{{ Set c , 0.9}, { Set mfp , 0.3}}
```

```
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
maxr = data[[2, 5]];
dr = data[[2, 7]];
pointsFluence = ppoints[data[[6]], dr, maxr];
plotϕshallow = Quiet[Show[
     ListPlot[pointsFluence[[1 ;; 60]],
      PlotRange → All, PlotStyle → PointSize[.01]],
     Plot[{
        4 Pi r^2 ϕGrosjean[r, 1 / mfp, c],
        4 Pi r^2 ϕDiffusion[r, 1 / mfp, c]
      }
      , {r, 0, maxr}, PlotRange → All, PlotStyle → {Red, Dashed},
      PlotLegends → Placed[{"Grosjean", "P1"}, {0.5, .2}]],
     Frame → True,
     FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
    ]];
logplotϕ = Quiet[Show[
     ListLogPlot[pointsFluence[[1 ;; -1 ;; 5]],
      PlotRange → All, PlotStyle → PointSize[.01]],
     LogPlot[{
        4 Pi r^2 ϕGrosjean[r, 1 / mfp, c],
        4 Pi r^2 ϕDiffusion[r, 1 / mfp, c]
      }
      , {r, 0, maxr}, PlotRange → All, PlotStyle → {Red, Dashed},
      PlotLegends → Placed[{"Grosjean", "P1"}, {0.3, .2}]],
     Frame → True,
     FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
    ]];
Show[GraphicsGrid[{{plotϕshallow, logplotϕ}}, ImageSize → 800],
  PlotLabel -> "Diffusion Approximations\nInfinite 3D, isotropic
     point source, isotropic scattering, fluence ϕ[r], c = " <>
   ToString[c] <> ", Σt = " <> ToString[1 / mfp]]
```

Diffusion Approximations
Infinite 3D, isotropic point source, isotropic scattering, fluence ϕ[r], c = 0.95, Σ_t = 3.33333

Out[1940]=





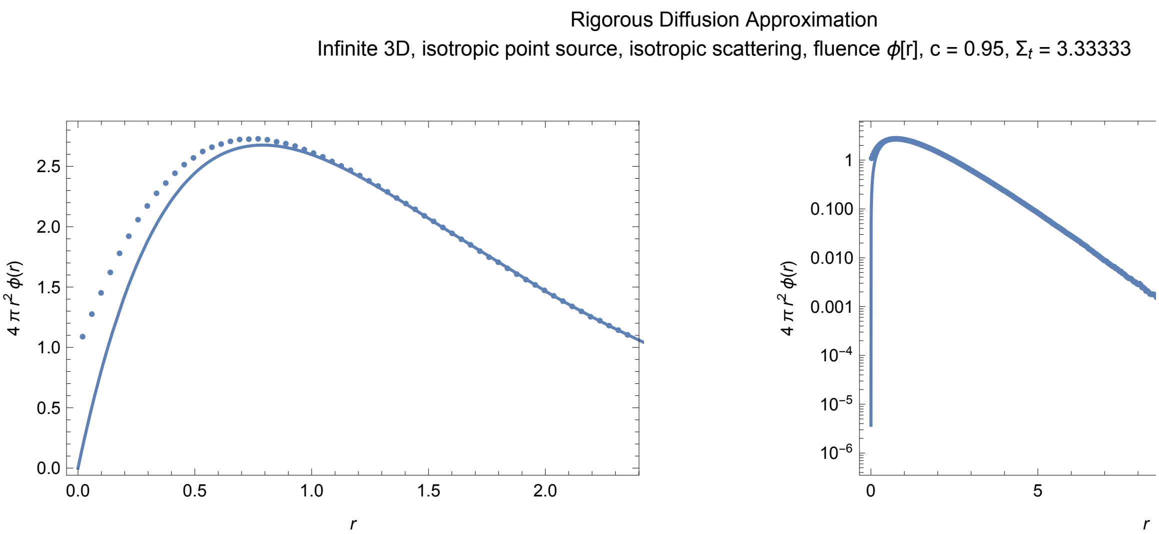
Fluence - Diffusion approximation (Rigorous) comparison to MC

In[∘]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :> (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :> (mfp = #;) & /@ mfps],
  Dynamic[mfp]}}
```

Out[∘]=
```
{{Set c, 0.9}, {Set mfp, 0.3}}
```

In[1941]:=
```
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
maxr = data[[2, 5]];
dr = data[[2, 7]];
pointsFluence = ppoints[data[[6]], dr, maxr];
plotϕshallow = Quiet[Show[
    ListPlot[pointsFluence[[1 ;; 60]],
     PlotRange → All, PlotStyle → PointSize[.01]],
    Plot[4 Pi r^2 ϕrigourousDiffusion[r, 1 / mfp, c], {r, 0, maxr}, PlotRange → All],
    Frame → True,
    FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
   ]];
logplotϕ = Quiet[Show[
    ListLogPlot[pointsFluence, PlotRange → All, PlotStyle → PointSize[.01]],
    LogPlot[
     4 Pi r^2 ϕrigourousDiffusion[r, 1 / mfp, c], {r, 0, maxr}, PlotRange → All],
    Frame → True,
    FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
   ]];
Show[GraphicsGrid[{{plotϕshallow, logplotϕ}}, ImageSize → 800],
 PlotLabel -> "Rigorous Diffusion Approximation\nInfinite 3D, isotropic
    point source, isotropic scattering, fluence ϕ[r], c = " <>
   ToString[c] <> ", Σt = " <> ToString[1 / mfp]]
```

Out[1947]=



N-th order fluence / scalar flux

N-th collided Fluence - Exact solution (1) comparison to MC

In[1950]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :> (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :> (mfp = #;) & /@ mfps],
  Dynamic[mfp]},
 {ActionMenu["Set collision order",
   "collisionOrder = " <> ToString[#] :> (collisionOrder = #;) & /@
    Range[0, numcollorders - 1]], Dynamic[collisionOrder]}}
```

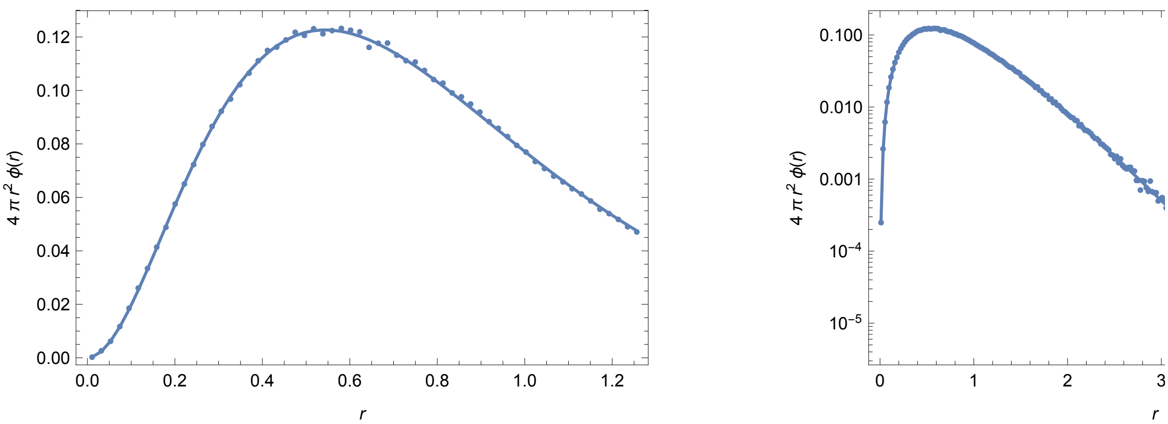
Out[1950]=
```
{{Set c, 0.9}, {Set mfp, 0.3}, {Set collision order, 5}}
```

In[2115]:=
```
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
maxr = data[[2, 5]];
dr = data[[2, 7]];
fluencei = 3 numcollorders + 15 + collisionOrder;

pointsFluence = ppoints[data[[fluencei]], dr, maxr];
exact1FluenceShallow =
  Quiet[{#[[1]], 4 Pi #[[1]]^2 ϕexact1[#[[1]], 1/mfp, c, collisionOrder]} & /@
   pointsFluence[[1 ;; 60]];
exact1Fluence = Quiet[{#[[1]], 4 Pi #[[1]]^2 ϕexact1[#[[1]], 1/mfp,
        c, collisionOrder]}] & /@ pointsFluence[[61 ;; -1 ;; 10]];
plotϕshallow = Quiet[Show[
    ListPlot[pointsFluence[[1 ;; 60]],
     PlotRange → All, PlotStyle → PointSize[.01]],
    ListPlot[exact1FluenceShallow, PlotRange → All, Joined → True],
    Frame → True,
    FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
   ]];
logplotϕ = Quiet[Show[
    ListLogPlot[pointsFluence, PlotRange → All, PlotStyle → PointSize[.01]],
    ListLogPlot[exact1FluenceShallow, PlotRange → All, Joined → True],
    ListLogPlot[exact1Fluence, PlotRange → All, Joined → True],
    Frame → True,
    FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
   ]];
Show[GraphicsGrid[{{plotϕshallow, logplotϕ}}, ImageSize → 800],
 PlotLabel -> "Exact solution (1)\nInfinite 3D medium, isotropic point source,
    isotropic scattering, n-th scattered fluence ϕ[r|" <>
   ToString[collisionOrder] <> "], c =" <> ToString[c] <>
   ", Σt = " <> ToString[1/mfp]]
```

Out[2124]=

Exact solution (1)
Infinite 3D medium, isotropic point source, isotropic scattering, n−th scattered fluence ϕ[r|4], c =0.8, Σ_t =



N-th collided Fluence - Exact solution (2) comparison to MC

In[∘]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :> (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :> (mfp = #;) & /@ mfps],
  Dynamic[mfp]},
 {ActionMenu["Set collision order",
   "collisionOrder = " <> ToString[#] :> (collisionOrder = #;) & /@
    Range[0, numcollorders - 1]], Dynamic[collisionOrder]}}
```

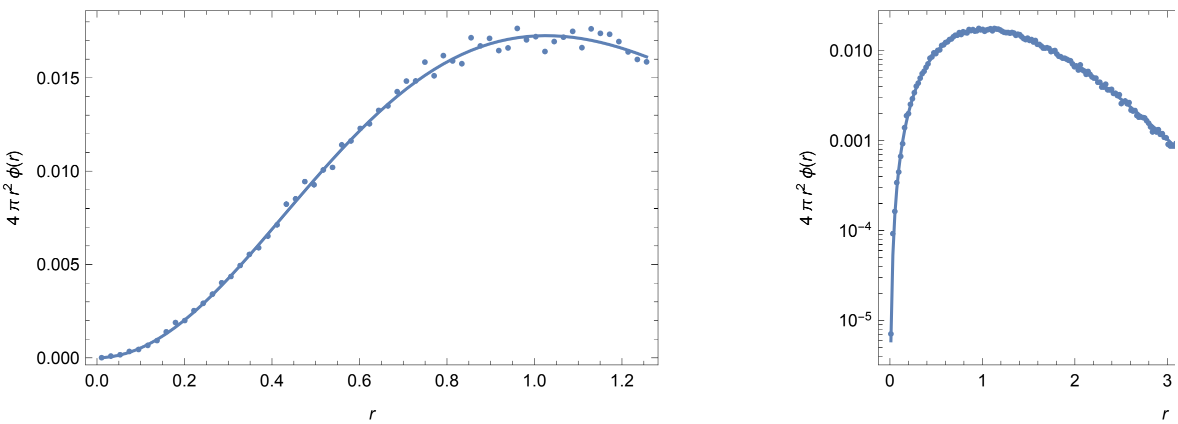
Out[∘]=
```
{{Set c, 0.9}, {Set mfp, 0.3}, {Set collision order, 5}}
```

In[2135]:=
```
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
maxr = data[[2, 5]];
dr = data[[2, 7]];
fluencei = 3 numcollorders + 15 + collisionOrder;

pointsFluence = ppoints[data[[fluencei]], dr, maxr];
exact1FluenceShallow =
  Quiet[{#[[1]], 4 Pi #[[1]]^2 ϕexact2[#[[1]], 1/mfp, c, collisionOrder]} & /@
   pointsFluence[[1 ;; 60]];
exact1Fluence = Quiet[{#[[1]], 4 Pi #[[1]]^2 ϕexact2[#[[1]], 1/mfp,
        c, collisionOrder]}] & /@ pointsFluence[[61 ;; -1 ;; 10]];
plotϕshallow = Quiet[Show[
    ListPlot[pointsFluence[[1 ;; 60]],
     PlotRange → All, PlotStyle → PointSize[.01]],
    ListPlot[exact1FluenceShallow, PlotRange → All, Joined → True],
    Frame → True,
    FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
   ]];
logplotϕ = Quiet[Show[
    ListLogPlot[pointsFluence, PlotRange → All, PlotStyle → PointSize[.01]],
    ListLogPlot[exact1FluenceShallow, PlotRange → All, Joined → True],
    ListLogPlot[exact1Fluence, PlotRange → All, Joined → True],
    Frame → True,
    FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
   ]];
Show[GraphicsGrid[{{plotϕshallow, logplotϕ}}, ImageSize → 800],
 PlotLabel -> "Exact solution (2)\nInfinite 3D medium, isotropic point source,
    isotropic scattering, n-th scattered fluence ϕ[r|" <>
   ToString[collisionOrder] <> "], c =" <> ToString[c] <>
   ", Σt = " <> ToString[1/mfp]]
```

Out[2144]=

Exact solution (2)
Infinite 3D medium, isotropic point source, isotropic scattering, n–th scattered fluence ϕ[r|11], c =0.8, Σ_t :



N-th collided Fluence - Approximations

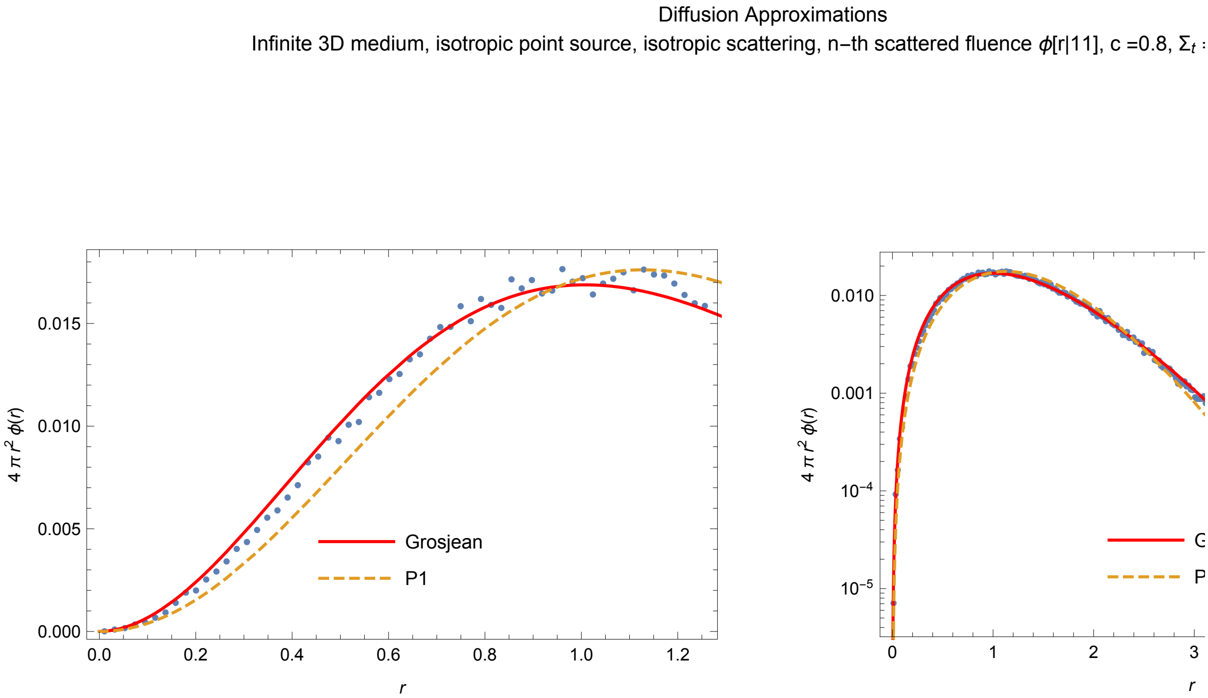
In[∘]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :> (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :> (mfp = #;) & /@ mfps],
  Dynamic[mfp]},
 {ActionMenu["Set collision order",
   "collisionOrder = " <> ToString[#] :> (collisionOrder = #;) & /@
    Range[0, numcollorders - 1]], Dynamic[collisionOrder]}}
```

Out[∘]=
```
{{ Set c , 0.9}, { Set mfp , 0.3}, { Set collision order , 5}}
```

In[2205]:=
```
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
maxr = data[[2, 5]];
dr = data[[2, 7]];
fluencei = 3 numcollorders + 15 + collisionOrder;

pointsFluence = ppoints[data[[fluencei]], dr, maxr];
seriesclassical = c^collisionOrder
   SeriesCoefficient[ϕDiffusion[r, 1 / mfp, C], {C, 0, collisionOrder}];
seriesG = c^collisionOrder SeriesCoefficient[
    ϕGrosjean[r, 1 / mfp, C], {C, 0, collisionOrder}];
plotϕshallow = Quiet[Show[
     ListPlot[pointsFluence[[1 ;; 60]],
      PlotRange → All, PlotStyle → PointSize[.01]],
     Plot[{4 Pi r^2 seriesG, 4 Pi r^2 seriesclassical}, {r, 0, maxr},
      PlotRange → All, PlotStyle → {Red, Dashed},
      PlotLegends → Placed[{"Grosjean", "P1"}, {0.5, .2}]],
     Frame → True,
     FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
    ]];
logplotϕ = Quiet[Show[
     ListLogPlot[pointsFluence, PlotRange → All, PlotStyle → PointSize[.01]],
     LogPlot[{4 Pi r^2 seriesG, 4 Pi r^2 seriesclassical},
      {r, 0, maxr}, PlotRange → All, PlotStyle → {Red, Dashed},
      PlotLegends → Placed[{"Grosjean", "P1"}, {0.5, .2}]],
     Frame → True,
     FrameLabel -> {{4 Pi r^2 ϕ[r],}, {r,}}
    ]];
Show[GraphicsGrid[{{plotϕshallow, logplotϕ}}, ImageSize → 800], PlotLabel ->
  "Diffusion Approximations\nInfinite 3D medium, isotropic point source,
     isotropic scattering, n-th scattered fluence ϕ[r|" <>
   ToString[collisionOrder] <> "], c =" <> ToString[c] <>
   ", Σt = " <> ToString[1 / mfp]]
```

Out[2214]=



Compare moments of ϕ

```
In[•]:= {{ActionMenu["Set c", "c = " <> ToString[#] :> (c = #;) & /@ cs], Dynamic[c]},
  {ActionMenu["Set mfp", "mfp = " <> ToString[#] :> (mfp = #;) & /@ mfps],
   Dynamic[mfp]}}

Out[•]= {{Set c, 0.9}, {Set mfp, 0.3}}

In[2287]:= data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
nummoments = data[[2, 15]];
ϕmoments = {data[[10]]};
ks = Table[k, {k, 0, nummoments - 1}];
analytic = Table[ϕm[c, 1 / mfp, k], {k, ks}];
j = Join[{ks}, {analytic}, ϕmoments];
TableForm[
 Join[{{"n", "analytic", "MC"}}, Transpose[j]]
]
```

Out[2293]//TableForm=

n	analytic	MC
0	6.	6.00171
1	0.	9.16206
2	21.6	21.6302
3	0.	68.5295
4	269.568	272.563

n-th collided moments of ϕ

In[2265]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :→ (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :→ (mfp = #;) & /@ mfps],
  Dynamic[mfp]},
 {ActionMenu["Set collision order",
   "collisionOrder = " <> ToString[#] :→ (collisionOrder = #;) & /@
    Range[0, numcollorders - 1]], Dynamic[collisionOrder]}}
```

Out[2265]=
```
{{Set c, 0.9}, {Set mfp, 0.3}, {Set collision order, 5}}
```

In[2273]:=
```
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
nummoments = data[[2, 15]];
ϕmoments = N[{data[[numcollorders + 13 + collisionOrder]]}];
ks = Table[k, {k, 0, nummoments - 1}];
analytic = Table[ϕm[c, 1 / mfp, k, collisionOrder], {k, ks}];
j = Join[{ks}, {analytic}, ϕmoments];
TableForm[
 Join[{{"n", "analytic", "MC"}}, Transpose[j]]
]
```

Out[2279]//TableForm=

n	analytic	MC
0	0.03125	0.031262
1	0. + 0. i	0.094565
2	0.375	0.374203
3	0.	1.83103
4	10.75	10.681

Angular Distributions

In[ ]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :→ (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :→ (mfp = #;) & /@ mfps],
  Dynamic[mfp]}}
```

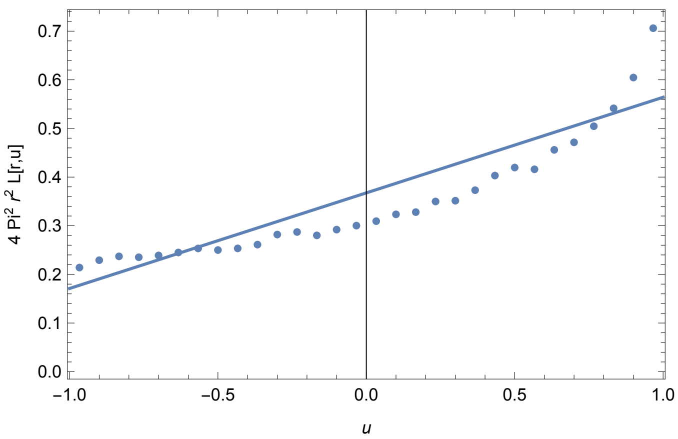
Out[ ]=
```
{{Set c, 0.9}, {Set mfp, 0.3}}
```

In[2437]:=
```
depthi = 52
```

Out[2437]=
```
52
```

In[2438]:=
```
Clear[u];
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
du = data[[2, 9]];
maxr = data[[2, 5]];
dr = data[[2, 7]];
fluxi = 17 + 4 numcollorders + Floor[maxr / dr];
angularFlux  = ppointsu[data[[fluxi + depthi]], du, 1];
r  = dr * depthi - 0.5 dr;
Show[
 ListPlot[angularFlux, PlotRange → All,
  Frame → True,
  FrameLabel -> {{"4 Pi² r² L[r,u]",}, {u,}}],
 Plot[4 Pi r² Pi Ldiffusion[r, u, 1 / mfp, c], {u, -1, 1}, PlotRange → All]
]
```

Out[2446]=



Angular Distribution: Integral of Grosjean’s Diffusion Approximation

In[∘]:=
```
{{ActionMenu["Set c", "c = " <> ToString[#] :→ (c = #;) & /@ cs], Dynamic[c]},
 {ActionMenu["Set mfp", "mfp = " <> ToString[#] :→ (mfp = #;) & /@ mfps],
  Dynamic[mfp]}}
```

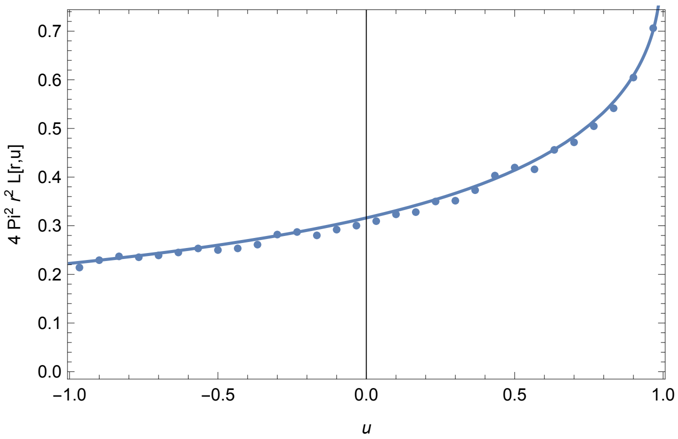
Out[∘]= {{ Set c , 0.9}, { Set mfp , 0.3}}

In[∘]:=
```
depthi = 52
```

Out[∘]= 52

In[2447]:=
```
Clear[u];
data = SelectFirst[simulations, #[[1]] == c && #[[2]] == mfp &][[3]];
du = data[[2, 9]];
maxr = data[[2, 5]];
dr = data[[2, 7]];
fluxi = 17 + 4 numcollorders + Floor[maxr / dr];
angularFlux  = ppointsu[data[[fluxi + depthi]], du, 1];
r  = dr * depthi - 0.5 dr;
Show[
 ListPlot[angularFlux, PlotRange → All,
  Frame → True,
  FrameLabel -> {{"4 Pi^2  r^2  L[r,u]",}, {u,}}],
 Plot[4 Pi r^2 Pi Lintegral[r, u, 1 / mfp, c, ϕGrosjean], {u, -1, 1}, PlotRange → All]
]
```

Out[2455]=



End context

```
End[]
```