Scattering Kernels in 3D

This is code to accompany the book:

A Hitchhiker’s Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

Draine

Draine, B.T. (2003) ‘*Scattering by interstellar dust grains. 1: Optical and ultraviolet*’, ApJ., 598, 1017–25.

In[10790]:=
$$\text{pDraine}[u\_, g\_, \alpha\_] := \frac{1}{4\,\text{Pi}} \left(\frac{1 - g^2}{\left(1 + g^2 - 2\,g\,u\right)^{3/2}} \frac{1 + \alpha\, u^2}{1 + \alpha \left(1 + 2\,g^2\right) / 3} \right)$$

Normalization condition

```
In[]:= Integrate[2 Pi pDraine[u, g, a], {u, -1, 1}, Assumptions → 0 < a < 1 && -1 < g < 1]
Out[]= 1
```

Mean-cosine

```
In[]:= Integrate[2 Pi pDraine[u, g, a] u, {u, -1, 1}, Assumptions → 0 < a < 1 && -1 < g < 1]
```

Out[]=
$$\frac{3}{5} \left(g + \frac{2\,(1 + a)\,g}{3 + a + 2\,a\,g^2} \right)$$

In[]:=
$$\frac{3}{5} \left(g + \frac{2\,(1 + a)\,g}{3 + a + 2\,a\,g^2} \right) \text{ /. } a \to 0$$

Out[]= g

Legendre expansion coefficients

```
In[]:= Integrate[2 Pi (2 k + 1) pDraine[Cos[y], g, a] LegendreP[k, Cos[y]] Sin[y] /. k → 0,
        {y, 0, Pi}, Assumptions → 0 < a < 1 && -1 < g < 1]
Out[]= 1
```

```
In[]:= Integrate[2 Pi (2 k + 1) pDraine[Cos[y], g, a] LegendreP[k, Cos[y]] Sin[y] /. k → 1,
        {y, 0, Pi}, Assumptions → 0 < a < 1 && -1 < g < 1]
```

Out[]=
$$\frac{9\,g\,\left(5 + a\,\left(3 + 2\,g^2\right)\right)}{5\,\left(3 + a + 2\,a\,g^2\right)}$$

In[•]:=
```
Integrate[2 Pi (2 k + 1) pDraine[Cos[y], g, a] LegendreP[k, Cos[y]] Sin[y] /. k → 2,
  {y, 0, Pi}, Assumptions → 0 < a < 1 && -1 < g < 1]
```

Out[•]=
$$\frac{14\,a + 5\,(21 + 11\,a)\,g^2 + 36\,a\,g^4}{7\,(3 + a + 2\,a\,g^2)}$$

In[•]:=
```
Integrate[2 Pi (2 k + 1) pDraine[Cos[y], g, a] LegendreP[k, Cos[y]] Sin[y] /. k → 3,
  {y, 0, Pi}, Assumptions → 0 < a < 1 && -1 < g < 1]
```

Out[•]=
$$\frac{g\,(54\,a + 7\,(45 + 23\,a)\,g^2 + 100\,a\,g^4)}{15\,(3 + a + 2\,a\,g^2)}$$

sampling

In[•]:=
```
cdf = Integrate[2 Pi pDraine[u, g, a],
   {u, -1, x}, Assumptions → 0 < a < 1 && -1 < g < 1 && -1 < x < 1]
```

Out[•]=
$$\Big(3\,(-1 + g)\,g^2\,\Big(-1 - g + \sqrt{1 + g^2 - 2\,g\,x}\Big) + a\,\Big(2 - 2\,g^6 - 2\,g\,x - 2\,\sqrt{1 + g^2 - 2\,g\,x} + g^3\,\sqrt{1 + g^2 - 2\,g\,x} + g^4\,(-2 + x^2) + 2\,g^5\,\Big(x + \sqrt{1 + g^2 - 2\,g\,x}\Big) - g^2\,\Big(-2 + x^2 + \sqrt{1 + g^2 - 2\,g\,x}\Big)\Big)\Big) \Big/ \Big(2\,g^3\,(3 + a + 2\,a\,g^2)\,\sqrt{1 + g^2 - 2\,g\,x}\Big)$$

special case g = 1/2, a = 12 (useful for approximating Mie scattering of water spheres in air)

In[•]:=
```
pDraine[μ, 1/2, 12] // FullSimplify
```

Out[•]=
$$\frac{3\,(1 + 12\,\mu^2)}{14\,\pi\,(5 - 4\,\mu)^{3/2}}$$

Simplification of an exact CDF inverse:

In[•]:=
```
sampleDraineFog[xi_] := Module[{T1, T2, T3},
  T1 = Sqrt[(67 + 14 xi)^4 (5239 - 102376 xi + 492072 xi^2 + 105056 xi^3 + 5488 xi^4)];
  T2 = (883297 - 8820952 xi + 2597784 xi^2 + 735392 xi^3 + 38416 xi^4 + Sqrt[7] T1)^(1/3);
  T3 = Sqrt[((3954 × 2^(2/3) × 3^(1/3) - 56280 × 2^(2/3) × 3^(1/3) xi -
        5880 × 2^(2/3) × 3^(1/3) xi^2 + 912 T2 + 2^(1/3) × 3^(2/3) T2^2) / T2)];
  1/12 (-30 - T3 + Sqrt[(1824 + (6 × 2^(2/3) × 3^(1/3) (-659 + 9380 xi + 980 xi^2)) / T2 -
        2^(1/3) × 3^(2/3) T2 + (12 (67 + 14 xi)^2) / (T3))])
 ]
```

In[∘]:=
```
Show[Histogram[Table[sampleDraineFog[RandomReal[]], {i, 1, 100 000}],
   100, "PDF", ScalingFunctions → "Log"],
  LogPlot[2 Pi pDraine[u, .5, 12.], {u, -1, 1}, PlotRange → All]

]
```

Out[∘]=