Scattering Kernels in dD

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

Isotropic Scattering

Isotropic scattering in dD can be sampled by taking the norm of an array of d normal random variates. However, in odd dimensions, the cosine can be sampled in one step, given the roots to certain polynomials.

In[∘]:= $G[\mu\_, d\_] := \dfrac{\left(1-\mu^2\right)^{\frac{1}{2}(-3+d)}}{\frac{1}{2}\sqrt{\pi}\ \mathrm{Gamma}\left[\frac{1}{2}(-1+d)\right]\ \mathrm{Gamma}\left[\frac{d}{2}\right]^{-1}}$

In[∘]:= $\mathrm{Integrate}\left[\frac{1}{2} G[u, d], \{u, -1, k\}, \mathrm{Assumptions} \to d > 1\ \&\&\ -1 < k < 1\right]$

Out[∘]= $\dfrac{1}{2} + \dfrac{k\ \mathrm{Gamma}\left[\frac{d}{2}\right]\ \mathrm{Hypergeometric2F1}\left[\frac{1}{2}, \frac{3-d}{2}, \frac{3}{2}, k^2\right]}{\sqrt{\pi}\ \mathrm{Gamma}\left[\frac{1}{2}(-1+d)\right]}$

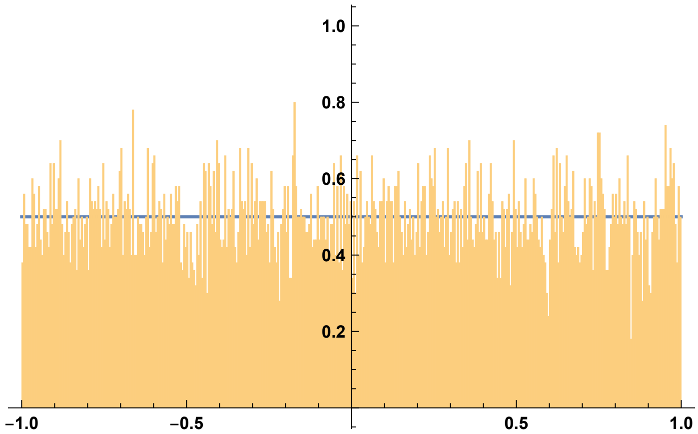
3D

In[∘]:= $\mathrm{Solve}\left[\dfrac{1}{2} + \dfrac{k\ \mathrm{Gamma}\left[\frac{d}{2}\right]\ \mathrm{Hypergeometric2F1}\left[\frac{1}{2}, \frac{3-d}{2}, \frac{3}{2}, k^2\right]}{\sqrt{\pi}\ \mathrm{Gamma}\left[\frac{1}{2}(-1+d)\right]} == x\ /.\ d \to 3, k\right]$

Out[∘]= $\{\{k \to -1 + 2\,x\}\}$

```
In[•]:= With[{d = 3},
  Show[
   Plot[1/2 G[u, d] , {u, -1, 1}],
   Histogram[
    Table[
     -1 + 2 RandomReal[]
     , {i, Range[10 000]}]
    , 500, "PDF"]
  ]
 ]
```

Out[•]=



5D

In[•]:= $\text{Solve}\left[\frac{1}{2}+\frac{k\,\text{Gamma}\left[\frac{d}{2}\right]\,\text{Hypergeometric2F1}\left[\frac{1}{2},\frac{3-d}{2},\frac{3}{2},k^2\right]}{\sqrt{\pi}\,\text{Gamma}\left[\frac{1}{2}(-1+d)\right]}==x\ /.\ d\to 5,\ k\right]\ /.\ x\to \#$

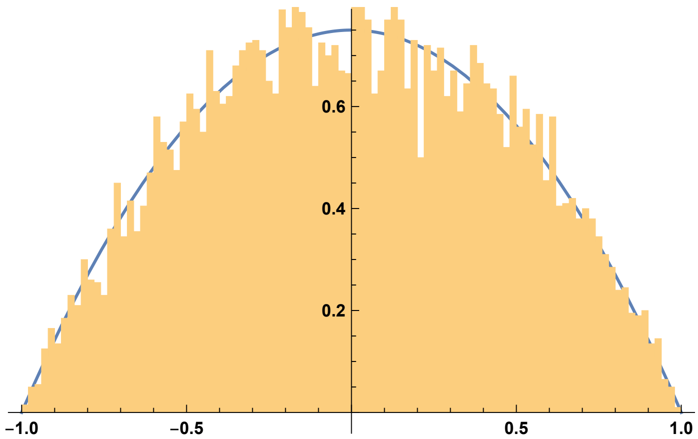
Out[•]= $\left\{\left\{k\to -\frac{1}{\left(-1+2\,\#1+2\sqrt{-\#1+\#1^2}\right)^{1/3}}-\left(-1+2\,\#1+2\sqrt{-\#1+\#1^2}\right)^{1/3}\right\},\right.$

$\left\{k\to \frac{1+i\sqrt{3}}{2\left(-1+2\,\#1+2\sqrt{-\#1+\#1^2}\right)^{1/3}}+\frac{1}{2}\left(1-i\sqrt{3}\right)\left(-1+2\,\#1+2\sqrt{-\#1+\#1^2}\right)^{1/3}\right\},$

$\left.\left\{k\to \frac{1-i\sqrt{3}}{2\left(-1+2\,\#1+2\sqrt{-\#1+\#1^2}\right)^{1/3}}+\frac{1}{2}\left(1+i\sqrt{3}\right)\left(-1+2\,\#1+2\sqrt{-\#1+\#1^2}\right)^{1/3}\right\}\right\}$

In[∘]:=
```
With[{d = 5},
 Show[
  Plot[1/2 G[u, d] , {u, -1, 1}],
  Histogram[
   Table[
    Chop[(1 - ⅈ Sqrt[3])/(2 (-1 + 2 #1 + 2 Sqrt[-#1 + #1^2])^(1/3)) + 1/2 (1 + ⅈ Sqrt[3]) (-1 + 2 #1 + 2 Sqrt[-#1 + #1^2])^(1/3) &[
      RandomReal[]]]
    , {i, Range[10 000]}]
   , 100, "PDF"]
  ]
 ]
```

Out[∘]=



7D

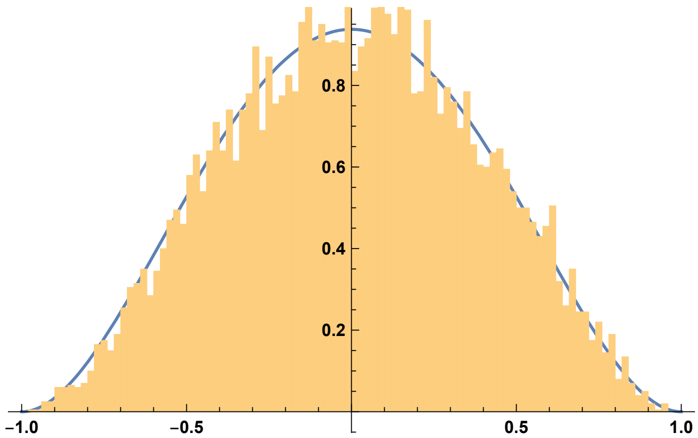
In[∘]:=
```
Solve[1/2 + (k Gamma[d/2] Hypergeometric2F1[1/2, (3-d)/2, 3/2, k^2])/(Sqrt[π] Gamma[1/2 (-1 + d)]) == x /. d → 7, k]
```

Out[∘]=
```
{{k → Root[8 - 16 x + 15 #1 - 10 #1^3 + 3 #1^5 &, 1]},
 {k → Root[8 - 16 x + 15 #1 - 10 #1^3 + 3 #1^5 &, 2]},
 {k → Root[8 - 16 x + 15 #1 - 10 #1^3 + 3 #1^5 &, 3]},
 {k → Root[8 - 16 x + 15 #1 - 10 #1^3 + 3 #1^5 &, 4]},
 {k → Root[8 - 16 x + 15 #1 - 10 #1^3 + 3 #1^5 &, 5]}}
```

In[•]:=

```
With[{d = 7},
 Show[
  Plot[1/2 G[u, d] , {u, -1, 1}],
  Histogram[
   Table[
    Chop[Root[8 - 16 RandomReal[] + 15 #1 - 10 #1^3 + 3 #1^5 &, 1]]
    , {i, Range[10 000]}]
   , 100, "PDF"]
 ]
]
```

Out[•]=



9D

In[•]:=

$$\text{Solve}\left[\frac{1}{2}+\frac{k\,\text{Gamma}\left[\frac{d}{2}\right]\,\text{Hypergeometric2F1}\left[\frac{1}{2},\frac{3-d}{2},\frac{3}{2},k^2\right]}{\sqrt{\pi}\,\text{Gamma}\left[\frac{1}{2}(-1+d)\right]}==x\;/.\;d\to 9,\;k\right]$$

Out[•]=

```
{{k → Root[-16 + 32 x - 35 #1 + 35 #1^3 - 21 #1^5 + 5 #1^7 &, 1]},
 {k → Root[-16 + 32 x - 35 #1 + 35 #1^3 - 21 #1^5 + 5 #1^7 &, 2]},
 {k → Root[-16 + 32 x - 35 #1 + 35 #1^3 - 21 #1^5 + 5 #1^7 &, 3]},
 {k → Root[-16 + 32 x - 35 #1 + 35 #1^3 - 21 #1^5 + 5 #1^7 &, 4]},
 {k → Root[-16 + 32 x - 35 #1 + 35 #1^3 - 21 #1^5 + 5 #1^7 &, 5]},
 {k → Root[-16 + 32 x - 35 #1 + 35 #1^3 - 21 #1^5 + 5 #1^7 &, 6]},
 {k → Root[-16 + 32 x - 35 #1 + 35 #1^3 - 21 #1^5 + 5 #1^7 &, 7]}}
```

In[ ]:=
```
With[{d = 9},
 Show[
  Plot[1/2 G[u, d] , {u, -1, 1}],
  Histogram[
   Table[
    Chop[Root[-16 + 32 RandomReal[] - 35 #1 + 35 #1^3 - 21 #1^5 + 5 #1^7 &, 2]]
    , {i, Range[10 000]}]
   , 100, "PDF"]
 ]
]
```

Out[ ]=