GTR NDF

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

notation

u = **m** . **n** = Cos[θ_m]

α = roughness

Definitions and derivations

In[•]:=
$$\text{GTR`D}[u\_, \alpha\_, \gamma\_] := \frac{\alpha^{2\gamma}(-1+\alpha^2)(-1+\gamma)}{\pi(-\alpha^2+\alpha^{2\gamma})} \frac{1}{(1+u^2(-1+\alpha^2))^{\gamma}}$$

In[•]:=
$$\text{GTR`D}[u\_, \alpha\_, 1] := \frac{-1+\alpha^2}{2\pi(1+u^2(-1+\alpha^2))\,\text{Log}[\alpha]}$$

Similar to the Henyey Greenstein phase function:

In[•]:=
```
FullSimplify[GTR`D[u, α, 3/2], Assumptions → 0 < u < 1 && 0 < α < 1]
```

Out[•]=
$$\frac{\alpha(1+\alpha)}{2\pi(1+u^2(-1+\alpha^2))^{3/2}}$$

In[•]:=
$$\text{GTR`D}[u\_, \alpha\_, 3/2] := \frac{\alpha(1+\alpha)}{2\pi(1+u^2(-1+\alpha^2))^{3/2}}$$

In[•]:=
$$\text{GTR`}\Lambda[u\_, \alpha\_, 1] := \frac{-1+u\sqrt{1+\left(-1+\frac{1}{u^2}\right)\alpha^2}+u\,\text{Log}\left[1+\frac{1}{u}\right]-u\,\text{Log}\left[1+\sqrt{1+\left(-1+\frac{1}{u^2}\right)\alpha^2}\right]}{2u\,\text{Log}[\alpha]}$$

In[•]:=
$$\text{GTR`}\sigma[u\_, \alpha\_, 1] := \left(\frac{-1+\sqrt{u^2+\alpha^2-u^2\alpha^2}+u\,\text{ArcSinh}\left[\frac{u}{\sqrt{1-u^2}}\right]+2u\,\text{Log}[\alpha]-u\,\text{Log}\left[\frac{u+\sqrt{u^2+\alpha^2-u^2\alpha^2}}{\sqrt{1-u^2}}\right]}{2\,\text{Log}[\alpha]}\right)$$

height field normalization

In[•]:=
```
Integrate[2 Pi u GTR`D[u, α, γ], {u, 0, 1}, Assumptions → 0 < α < 1 && 0 < γ]
```

Out[•]=
1

distribution of slopes

In[∘]:=
```
FullSimplify[GTR`D[1/Sqrt[p^2 + q^2 + 1], α, γ] (1/Sqrt[p^2 + q^2 + 1])^4,
 Assumptions → 0 < α < 1 && p > 0 && q > 0]
```

Out[∘]=
$$\frac{\left(1+p^2+q^2\right)^{-2+\gamma}\alpha^{2\gamma}\left(-1+\alpha^2\right)\left(p^2+q^2+\alpha^2\right)^{-\gamma}(-1+\gamma)}{\pi\left(-\alpha^2+\alpha^{2\gamma}\right)}$$

In[∘]:=
```
FullSimplify[GTR`D[1/Sqrt[p^2 + q^2 + 1], α, 3/2] (1/Sqrt[p^2 + q^2 + 1])^4,
 Assumptions → 0 < α < 1 && p > 0 && q > 0]
```

Out[∘]=
$$\frac{\alpha(1+\alpha)}{2\pi\sqrt{1+p^2+q^2}\left(p^2+q^2+\alpha^2\right)^{3/2}}$$

In[∘]:=
$$\text{GTR`P22}[p\_, q\_, \alpha\_, \gamma\_] := \frac{\left(1+p^2+q^2\right)^{-2+\gamma}\alpha^{2\gamma}\left(-1+\alpha^2\right)\left(p^2+q^2+\alpha^2\right)^{-\gamma}(-1+\gamma)}{\pi\left(-\alpha^2+\alpha^{2\gamma}\right)}$$

In[∘]:=
$$\text{GTR`P22}[p\_, q\_, \alpha\_, 1] := \frac{-1+\alpha^2}{2\pi\left(1+p^2+q^2\right)\left(p^2+q^2+\alpha^2\right)\text{Log}[\alpha]}$$

In[∘]:=
$$\text{GTR`P22}[p\_, q\_, \alpha\_, 3/2] := \frac{\alpha(1+\alpha)}{2\pi\sqrt{1+p^2+q^2}\left(p^2+q^2+\alpha^2\right)^{3/2}}$$

Distribution of slope normalization test:

In[∘]:=
```
Integrate[GTR`P22[p, q, α, γ], {p, -Infinity, Infinity},
 {q, -Infinity, Infinity}, Assumptions → 0 < α < 1 && γ > 0]
```

Out[∘]= 1

Marginal slope distribution:

In[∘]:=
```
Integrate[GTR`P22[p, q, α, γ],
 {q, -Infinity, Infinity}, Assumptions → α > 0 && p > -7 && γ > 1]
```

Out[∘]=
$$\frac{1}{2\sqrt{\pi}\left(-\alpha^2+\alpha^{2\gamma}\right)}\alpha^{2\gamma}\left(-1+\alpha^2\right)(-1+\gamma)\Bigg(-\Bigg(\Bigg(\text{Gamma}\left[-\frac{3}{2}+\gamma\right]\Bigg(2\left(p^2+\alpha^2\right)(-2+\gamma)\,\text{Hypergeometric2F1}\left[-\frac{1}{2},2-\gamma,\frac{5}{2}-\gamma,\frac{1+p^2}{p^2+\alpha^2}\right]-\left(-1+\alpha^2\right)(-3+2\gamma)\,\text{Hypergeometric2F1}\left[\frac{1}{2},2-\gamma,\frac{5}{2}-\gamma,\frac{1+p^2}{p^2+\alpha^2}\right]\Bigg)\Bigg)\Bigg/\left(\left(-1+\alpha^2\right)\left(p^2+\alpha^2\right)^{3/2}\text{Gamma}[\gamma]\right)\Bigg)+\frac{2\left(1+p^2\right)^{-\frac{3}{2}+\gamma}\left(p^2+\alpha^2\right)^{-\gamma}\text{Gamma}\left[\frac{3}{2}-\gamma\right]\text{Hypergeometric2F1}\left[\frac{1}{2},\gamma,-\frac{1}{2}+\gamma,\frac{1+p^2}{p^2+\alpha^2}\right]}{\text{Gamma}[2-\gamma]}\Bigg)$$

In[∘]:=
$$\text{GTR`P2}[p\_, \alpha\_, \gamma\_] := \frac{1}{2\sqrt{\pi}\left(-\alpha^2+\alpha^{2\gamma}\right)}\alpha^{2\gamma}\left(-1+\alpha^2\right)(-1+\gamma)\left(-\left(\left(\text{Gamma}\left[-\frac{3}{2}+\gamma\right]\left(2\left(p^2+\alpha^2\right)(-2+\gamma)\,\text{Hypergeometric2F1}\left[-\frac{1}{2}, 2-\gamma, \frac{5}{2}-\gamma, \frac{1+p^2}{p^2+\alpha^2}\right] - \left(-1+\alpha^2\right)(-3+2\gamma)\,\text{Hypergeometric2F1}\left[\frac{1}{2}, 2-\gamma, \frac{5}{2}-\gamma, \frac{1+p^2}{p^2+\alpha^2}\right]\right)\right)\Big/\left(\left(-1+\alpha^2\right)\left(p^2+\alpha^2\right)^{3/2}\text{Gamma}[\gamma]\right)\right) + \frac{2\left(1+p^2\right)^{-\frac{3}{2}+\gamma}\left(p^2+\alpha^2\right)^{-\gamma}\text{Gamma}\left[\frac{3}{2}-\gamma\right]\text{Hypergeometric2F1}\left[\frac{1}{2}, \gamma, -\frac{1}{2}+\gamma, \frac{1+p^2}{p^2+\alpha^2}\right]}{\text{Gamma}[2-\gamma]}\right)$$

In[∘]:=
$$\text{Integrate}[\text{GTR`P22}[p, q, \alpha, 1], \{q, -\text{Infinity}, \text{Infinity}\}, \text{Assumptions} \to \alpha > 0 \,\&\&\, p > -7]$$

Out[∘]=
$$\frac{\frac{1}{\sqrt{1+p^2}} - \frac{1}{\sqrt{p^2+\alpha^2}}}{\text{Log}\left[\alpha^2\right]}$$

In[∘]:=
$$\text{GTR`P2}[q\_, \alpha\_, 1] := \frac{\frac{1}{\sqrt{1+q^2}} - \frac{1}{\sqrt{q^2+\alpha^2}}}{\text{Log}\left[\alpha^2\right]}$$

In[∘]:=
$$\text{Integrate}\left[\text{GTR`P22}\left[p, q, \alpha, 3/2\right], \{q, -\text{Infinity}, \text{Infinity}\}, \text{Assumptions} \to \alpha > 0 \,\&\&\, p > -7\right]$$

Out[∘]=
$$\frac{\alpha\left(-\text{EllipticE}\left[\frac{-1+\alpha^2}{p^2+\alpha^2}\right] + \text{EllipticK}\left[\frac{-1+\alpha^2}{p^2+\alpha^2}\right]\right)}{\pi(-1+\alpha)\sqrt{p^2+\alpha^2}}$$

In[∘]:=
$$\text{GTR`P2}\left[q\_, \alpha\_, 3/2\right] := \frac{\alpha\left(-\text{EllipticE}\left[\frac{-1+\alpha^2}{p^2+\alpha^2}\right] + \text{EllipticK}\left[\frac{-1+\alpha^2}{p^2+\alpha^2}\right]\right)}{\pi(-1+\alpha)\sqrt{p^2+\alpha^2}}$$

derivation of Λ(u)

In[∘]:=
$$\text{FullSimplify}\left[\frac{\sqrt{1-u^2}}{u}\,\text{Integrate}\left[\left(q - \frac{u}{\sqrt{1-u^2}}\right)\text{GTR`P2}[q, \alpha, 1], \left\{q, \frac{u}{\sqrt{1-u^2}}, \text{Infinity}\right\}, \text{Assumptions} \to 0 < u < 1 \,\&\&\, 0 < \alpha < 1\right], \text{Assumptions} \to 0 < u < 1 \,\&\&\, 0 < \alpha < 1\right]$$

Out[∘]=
$$\frac{-1 + u\sqrt{1+\left(-1+\frac{1}{u^2}\right)\alpha^2} + u\,\text{Log}\left[1+\frac{1}{u}\right] - u\,\text{Log}\left[1+\sqrt{1+\left(-1+\frac{1}{u^2}\right)\alpha^2}\right]}{2u\,\text{Log}[\alpha]}$$

Cross section σ(u) derivation

sigma derivation

In[•]:=
```
Integrate[(u - p √(1 - u^2)) GTR`P2[p, a, 1],
  {p, -Infinity, u/√(1 - u^2)}, Assumptions → 0 < u < 1 && 0 < a < 1]
```

Out[•]=
$$\frac{-1+\sqrt{a^2+u^2-a^2 u^2}+u\,\text{ArcSinh}\left[\frac{u}{\sqrt{1-u^2}}\right]+2\,u\,\text{Log}[a]-u\,\text{Log}\left[\frac{u+\sqrt{a^2+u^2-a^2 u^2}}{\sqrt{1-u^2}}\right]}{2\,\text{Log}[a]}$$

In[•]:=
```
Delta`σ[u_, ui_] := Re[2 (√(1 - u^2 - ui^2) + u ui ArcCos[-(u ui)/(√(1 - u^2) √(1 - ui^2))])]
```

In[•]:=
```
With[{α = .1},
  Plot[{
     Quiet[NIntegrate[GTR`D[ui, α, 1] × Delta`σ[u, ui], {ui, 0, 1}]],
     ((-1 + √(u^2 + α^2 - u^2 α^2) + u ArcSinh[u/√(1-u^2)] + 2 u Log[α] - u Log[(u+√(u^2+α^2-u^2 α^2))/√(1-u^2)]) / (2 Log[α]))
   }, {u, -1, 1}]
]
```

Out[•]=