Ground Glass (GGX) NDF

notation

u = **m** . **n** = Cos[θ_m]
α = roughness

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definitions and derivations

In[2744]:= $\text{GGX`D}[u\_, \alpha\_] := \dfrac{\alpha^2}{\pi \left(1 + u^2 \left(-1 + \alpha^2\right)\right)^2} \text{HeavisideTheta}[u]$

In[490]:= $\text{GGX`}\sigma[u\_, \alpha\_] := \dfrac{1}{2}\left(\sqrt{\alpha^2 - \alpha^2 u^2 + u^2} + u\right)$

In[2645]:= $\text{GGX`}\Lambda[u\_, \alpha\_] := \dfrac{1}{2}\left(-1 + \dfrac{\sqrt{\alpha^2 - u^2 \left(-1 + \alpha^2\right)}}{u}\right)$

In[538]:= $\left(1 + \text{GGX`}\Lambda[u, \alpha]\right) u == \text{GGX`}\sigma[u, \alpha]$ // FullSimplify

Out[538]= True

In[561]:= $\left(\text{GGX`}\Lambda[u, \alpha]\right) u == \text{GGX`}\sigma[-u, \alpha]$ // FullSimplify

Out[561]= True

In[2646]:= $\text{FullSimplify}\left[\text{GGX`}\Lambda\left[u, \dfrac{u}{\sqrt{1-u^2}\, x}\right], \text{Assumptions} \to 0 < u < 1 \,\&\&\, x > 0\right]$

Out[2646]= $\dfrac{1}{2}\left(-1 + \sqrt{1 + \dfrac{1}{x^2}}\right)$

shape invariant f(x)

In[1237]:= $\text{FullSimplify}\left[\text{GGX`D}[u, \alpha]\, u^4 \alpha^2 \text{ /. } u \text{ -> } \dfrac{1}{\sqrt{1 + x^2 \alpha^2}}, \text{Assumptions} \to 1 - \dfrac{1}{\sqrt{1 + x^2 \alpha^2}} > 0\right]$

Out[1237]= $\dfrac{1}{\pi \left(1 + x^2\right)^2}$

height field normalization

In[510]:= $\text{Integrate}[2\,\text{Pi}\, u\, \text{GGX`D}[u, \alpha], \{u, 0, 1\}, \text{Assumptions} \to 0 < \alpha < 1]$

Out[510]= 1

distribution of slopes

In[516]:= $$\texttt{FullSimplify}\left[\texttt{GGX`D}\left[\frac{1}{\sqrt{p^2+q^2+1}},\alpha\right]\left(\frac{1}{\sqrt{p^2+q^2+1}}\right)^4,\ \texttt{Assumptions}\to 0<\alpha<1\ \&\&\ p>0\ \&\&\ q>0\right]$$

Out[516]= $$\frac{\alpha^2}{\pi\left(p^2+q^2+\alpha^2\right)^2}$$

In[519]:= $$\texttt{GGX`P22[p\_, q\_, }\alpha\texttt{\_] :=}\ \frac{\alpha^2}{\pi\left(p^2+q^2+\alpha^2\right)^2}$$

In[520]:= `Integrate[GGX`P22[p, q, α], {p, -Infinity, Infinity}, {q, -Infinity, Infinity}, Assumptions → 0 < α < 1]`

Out[520]= 1

In[522]:= `Integrate[GGX`P22[p, q, α], {q, -Infinity, Infinity}, Assumptions → α > 0 && Im[p] == 0]`

Out[522]= $$\frac{\alpha^2}{2\left(p^2+\alpha^2\right)^{3/2}}$$

In[525]:= $$\texttt{GGX`P2[p\_, }\alpha\texttt{\_] :=}\ \frac{\alpha^2}{2\left(p^2+\alpha^2\right)^{3/2}}$$

derivation of Λ(u)

In[532]:= $$\texttt{FullSimplify}\left[\frac{\sqrt{1-u^2}}{u}\,\texttt{Integrate}\left[\left(q-\frac{u}{\sqrt{1-u^2}}\right)\texttt{GGX`P2}[q,\alpha],\ \left\{q,\frac{u}{\sqrt{1-u^2}},\texttt{Infinity}\right\},\ \texttt{Assumptions}\to 0<u<1\ \&\&\ 0<\alpha<1\right],\ \texttt{Assumptions}\to 0<u<1\ \&\&\ 0<\alpha<1\right]$$

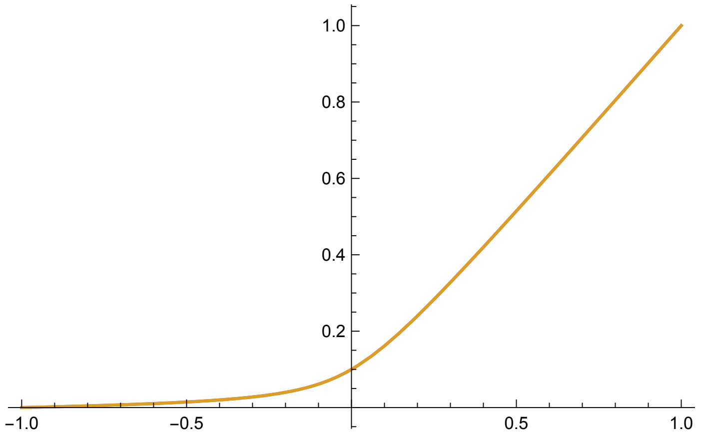
Out[532]= $$\frac{1}{2}\left(-1+\frac{\sqrt{\alpha^2-u^2\left(-1+\alpha^2\right)}}{u}\right)$$

compare σ to delta integral:

$$\texttt{Delta`}\sigma\texttt{[u\_, ui\_] :=}\ \texttt{Re}\left[2\left(\sqrt{1-u^2-ui^2}+u\,ui\,\texttt{ArcCos}\left[-\frac{u\,ui}{\sqrt{1-u^2}\sqrt{1-ui^2}}\right]\right)\right]$$

In[499]:=
```
With[{α = .2},
 Plot[{
   Quiet[NIntegrate[GGX`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
   GGX`σ[u, α]
  }, {u, -1, 1}]
]
```

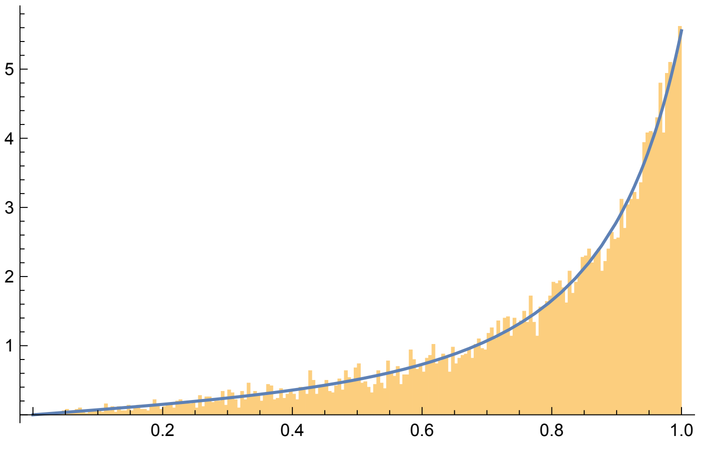
Out[499]=



importance sampling

In[4481]:=
```
With[{α = .6},
 Show[
  Histogram[Table[Sqrt[1/(1 - α^2 + α^2/RandomReal[])], {i, Range[10 000]}], 200, "PDF"],
  Plot[GGX`D[u, α] 2 Pi u, {u, 0, 1}, PlotRange → All]
 ]
]
```

Out[4481]=



As superposition of Beckmann NDFs:

Frechet-2 superposition:

In[631]:= $\text{Beckmann`D}[u\_, \alpha\_] := \frac{e^{-\frac{-1+\frac{1}{u^2}}{\alpha^2}}}{\alpha^2 \pi u^4}$

In[635]:= `PDF[FrechetDistribution[2, α]][αB]`

Out[635]= $\begin{cases} \frac{2 e^{-\frac{\alpha^2}{\alpha B^2}} \alpha^2}{\alpha B^3} & \alpha B > 0 \\ 0 & \text{True} \end{cases}$

The GGX NDF is a Frechet-2 distribution of Beckmann NDFs:

In[634]:= $\text{FullSimplify}\left[\text{Integrate}\left[\frac{2 e^{-\frac{\alpha^2}{\alpha B^2}} \alpha^2}{\alpha B^3} \text{Beckmann`D}[u, \alpha B], \{\alpha B, 0, \text{Infinity}\}, \text{Assumptions} \to \alpha > 0 \,\&\&\, 0 < u < 1\right] == \text{GGX`D}[u, \alpha], \text{Assumptions} \to 0 < u < 1\right]$

Out[634]= True

In[1715]:= $\text{FullSimplify}\left[\text{Integrate}\left[\frac{2 e^{-\frac{1}{\alpha B^2}}}{\alpha B^3} \text{Beckmann`D}[u, \alpha\,\alpha B], \{\alpha B, 0, \text{Infinity}\}, \text{Assumptions} \to \alpha > 0 \,\&\&\, 0 < u < 1\right] == \text{GGX`D}[u, \alpha], \text{Assumptions} \to 0 < u < 1\right]$

Out[1715]= True

Which yields a new derivation of GGX Λ

In[661]:= $\text{FullSimplify}\left[\text{Integrate}\left[\frac{2 e^{-\frac{\alpha^2}{\alpha B^2}} \alpha^2}{\alpha B^3} \text{Beckmann`}\Lambda[u, \alpha B], \{\alpha B, 0, \text{Infinity}\}, \text{Assumptions} \to \alpha > 0 \,\&\&\, 0 < u < 1\right] == \text{GGX`}\Lambda[u, \alpha], \text{Assumptions} \to 0 < u < 1\right]$

Out[661]= True

In[638]:= $\text{Integrate}\left[\frac{2 e^{-\frac{\alpha^2}{\alpha B^2}} \alpha^2}{\alpha B^3} \alpha B, \{\alpha B, 0, \text{Infinity}\}, \text{Assumptions} \to \alpha > 0\right]$

Out[638]= $\sqrt{\pi}\ \alpha$

The mean squared Beckmann roughness in the superposition is unbounded:

In[639]:= $\text{Integrate}\left[\frac{2 e^{-\frac{\alpha^2}{\alpha B^2}} \alpha^2}{\alpha B^3} \alpha B^2, \{\alpha B, 0, \text{Infinity}\}, \text{Assumptions} \to \alpha > 0\right]$

Out[639]= $\text{Integrate}\left[\frac{2 e^{-\frac{\alpha^2}{\alpha B^2}} \alpha^2}{\alpha B}, \{\alpha B, 0, \infty\}, \text{Assumptions} \to \alpha > 0\right]$

Gamma-1 superposition

In[655]:= `PDF[GammaDistribution[1, 1]][αB]`

Out[655]= $\begin{cases} e^{-\alpha B} & \alpha B > 0 \\ 0 & \text{True} \end{cases}$

In[657]:=
```
FullSimplify[Integrate[ⅇ^-αB Beckmann`D[u, α/√αB], {αB, 0, Infinity},
    Assumptions → α > 0 && 0 < u < 1] == GGX`D[u, α], Assumptions → 0 < u < 1]
```

Out[657]= True

In[663]:=
```
FullSimplify[Integrate[ⅇ^-αB Beckmann`Λ[u, α/√αB], {αB, 0, Infinity},
    Assumptions → α > 0 && 0 < u < 1] == GGX`Λ[u, α], Assumptions → 0 < u < 1]
```

Out[663]= True