Common Quantities and Functions

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com

Surface Area of Unit Sphere in d Dimensions

$$\Omega[d\_, r\_] := \frac{d\,\pi^{d/2}\,r^{d-1}}{\text{Gamma}\left[\frac{d}{2}+1\right]}$$

Spherical Diffusion Mode in d Dimensions

$$\text{diffusionMode}[v\_, d\_, r\_] := (2\pi)^{-\frac{d}{2}}\,r^{1-\frac{d}{2}}\,v^{-1-\frac{d}{2}}\,\text{BesselK}\left[\frac{1}{2}(-2+d), \frac{r}{v}\right]$$

```
Table[{d, FullSimplify[diffusionMode[v, d, r], Assumptions → v > 0 && r > 0]},
  {d, 1, 3}] // TableForm
```

1	$\frac{e^{-\frac{r}{v}}}{2v}$
2	$\frac{\text{BesselK}\left[0,\frac{r}{v}\right]}{2\pi v^2}$
3	$\frac{e^{-\frac{r}{v}}}{4\pi r v^2}$

```
Integrate[Ω[d, r] diffusionMode[v, d, r],
 {r, 0, Infinity}, Assumptions → v > 0 && d ≥ 1]
```

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Caseology Quantities

Definitions

$$\text{CaseN0}[c\_, v0\_] := \frac{1}{2}\,c\,v0^3\left(\frac{c}{v0^2-1}-\frac{1}{v0^2}\right)$$

In[3187]:=
$$\text{Casev0}[c\_?\text{NumericQ}] := \text{FindRoot}\left[c\,v\,\text{ArcTanh}\left[\frac{1}{v}\right]-1, \left\{v, 1+10^{-10}, 10^{10}\right\}, \text{Method}\to\text{"Brent"}\right][[1]][[2]]$$

In[3190]:=
$$\text{Casev0}[c\_, prec\_] := \text{ReplaceAll}\left[\text{Abs}[v], \text{First}\left[\text{FindRoot}\left[c\,v\,\text{ArcTanh}\left[\frac{1}{v}\right]-1, \{v, 2\}, \text{WorkingPrecision}\to prec\right]\right]\right];$$

$$\text{CaseN}[c\_,\ v\_] := v\left(\text{Case}\lambda[v,\ c]^{2} + \left(\frac{\pi\, c\, v}{2}\right)^{2}\right)$$

$$\text{Case}\lambda[v\_,\ c\_] := 1 - c\, v\, \text{ArcTanh}[v]$$

Approximations

Approximation from [Case and Zweifel 1967]

In[3428]:=
$$\text{k0low}[c\_] := 1 - 2\, E^{-2/c}\left(1 + \frac{4 - c}{c}\, E^{-2/c} + \frac{24 - 12\, c + c^{2}}{c^{2}}\, E^{-4/c} + \frac{512 - 384\, c + 72\, c^{2} - 3\, c^{3}}{3\, c^{3}}\, E^{-6/c}\right);$$

$$\text{k0high}[c\_] := \sqrt{3\,(1 - c)}\,\left(1 - \frac{2}{5}\,(1 - c) - \frac{12}{175}\,(1 - c)^{2} - \frac{2}{125}\,(1 - c)^{3} + \frac{166}{67\,375}\,(1 - c)^{4}\right);$$

$$\text{Casev0approx}[c\_] := \text{If}\left[c > 0.56,\ \frac{1}{\text{k0high}[c]},\ \frac{1}{\text{k0low}[c]}\right]$$

Approximation [d'Eon 2017]

In[3456]:=
$$\text{Casev0approx2}[c\_] := 1 \Big/ \sqrt{1 - c^{2.4429445001914587\text{`} + \frac{0.5786368322364553\text{`}}{c} - 0.021581332427913873\text{`}\, c}}$$

Benchmark Values for Discrete Eigenvalue v_0

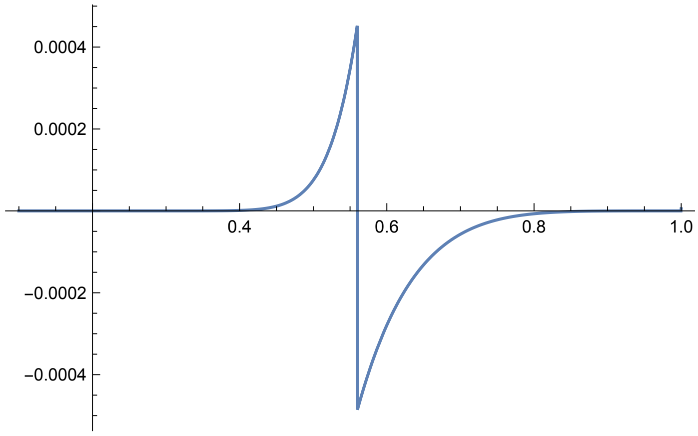
$$\text{v0BenchTable} = \text{TableForm}\Big[\text{Join}\Big[\{\{\text{"}\alpha\text{"},\ \text{"}v_0\text{"}\}\},\ \text{Map}\Big[\{\text{N}[\#],\ \text{Casev0}[\#,\ 40]\}\ \&,\ \Big\{\frac{1}{100},\ \frac{5}{100},\ \frac{1}{10},\ \frac{2}{10},\ \frac{3}{10},\ \frac{5}{10},\ \frac{7}{10},\ \frac{8}{10},\ \frac{85}{100},\ \frac{9}{10},\ \frac{95}{100},\ \frac{98}{100},\ \frac{99}{100},\ \frac{995}{1000},\ \frac{999}{1000},\ \frac{9999}{10\,000},\ \frac{99\,999}{100\,000},\ \frac{999\,999}{1\,000\,000}\Big\}\Big]\Big]\Big]$$

FindRoot::cvmit : Failed to converge to the requested accuracy or precision within 100 iterations. >>

α	v_0
0.01	1.0000000000000000000000000000000093600630
0.05	1.0000000000000000008496708510583180914518
0.1	1.000000004122307593242207339133885345957
0.2	1.000090886544380710821109192160326963735
0.3	1.002592888793223199142982501642964092168
0.5	1.044382033760833484984013906344747760869
0.7	1.206804253985286033572144537105448397639
0.8	1.407634309062772015890071825808163836056
0.85	1.588558625363179696428421317704501663412
0.9	1.903204856044847718980561237457780816825
0.95	2.635148834268739177311679967586549522622
0.98	4.115520476316445421271431792682995753409
0.99	5.796729451302002309775836365597598793316
0.995	8.181342535857420321730013033380917475302
0.999	18.26472572652667373356350462926948043553
0.9999	57.73733645201289717419088459805147261345
0.99999	182.5749161359718602430336283413298737341
0.999999	577.3505001298654062131292059610773432721

Evaluate Case approximation

```
p = Plot[Casev0[c] - Casev0approx[c], {c, 0.1, 1}, PlotRange → All]
```



In[3458]:=
```
p = Plot[{Casev0[c] / Casev0approx2[c], Casev0[c] / Casev0approx[c]},
  {c, 0.1, 1}, PlotRange → All]
```

Out[3458]=