Ei NDF

This is code to accompany the book:

A Hitchhiker’s Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

notation

u = **m** . **n** = Cos[θ_m]
α = roughness

Definitions and derivations

In[2746]:=
$$\text{Ei`D}[u\_, \alpha\_] := \frac{\text{Gamma}\left[0, \frac{-1+\frac{1}{u^2}}{\alpha^2}\right]}{\pi\, u^4\, \alpha^2}\, \text{HeavisideTheta}[u]$$

In[2623]:=
$$\text{Ei`}\Lambda[u\_, \alpha\_] := \frac{2\, e^{\frac{u^2}{(-1+u^2)\,\alpha^2}} \sqrt{1-u^2}\, \alpha \left(-\alpha^2 + u^2 \left(-1+\alpha^2\right)\right) + \sqrt{\pi}\, u \left(3\,\alpha^2 + u^2 \left(2 - 3\,\alpha^2\right)\right) \text{Erfc}\left[\frac{u}{\sqrt{1-u^2}\,\alpha}\right]}{6 \sqrt{\pi}\, u \left(-1+u^2\right) \alpha^2}$$

In[1331]:=
$$\text{Ei`}\sigma[u\_, \alpha\_] := \frac{1}{6\sqrt{\pi}\left(-1+u^2\right)\alpha^2} \Bigg(\alpha \left(3\sqrt{\pi}\, u \left(-1+u^2\right) \alpha + 2\, e^{\frac{u^2}{(-1+u^2)\,\alpha^2}} \sqrt{1-u^2} \left(-\alpha^2 + u^2 \left(-1+\alpha^2\right)\right)\right) + 3\sqrt{\pi}\, u \left(-\alpha^2 + u^2\left(-2+\alpha^2\right)\right) \text{Erf}\left[\frac{u}{\sqrt{1-u^2}\,\alpha}\right] + 2\sqrt{\pi}\, u^2\, \text{Abs}[u] \left(1 + 2\, \text{Erf}\left[\frac{u^2 \sqrt{1-u^2}}{\alpha\, \text{Abs}[u] - u^2\, \alpha\, \text{Abs}[u]}\right]\right)\Bigg)$$

In[2624]:=
$$\text{Ei`G1}[u\_, a\_] := \frac{1}{1 + \text{Ei`}\Lambda[u, a]}$$

In[2689]:=
$$\text{FullSimplify}\left[\text{Ei`}\Lambda\left[u, \frac{u}{\sqrt{1-u^2}\, x}\right], \text{Assumptions} \to 0 < u < 1 \,\&\&\, x > 0\right]$$

Out[2689]=
$$\frac{e^{-x^2}\left(1+x^2\right)}{3\sqrt{\pi}\, x} - \frac{1}{6}\left(3 + 2\,x^2\right) \text{Erfc}[x]$$

derivation

$$\text{Beckmann`D}[u\_,\ \alpha\_] := \frac{e^{-\frac{-1+\frac{1}{u^2}}{\alpha^2}}}{\alpha^2\,\pi\,u^4}\ \text{HeavisideTheta}[u]$$

In[1304]:= $$\text{Integrate}\left[\text{Beckmann`D}\left[u,\ \alpha\sqrt{m}\right],\ \{m,\ 0,\ 1\},\ \text{Assumptions} \to 0 < u < 1\ \&\&\ \alpha > 0\right]$$

Out[1304]= $$\frac{\text{Gamma}\left[0,\ \frac{-1+\frac{1}{u^2}}{\alpha^2}\right]}{\pi\,u^4\,\alpha^2}$$

In[1326]:= $$\text{Integrate}\left[\text{Beckmann`}\sigma\left[u,\ \alpha\sqrt{m}\right],\ \{m,\ 0,\ 1\},\ \text{Assumptions} \to -1 < u < 1\ \&\&\ \alpha > 0\right]$$

Out[1326]= $$\frac{1}{6\sqrt{\pi}\,\left(-1+u^2\right)\alpha^2}\Bigg(\alpha\left(3\sqrt{\pi}\,u\left(-1+u^2\right)\alpha + 2\,e^{\frac{u^2}{\left(-1+u^2\right)\alpha^2}}\sqrt{1-u^2}\left(-\alpha^2+u^2\left(-1+\alpha^2\right)\right)\right) + 3\sqrt{\pi}\,u\left(-\alpha^2+u^2\left(-2+\alpha^2\right)\right)\text{Erf}\left[\frac{u}{\sqrt{1-u^2}\,\alpha}\right] + 2\sqrt{\pi}\,u^2\,\text{Abs}[u]\left(1+2\,\text{Erf}\left[\frac{u^2\sqrt{1-u^2}}{\alpha\,\text{Abs}[u] - u^2\,\alpha\,\text{Abs}[u]}\right]\right)\Bigg)$$

In[1308]:= $$\text{Integrate}\left[\text{Beckmann`}\Lambda\left[u,\ \alpha\sqrt{m}\right],\ \{m,\ 0,\ 1\},\ \text{Assumptions} \to 0 < u < 1\ \&\&\ \alpha > 0\right]$$

Out[1308]= $$\frac{2\,e^{\frac{u^2}{\left(-1+u^2\right)\alpha^2}}\sqrt{1-u^2}\,\alpha\left(-\alpha^2+u^2\left(-1+\alpha^2\right)\right) + \sqrt{\pi}\,u\left(3\,\alpha^2+u^2\left(2-3\,\alpha^2\right)\right)\text{Erfc}\left[\frac{u}{\sqrt{1-u^2}\,\alpha}\right]}{6\sqrt{\pi}\,u\left(-1+u^2\right)\alpha^2}$$

shape invariant f(x)

In[1314]:= $$\text{FullSimplify}\left[\text{Ei`D}[u,\ \alpha]\ u^4\,\alpha^2\ /.\ u \to \frac{1}{\sqrt{1+x^2\,\alpha^2}},\ \text{Assumptions} \to 1 - \frac{1}{\sqrt{1+x^2\,\alpha^2}} > 0\ \&\&\ x > 0\ \&\&\ \alpha > 0\right]$$

Out[1314]= $$\frac{\text{Gamma}\left[0,\ x^2\right]}{\pi}$$

height field normalization

In[1315]:= $$\text{Integrate}[2\,\text{Pi}\,u\,\text{Ei`D}[u,\ \alpha],\ \{u,\ 0,\ 1\},\ \text{Assumptions} \to 0 < \alpha < 1]$$

Out[1315]= 1

distribution of slopes

In[1316]:=
$$\text{FullSimplify}\left[\text{Ei`D}\left[\frac{1}{\sqrt{p^2+q^2+1}}, \alpha\right]\left(\frac{1}{\sqrt{p^2+q^2+1}}\right)^4, \text{Assumptions} \to 0 < \alpha < 1 \,\&\&\, p > 0 \,\&\&\, q > 0\right]$$

Out[1316]=
$$\frac{\text{Gamma}\left[0, \frac{p^2+q^2}{\alpha^2}\right]}{\pi\,\alpha^2}$$

In[1318]:=
$$\text{Ei`P22}[p\_, q\_, \alpha\_] := \frac{\text{Gamma}\left[0, \frac{p^2+q^2}{\alpha^2}\right]}{\pi\,\alpha^2}$$

In[1319]:=
```
Integrate[Ei`P22[p, q, α], {p, -Infinity, Infinity},
  {q, -Infinity, Infinity}, Assumptions → 0 < α < 1]
```

Out[1319]= 1

compare σ to delta integral:

In[1322]:=
$$\text{Delta`}\sigma[u\_, ui\_] := \text{Re}\left[2\left(\sqrt{1-u^2-ui^2} + u\,ui\,\text{ArcCos}\left[-\frac{u\,ui}{\sqrt{1-u^2}\,\sqrt{1-ui^2}}\right]\right)\right]$$

In[1332]:=
```
With[{α = .7},
  Plot[{
     Quiet[NIntegrate[Ei`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
     Quiet[Ei`σ[u, α]]
   }, {u, -1, 1}]
]
```

Out[1332]=