Scattering Kernels in 3D

This is code to accompany the book:

A Hitchhiker’s Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

Isotropic Scattering

```
pIsotropic[u_] := 1/(4 Pi)
```

Normalization condition

```
Integrate[2 Pi pIsotropic[u], {u, -1, 1}]
```

```
1
```

Mean-cosine

```
Integrate[2 Pi pIsotropic[u] u, {u, -1, 1}]
```

```
0
```

Legendre expansion coefficients

```
Integrate[
 2 Pi (2 k + 1) pIsotropic[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 0, {y, 0, Pi}]
```

```
1
```

```
Integrate[
 2 Pi (2 k + 1) pIsotropic[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 1, {y, 0, Pi}]
```

```
0
```

sampling

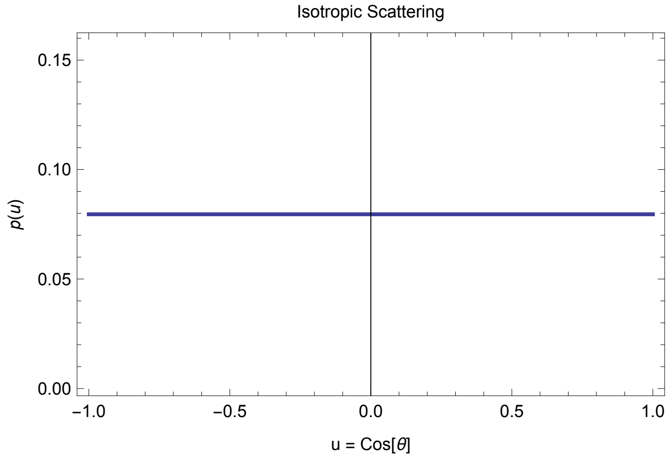
```
cdf = Integrate[2 Pi pIsotropic[u], {u, -1, x}]
```

$$\frac{1+x}{2}$$

```
Solve[cdf == e, x]
```

$$\{\{x \to -1 + 2 e\}\}$$

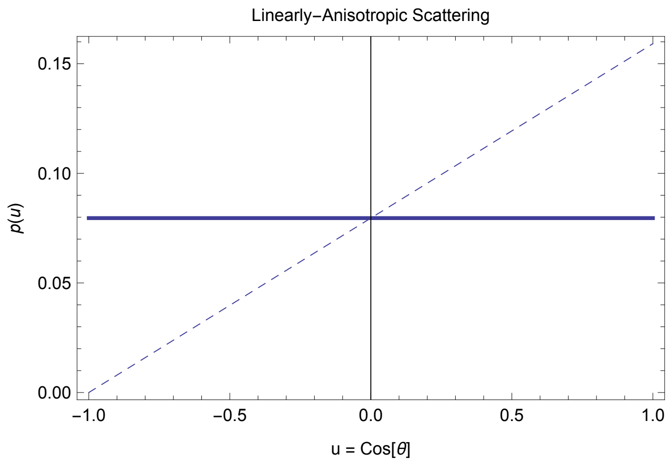
```
Clear[u]; Show[
 Plot[pIsotropic[u], {u, -1, 1}, PlotStyle → Thick]
 , Frame → True,
 FrameLabel → {{p[u],}, {"u = Cos[θ]", "Isotropic Scattering"}}]
```



Linearly-Anisotropic Scattering (Eddington)

$$\text{pLinaniso[u\_, b\_] := } \frac{1}{4\,\text{Pi}}\,(1 + b\,u)$$

```
Clear[u];
Show[
 Plot[pIsotropic[u], {u, -1, 1}, PlotStyle → Thick],
 Plot[pLinaniso[u, 1], {u, -1, 1}, PlotStyle → Dashed]
 , Frame → True,
 FrameLabel → {{p[u],}, {"u = Cos[θ]", "Linearly-Anisotropic Scattering"}}]
```



Normalization condition

```
Integrate[2 Pi pLinaniso[u, b], {u, -1, 1}, Assumptions → b > -1 && b < 1]
1
```

Mean cosine (g)

```
Integrate[2 Pi pLinaniso[u, b] u, {u, -1, 1}, Assumptions → b > -1 && b < 1]
```

$$\frac{b}{3}$$

Legendre expansion coefficients

```
Integrate[
 2 Pi (2 k + 1) pLinaniso[Cos[y], b] LegendreP[k, Cos[y]] Sin[y] /. k → 0, {y, 0, Pi}]
```

1

```
Integrate[
 2 Pi (2 k + 1) pLinaniso[Cos[y], b] LegendreP[k, Cos[y]] Sin[y] /. k → 1, {y, 0, Pi}]
```

b

sampling

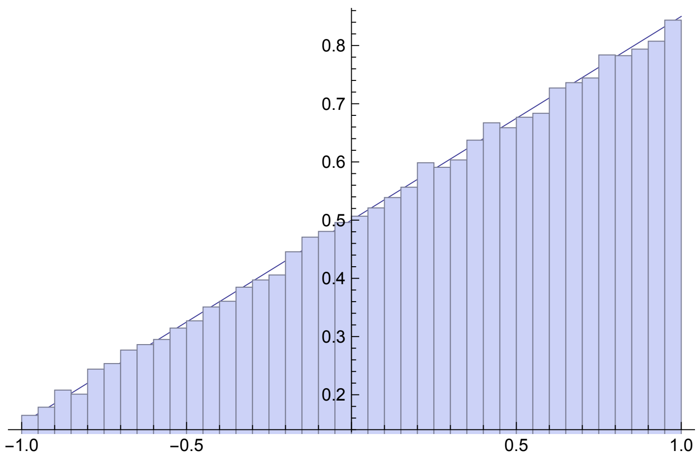
```
cdf = Integrate[2 Pi pLinaniso[u, b], {u, -1, x}]
```

$$\frac{1}{2} - \frac{b}{4} + \frac{x}{2} + \frac{b\,x^2}{4}$$

```
Solve[cdf == e, x]
```

$$\left\{\left\{x \to \frac{-1 - \sqrt{1 - 2\,b + b^2 + 4\,b\,e}}{b}\right\}, \left\{x \to \frac{-1 + \sqrt{1 - 2\,b + b^2 + 4\,b\,e}}{b}\right\}\right\}$$

```
b = 0.7;
Show[
 Plot[2 Pi pLinaniso[u, b], {u, -1, 1}],
 Histogram[
  Map[(-1 + Sqrt[1 - 2 b + b^2 + 4 b #])/b &, Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"]
]
Clear[b];
```



Rayleigh Scattering

General form:

$$\text{pRayleigh}[u\_, \gamma\_] := \frac{1}{4\,\text{Pi}}\,\frac{3}{4\,(1+2\,\gamma)}\,\left((1+3\,\gamma)+(1-\gamma)\,u^2\right)$$

Common special case ($\gamma = 0$):

$$\text{pRayleigh}[u\_] := \left(1+u^2\right)\frac{3}{16\,\text{Pi}}$$

Normalization condition

```
Integrate[2 Pi pRayleigh[u], {u, -1, 1}]
```

1

```
Integrate[2 Pi pRayleigh[u, y], {u, -1, 1}, Assumptions → y > 0] // Simplify
```

1

Mean cosine (g)

```
Integrate[2 Pi pRayleigh[u] u, {u, -1, 1}]
```

0

```
Integrate[2 Pi pRayleigh[u, y] u, {u, -1, 1}, Assumptions → y > 0] // Simplify
```

0

Legendre expansion coefficients

```
Integrate[
 2 Pi (2 k + 1) pRayleigh[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 0, {y, 0, Pi}]
```

1

```
Integrate[
 2 Pi (2 k + 1) pRayleigh[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 1, {y, 0, Pi}]
```

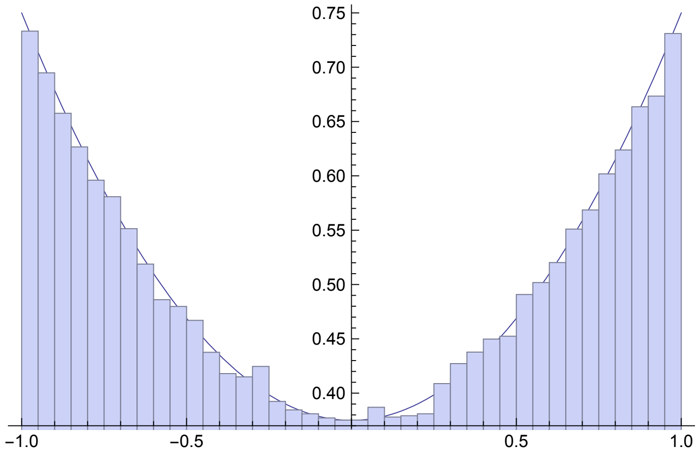
0

```
Integrate[
 2 Pi (2 k + 1) pRayleigh[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 2, {y, 0, Pi}]
```

$\frac{1}{2}$

sampling

```
Show[
 Plot[2 Pi pRayleigh[u], {u, -1, 1}],
 Histogram[Map[(1 - (2 - 4 # + Sqrt[5 + 16 (-1 + #) #])^(2/3))/(2 - 4 # + Sqrt[5 + 16 (-1 + #) #])^(1/3) &,
   Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"]
]
Clear[b];
```



Lambertian Sphere

geometrical optics far-field phase function of a white Lambertian sphere in 3D:
[Schoenberg 1929] - **doi**: 10.1007/978-3-642-90703-6_1
[Esposito and Lumme 1977, Blinn 1982, Porco et al. 2008]

In[∘]:=
```
pLambertSphere[u_] := 2 (Sqrt[1 - u^2] - u ArcCos[u])/(3 π^2)
```

$$\text{pLambertSphere}[u\_] := \frac{2\left(\sqrt{1-u^2} - u\,\text{ArcCos}[u]\right)}{3\pi^2}$$

MC testing

Normalization condition

In[∘]:=
```
Integrate[2 Pi pLambertSphere[u], {u, -1, 1}]
```

Out[∘]= 1

forward scattering probability

In[∘]:=
```
Clear[u]; Integrate[2 Pi pLambertSphere[u], {u, 0, 1}]
```

Out[∘]= $\frac{1}{6}$

Mean cosine (g)

In[•]:=
```
Integrate[2 Pi pLambertSphere[u] u, {u, -1, 1}]
```

Out[•]= $-\frac{4}{9}$

Mean square cosine

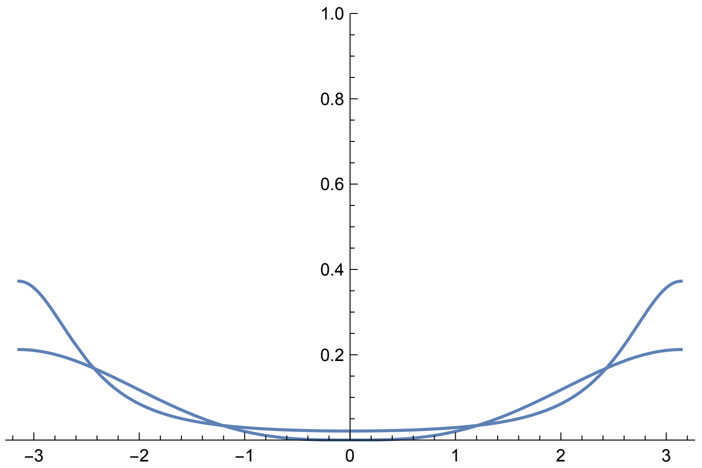
In[•]:=
```
Integrate[2 Pi pLambertSphere[u] u^2, {u, -1, 1}]
```

Out[•]= $\frac{3}{8}$

This phase function is not particularly well approximated by Henyey Greenstein:

In[•]:=
```
Show[
 Plot[pHG[Cos[t], -4/9], {t, -Pi, Pi}, PlotRange → {0, 1}],
 Plot[pLambertSphere[Cos[t]], {t, -Pi, Pi}, PlotRange → All]
]
```

Out[•]=



Legendre expansion coefficients

In[•]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 0,
 {y, 0, Pi}]
```

Out[•]= 1

In[•]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 1,
 {y, 0, Pi}]
```

Out[•]= $-\frac{4}{3}$

In[•]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 2,
 {y, 0, Pi}]
```

Out[•]= $\frac{5}{16}$

In[ ]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 3,
  {y, 0, Pi}]
```
Out[ ]= 0

In[ ]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 4,
  {y, 0, Pi}]
```
Out[ ]= $\frac{1}{64}$

In[ ]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 6,
  {y, 0, Pi}]
```
Out[ ]= $\frac{13}{4096}$

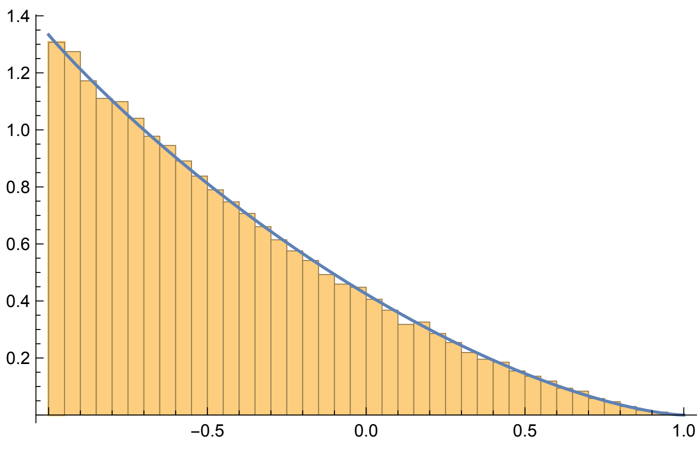
In[ ]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 8,
  {y, 0, Pi}]
```
Out[ ]= $\frac{17}{16\,384}$

In[ ]:=
```
Integrate[2 Pi (2 k + 1) pLambertSphere[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 10,
  {y, 0, Pi}]
```
Out[ ]= $\frac{343}{786\,432}$

Importance sampling:

The cosine of deflection can be sampled from:

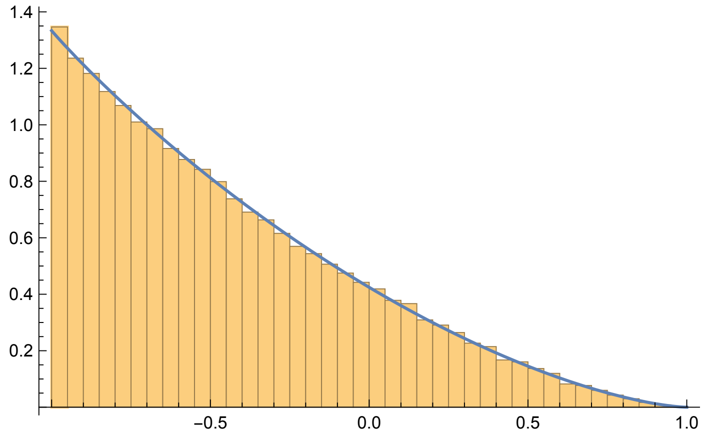
In[ ]:=
```
Show[
 Histogram[Table[
   Sin[2 Pi RandomReal[]] √((1 - #1) (1 - #2)) - √(#1 #2) &[RandomReal[], RandomReal[]]
   , {i, Range[100 000]}], 50, "PDF"],
 Plot[2 Pi pLambertSphere[u], {u, -1, 1}]
]
```
Out[ ]=



Approximate CDF inverse:

In[]:=
```
lambertSphereApproxCDFi[x_] := 1 - 2 (1 - x^(1.01938` + 0.0401885` x))^0.397225`
```

In[]:=
```
Show[
 Histogram[Table[
   lambertSphereApproxCDFi[RandomReal[]]
   , {i, Range[100 000]}], 50, "PDF"],
 Plot[2 Pi pLambertSphere[u], {u, -1, 1}]
]
```

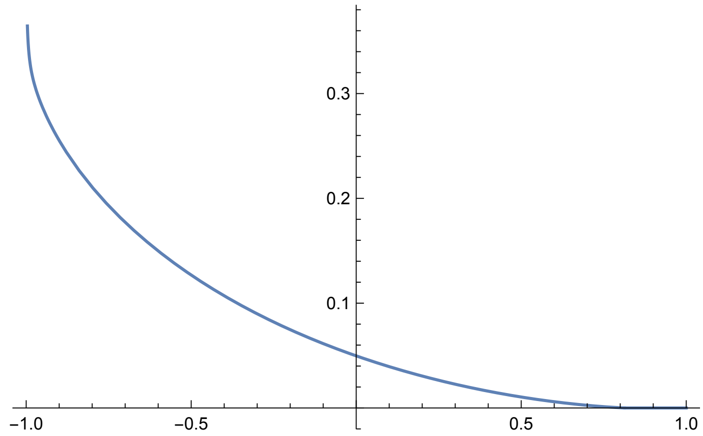
Out[]=


Callisto

[Porco et al. 2008] - **doi**: 10.1088/0004-6256/136/5/2172

In[]:=
```
pCallisto[u_] := HeavisideTheta[2.521 - ArcCos[-u]]
  2.2 / (4 Pi (1.0004369822233856`)) (2 - 0.79333 ArcCos[-u] + Exp[-21.2 ArcCos[-u]])
  (1 + Sin[ArcCos[-u]/2] Tan[ArcCos[-u]/2] Log[Tan[ArcCos[-u]/4]])
```

In[]:=
```
Plot[pCallisto[u], {u, -1, 1}]
```

Out[]=


Normalization condition

In[•]:=
```
NIntegrate[ 2 Pi pCallisto[u] , {u, -1, 1}]
```
Out[•]= 1.

Mean cosine (g)

In[•]:=
```
NIntegrate[2 Pi pCallisto[u] u, {u, -1, 1}]
```
Out[•]= -0.560001

Legendre expansion coefficients

In[•]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 0, {y, 0, Pi}]
```
Out[•]= 1.

In[•]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 1, {y, 0, Pi}]
```
Out[•]= -1.68

In[•]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 2, {y, 0, Pi}]
```
Out[•]= 0.851712

In[•]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 3, {y, 0, Pi}]
```
Out[•]= -0.285211

In[•]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 4, {y, 0, Pi}]
```
Out[•]= 0.182995

In[•]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 6, {y, 0, Pi}]
```
Out[•]= 0.0908047

In[•]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 8, {y, 0, Pi}]
```
Out[•]= 0.064234

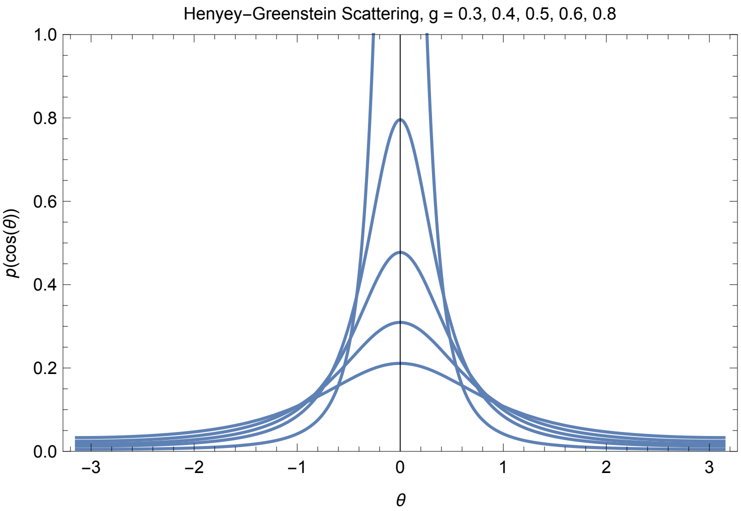
In[•]:=
```
NIntegrate[
  2 Pi (2 k + 1) pCallisto[Cos[y]] LegendreP[k, Cos[y]] Sin[y] /. k → 10, {y, 0, Pi}]
```
Out[•]= 0.0552028

Henyey-greenstein Scattering

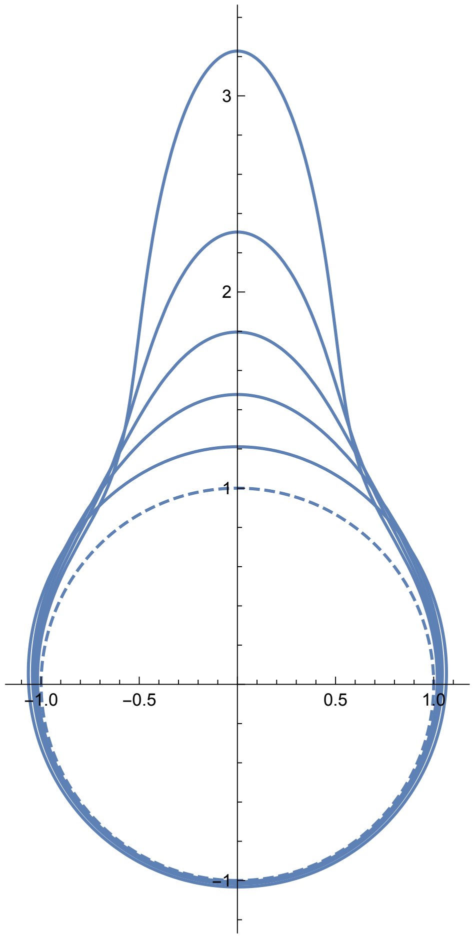
[Henyey and Greestein 1940] - "Diffuse radiation in the Galaxy"

In[ ]:= Clear[pHG]; pHG[dot_, g_] := $\frac{1}{4\,\text{Pi}}\frac{\left(1-g^2\right)}{\left(1+g^2-2\,g\,\text{dot}\right)^{\frac{3}{2}}}$

```
pHGplot = Show[
  Plot[pHG[Cos[t], .8], {t, -Pi, Pi}, PlotRange → {0, 1}],
  Plot[pHG[Cos[t], .6], {t, -Pi, Pi}, PlotRange → All],
  Plot[pHG[Cos[t], .5], {t, -Pi, Pi}, PlotRange → All],
  Plot[pHG[Cos[t], .4], {t, -Pi, Pi}, PlotRange → All],
  Plot[pHG[Cos[t], .3], {t, -Pi, Pi}, PlotRange → All],
  Frame → True,
  ImageSize → 400,
  FrameLabel → {{p[Cos[θ]],},
    {θ, "Henyey-Greenstein Scattering, g = 0.3, 0.4, 0.5, 0.6, 0.8"}}]
```



```
Show[
 ParametricPlot[{Sin[t], Cos[t]} (1),
  {t, -Pi, Pi}, PlotRange → All, PlotStyle → Dashed],
 ParametricPlot[{Sin[t], Cos[t]} (1 + pHG[Cos[t], 0.75]),
  {t, -Pi, Pi}, PlotRange → All],
 ParametricPlot[{Sin[t], Cos[t]} (1 + pHG[Cos[t], 0.68]),
  {t, -Pi, Pi}, PlotRange → All],
 ParametricPlot[{Sin[t], Cos[t]} (1 + pHG[Cos[t], 0.6]),
  {t, -Pi, Pi}, PlotRange → All],
 ParametricPlot[{Sin[t], Cos[t]} (1 + pHG[Cos[t], 0.5]),
  {t, -Pi, Pi}, PlotRange → All],
 ParametricPlot[{Sin[t], Cos[t]} (1 + pHG[Cos[t], 0.3]),
  {t, -Pi, Pi}, PlotRange → All]
]
```



Normalization condition

```
Integrate[2 Pi pHG[u, g], {u, -1, 1}, Assumptions → g > -1 && g < 1]
1
```

Legendre expansion coefficients

```
Integrate[2 Pi (2 k + 1) pHG[u, g] LegendreP[k, u] /. k → 0,
  {u, -1, 1}, Assumptions → g > -1 && g < 1]
```

1

```
Integrate[2 Pi (2 k + 1) pHG[u, g] LegendreP[k, u] /. k → 1,
  {u, -1, 1}, Assumptions → g > -1 && g < 1]
```

3 g

```
In[•]:= Integrate[2 Pi (2 k + 1) pHG[u, g] LegendreP[k, u] /. k → 2,
  {u, -1, 1}, Assumptions → g > -1 && g < 1]
```

Out[•]= $5\,g^2$

```
In[•]:= Integrate[2 Pi (2 k + 1) pHG[u, g] LegendreP[k, u] /. k → 3,
  {u, -1, 1}, Assumptions → g > -1 && g < 1]
```

Out[•]= $7\,g^3$

```
In[•]:= Integrate[2 Pi (2 k + 1) pHG[u, g] LegendreP[k, u] /. k → 4,
  {u, -1, 1}, Assumptions → g > -1 && g < 1]
```

Out[•]= $9\,g^4$

sampling

```
cdf = Integrate[2 Pi pHG[u, g], {u, -1, x}, Assumptions → g > -1 && g < 1 && x < 1]
```

$$\frac{(-1+g)\left(-1-g+\sqrt{1+g^2-2\,g\,x}\right)}{2\,g\,\sqrt{1+g^2-2\,g\,x}}$$

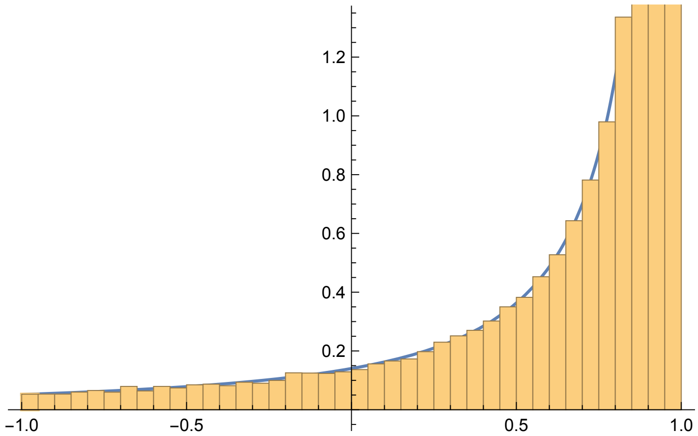
```
Solve[cdf == e, x]
```

$$\left\{\left\{x \to \frac{-1+2\,e+2\,g-2\,e\,g+2\,e^2\,g-g^2+2\,e\,g^2-2\,e\,g^3+2\,e^2\,g^3}{(1-g+2\,e\,g)^2}\right\}\right\}$$

```
FullSimplify[%]
```

$$\left\{\left\{x \to -\frac{(-1+g)^2+2\,e\,(-1+g)\,(1+g^2)-2\,e^2\,(g+g^3)}{(1+(-1+2\,e)\,g)^2}\right\}\right\}$$

```
g = 0.7;
Show[
 Plot[2 Pi pHG[u, g], {u, -1, 1}],
 Histogram[Map[-((-1 + g)^2 + 2 # (-1 + g) (1 + g^2) - 2 #^2 (g + g^3))/((1 + (-1 + 2 #) g)^2) &,
   Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"]
]
Clear[b, g];
```



When cosine u has been sampled with random variable ξ, what is the PDF at the sampled direction in terms of ξ?

In[•]:=
```
FullSimplify[pHG[-((-1 + g)^2 + 2 (-1 + g) (1 + g^2) ξ - 2 (g + g^3) ξ^2)/((1 + g (-1 + 2 ξ))^2), g],
 Assumptions → -1 < g < 1 && 0 < ξ < 1]
```

Out[•]=
$$\frac{\left(1+g\left(-1+2\,\xi\right)\right)^3}{4\left(-1+g^2\right)^2\pi}$$

Haltrin

[Haltrin 1988] - a phase function such that the asymptotic mode in plane geometry has an exact solution:

Consequently, the phase function $p(\cos\chi) = p_H(\cos\chi)$, where

$$p_H(\cos\chi) \equiv 2g\,\delta(1-\cos\chi) + (1-g)\left[2(1-\cos\chi)\right]^{-1/2}, \quad (21)$$

In[•]:=
```
pHaltrin[u_, g_] := 1/(4 Pi) (2 g DiracDelta[1 - u] + (1 - g)/Sqrt[2 (1 - u)])
```

Normalization condition

In[•]:=
```
Integrate[2 Pi pHaltrin[u, g], {u, -1, 1}, Assumptions → g > -1 && g < 1] /.
  HeavisideTheta[0] → 1
```
Out[•]= 1

Mean cosine (g)

In[•]:=
```
Integrate[2 Pi pHaltrin[u, g] u, {u, -1, 1}] /. HeavisideTheta[0] → 1 // FullSimplify
```
Out[•]= $\frac{1}{3}(1+2\,g)$

Legendre expansion coefficients

In[•]:=
```
Integrate[2 Pi (2 k + 1) pHaltrin[u, g] LegendreP[k, u] /. k → 0,
   {u, -1, 1}, Assumptions → g > -1 && g < 1] /. HeavisideTheta[0] → 1
```
Out[•]= 1

In[•]:=
```
Integrate[2 Pi (2 k + 1) pHaltrin[u, g] LegendreP[k, u] /. k → 1,
   {u, -1, 1}, Assumptions → g > -1 && g < 1] /. HeavisideTheta[0] → 1
```
Out[•]= $1+2\,g$

In[•]:=
```
Integrate[2 Pi (2 k + 1) pHaltrin[u, g] LegendreP[k, u] /. k → 2,
   {u, -1, 1}, Assumptions → g > -1 && g < 1] /. HeavisideTheta[0] → 1
```
Out[•]= $1+4\,g$

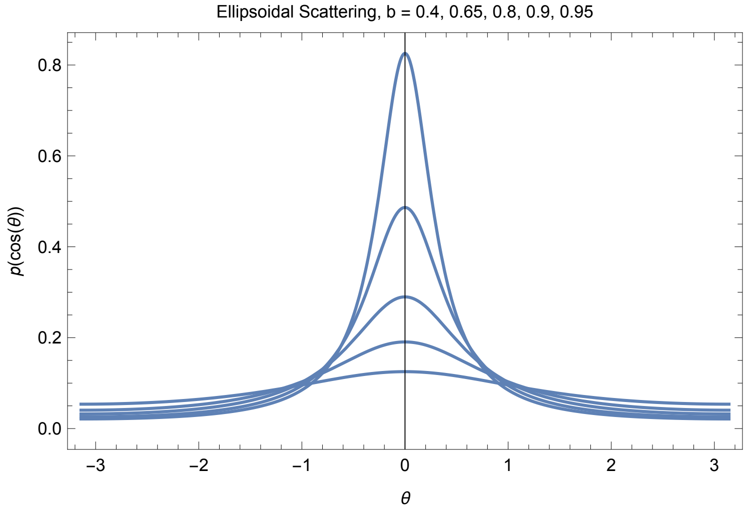
In[•]:=
```
Integrate[2 Pi (2 k + 1) pHaltrin[u, g] LegendreP[k, u] /. k → 3,
   {u, -1, 1}, Assumptions → g > -1 && g < 1] /. HeavisideTheta[0] → 1
```
Out[•]= $1+6\,g$

In[•]:=
```
Integrate[2 Pi (2 k + 1) pHaltrin[u, g] LegendreP[k, u] /. k → 4,
   {u, -1, 1}, Assumptions → g > -1 && g < 1] /. HeavisideTheta[0] → 1
```
Out[•]= $1+8\,g$

Kagiwada-Kalaba (Ellipsoidal) Scattering

In[•]:=
```
pEllipsoidal[u_, b_] := b (2 Pi Log[(1 + b) / (1 - b)] (1 - b u))^-1
```

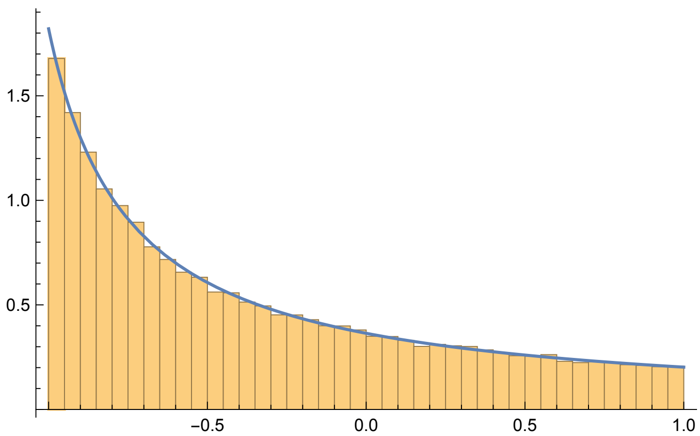
```
pEllplot = Show[
  Plot[pEllipsoidal[Cos[t], .9], {t, -Pi, Pi}, PlotRange → All],
  Plot[pEllipsoidal[Cos[t], .8], {t, -Pi, Pi}, PlotRange → All],
  Plot[pEllipsoidal[Cos[t], .65], {t, -Pi, Pi}, PlotRange → All],
  Plot[pEllipsoidal[Cos[t], .4], {t, -Pi, Pi}, PlotRange → All],
  Plot[pEllipsoidal[Cos[t], .95], {t, -Pi, Pi}, PlotRange → All],
  Frame → True,
  ImageSize → 400,
  FrameLabel → {{p[Cos[θ]],},
    {θ, "Ellipsoidal Scattering, b = 0.4, 0.65, 0.8, 0.9, 0.95"}}]
```



sampling

```
b = -0.8;
Show[Histogram[
  Map[(1 - (1 + b) ((1+b)/(1-b))^-#)/b &, Table[RandomReal[], {i, 1, 100000}]], 50, "PDF"],
 Plot[2 Pi pEllipsoidal[u, b], {u, -1, 1}]

]
Clear[b];
```



When cosine u has been sampled with random variable ξ, what is the PDF at the sampled direction in terms of ξ?

In[∘]:=
```
FullSimplify[pEllipsoidal[(1 - (1 + b) ((1+b)/(1-b))^-#)/b &[ξ], b],
  Assumptions → 0 < b < 1 && 0 < ξ < 1]
```

Out[∘]= $\dfrac{(1-b)^{-\xi}\, b\, (1+b)^{-1+\xi}}{2\,\pi\, \mathrm{Log}\left[\frac{1+b}{1-b}\right]}$

Expansion coefficients

In[∘]:=
```
Integrate[2 Pi (2 k + 1) pEllipsoidal[u, b] LegendreP[k, u] /. k → 0,
   {u, -1, 1}, Assumptions → 0 < b < 1] /. Log[(1 + b)/(1 - b)] -> 2 ArcTanh[b]
```

Out[∘]= 1

In[∘]:=
```
Integrate[2 Pi (2 k + 1) pEllipsoidal[u, b] LegendreP[k, u] /. k → 1, {u, -1, 1},
    Assumptions → 0 < b < 1] /. Log[(1 + b)/(1 - b)] -> 2 ArcTanh[b] // FullSimplify
```

Out[∘]= $\dfrac{3}{b} - \dfrac{3}{\mathrm{ArcTanh}[b]}$

In[∘]:=
```
Integrate[2 Pi (2 k + 1) pEllipsoidal[u, b] LegendreP[k, u] /. k → 2, {u, -1, 1},
    Assumptions → 0 < b < 1] /. Log[(1 + b)/(1 - b)] -> 2 ArcTanh[b] // FullSimplify
```

Out[∘]= $\dfrac{5}{2}\left(-1 + \dfrac{3}{b^2} - \dfrac{3}{b\, \mathrm{ArcTanh}[b]}\right)$

In[∘]:=
```
Integrate[2 Pi (2 k + 1) pEllipsoidal[u, b] LegendreP[k, u] /. k → 3, {u, -1, 1},
    Assumptions → 0 < b < 1] /. Log[(1 + b)/(1 - b)] -> 2 ArcTanh[b] // FullSimplify
```

Out[∘]= $\dfrac{7\left(b\left(-15 + 4\, b^2\right) + \left(15 - 9\, b^2\right) \mathrm{ArcTanh}[b]\right)}{6\, b^3\, \mathrm{ArcTanh}[b]}$

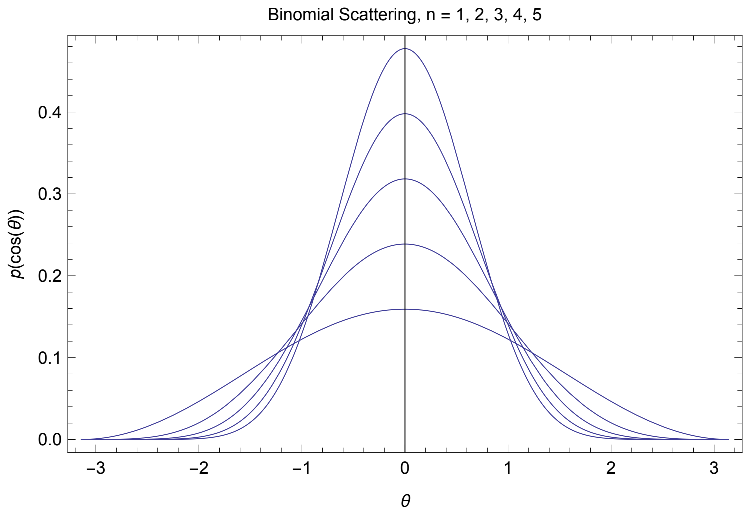
In[∘]:=
```
Integrate[2 Pi (2 k + 1) pEllipsoidal[u, b] LegendreP[k, u] /. k → 4, {u, -1, 1},
    Assumptions → 0 < b < 1] /. Log[(1 + b)/(1 - b)] -> 2 ArcTanh[b] // FullSimplify
```

Out[∘]= $\dfrac{15\, b\left(-21 + 11\, b^2\right) + 9\left(35 - 30\, b^2 + 3\, b^4\right) \mathrm{ArcTanh}[b]}{8\, b^4\, \mathrm{ArcTanh}[b]}$

Binomial Scattering

In[∘]:=
```
pBinomial[u_, n_] := Pi^-1 ((n + 1) / 2^(n+2)) (1 + u)^n
```

```
pBinplot = Show[
  Plot[pBinomial[Cos[t], 1], {t, -Pi, Pi}, PlotRange → All],
  Plot[pBinomial[Cos[t], 2], {t, -Pi, Pi}, PlotRange → All],
  Plot[pBinomial[Cos[t], 3], {t, -Pi, Pi}, PlotRange → All],
  Plot[pBinomial[Cos[t], 4], {t, -Pi, Pi}, PlotRange → All],
  Plot[pBinomial[Cos[t], 5], {t, -Pi, Pi}, PlotRange → All],
  Frame → True,
  ImageSize → 400,
  FrameLabel → {{p[Cos[θ]],}, {θ, "Binomial Scattering, n = 1, 2, 3, 4, 5"}}]
```



Normalization condition

```
Integrate[2 Pi pBinomial[u, n], {u, -1, 1}, Assumptions → n ≥ 0]
```

1

Mean cosine (g)

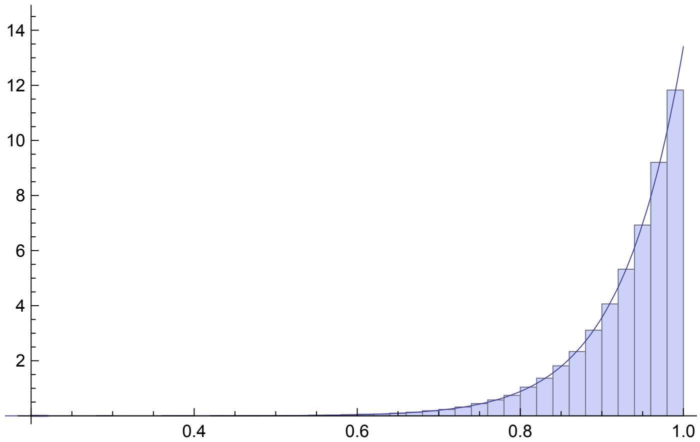
```
Integrate[2 Pi pBinomial[u, n] u, {u, -1, 1}, Assumptions → n ≥ 0]
```

$$\frac{n}{2+n}$$

sampling

```
n = 25.8;
Show[
 Histogram[Map[-1 + (2^(1+n) #)^(1/(1+n)) &, Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"],
 Plot[2 Pi pBinomial[u, n], {u, -1, 1}, PlotRange → All]

]
Clear[b];
```



In[•]:=
```
Integrate[2 Pi (2 k + 1) pBinomial[u, n] LegendreP[k, u] /. k → 0,
 {u, -1, 1}, Assumptions → n > 1]
```

Out[•]= 1

In[•]:=
```
Integrate[2 Pi (2 k + 1) pBinomial[u, n] LegendreP[k, u] /. k → 1,
 {u, -1, 1}, Assumptions → n > 1]
```

Out[•]= $\frac{3\,n}{2+n}$

In[•]:=
```
Integrate[2 Pi (2 k + 1) pBinomial[u, n] LegendreP[k, u] /. k → 2,
 {u, -1, 1}, Assumptions → n > 1]
```

Out[•]= $\frac{5\,(-1+n)\,n}{6+5\,n+n^2}$

In[•]:=
```
Integrate[2 Pi (2 k + 1) pBinomial[u, n] LegendreP[k, u] /. k → 3,
 {u, -1, 1}, Assumptions → n > 1]
```

Out[•]= $\frac{7\,(-2+n)\,(-1+n)\,n}{(2+n)\,(3+n)\,(4+n)}$

In[•]:=
```
Integrate[2 Pi (2 k + 1) pBinomial[u, n] LegendreP[k, u] /. k → 4,
 {u, -1, 1}, Assumptions → n > 1]
```

Out[•]= $\frac{9\,(-3+n)\,(-2+n)\,(-1+n)\,n}{(2+n)\,(3+n)\,(4+n)\,(5+n)}$

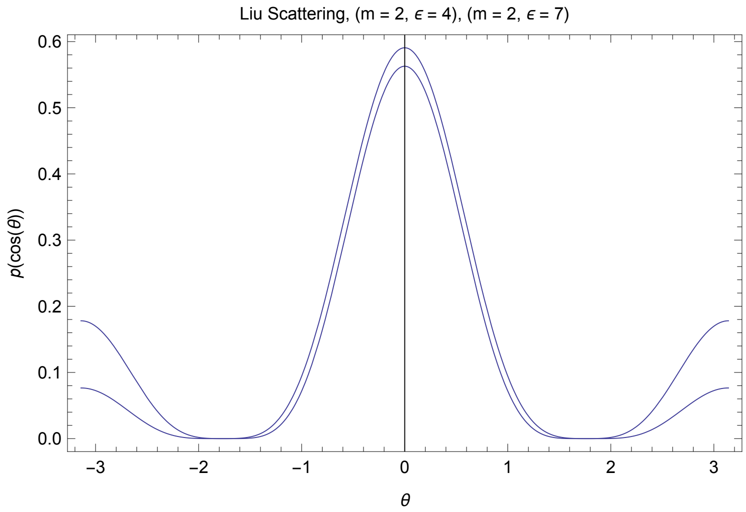
In[•]:=
```
Integrate[2 Pi (2 k + 1) pBinomial[u, n] LegendreP[k, u] /. k → 11,
   {u, -1, 1}, Assumptions → n > 1] /
  ((1 + 2 j) Gamma[2 + n] / (Gamma[1 - j + n] Pochhammer[1 + n, 1 + j]) /. j → 11) // FullSimplify
```
Out[•]= 1

Liu Scattering

In[•]:=
```
pLiu[u_, e_, m_] := e (2 m + 1) (1 + e u)^(2 m) / (2 Pi ((1 + e)^(2 m+1) - (1 - e)^(2 m+1)))

Clear[m]

pLiuplot = Show[
  Plot[pLiu[Cos[t], 4, 2], {t, -Pi, Pi}, PlotRange → All],
  Plot[pLiu[Cos[t], 7, 2], {t, -Pi, Pi}, PlotRange → All],
  Frame → True,
  ImageSize → 400,
  FrameLabel →
   {{p[Cos[θ]],}, {θ, "Liu Scattering, (m = 2, ϵ = 4), (m = 2, ϵ = 7)"}}]
```



Normalization condition

```
Integrate[2 Pi pLiu[u, e, m], {u, -1, 1}, Assumptions → e > 0 && m > 0 && m ∈ Integers]
```
1

Mean cosine (g)

```
Integrate[2 Pi u pLiu[u, e, m], {u, -1, 1},
 Assumptions → e > 0 && m > 0 && m ∈ Integers && e < 1]
```
$$\frac{(1+e)^{1+2m}\left(-1+e+2em\right)+(1-e)^{1+2m}\left(1+e+2em\right)}{2e\left(-(1-e)^{1+2m}+(1+e)^{1+2m}\right)(1+m)}$$

Legendre expansion coefficients

```
Integrate[2 Pi (2 k + 1) pLiu[u, e, m] LegendreP[k, u] /. k → 0, {u, -1, 1},
 Assumptions → m > 0 && m ∈ Integers && e ∈ Reals && e ≠ 0 && Abs[e] < 1]
```

1

```
Integrate[2 Pi (2 k + 1) pLiu[u, e, m] LegendreP[k, u] /. k → 2, {u, -1, 1},
 Assumptions → m > 0 && m ∈ Integers && e ∈ Reals && e ≠ 0 && Abs[e] < 1]
```

$$\left(5\left((1+e)^{1+2m}\left(3+e\left(-3+2m\left(-3+2e\,(1+m)\right)\right)\right)+(1-e)^{2m}\,(-1+e)\,\left(3+e\left(3+2m\left(3+2e\,(1+m)\right)\right)\right)\right)\right)\Big/\left(2e^2\left(-(1-e)^{1+2m}+(1+e)^{1+2m}\right)(1+m)\,(3+2m)\right)$$

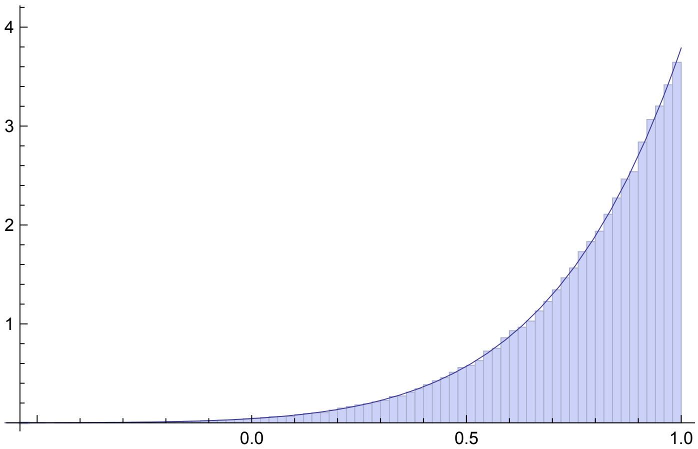
sampling

```
m = 3.5;
ϵ = 0.9;
```

$$\texttt{Show[Histogram[Map[}\frac{-1+\left((-1+\#)\,(1-\epsilon)^{2m}\,(-1+\epsilon)+\#\,(1+\epsilon)^{1+2m}\right)^{\frac{1}{1+2m}}}{\epsilon}\texttt{\&,}$$

```
    Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"],
 Plot[2 Pi pLiu[u, ϵ, m], {u, -1, 1}, PlotRange → All]

]
Clear[m, ϵ];
```



When cosine u has been sampled with random variable ξ, what is the PDF at the sampled direction in terms of ξ?

In[∘]:=

$$\texttt{FullSimplify[pLiu[}\frac{-1+\left((-1+\#)\,(1-\epsilon)^{2m}\,(-1+\epsilon)+\#\,(1+\epsilon)^{1+2m}\right)^{\frac{1}{1+2m}}}{\epsilon}\texttt{\&[}\xi\texttt{], }\epsilon\texttt{, m],}$$

```
 Assumptions → ϵ > 0 && m > 0 && 0 < ξ < 1]
```

Out[∘]=

$$\frac{(1+2m)\,\epsilon\,\left((1-\epsilon)^{2m}\,(-1+\epsilon)\,(-1+\xi)+(1+\epsilon)^{1+2m}\,\xi\right)^{\frac{2m}{1+2m}}}{2\pi\left(-(1-\epsilon)^{1+2m}+(1+\epsilon)^{1+2m}\right)}$$

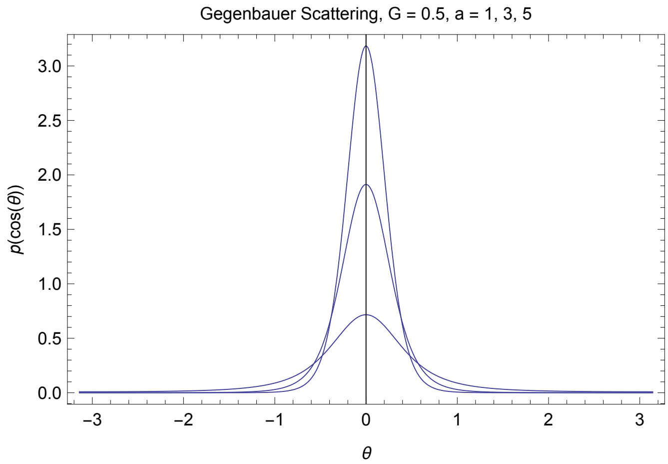
Gegenbauer Scattering

In[∘]:=

```
pGegenbauer[u_, g_, a_] := (1 + g^2 - 2 g u)^-(a+1) / (((1-g)^-2 a - (1+g)^-2 a) π / (a g))

Show[
 Plot[pGegenbauer[Cos[t], 0.5, 1], {t, -Pi, Pi}, PlotRange → All],
 Plot[pGegenbauer[Cos[t], 0.5, 3], {t, -Pi, Pi}, PlotRange → All],
 Plot[pGegenbauer[Cos[t], 0.5, 5], {t, -Pi, Pi}, PlotRange → All],

 Frame → True,
 FrameLabel →
  {{p[Cos[θ]],}, {θ, "Gegenbauer Scattering, G = 0.5, a = 1, 3, 5"}}]
```



Normalization condition

```
Integrate[2 Pi pGegenbauer[u, g, a], {u, -1, 1}, Assumptions → -1 ≤ g ≤ 1 && a > 0]
1
```

Mean cosine (g)

```
Integrate[2 Pi u pGegenbauer[u, g, a], {u, -1, 1}, Assumptions → -1 ≤ g ≤ 1 && a > 0]
```

$$\frac{(1+g)^{2a}\left(1-2ag+g^2\right)-(1-g)^{2a}\left(1+2ag+g^2\right)}{2(-1+a)g\left((1-g)^{2a}-(1+g)^{2a}\right)}$$

Legendre expansion coefficients

```
Integrate[2 Pi (2 k + 1) pGegenbauer[u, g, a] LegendreP[k, u] /. k → 0,
 {u, -1, 1}, Assumptions → -1 ≤ g ≤ 1 && a > 0]
1
```

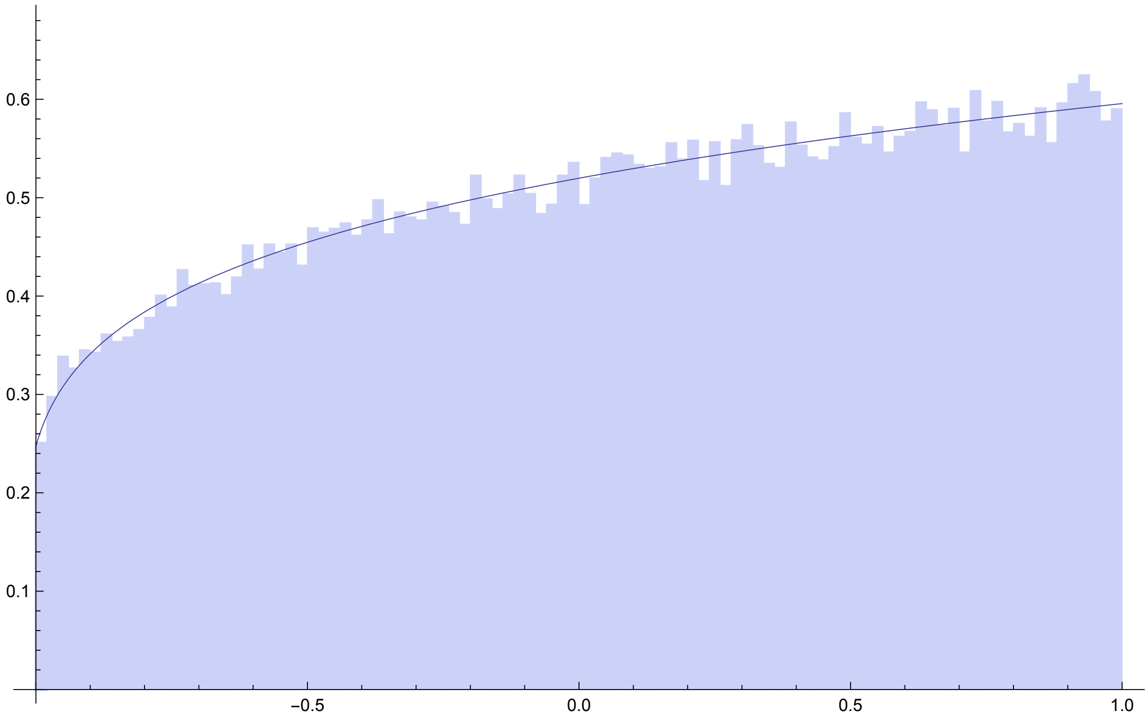
```
FullSimplify[Integrate[2 Pi (2 k + 1) pGegenbauer[u, g, a] LegendreP[k, u] /. k → 3,
   {u, -1, 1}, Assumptions → -1 ≤ g ≤ 1 && a > 0]]
```

$$-\left(7\left(24 a^2 g^2\left(1+g^2\right)\left((1-g)^{2a}-(1+g)^{2a}\right)+3\left(5+3 g^2+3 g^4+5 g^6\right)\left((1-g)^{2a}-(1+g)^{2a}\right)+8 a^3 g^3\left((1-g)^{2a}+(1+g)^{2a}\right)+2 a g\left(15+14 g^2+15 g^4\right)\left((1-g)^{2a}+(1+g)^{2a}\right)\right)\right)\Big/\left(8(-3+a)(-2+a)(-1+a) g^3\left((1-g)^{2a}-(1+g)^{2a}\right)\right)$$

sampling

```
g = -0.8;
a = -1.2;
Show[Histogram[Map[(1 + g^2 - (# (1 - g)^(-2 a) - (-1 + #) (1 + g)^(-2 a))^(-1/a))/(2 g) &,
    Table[RandomReal[], {i, 1, 100 000}]], 100, "PDF"],
 Plot[2 Pi pGegenbauer[u, g, a], {u, -1, 1}, PlotRange → All]

]
Clear[g, a];
```



When cosine u has been sampled with random variable ξ, what is the PDF at the sampled direction in terms of ξ?

In[∘]:=
```
FullSimplify[pGegenbauer[(1 + g^2 - (# (1 - g)^(-2 a) - (-1 + #) (1 + g)^(-2 a))^(-1/a))/(2 g) &[ξ], g, a],
 Assumptions → a > 0 && -1 < g < 1 && 0 < ξ < 1]
```

Out[∘]=
$$\frac{a\, g\left(\left(-(1+g)^{-2a}(-1+\xi)+(1-g)^{-2a}\xi\right)^{-1/a}\right)^{-1-a}}{\left((1-g)^{-2a}-(1+g)^{-2a}\right)\pi}$$

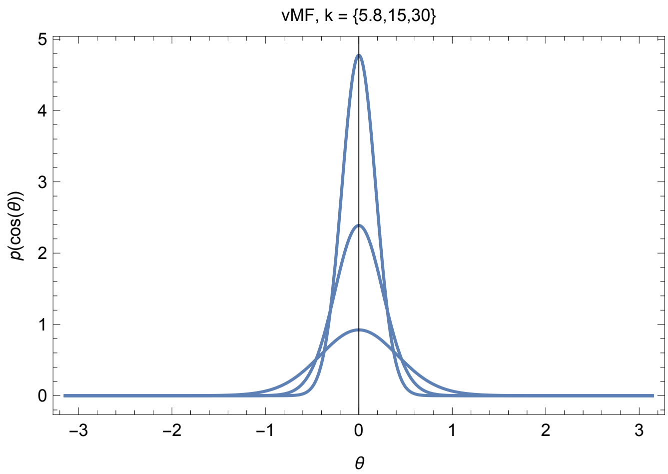
vMF (spherical Gaussian) Scattering

[Pomraning and Prinja 1995] - “Transverse Diffusion of a Collimated Particle Beam”
https://doi.org/10.1007/BF02178551

In[1480]:=
```
pVMF[u_, k_] := k/(4 Pi Sinh[k]) Exp[k u]

Show[
 Plot[pVMF[Cos[t], 5.8], {t, -Pi, Pi}, PlotRange → All],
 Plot[pVMF[Cos[t], 15], {t, -Pi, Pi}, PlotRange → All],
 Plot[pVMF[Cos[t], 30], {t, -Pi, Pi}, PlotRange → All],

 Frame → True,
 FrameLabel → {{p[Cos[θ]],}, {θ, "vMF, k = {5.8,15,30}"}}]
```



Normalization condition

```
Integrate[2 Pi pVMF[u, k], {u, -1, 1}, Assumptions → k > 0]
```

1

Mean cosine (g)

```
Integrate[2 Pi u pVMF[u, k], {u, -1, 1}, Assumptions → k > 0]
```

$-\frac{1}{k} + \text{Coth}[k]$

Legendre expansion coefficients

In[∘]:=
```
Integrate[2 Pi (2 o + 1) pVMF[u, k] LegendreP[o, u] /. o → 0,
 {u, -1, 1}, Assumptions → k > 0]
```

Out[∘]= 1

In[•]:=
```
Integrate[2 Pi (2 o + 1) pVMF[u, k] LegendreP[o, u] /. o → 1,
 {u, -1, 1}, Assumptions → k > 0]
```

Out[•]= $-\frac{3}{k} + 3\,\text{Coth}[k]$

In[•]:=
```
Integrate[2 Pi (2 o + 1) pVMF[u, k] LegendreP[o, u] /. o → 2,
 {u, -1, 1}, Assumptions → k > 0]
```

Out[•]= $\frac{5\left(3 + k^2 - 3\,k\,\text{Coth}[k]\right)}{k^2}$

In[•]:=
```
Integrate[2 Pi (2 o + 1) pVMF[u, k] LegendreP[o, u] /. o → 3,
 {u, -1, 1}, Assumptions → k > 0]
```

Out[•]= $\frac{7\left(-3\left(5 + 2\,k^2\right) + k\left(15 + k^2\right)\text{Coth}[k]\right)}{k^3}$

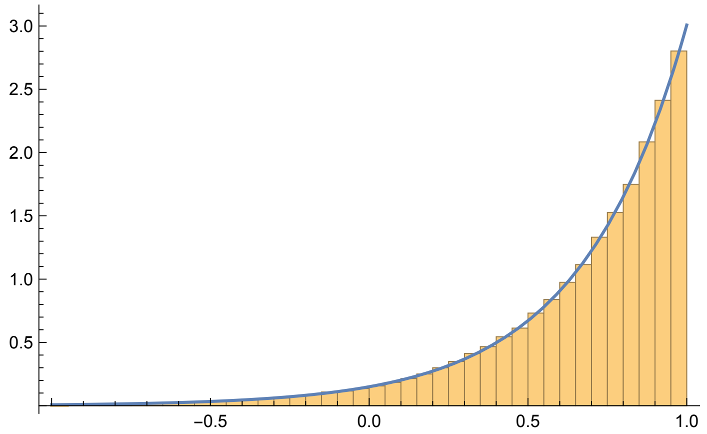
```
Integrate[2 Pi (2 o + 1) pVMF[u, k] LegendreP[o, u] /. o → 4,
 {u, -1, 1}, Assumptions → k > 0]
```

$\frac{9\left(105 + 45\,k^2 + k^4 - 5\,k\left(21 + 2\,k^2\right)\text{Coth}[k]\right)}{k^4}$

sampling

```
k = 3;
Show[Histogram[
  Map[Log[E^-k (1 - #) + E^k #] / k &, Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"],
 Plot[2 Pi pVMF[u, k], {u, -1, 1}, PlotRange → All]

]
Clear[k];
```



When cosine u has been sampled with random variable ξ, what is the PDF at the sampled direction in terms of ξ?

In[•]:= FullSimplify $\left[\text{pVMF}\left[\frac{\text{Log}\left[\text{E}^{-k}(1-\#)+\text{E}^{k}\#\right]}{k}\&[\xi],k\right],\text{Assumptions}\to k>0\,\&\&\,0<\xi<1\right]$

Out[•]= $\frac{k\,(-1+2\,\xi+\text{Coth}[k])}{4\,\pi}$

Klein-Nishina

Normalized variant of Klein-Nishina - energy parameter “e” = $\frac{E_{\gamma}}{m_e\,c^2}$

$$\text{pKleinNishina}[u\_,e\_] := \frac{1}{1+e\,(1-u)}\;\frac{1}{\frac{2\,\pi\,\text{Log}[1+2\,e]}{e}}$$

Normalization condition

In[•]:= Integrate[2 Pi pKleinNishina[u, e], {u, -1, 1}, Assumptions → e > 0]

Out[•]= 1

Mean-cosine

In[•]:= Integrate[2 Pi pKleinNishina[u, e] u, {u, -1, 1}, Assumptions → e > 0]

Out[•]= $1+\frac{1}{e}-\frac{2}{\text{Log}[1+2\,e]}$

Legendre expansion coefficients

In[•]:= Integrate[
2 Pi (2 k + 1) pKleinNishina[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 0,
{y, 0, Pi}, Assumptions → e > 0]

Out[•]= 1

In[•]:= Integrate[
2 Pi (2 k + 1) pKleinNishina[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 1,
{y, 0, Pi}, Assumptions → e > 0]

Out[•]= $3+\frac{3}{e}-\frac{6}{\text{Log}[1+2\,e]}$

In[•]:= Integrate[
2 Pi (2 k + 1) pKleinNishina[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 2,
{y, 0, Pi}, Assumptions → e > 0]

Out[•]= $\frac{5}{4}\left(1+\frac{3\left(2+4\,e+e^2-\frac{4\,e\,(1+e)}{\text{Log}[1+2\,e]}\right)}{e^2}\right)$

In[•]:=
```
Integrate[
 2 Pi (2 k + 1) pKleinNishina[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 3,
 {y, 0, Pi}, Assumptions → e > 0]
```

Out[•]=
$$\frac{7\left(15+45\,e+36\,e^2+6\,e^3-\frac{2\,e\,(15+30\,e+11\,e^2)}{\mathrm{Log}[1+2\,e]}\right)}{6\,e^3}$$

sampling

In[•]:=
```
cdf = Integrate[2 Pi pKleinNishina[u, e], {u, -1, x}, Assumptions → e > 0 && 0 < x < 1]
```

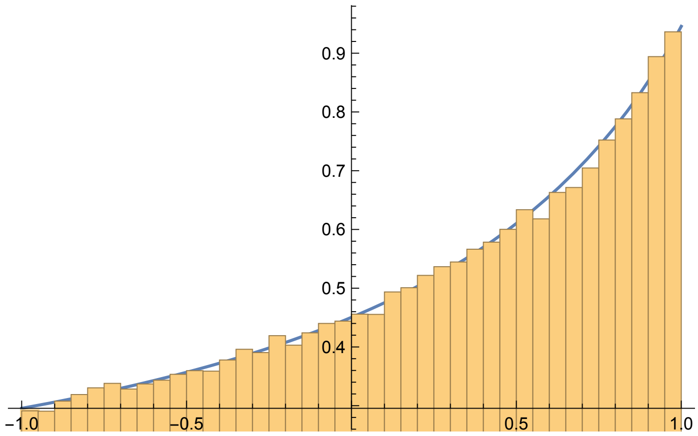
Out[•]=
$$1-\frac{\mathrm{Log}[1+e-e\,x]}{\mathrm{Log}[1+2\,e]}$$

In[•]:=
```
Solve[cdf == k, x]
```

Out[•]=
$$\left\{\left\{x\to\mathrm{ConditionalExpression}\left[\frac{1+e-(1+2\,e)^{1-k}}{e},\ -\pi\le\mathrm{Im}\left[(-1+k)\,\mathrm{Log}[1+2\,e]\right]<\pi\right]\right\}\right\}$$

In[•]:=
```
With[{e = 1.1},

 Show[
  Plot[2 Pi pKleinNishina[u, e], {u, -1, 1}],
  Histogram[
   Map[(1 + e - (1 + 2 e)^(1-#))/e &, Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"]
 ]
]
```

Out[•]=


Cornette-Shanks

[Cornette and Shanks 1992] - *Physically reasonable analytic expression for the single-scattering phase function.*

Independently proposed [Liu and Weng 2006]

In[∘]:= pCornetteShanks[u_, g_] := $\frac{3}{8\,\text{Pi}}\,\frac{\left(1-g^2\right)\left(1+u^2\right)}{\left(2+g^2\right)\left(1+g^2-2\,g\,u\right)^{3/2}}$

Normalization condition

```
In[∘]:= Integrate[2 Pi pCornetteShanks[u, g], {u, -1, 1}, Assumptions → -1 < g < 1]
Out[∘]= 1
```

Mean-cosine

```
In[∘]:= Integrate[2 Pi pCornetteShanks[u, g] u, {u, -1, 1}, Assumptions → -1 < g < 1]
```

Out[∘]= $\frac{3\,g\left(4+g^2\right)}{5\left(2+g^2\right)}$

Legendre expansion coefficients

```
In[∘]:= Integrate[
   2 Pi (2 k + 1) pCornetteShanks[Cos[y], g] LegendreP[k, Cos[y]] Sin[y] /. k → 0,
   {y, 0, Pi}, Assumptions → -1 < g < 1]
Out[∘]= 1
```

```
In[∘]:= Integrate[
   2 Pi (2 k + 1) pCornetteShanks[Cos[y], g] LegendreP[k, Cos[y]] Sin[y] /. k → 1,
   {y, 0, Pi}, Assumptions → -1 < g < 1]
```

Out[∘]= $\frac{9\,g\left(4+g^2\right)}{5\left(2+g^2\right)}$

```
In[∘]:= Integrate[
   2 Pi (2 k + 1) pCornetteShanks[Cos[y], g] LegendreP[k, Cos[y]] Sin[y] /. k → 2,
   {y, 0, Pi}, Assumptions → -1 < g < 1]
```

Out[∘]= $\frac{7+80\,g^2+18\,g^4}{14+7\,g^2}$

```
In[∘]:= Integrate[
   2 Pi (2 k + 1) pCornetteShanks[Cos[y], g] LegendreP[k, Cos[y]] Sin[y] /. k → 3,
   {y, 0, Pi}, Assumptions → -1 < g < 1]
```

Out[∘]= $\frac{g\left(27+238\,g^2+50\,g^4\right)}{15\left(2+g^2\right)}$

sampling

In[∘]:=
```
cdf = Integrate[2 Pi pCornetteShanks[u, g],
  {u, -1, x}, Assumptions → -1 < g < 1 && 0 < x < 1]
```

Out[∘]=
$$\frac{1}{4\,g^3\,(2+g^2)\,\sqrt{1+g^2-2\,g\,x}}\left(2-2\,g^6-2\,g\,x-2\,\sqrt{1+g^2-2\,g\,x}+4\,g^3\,\sqrt{1+g^2-2\,g\,x}+g^4\,(-5+x^2)+2\,g^5\,\left(x+\sqrt{1+g^2-2\,g\,x}\right)-g^2\,\left(-5+x^2+4\,\sqrt{1+g^2-2\,g\,x}\right)\right)$$

Draine

Draine, B.T. (2003) '*Scattering by interstellar dust grains. 1: Optical and ultraviolet*', ApJ., 598, 1017–25.

In[∘]:=
$$\texttt{pDraine[u\_, g\_, }\alpha\texttt{\_] :=}\ \frac{1}{4\,\mathrm{Pi}}\left(\frac{1-g^2}{(1+g^2-2\,g\,u)^{3/2}}\ \frac{1+\alpha\,u^2}{1+\alpha\,(1+2\,g^2)/3}\right)$$

Normalization condition

In[∘]:=
```
Integrate[2 Pi pDraine[u, g, a], {u, -1, 1}, Assumptions → 0 < a < 1 && -1 < g < 1]
```

Out[∘]= 1

Mean-cosine

In[∘]:=
```
Integrate[2 Pi pDraine[u, g, a] u, {u, -1, 1}, Assumptions → 0 < a < 1 && -1 < g < 1]
```

Out[∘]=
$$\frac{3}{5}\left(g+\frac{2\,(1+a)\,g}{3+a+2\,a\,g^2}\right)$$

In[∘]:=
$$\frac{3}{5}\left(g+\frac{2\,(1+a)\,g}{3+a+2\,a\,g^2}\right)\ \texttt{/. a → 0}$$

Out[∘]= g

Legendre expansion coefficients

In[∘]:=
```
Integrate[2 Pi (2 k + 1) pDraine[Cos[y], g, a] LegendreP[k, Cos[y]] Sin[y] /. k → 0,
 {y, 0, Pi}, Assumptions → 0 < a < 1 && -1 < g < 1]
```

Out[∘]= 1

In[∘]:=
```
Integrate[2 Pi (2 k + 1) pDraine[Cos[y], g, a] LegendreP[k, Cos[y]] Sin[y] /. k → 1,
 {y, 0, Pi}, Assumptions → 0 < a < 1 && -1 < g < 1]
```

Out[∘]=
$$\frac{9\,g\,\left(5+a\,(3+2\,g^2)\right)}{5\,(3+a+2\,a\,g^2)}$$

In[•]:=
```
Integrate[2 Pi (2 k + 1) pDraine[Cos[y], g, a] LegendreP[k, Cos[y]] Sin[y] /. k → 2,
  {y, 0, Pi}, Assumptions → 0 < a < 1 && -1 < g < 1]
```

Out[•]= $$\frac{14\,a+5\,(21+11\,a)\,g^2+36\,a\,g^4}{7\,(3+a+2\,a\,g^2)}$$

In[•]:=
```
Integrate[2 Pi (2 k + 1) pDraine[Cos[y], g, a] LegendreP[k, Cos[y]] Sin[y] /. k → 3,
  {y, 0, Pi}, Assumptions → 0 < a < 1 && -1 < g < 1]
```

Out[•]= $$\frac{g\,(54\,a+7\,(45+23\,a)\,g^2+100\,a\,g^4)}{15\,(3+a+2\,a\,g^2)}$$

sampling

In[•]:=
```
cdf = Integrate[2 Pi pDraine[u, g, a],
   {u, -1, x}, Assumptions → 0 < a < 1 && -1 < g < 1 && -1 < x < 1]
```

Out[•]= $$\Big(3\,(-1+g)\,g^2\,\left(-1-g+\sqrt{1+g^2-2\,g\,x}\right)+ a\,\Big(2-2\,g^6-2\,g\,x-2\,\sqrt{1+g^2-2\,g\,x}+g^3\,\sqrt{1+g^2-2\,g\,x}+g^4\,(-2+x^2)+ 2\,g^5\,\left(x+\sqrt{1+g^2-2\,g\,x}\right)-g^2\,\left(-2+x^2+\sqrt{1+g^2-2\,g\,x}\right)\Big)\Big)\Big/ \left(2\,g^3\,(3+a+2\,a\,g^2)\,\sqrt{1+g^2-2\,g\,x}\right)$$

Schlick

In[•]:= pSchlick[u_, k_] := $\frac{1}{4\,Pi}\left(\frac{1-k^2}{(1+k\,u)^2}\right)$

Normalization condition

In[•]:=
```
Integrate[2 Pi pSchlick[u, k], {u, -1, 1}, Assumptions → -1 < k < 1]
```

Out[•]= 1

Mean-cosine

In[•]:=
```
Integrate[2 Pi pSchlick[u, k] u, {u, -1, 1}, Assumptions → -1 < k < 1]
```

Out[•]= $$-\frac{k-\mathrm{ArcTanh}[k]+k^2\,\mathrm{ArcTanh}[k]}{k^2}$$

Legendre expansion coefficients

In[•]:=
```
Integrate[2 Pi (2 k + 1) pSchlick[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 0,
  {y, 0, Pi}, Assumptions → -1 < e < 1]
```

Out[•]= ConditionalExpression[1, e ≠ 0]

In[•]:= Integrate[2 Pi (2 k + 1) pSchlick[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 1, {y, 0, Pi}, Assumptions → -1 < e < 1]

Out[•]= $\text{ConditionalExpression}\left[-\frac{3\,(e + (-1 + e^2)\,\text{ArcTanh}[e])}{e^2},\ e \neq 0\right]$

In[•]:= Integrate[2 Pi (2 k + 1) pSchlick[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 2, {y, 0, Pi}, Assumptions → -1 < e < 1]

Out[•]= $\text{ConditionalExpression}\left[-\frac{5\,(-6\,e + 4\,e^3 - 6\,(-1 + e^2)\,\text{ArcTanh}[e])}{2\,e^3},\ e \neq 0\right]$

In[•]:= Integrate[2 Pi (2 k + 1) pSchlick[Cos[y], e] LegendreP[k, Cos[y]] Sin[y] /. k → 3, {y, 0, Pi}, Assumptions → -1 < e < 1]

Out[•]= $\text{ConditionalExpression}\left[-\frac{7\,(30\,e - 26\,e^3 - 6\,(5 - 6\,e^2 + e^4)\,\text{ArcTanh}[e])}{4\,e^4},\ e \neq 0\right]$

sampling

In[•]:= cdf = Integrate[2 Pi pSchlick[u, e], {u, -1, x}, Assumptions → -1 < e < 1 && 0 < x < 1]

Out[•]= $\frac{(1 + e)\,(1 + x)}{2 + 2\,e\,x}$

In[•]:= Solve[cdf == k, x]

Out[•]= $\left\{\left\{x \to \frac{1 + e - 2\,k}{-1 - e + 2\,e\,k}\right\}\right\}$

In[•]:=
```
With[{e = -.7},
 Show[
  Plot[2 Pi pSchlick[u, e], {u, -1, 1}],
  Histogram[Map[(1 + e - 2 #)/(-1 - e + 2 e #) &, Table[RandomReal[], {i, 1, 100 000}]], 50, "PDF"]
 ]
]
```

Out[•]=