

Erfc NDF

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

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notation

$u = \mathbf{m} \cdot \mathbf{n} = \cos[\theta_m]$

$\alpha = \text{roughness}$

Definitions and derivations

$$\text{In}[4502]:= \text{Erfc}^{\text{D}}[u_, \alpha_] := \frac{\text{Erfc}\left[\frac{\sqrt{1-u^2}}{\alpha u}\right]}{2 \alpha u^3 \sqrt{\pi - \pi u^2}} \text{HeavisideTheta}[u]$$

$$\text{In}[1284]:= \text{Erfc}^{\sigma}[u_, \alpha_] := \frac{1}{4} \left(2 u + \frac{e^{\frac{u^2}{(-1+u^2) \alpha^2}} \sqrt{1-u^2} \alpha}{\sqrt{\pi}} + 2 u \text{Erf}\left[\frac{u}{\sqrt{1-u^2} \alpha}\right] + \frac{u^2 \text{Gamma}\left[0, -\frac{u^2}{(-1+u^2) \alpha^2}\right]}{\sqrt{\pi - \pi u^2} \alpha} \right)$$

$$\text{In}[2692]:= \text{Erfc}^{\Lambda}[u_, \text{roughness}_] := \frac{1}{4} \left(-2 \text{Erfc}\left[\frac{u}{\text{roughness} \sqrt{1-u^2}}\right] + \frac{\sqrt{1-u^2} \left(e^{-\frac{u^2}{\text{roughness}^2 (1-u^2)}} \text{roughness}^2 + \frac{u^2 \text{ExpIntegralE}\left[1, \frac{u^2}{\text{roughness}^2 (1-u^2)}\right]}{1-u^2} \right)}{\sqrt{\pi} \text{roughness} u} \right)$$

$$\text{In}[1286]:= (1 + \text{Erfc}^{\Lambda}[u, \alpha]) u == \text{Erfc}^{\sigma}[u, \alpha] // \text{FullSimplify}$$

Out[1286]= True

$$\text{In}[1287]:= (\text{Erfc}^{\Lambda}[u, \alpha]) u == \text{Erfc}^{\sigma}[-u, \alpha] // \text{FullSimplify}$$

Out[1287]= True

In[2693]:= FullSimplify[Erfc`D[u, $\frac{u}{\sqrt{1-u^2} x}$], Assumptions $\rightarrow 0 < u < 1 \&\& x > 0$]

Out[2693]= $\frac{1}{4} \left(-2 \operatorname{Erfc}[x] + \frac{e^{-x^2} + x^2 \operatorname{ExpIntegralE}[1, x^2]}{\sqrt{\pi} x} \right)$

derivation

Beckmann`D[u_, α _] := $\frac{e^{-1 + \frac{1}{\alpha^2 u^2}}}{\alpha^2 \pi u^4} \operatorname{HeavisideTheta}[u]$

In[*]:= Integrate[Beckmann`D[u, α m], {m, 0, 1}, Assumptions $\rightarrow 0 < u < 1 \&\& \alpha > 0$]

Out[*]= $\frac{\operatorname{Erfc}\left[\frac{\sqrt{1-u^2}}{u \alpha}\right]}{2 u^3 \sqrt{\pi - \pi u^2} \alpha}$

shape invariant f(x)

In[1292]:= FullSimplify[Erfc`D[u, α] $u^4 \alpha^2 / . u \rightarrow \frac{1}{\sqrt{1+x^2} \alpha^2}$,
Assumptions $\rightarrow 1 - \frac{1}{\sqrt{1+x^2} \alpha^2} > 0 \&\& x > 0 \&\& \alpha > 0$]

Out[1292]= $\frac{\operatorname{Erfc}[x]}{2 \sqrt{\pi} x}$

height field normalization

In[1293]:= Integrate[2 Pi u Erfc`D[u, α], {u, 0, 1}, Assumptions $\rightarrow 0 < \alpha < 1$]

Out[1293]= 1

distribution of slopes

In[1294]:= FullSimplify[Erfc`D[$\frac{1}{\sqrt{p^2+q^2+1}}$, α] $\left(\frac{1}{\sqrt{p^2+q^2+1}}\right)^4$,
Assumptions $\rightarrow 0 < \alpha < 1 \&\& p > 0 \&\& q > 0$]

Out[1294]= $\frac{\operatorname{Erfc}\left[\frac{\sqrt{p^2+q^2}}{\alpha}\right]}{2 \sqrt{\pi} \sqrt{p^2+q^2} \alpha}$

In[1295]:= Erfc`P22[p_, q_, α _] := $\frac{\operatorname{Erfc}\left[\frac{\sqrt{p^2+q^2}}{\alpha}\right]}{2 \sqrt{\pi} \sqrt{p^2+q^2} \alpha}$

In[1296]:= Integrate[Erfc`P22[p, q, α], {p, -Infinity, Infinity},
{q, -Infinity, Infinity}, Assumptions $\rightarrow 0 < \alpha < 1$]

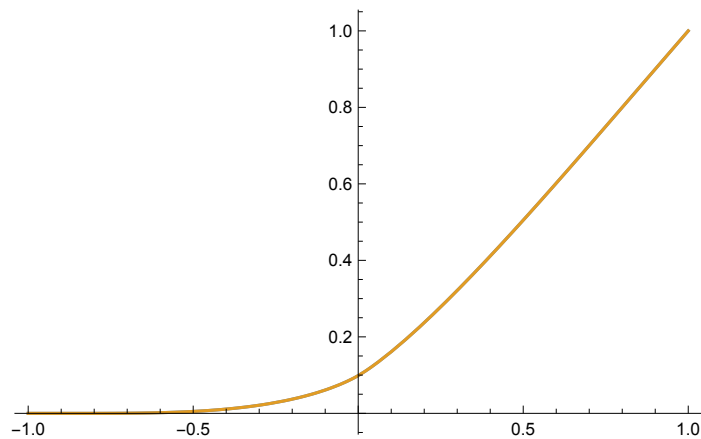
Out[1296]= 1

compare σ to delta integral:

```
In[1298]:= Delta`σ[u_, ui_] := Re[2  $\left( \sqrt{1 - u^2 - ui^2} + u ui \operatorname{ArcCos}\left[-\frac{u ui}{\sqrt{1 - u^2} \sqrt{1 - ui^2}}\right] \right)$ ]
```

```
In[1301]:= With[{α = .7},
  Plot[{
    Quiet[NIntegrate[Erfc`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
    Quiet[Erfc`σ[u, α]]
  }, {u, -1, 1}]
]
```

Out[1301]=



importance sampling

```
In[4507]:= With[{α = 1.7},
  Show[
    Histogram[Table[1 /  $\left( \sqrt{1 - (\alpha \operatorname{RandomReal}[])^2} (\operatorname{Log}[\operatorname{RandomReal}[]]) \right)$ ],
      {i, Range[10 000]}], 200, "PDF",
    Plot[Erfc`D[u, α] 2 Pi u, {u, 0, .999}, PlotRange → All]
  ]
]
```

Out[4507]=

