Student-T NDF

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

notation

u = **m** . **n** = Cos[θ_m]
α = roughness

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definitions and derivations

In[2785]:=
$$\text{ST`D}[u\_, \alpha\_, \gamma\_] := \frac{\left(1 + \frac{1-u^2}{u^2 \alpha^2 (-1+\gamma)}\right)^{-\gamma}}{\pi u^4 \alpha^2} \text{HeavisideTheta}[u]$$

In[1183]:=
$$\text{ST`}\sigma[u\_, \alpha\_, \gamma\_] := \frac{u}{2} + \frac{\alpha \left(1 - \frac{u^2}{(-1+u^2) \alpha^2 (-1+\gamma)}\right)^{\frac{3}{2}-\gamma} \sqrt{(1-u^2)(-1+\gamma)}\ \text{Gamma}\left[-\frac{3}{2}+\gamma\right]}{2\sqrt{\pi}\ \text{Gamma}[-1+\gamma]} + \frac{u^2\ \text{Gamma}\left[-\frac{1}{2}+\gamma\right] \text{Hypergeometric2F1}\left[\frac{1}{2}, -\frac{1}{2}+\gamma, \frac{3}{2}, \frac{u^2}{(-1+u^2)\alpha^2(-1+\gamma)}\right]}{\sqrt{\pi}\ \alpha \sqrt{(1-u^2)(-1+\gamma)}\ \text{Gamma}[-1+\gamma]}$$

In[1184]:=
$$\text{ST`}\Lambda[u\_, \alpha\_, \gamma\_] := \frac{1}{4\sqrt{\pi}\ u\ \text{Gamma}[-1+\gamma]} \left(2 \alpha^{-2+2\gamma} \left(u^2 - (-1+u^2)\alpha^2(-1+\gamma)\right)^{\frac{3}{2}-\gamma} \left((1-u^2)(-1+\gamma)\right)^{-1+\gamma} + (-1)^{-\gamma} u (-3+2\gamma)\ \text{Beta}\left[\frac{(-1+u^2)\alpha^2(-1+\gamma)}{u^2}, -1+\gamma, \frac{3}{2}-\gamma\right]\right) \text{Gamma}\left[-\frac{3}{2}+\gamma\right]$$

In[1185]:=
```
(1 + ST`Λ[u, α, γ]) u == ST`σ[u, α, γ] /. γ → 3 /. u → 1/2 /. α → 1/3 // FullSimplify
```

Out[1185]= True

In[1187]:=
```
ST`Λ[u, α, γ] u == ST`σ[-u, α, γ] /. γ → 3 /. u → 1/2 /. α → 1/3 // FullSimplify
```

Out[1187]= True

In[2705]:= FullSimplify[ST`Λ[u, $\frac{u}{\sqrt{1-u^2}\ x}$, γ], Assumptions → 0 < u < 1 && x > 0 && γ > 3/2]

Out[2705]=
$$\frac{1}{4\sqrt{\pi}\ x}(-1+\gamma)^{\gamma}\,\text{Gamma}\left[-\frac{3}{2}+\gamma\right]\left(\frac{2\left(-1+x^2+\gamma\right)^{\frac{3}{2}-\gamma}}{\text{Gamma}[\gamma]}-\frac{x^{3-2\gamma}\left(-3+2\gamma\right)\text{Hypergeometric2F1Regularized}\left[-1+\gamma,-\frac{1}{2}+\gamma,\gamma,\frac{1-\gamma}{x^2}\right]}{-1+\gamma}\right)$$

shape invariant f(x)

In[1240]:= FullSimplify[ST`D[u, α, γ] u^4 $α^2$ /. u -> $\frac{1}{\sqrt{1+x^2\alpha^2}}$,

Assumptions → 1 - $\frac{1}{\sqrt{1+x^2\alpha^2}}$ > 0 && α > 2]

Out[1240]=
$$\frac{\left(\frac{-1+\gamma}{-1+x^2+\gamma}\right)^{\gamma}}{\pi}$$

Relationship to Beckmann NDF:

In[2766]:= Integrate[PDF[GammaDistribution[γ - 1, 1]][αB] × Beckmann`D[u, α $\sqrt{\frac{\gamma-1}{\alpha B}}$],

{αB, 0, Infinity}, Assumptions → α > 0 && 0 < u < 1 && γ > 3/2]

Out[2766]=
$$\frac{\left(1+\frac{1-u^2}{u^2\alpha^2\,(-1+\gamma)}\right)^{-\gamma}}{\pi\,u^4\,\alpha^2}$$

height field normalization

In[2783]:= Integrate[2 Pi u ST`D[u, α, γ], {u, 0, 1}, Assumptions → 0 < α && γ > 3/2]

Out[2783]= 1

distribution of slopes

In[1206]:= FullSimplify[ST`D[$\frac{1}{\sqrt{p^2+q^2+1}}$, α, γ] $\left(\frac{1}{\sqrt{p^2+q^2+1}}\right)^4$,

Assumptions → 0 < α < 1 && p > 0 && q > 0]

Out[1206]=
$$\frac{\alpha^{-2+2\gamma}\left(p^2+q^2+\alpha^2\,(-1+\gamma)\right)^{-\gamma}(-1+\gamma)^{\gamma}}{\pi}$$

In[1208]:= ST`P22[p_, q_, α_, γ_] := $\frac{\alpha^{-2+2\gamma}\left(p^2+q^2+\alpha^2\,(-1+\gamma)\right)^{-\gamma}(-1+\gamma)^{\gamma}}{\pi}$

In[1209]:=
```
Integrate[ST`P22[p, q, α, γ], {p, -Infinity, Infinity},
 {q, -Infinity, Infinity}, Assumptions → 0 < α < 1 && γ > 2]
```

Out[1209]= 1

In[1210]:=
```
Integrate[ST`P22[p, q, α, γ],
 {q, -Infinity, Infinity}, Assumptions → 0 < α < 1 && γ > 2]
```

Out[1210]= ConditionalExpression[

$$\frac{\alpha^{-2+2\gamma}\left(p^2+\alpha^2(-1+\gamma)\right)^{\frac{1}{2}-\gamma}(-1+\gamma)^{\gamma}\,\text{Gamma}\left[-\frac{1}{2}+\gamma\right]}{\sqrt{\pi}\,\text{Gamma}[\gamma]},\ \alpha^2\gamma+\text{Re}\left[p^2\right]>\alpha^2\Big]$$

In[1211]:=

$$\text{ST`P2}[p\_,\alpha\_,\gamma\_] := \frac{\alpha^{-2+2\gamma}\left(p^2+\alpha^2(-1+\gamma)\right)^{\frac{1}{2}-\gamma}(-1+\gamma)^{\gamma}\,\text{Gamma}\left[-\frac{1}{2}+\gamma\right]}{\sqrt{\pi}\,\text{Gamma}[\gamma]}$$

compare σ to delta integral:

In[1157]:=

$$\text{Delta`}\sigma[u\_,ui\_] := \text{Re}\left[2\left(\sqrt{1-u^2-ui^2}+u\,ui\,\text{ArcCos}\left[-\frac{u\,ui}{\sqrt{1-u^2}\sqrt{1-ui^2}}\right]\right)\right]$$

In[1166]:=
```
With[{α = .7, γ = 3},
 Plot[{
   Quiet[NIntegrate[ST`D[ui, α, γ] × Delta`σ[u, ui], {ui, 0, 1}]],
   ST`σ[u, α, γ]
  }, {u, -1, 1}]
]
```

Out[1166]=