

Bessel K NDF

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

© 2020 Eugene d'Eon

www.eugenedeon.com/hitchhikers

notation

$u = \mathbf{m} \cdot \mathbf{n} = \cos[\theta_m]$

α = roughness

Definitions and derivations

In[4438]:= $\text{BesselK`D}[u_, \alpha_, a_] :=$

$$\frac{2 (1 - u^2)^{-\frac{1}{2} + \frac{a}{2}} (u \alpha)^{-a} \text{BesselK}\left[1 - a, \frac{2 \sqrt{1 - u^2}}{u \alpha}\right]}{\pi u^3 \alpha \text{Gamma}[a]} \text{HeavisideTheta}[u]$$

derivation

$$\text{Beckmann`D}[u_, \alpha_] := \frac{e^{-1 + \frac{1}{\alpha^2}}}{\alpha^2 \pi u^4} \text{HeavisideTheta}[u]$$

In[1489]:= $\text{Integrate}[\text{Beckmann`D}[u, \alpha \sqrt{m}] \times \text{PDF}[\text{GammaDistribution}[a, 1]][m],$
 $\{m, 0, \text{Infinity}\}, \text{Assumptions} \rightarrow 0 < u < 1 \&\& \alpha > 0 \&\& a > 0]$

Out[1489]=
$$\frac{2 (1 - u^2)^{-\frac{1}{2} + \frac{a}{2}} (u \alpha)^{-a} \text{BesselK}\left[1 - a, \frac{2 \sqrt{1 - u^2}}{u \alpha}\right]}{\pi u^3 \alpha \text{Gamma}[a]}$$

In[2527]:= $\text{FullSimplify}[\text{Integrate}[\text{Beckmann`D}[u, \frac{\alpha}{\sqrt{2}} m] \times \text{PDF}[\text{ChiDistribution}[2 a]][m], \{m, 0, \text{Infinity}\},$
 $\text{Assumptions} \rightarrow 0 < u < 1 \&\& \alpha > 0 \&\& a > 0], \text{Assumptions} \rightarrow a > 0 \&\& \alpha > 0]$

Out[2527]=
$$\frac{2 (1 - u^2)^{\frac{1}{2} (-1 + a)} (u \alpha)^{-1 - a} \text{BesselK}\left[-1 + a, \frac{2 \sqrt{1 - u^2}}{u \alpha}\right]}{\pi u^2 \text{Gamma}[a]}$$

shape invariant f(x)

```
In[1493]:= FullSimplify[BesselK`D[u, α, a] u^4 α^2 /. u ->  $\frac{1}{\sqrt{1+x^2 \alpha^2}}$ ,
Assumptions -> 1 -  $\frac{1}{\sqrt{1+x^2 \alpha^2}} > 0 \&\& x > 0 \&\& \alpha > 0 \&\& a > 0]$ 
Out[1493]=  $\frac{2 x^{-1+a} \text{BesselK}[-1+a, 2 x]}{\pi \text{Gamma}[a]}$ 
```

height field normalization

```
In[2549]:= NIntegrate[2 Pi u BesselK`D[u, .6, 1.6], {u, 0, 1}]
Out[2549]= 1.
```

distribution of slopes

```
In[1494]:= FullSimplify[BesselK`D[ $\frac{1}{\sqrt{p^2+q^2+1}}$ , α, a]  $\left(\frac{1}{\sqrt{p^2+q^2+1}}\right)^4$ ,
Assumptions -> 0 < α < 1 && p > 0 && q > 0]
Out[1494]=  $\frac{2 (p^2+q^2)^{\frac{1}{2}(-1+a)} \alpha^{-1-a} \text{BesselK}[-1+a, \frac{2\sqrt{p^2+q^2}}{\alpha}]}{\pi \text{Gamma}[a]}$ 

In[1495]:= BesselK`P22[p_, q_, α_, a_] :=  $\frac{2 (p^2+q^2)^{\frac{1}{2}(-1+a)} \alpha^{-1-a} \text{BesselK}[-1+a, \frac{2\sqrt{p^2+q^2}}{\alpha}]}{\pi \text{Gamma}[a]}$ 

In[2550]:= Integrate[BesselK`P22[p, q, α, a], {p, -Infinity, Infinity},
{q, -Infinity, Infinity}, Assumptions -> 0 < α < 1 && a > 0]
Out[2550]= 1
```

importance sampling

```

In[4472]:= With[{α = .7, a = 1.3},
  Show[
    Histogram[Table[
      1 /  $\left( \sqrt{1 - \left( \frac{\alpha}{\sqrt{2}} \text{RandomVariate}[\text{ChiDistribution}[2 a]] \right)^2} (\text{Log}[\text{RandomReal}[]]) \right)$ ,
      {i, Range[10 000]}], 200, "PDF"],
    Plot[BesselK`D[u, α, a] 2 Pi u, {u, 0, 1}, PlotRange → All]
  ]
]

```

