Ground Glass (GGX) NDF

notation

u = **m** . **n** = Cos[θ_m]
α = roughness

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definitions and derivations

In[2744]:= $\texttt{GGX`D[u\_, }\alpha\texttt{\_] :=} \dfrac{\alpha^2}{\pi\left(1+u^2\left(-1+\alpha^2\right)\right)^2}\ \texttt{HeavisideTheta[u]}$

In[490]:= $\texttt{GGX`}\sigma\texttt{[u\_, }\alpha\texttt{\_] :=} \dfrac{1}{2}\left(\sqrt{\alpha^2-\alpha^2 u^2+u^2}+u\right)$

In[2645]:= $\texttt{GGX`}\Lambda\texttt{[u\_, }\alpha\texttt{\_] :=} \dfrac{1}{2}\left(-1+\dfrac{\sqrt{\alpha^2-u^2\left(-1+\alpha^2\right)}}{u}\right)$

In[538]:= $\left(1+\texttt{GGX`}\Lambda[u,\alpha]\right) u == \texttt{GGX`}\sigma[u,\alpha]$ // FullSimplify

Out[538]= True

In[561]:= $\left(\texttt{GGX`}\Lambda[u,\alpha]\right) u == \texttt{GGX`}\sigma[-u,\alpha]$ // FullSimplify

Out[561]= True

In[2646]:= $\texttt{FullSimplify}\left[\texttt{GGX`}\Lambda\left[u, \dfrac{u}{\sqrt{1-u^2}\ x}\right], \texttt{Assumptions} \to 0<u<1\ \&\&\ x>0\right]$

Out[2646]= $\dfrac{1}{2}\left(-1+\sqrt{1+\dfrac{1}{x^2}}\right)$

shape invariant f(x)

In[1237]:= $\texttt{FullSimplify}\left[\texttt{GGX`D}[u,\alpha]\ u^4\alpha^2\ \texttt{/. u ->} \dfrac{1}{\sqrt{1+x^2\alpha^2}}, \texttt{Assumptions} \to 1-\dfrac{1}{\sqrt{1+x^2\alpha^2}}>0\right]$

Out[1237]= $\dfrac{1}{\pi\left(1+x^2\right)^2}$

height field normalization

In[510]:= Integrate[2 Pi u GGX`D[u, α], {u, 0, 1}, Assumptions → 0 < α < 1]

Out[510]= 1

distribution of slopes

In[516]:=
```
FullSimplify[GGX`D[1/Sqrt[p^2 + q^2 + 1], α] (1/Sqrt[p^2 + q^2 + 1])^4,
  Assumptions → 0 < α < 1 && p > 0 && q > 0]
```

Out[516]=
$$\frac{\alpha^2}{\pi\left(p^2+q^2+\alpha^2\right)^2}$$

In[519]:=
```
GGX`P22[p_, q_, α_] := α^2/(π (p^2 + q^2 + α^2)^2)
```

In[520]:=
```
Integrate[GGX`P22[p, q, α], {p, -Infinity, Infinity},
  {q, -Infinity, Infinity}, Assumptions → 0 < α < 1]
```

Out[520]= 1

In[522]:=
```
Integrate[GGX`P22[p, q, α],
  {q, -Infinity, Infinity}, Assumptions → α > 0 && Im[p] == 0]
```

Out[522]=
$$\frac{\alpha^2}{2\left(p^2+\alpha^2\right)^{3/2}}$$

In[525]:=
```
GGX`P2[p_, α_] := α^2/(2 (p^2 + α^2)^(3/2))
```

derivation of Λ(u)

In[532]:=
```
FullSimplify[Sqrt[1 - u^2]/u Integrate[(q - u/Sqrt[1 - u^2]) GGX`P2[q, α], {q, u/Sqrt[1 - u^2], Infinity},
    Assumptions → 0 < u < 1 && 0 < α < 1], Assumptions → 0 < u < 1 && 0 < α < 1]
```

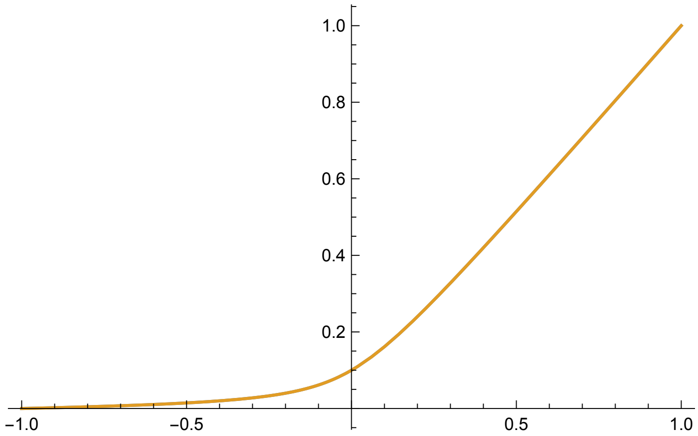
Out[532]=
$$\frac{1}{2}\left(-1+\frac{\sqrt{\alpha^2-u^2\left(-1+\alpha^2\right)}}{u}\right)$$

compare σ to delta integral:

```
Delta`σ[u_, ui_] := Re[2 (Sqrt[1 - u^2 - ui^2] + u ui ArcCos[-(u ui)/(Sqrt[1 - u^2] Sqrt[1 - ui^2])])]
```

In[499]:=
```
With[{α = .2},
 Plot[{
   Quiet[NIntegrate[GGX`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
   GGX`σ[u, α]
  }, {u, -1, 1}]
]
```

Out[499]=



As superposition of Beckmann NDFs:

Frechet-2 superposition:

In[631]:= Beckmann`D[u_, α_] :=
$$\frac{e^{-\frac{-1+\frac{1}{u^2}}{\alpha^2}}}{\alpha^2\,\pi\,u^4}$$

In[635]:=
```
PDF[FrechetDistribution[2, α]][αB]
```

Out[635]=
$$\begin{cases} \frac{2\,e^{-\frac{\alpha^2}{\alpha B^2}}\,\alpha^2}{\alpha B^3} & \alpha B > 0 \\ 0 & \text{True} \end{cases}$$

The GGX NDF is a Frechet-2 distribution of Beckmann NDFs:

In[634]:= FullSimplify[Integrate[$\frac{2\,e^{-\frac{\alpha^2}{\alpha B^2}}\,\alpha^2}{\alpha B^3}$ Beckmann`D[u, αB], {αB, 0, Infinity},
 Assumptions → α > 0 && 0 < u < 1] == GGX`D[u, α], Assumptions → 0 < u < 1]

Out[634]= True

In[1715]:= FullSimplify[Integrate[$\frac{2\,e^{-\frac{1}{\alpha B^2}}}{\alpha B^3}$ Beckmann`D[u, α αB], {αB, 0, Infinity},
 Assumptions → α > 0 && 0 < u < 1] == GGX`D[u, α], Assumptions → 0 < u < 1]

Out[1715]= True

Which yields a new derivation of GGX Λ

In[661]:= $\texttt{FullSimplify}\left[\texttt{Integrate}\left[\frac{2\, e^{-\frac{\alpha^2}{\alpha B^2}}\, \alpha^2}{\alpha B^3}\, \texttt{Beckmann`}\Lambda[u, \alpha B], \{\alpha B, 0, \texttt{Infinity}\},\right.\right.$
$\left.\left.\texttt{Assumptions} \to \alpha > 0 \,\&\&\, 0 < u < 1\right] == \texttt{GGX`}\Lambda[u, \alpha], \texttt{Assumptions} \to 0 < u < 1\right]$

Out[661]= True

In[638]:= $\texttt{Integrate}\left[\frac{2\, e^{-\frac{\alpha^2}{\alpha B^2}}\, \alpha^2}{\alpha B^3}\, \alpha B, \{\alpha B, 0, \texttt{Infinity}\}, \texttt{Assumptions} \to \alpha > 0\right]$

Out[638]= $\sqrt{\pi}\ \alpha$

The mean squared Beckmann roughness in the superposition is unbounded:

In[639]:= $\texttt{Integrate}\left[\frac{2\, e^{-\frac{\alpha^2}{\alpha B^2}}\, \alpha^2}{\alpha B^3}\, \alpha B^2, \{\alpha B, 0, \texttt{Infinity}\}, \texttt{Assumptions} \to \alpha > 0\right]$

Out[639]= $\texttt{Integrate}\left[\frac{2\, e^{-\frac{\alpha^2}{\alpha B^2}}\, \alpha^2}{\alpha B}, \{\alpha B, 0, \infty\}, \texttt{Assumptions} \to \alpha > 0\right]$

Gamma-1 superposition

In[655]:= $\texttt{PDF}[\texttt{GammaDistribution}[1, 1]][\alpha B]$

Out[655]= $\begin{cases} e^{-\alpha B} & \alpha B > 0 \\ 0 & \texttt{True} \end{cases}$

In[657]:= $\texttt{FullSimplify}\left[\texttt{Integrate}\left[e^{-\alpha B}\, \texttt{Beckmann`D}\left[u, \alpha \middle/ \sqrt{\alpha B}\right], \{\alpha B, 0, \texttt{Infinity}\},\right.\right.$
$\left.\left.\texttt{Assumptions} \to \alpha > 0 \,\&\&\, 0 < u < 1\right] == \texttt{GGX`D}[u, \alpha], \texttt{Assumptions} \to 0 < u < 1\right]$

Out[657]= True

In[663]:= $\texttt{FullSimplify}\left[\texttt{Integrate}\left[e^{-\alpha B}\, \texttt{Beckmann`}\Lambda\left[u, \alpha \middle/ \sqrt{\alpha B}\right], \{\alpha B, 0, \texttt{Infinity}\},\right.\right.$
$\left.\left.\texttt{Assumptions} \to \alpha > 0 \,\&\&\, 0 < u < 1\right] == \texttt{GGX`}\Lambda[u, \alpha], \texttt{Assumptions} \to 0 < u < 1\right]$

Out[663]= True