Double GGX NDF

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

notation

u = **m** . **n** = Cos[θ_m]
α = roughness

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definitions and derivations

The NDF that is to GGX as GGX is to Beckmann is:

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DoubleGGX`D[u_, α_] :=
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$$\frac{\alpha^2 \left(1 - u^2 + e^{-\frac{u^2 \alpha^2}{-1+u^2}} \left(1 + u^2 \left(-1 + \alpha^2\right)\right) \text{ExpIntegralEi}\left[\frac{u^2 \alpha^2}{-1+u^2}\right]\right)}{\pi \left(-1 + u^2\right)^3} \text{HeavisideTheta}[u]$$

In[683]:= DoubleGGX`σ[u_, α_] := $\frac{1}{2}\left(u + \sqrt{\pi}\ \text{Abs}[u]\ \text{HypergeometricU}\left[-\frac{1}{2}, 0, \left(-1 + \frac{1}{u^2}\right) \alpha^2\right]\right)$

In[2702]:= DoubleGGX`Λ[u_, α_] := $\frac{1}{2}\left(-1 + \frac{\sqrt{\pi}\ \text{Abs}[u]\ \text{HypergeometricU}\left[-\frac{1}{2}, 0, \left(-1 + \frac{1}{u^2}\right) \alpha^2\right]}{u}\right)$

In[686]:= (1 + DoubleGGX`Λ[u, α]) u == DoubleGGX`σ[u, α] // FullSimplify

Out[686]= True

In[687]:= FullSimplify[(DoubleGGX`Λ[u, α]) u == DoubleGGX`σ[-u, α],
Assumptions → 0 < α < 1 && -1 < u < 1]

Out[687]= True

In[690]:= FunctionExpand[HypergeometricU[$-\frac{1}{2}$, 0, $\left(-1 + \frac{1}{u^2}\right) \alpha^2$]]

Out[690]= $$-\frac{e^{\frac{1}{2}\left(-1+\frac{1}{u^2}\right)\alpha^2} \left(-1 + u^2\right) \alpha^2\ \text{BesselK}\left[0, \frac{1}{2}\left(-1 + \frac{1}{u^2}\right) \alpha^2\right]}{2\sqrt{\pi}\ u^2} - \frac{e^{\frac{1}{2}\left(-1+\frac{1}{u^2}\right)\alpha^2} \left(-1 + u^2\right) \alpha^2\ \text{BesselK}\left[1, \frac{1}{2}\left(-1 + \frac{1}{u^2}\right) \alpha^2\right]}{2\sqrt{\pi}\ u^2}$$

In[2703]:= $\text{FullSimplify}\left[\text{DoubleGGX`}\Lambda\left[u, \frac{u}{\sqrt{1-u^2}\ x}\right], \text{Assumptions} \to 0 < u < 1 \,\&\&\, x > 0\right]$

Out[2703]= $\frac{1}{2}\left(-1+\sqrt{\pi}\ \text{HypergeometricU}\left[-\frac{1}{2}, 0, \frac{1}{x^2}\right]\right)$

shape invariant f(x)

In[1233]:= $\text{FullSimplify}\left[\text{DoubleGGX`D}[u, \alpha]\ u^4\ \alpha^2\ /.\ u \to \frac{1}{\sqrt{1+x^2\,\alpha^2}}, \text{Assumptions} \to 1 - \frac{1}{\sqrt{1+x^2\,\alpha^2}} > 0\right]$

Out[1233]= $-\frac{x^2 + e^{\frac{1}{x^2}}\left(1+x^2\right)\ \text{ExpIntegralEi}\left[-\frac{1}{x^2}\right]}{\pi\ x^6}$

derivation

In[681]:= $\text{Integrate}\left[\text{Exp}[-\alpha B]\ \text{GGX`D}\left[u, \alpha \big/ \sqrt{\alpha B}\right], \{\alpha B, 0, \text{Infinity}\}, \text{Assumptions} \to 0 < u < 1 \,\&\&\, 0 < \alpha < 1\right]$

Out[681]= \$Aborted

In[682]:= $\text{Integrate}\left[\text{Exp}[-\alpha B]\ \text{GGX`}\sigma\left[u, \alpha \big/ \sqrt{\alpha B}\right], \{\alpha B, 0, \text{Infinity}\}, \text{Assumptions} \to -1 < u < 1 \,\&\&\, 0 < \alpha < 1\right]$

Out[682]= $\frac{1}{2}\left(u+\sqrt{\pi}\ \text{Abs}[u]\ \text{HypergeometricU}\left[-\frac{1}{2}, 0, \left(-1+\frac{1}{u^2}\right)\alpha^2\right]\right)$

In[684]:= $\text{Integrate}\left[\text{Exp}[-\alpha B]\ \text{GGX`}\Lambda\left[u, \alpha \big/ \sqrt{\alpha B}\right], \{\alpha B, 0, \text{Infinity}\}, \text{Assumptions} \to -1 < u < 1 \,\&\&\, 0 < \alpha < 1\right]$

Out[684]= $\frac{1}{2}\left(-1+\frac{\sqrt{\pi}\ \text{Abs}[u]\ \text{HypergeometricU}\left[-\frac{1}{2}, 0, \left(-1+\frac{1}{u^2}\right)\alpha^2\right]}{u}\right)$

distribution of slopes

In[692]:= $\text{FullSimplify}\left[\text{DoubleGGX`D}\left[\frac{1}{\sqrt{p^2+q^2+1}}, \alpha\right]\left(\frac{1}{\sqrt{p^2+q^2+1}}\right)^4, \text{Assumptions} \to 0 < \alpha < 1 \,\&\&\, p > 0 \,\&\&\, q > 0\right]$

Out[692]= $-\frac{\alpha^2\left(p^2+q^2+e^{\frac{\alpha^2}{p^2+q^2}}\left(p^2+q^2+\alpha^2\right)\ \text{ExpIntegralEi}\left[-\frac{\alpha^2}{p^2+q^2}\right]\right)}{\pi\left(p^2+q^2\right)^3}$

In[695]:= $\text{DoubleGGX`P22}[p\_, q\_, \alpha\_] := -\frac{\alpha^2\left(p^2+q^2+e^{\frac{\alpha^2}{p^2+q^2}}\left(p^2+q^2+\alpha^2\right)\ \text{ExpIntegralEi}\left[-\frac{\alpha^2}{p^2+q^2}\right]\right)}{\pi\left(p^2+q^2\right)^3}$

In[696]:=
```
Integrate[DoubleGGX`P22[p, q, α], {p, -Infinity, Infinity},
 {q, -Infinity, Infinity}, Assumptions → 0 < α < 1]
```

Out[696]= 1

In[698]:=
```
Integrate[DoubleGGX`P22[p, q, 1],
 {q, -Infinity, Infinity}, Assumptions → α > 0 && Im[p] == 0]
```

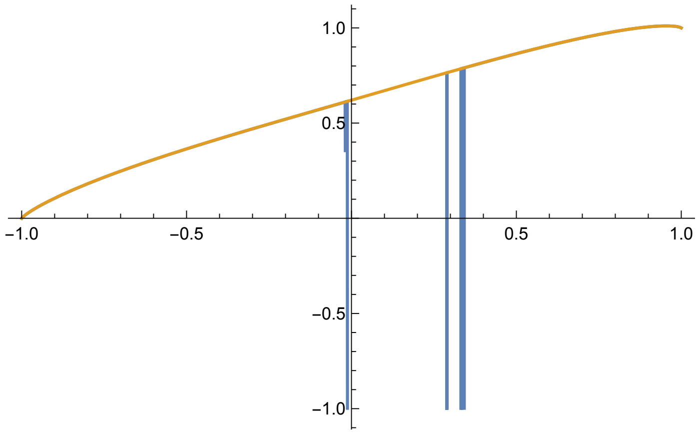
Out[698]= $Aborted

compare σ to delta integral:

In[699]:=
$$\text{Delta`}\sigma[\text{u\_}, \text{ui\_}] := \text{Re}\left[2\left(\sqrt{1-u^2-ui^2} + u\,ui\,\text{ArcCos}\left[-\frac{u\,ui}{\sqrt{1-u^2}\sqrt{1-ui^2}}\right]\right)\right]$$

In[705]:=
```
With[{α = .7},
 Plot[{
   Quiet[NIntegrate[DoubleGGX`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
   DoubleGGX`σ[u, α]
  }, {u, -1, 1}]
]
```

Out[705]=



importance sampling

In[4489]:=
```
With[{α = .9},
 Show[
  Histogram[Table[Sqrt[1/(1 - (α/Sqrt[#])^2 + (α/Sqrt[#])^2/RandomReal[])] &[-Log[RandomReal[]]],
    {i, Range[10 000]}], 200, "PDF"],
  Plot[DoubleGGX`D[u, α] 2 Pi u, {u, 0, .999}, PlotRange → All]
 ]
]
```

Out[4489]=