Double GGX NDF

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

notation

$u = \mathbf{m}\,.\,\mathbf{n} = \mathrm{Cos}[\theta_m]$

α = roughness

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definitions and derivations

The NDF that is to GGX as GGX is to Beckmann is:

In[2748]:= DoubleGGX\`D[u_, α_] :=

$$\frac{\alpha^2 \left(1 - u^2 + e^{-\frac{u^2 \alpha^2}{-1+u^2}} \left(1 + u^2 \left(-1 + \alpha^2\right)\right) \mathrm{ExpIntegralEi}\left[\frac{u^2 \alpha^2}{-1+u^2}\right]\right)}{\pi \left(-1 + u^2\right)^3} \mathrm{HeavisideTheta}[u]$$

In[683]:= DoubleGGX\`σ[u_, α_] :=

$$\frac{1}{2}\left(u + \sqrt{\pi}\,\mathrm{Abs}[u]\,\mathrm{HypergeometricU}\left[-\frac{1}{2}, 0, \left(-1 + \frac{1}{u^2}\right)\alpha^2\right]\right)$$

In[2702]:= DoubleGGX\`Λ[u_, α_] :=

$$\frac{1}{2}\left(-1 + \frac{\sqrt{\pi}\,\mathrm{Abs}[u]\,\mathrm{HypergeometricU}\left[-\frac{1}{2}, 0, \left(-1 + \frac{1}{u^2}\right)\alpha^2\right]}{u}\right)$$

In[686]:= (1 + DoubleGGX\`Λ[u, α]) u == DoubleGGX\`σ[u, α] // FullSimplify

Out[686]= True

In[687]:= FullSimplify[(DoubleGGX\`Λ[u, α]) u == DoubleGGX\`σ[-u, α], Assumptions → 0 < α < 1 && -1 < u < 1]

Out[687]= True

In[690]:= FunctionExpand[HypergeometricU[$-\frac{1}{2}$, 0, $\left(-1 + \frac{1}{u^2}\right)\alpha^2$]]

Out[690]=

$$-\frac{e^{\frac{1}{2}\left(-1+\frac{1}{u^2}\right)\alpha^2}\left(-1+u^2\right)\alpha^2\,\mathrm{BesselK}\left[0, \frac{1}{2}\left(-1+\frac{1}{u^2}\right)\alpha^2\right]}{2\sqrt{\pi}\,u^2} - \frac{e^{\frac{1}{2}\left(-1+\frac{1}{u^2}\right)\alpha^2}\left(-1+u^2\right)\alpha^2\,\mathrm{BesselK}\left[1, \frac{1}{2}\left(-1+\frac{1}{u^2}\right)\alpha^2\right]}{2\sqrt{\pi}\,u^2}$$

In[2703]:= $\text{FullSimplify}\left[\text{DoubleGGX`}\Lambda\left[u, \frac{u}{\sqrt{1-u^2}\ x}\right], \text{Assumptions} \to 0 < u < 1 \,\&\&\, x > 0\right]$

Out[2703]= $\frac{1}{2}\left(-1+\sqrt{\pi}\ \text{HypergeometricU}\left[-\frac{1}{2}, 0, \frac{1}{x^2}\right]\right)$

shape invariant f(x)

In[1233]:= $\text{FullSimplify}\left[\text{DoubleGGX`D}[u, \alpha]\ u^4\ \alpha^2\ /.\ u \to \frac{1}{\sqrt{1+x^2\ \alpha^2}},\right.$
$\left.\text{Assumptions} \to 1 - \frac{1}{\sqrt{1+x^2\ \alpha^2}} > 0\right]$

Out[1233]= $-\frac{x^2 + e^{\frac{1}{x^2}}\left(1+x^2\right)\ \text{ExpIntegralEi}\left[-\frac{1}{x^2}\right]}{\pi\ x^6}$

derivation

In[681]:= $\text{Integrate}\left[\text{Exp}[-\alpha B]\ \text{GGX`D}\left[u, \alpha \middle/ \sqrt{\alpha B}\right],\right.$
$\left.\{\alpha B, 0, \text{Infinity}\}, \text{Assumptions} \to 0 < u < 1 \,\&\&\, 0 < \alpha < 1\right]$

Out[681]= $Aborted

In[682]:= $\text{Integrate}\left[\text{Exp}[-\alpha B]\ \text{GGX`}\sigma\left[u, \alpha \middle/ \sqrt{\alpha B}\right],\right.$
$\left.\{\alpha B, 0, \text{Infinity}\}, \text{Assumptions} \to -1 < u < 1 \,\&\&\, 0 < \alpha < 1\right]$

Out[682]= $\frac{1}{2}\left(u + \sqrt{\pi}\ \text{Abs}[u]\ \text{HypergeometricU}\left[-\frac{1}{2}, 0, \left(-1+\frac{1}{u^2}\right)\alpha^2\right]\right)$

In[684]:= $\text{Integrate}\left[\text{Exp}[-\alpha B]\ \text{GGX`}\Lambda\left[u, \alpha \middle/ \sqrt{\alpha B}\right],\right.$
$\left.\{\alpha B, 0, \text{Infinity}\}, \text{Assumptions} \to -1 < u < 1 \,\&\&\, 0 < \alpha < 1\right]$

Out[684]= $\frac{1}{2}\left(-1 + \frac{\sqrt{\pi}\ \text{Abs}[u]\ \text{HypergeometricU}\left[-\frac{1}{2}, 0, \left(-1+\frac{1}{u^2}\right)\alpha^2\right]}{u}\right)$

distribution of slopes

In[692]:= $\text{FullSimplify}\left[\text{DoubleGGX`D}\left[\frac{1}{\sqrt{p^2+q^2+1}}, \alpha\right]\left(\frac{1}{\sqrt{p^2+q^2+1}}\right)^4,\right.$
$\left.\text{Assumptions} \to 0 < \alpha < 1 \,\&\&\, p > 0 \,\&\&\, q > 0\right]$

Out[692]= $-\frac{\alpha^2\left(p^2+q^2+e^{\frac{\alpha^2}{p^2+q^2}}\left(p^2+q^2+\alpha^2\right)\ \text{ExpIntegralEi}\left[-\frac{\alpha^2}{p^2+q^2}\right]\right)}{\pi\left(p^2+q^2\right)^3}$

In[695]:= $\text{DoubleGGX`P22}[p\_, q\_, \alpha\_] := -\frac{\alpha^2\left(p^2+q^2+e^{\frac{\alpha^2}{p^2+q^2}}\left(p^2+q^2+\alpha^2\right)\ \text{ExpIntegralEi}\left[-\frac{\alpha^2}{p^2+q^2}\right]\right)}{\pi\left(p^2+q^2\right)^3}$

In[696]:=
```
Integrate[DoubleGGX`P22[p, q, α], {p, -Infinity, Infinity},
 {q, -Infinity, Infinity}, Assumptions → 0 < α < 1]
```
Out[696]=
```
1
```

In[698]:=
```
Integrate[DoubleGGX`P22[p, q, 1],
 {q, -Infinity, Infinity}, Assumptions → α > 0 && Im[p] == 0]
```
Out[698]=
```
$Aborted
```

compare σ to delta integral:

In[699]:=

$$\text{Delta`}\sigma[\text{u\_}, \text{ui\_}] := \text{Re}\left[2\left(\sqrt{1-u^2-ui^2} + u\, ui\, \text{ArcCos}\left[-\frac{u\, ui}{\sqrt{1-u^2}\sqrt{1-ui^2}}\right]\right)\right]$$

In[705]:=
```
With[{α = .7},
 Plot[{
   Quiet[NIntegrate[DoubleGGX`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
   DoubleGGX`σ[u, α]
  }, {u, -1, 1}]
]
```
Out[705]=