Erfc NDF

This is code to accompany the book:

A Hitchhiker's Guide to Multiple Scattering

www.eugenedeon.com/hitchhikers

notation

u = **m** . **n** = Cos[θ_m]

α = roughness

Definitions and derivations

$$\text{Erfc`D}[u\_, \alpha\_] := \frac{\text{Erfc}\left[\frac{\sqrt{1-u^2}}{\alpha\, u}\right]}{2\,\alpha\, u^3 \sqrt{\pi - \pi u^2}}\ \text{HeavisideTheta}[u]$$

In[1284]:= $$\text{Erfc`}\sigma[u\_, \alpha\_] := \frac{1}{4}\left(2\,u + \frac{e^{\frac{u^2}{(-1+u^2)\,\alpha^2}}\sqrt{1-u^2}\ \alpha}{\sqrt{\pi}} + 2\,u\ \text{Erf}\left[\frac{u}{\sqrt{1-u^2}\ \alpha}\right] + \frac{u^2\ \text{Gamma}\left[0, -\frac{u^2}{(-1+u^2)\,\alpha^2}\right]}{\sqrt{\pi - \pi u^2}\ \alpha}\right)$$

In[2692]:= $$\text{Erfc`}\Lambda[u\_, \text{roughness}\_] := \frac{1}{4}\left(-2\ \text{Erfc}\left[\frac{u}{\text{roughness}\sqrt{1-u^2}}\right] + \frac{\sqrt{1-u^2}\left(e^{-\frac{u^2}{\text{roughness}^2\,(1-u^2)}}\ \text{roughness}^2 + \frac{u^2\ \text{ExpIntegralE}\left[1, \frac{u^2}{\text{roughness}^2\,(1-u^2)}\right]}{1-u^2}\right)}{\sqrt{\pi}\ \text{roughness}\ u}\right)$$

In[1286]:= $(1 + \text{Erfc`}\Lambda[u, \alpha])\ u == \text{Erfc`}\sigma[u, \alpha]$ // FullSimplify

Out[1286]= True

In[1287]:= $(\text{Erfc`}\Lambda[u, \alpha])\ u == \text{Erfc`}\sigma[-u, \alpha]$ // FullSimplify

Out[1287]= True

In[2693]:=
$$\texttt{FullSimplify}\left[\texttt{Erfc`}\Lambda\left[u, \frac{u}{\sqrt{1-u^2}\; x}\right], \texttt{Assumptions} \to 0 < u < 1 \;\&\&\; x > 0\right]$$

Out[2693]=
$$\frac{1}{4}\left(-2\,\texttt{Erfc}[x] + \frac{e^{-x^2} + x^2\,\texttt{ExpIntegralE}\left[1, x^2\right]}{\sqrt{\pi}\; x}\right)$$

derivation

$$\texttt{Beckmann`D}[u\_, \alpha\_] := \frac{e^{-\frac{-1+\frac{1}{u^2}}{\alpha^2}}}{\alpha^2\,\pi\,u^4}\,\texttt{HeavisideTheta}[u]$$

In[•]:=
$$\texttt{Integrate}[\texttt{Beckmann`D}[u, \alpha\, m], \{m, 0, 1\}, \texttt{Assumptions} \to 0 < u < 1 \;\&\&\; \alpha > 0]$$

Out[•]=
$$\frac{\texttt{Erfc}\left[\frac{\sqrt{1-u^2}}{u\,\alpha}\right]}{2\,u^3\,\sqrt{\pi - \pi\,u^2}\;\alpha}$$

shape invariant f(x)

In[1292]:=
$$\texttt{FullSimplify}\left[\texttt{Erfc`D}[u, \alpha]\,u^4\,\alpha^2 \;/.\; u \to \frac{1}{\sqrt{1 + x^2\,\alpha^2}},\right.$$
$$\left.\texttt{Assumptions} \to 1 - \frac{1}{\sqrt{1 + x^2\,\alpha^2}} > 0 \;\&\&\; x > 0 \;\&\&\; \alpha > 0\right]$$

Out[1292]=
$$\frac{\texttt{Erfc}[x]}{2\,\sqrt{\pi}\; x}$$

height field normalization

In[1293]:=
$$\texttt{Integrate}[2\,\texttt{Pi}\,u\,\texttt{Erfc`D}[u, \alpha], \{u, 0, 1\}, \texttt{Assumptions} \to 0 < \alpha < 1]$$

Out[1293]= 1

distribution of slopes

In[1294]:=
$$\texttt{FullSimplify}\left[\texttt{Erfc`D}\left[\frac{1}{\sqrt{p^2 + q^2 + 1}}, \alpha\right]\left(\frac{1}{\sqrt{p^2 + q^2 + 1}}\right)^4,\right.$$
$$\left.\texttt{Assumptions} \to 0 < \alpha < 1 \;\&\&\; p > 0 \;\&\&\; q > 0\right]$$

Out[1294]=
$$\frac{\texttt{Erfc}\left[\frac{\sqrt{p^2+q^2}}{\alpha}\right]}{2\,\sqrt{\pi}\,\sqrt{p^2 + q^2}\;\alpha}$$

In[1295]:=
$$\texttt{Erfc`P22}[p\_, q\_, \alpha\_] := \frac{\texttt{Erfc}\left[\frac{\sqrt{p^2+q^2}}{\alpha}\right]}{2\,\sqrt{\pi}\,\sqrt{p^2 + q^2}\;\alpha}$$

In[1296]:=
```
Integrate[Erfc`P22[p, q, α], {p, -Infinity, Infinity},
  {q, -Infinity, Infinity}, Assumptions → 0 < α < 1]
```

Out[1296]= 1

compare σ to delta integral:

In[1298]:=
```
Delta`σ[u_, ui_] := Re[2 (Sqrt[1 - u^2 - ui^2] + u ui ArcCos[-(u ui)/(Sqrt[1 - u^2] Sqrt[1 - ui^2])])]
```

$$\text{Delta`}\sigma[u\_,\, ui\_] := \text{Re}\left[2\left(\sqrt{1-u^2-ui^2} + u\, ui\, \text{ArcCos}\left[-\frac{u\, ui}{\sqrt{1-u^2}\,\sqrt{1-ui^2}}\right]\right)\right]$$

In[1301]:=
```
With[{α = .7},
 Plot[{
   Quiet[NIntegrate[Erfc`D[ui, α] × Delta`σ[u, ui], {ui, 0, 1}]],
   Quiet[Erfc`σ[u, α]]
  }, {u, -1, 1}]
]
```

Out[1301]=