MacDonald kernel

In[46]:=
```
MacDonald`K[x_] := 1/Pi BesselK[0, Abs[x]]
```

This kernel has a known explicit H-function [d'Eon and McCormick 2019]

Applications

The MacDonald kernel arises for isotropic scattering problems including:

- classical exponential random flights in Flatland
- BesselK0 random flights in the 1D rod
- $\frac{2\, s\, \mathrm{BesselK}[1,s]}{\pi}$ random flights in 3D
- $\frac{1}{2} e^{-s} (1 + s)$ random flights in 4D
- $\frac{2^{\frac{1}{2}-\frac{d}{2}}\, d\, s^{\frac{1}{2}(-1+d)}\, \mathrm{BesselK}\left[\frac{1}{2}(-1+d), s\right]}{\sqrt{\pi}\, \mathrm{Gamma}\left[1+\frac{d}{2}\right]}$ random flights in dD

References

- Fock, V. 1944. Some integral equations of mathematical physics. In: Doklady AN SSSR, vol. 26, 147–51, http://mi.mathnet.ru/eng/msb6183.
- Case, K. M. 1957. On Wiener-Hopf equations. Ann. Phys. (USA) 2(4): 384–405. doi:10.1016/0003-4916(57)90027-1
- Krein, M. G. 1962. Integral equations on a half-line with kernel depending upon the difference of the arguments. Amer. Math. Soc. Transl. 22: 163–288.
- Eugene d'Eon & M. M. R. Williams (2018): Isotropic Scattering in a Flatland Half-Space, *Journal of Computational and Theoretical Transport*, DOI: 10.1080/23324309.2018.1544566
- Eugene d'Eon & Norman J. McCormick (2019) Radiative Transfer in Half Spaces of Arbitrary Dimension, *Journal of Computational and Theoretical Transport*, 48:7, 280-337, DOI: 10.1080/23324309.2019.1696365

Normalization

In[47]:=
```
Integrate[MacDonald`K[x], {x, -Infinity, Infinity}]
```

Out[47]= 1

Fourier transform

Plane-parallel

In[48]:= $\sqrt{2\,\text{Pi}}$ FourierTransform[MacDonald`K[x], x, z]

Out[48]= $$\frac{1}{\sqrt{1+z^2}}$$

Radial symmetry

In[42]:= $$\text{pcMacDonaldIsotropic}[r\_, d\_] := \frac{2^{\frac{1}{2}-\frac{d}{2}}\, d\, r^{\frac{1}{2}(-1+d)}\, \text{BesselK}\left[\frac{1}{2}(-1+d), r\right]}{\sqrt{\pi}\,\text{Gamma}\left[1+\frac{d}{2}\right]}$$

In[45]:= TableForm[Table[pcMacDonaldIsotropic[r, d], {d, Range[10]}]]

Out[45]//TableForm=

$$\frac{2\,\text{BesselK}[0,r]}{\pi}$$
$$e^{-r}$$
$$\frac{2\,r\,\text{BesselK}[1,r]}{\pi}$$
$$\frac{1}{2}\,e^{-r}\left(1+\frac{1}{r}\right)r$$
$$\frac{2\,r^2\,\text{BesselK}[2,r]}{3\,\pi}$$
$$\frac{1}{8}\,e^{-r}\left(1+\frac{3}{r^2}+\frac{3}{r}\right)r^2$$
$$\frac{2\,r^3\,\text{BesselK}[3,r]}{15\,\pi}$$
$$\frac{1}{48}\,e^{-r}\left(1+\frac{15}{r^3}+\frac{15}{r^2}+\frac{6}{r}\right)r^3$$
$$\frac{2\,r^4\,\text{BesselK}[4,r]}{105\,\pi}$$
$$\frac{1}{384}\,e^{-r}\left(1+\frac{105}{r^4}+\frac{105}{r^3}+\frac{45}{r^2}+\frac{10}{r}\right)r^4$$

In[43]:= FullSimplify[πd[d, pcMacDonaldIsotropic[r, d]], Assumptions → z > 0]

Out[43]= $$\frac{1}{\sqrt{1+z^2}}$$

Laplace transform

In[49]:= LaplaceTransform[MacDonald`K[x], x, s]

Out[49]= $$\frac{\text{ArcCosh}[s]}{\pi\sqrt{-1+s^2}}$$