MacDonald kernel: **H-function**

Definition and application

This H function arises for isotropic scattering problems including:

- classical exponential random flights in Flatland
- BesselK0 random flights in the 1D rod
- $\frac{2\, s\, \text{BesselK}[1,s]}{\pi}$ random flights in 3D
- $\frac{1}{2}\, e^{-s}\, (1 + s)$ random flights in 4D
- $\frac{2^{\frac{1}{2}-\frac{d}{2}}\, d\, s^{\frac{1}{2}(-1+d)}\, \text{BesselK}\left[\frac{1}{2}(-1+d), s\right]}{\sqrt{\pi}\, \text{Gamma}\left[1+\frac{d}{2}\right]}$ random flights in dD

References

- Fock, V. 1944. Some integral equations of mathematical physics. In: Doklady AN SSSR, vol. 26, 147–51, http://mi.mathnet.ru/eng/msb6183.
- Case, K. M. 1957. On Wiener-Hopf equations. Ann. Phys. (USA) 2(4): 384–405. doi:10. 1016/0003-4916(57)90027-1
- Krein, M. G. 1962. Integral equations on a half-line with kernel depending upon the difference of the arguments. Amer. Math. Soc. Transl. 22: 163–288.
- Eugene d'Eon & M. M. R. Williams (2018): Isotropic Scattering in a Flatland Half-Space, *Journal of Computational and Theoretical Transport*, DOI: 10.1080/23324309.2018.1544566
- Eugene d’Eon & Norman J. McCormick (2019) Radiative Transfer in Half Spaces of Arbitrary Dimension, *Journal of Computational and Theoretical Transport*, 48:7, 280-337, DOI: 10.1080/23324309.2019.1696365

Explicit general solution

The H-function is known explicitly by adapting a derivation of V.A. Fock 1944 [d’Eon and McCormick 2019, Eq.(B.8)].

In[27]:= $\text{IFock}[x\_] := -2\,(x)\,\text{ArcTanh}\left[e^{i\,(x)}\right] - 2\,i\,\text{PolyLog}\left[2, e^{i\,(x)}\right] + \frac{1}{2}\,i\,\text{PolyLog}\left[2, e^{2\,i\,(x)}\right]$

In[52]:= $\text{MacDonald`H}[u\_, c\_] := \sqrt{\frac{1+u}{1+u\sqrt{1-c^2}}}\;\text{Abs}\left[\text{Exp}\left[\frac{1}{\text{Pi}}\,\text{IFock}[\text{ArcSec}[u] + \text{ArcSin}[c]]\right]\right]$

special case c = 1:

[d'Eon and Williams 2018]

In[134]:=
$$\text{MacDonald`Hc1[u\_, 1]} := \sqrt{1+u}\ \text{Exp}\left[\text{Re}\left[\frac{\text{HypergeometricPFQ}\left[\left\{\frac{1}{2}, 1, 1\right\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, \frac{1}{u^2}\right]}{\pi\, u}\right]\right]$$

In[135]:=
$$\text{MacDonald`Hc2[u\_, 1]} := \sqrt{1+u}\ \text{Exp}\left[\frac{\text{HypergeometricPFQ}\left[\left\{\frac{1}{2}, 1, 1\right\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, \frac{1}{u^2}\right]}{\pi\, u} + \frac{\text{ArcCosh[u]}}{2}\right]$$

In[136]:=
$$\text{MacDonald`Hc3[u\_, 1]} := \text{Chop}\left[\sqrt{(1+u)}\ e^{\frac{i\left(\text{PolyLog}\left[2,-\frac{i\left(1+\sqrt{1-u^2}\right)}{u}\right]-\text{PolyLog}\left[2,\frac{i\left(1+\sqrt{1-u^2}\right)}{u}\right]\right)}{\pi}}\left(-i\,u-\sqrt{1-u^2}\right)^{-\frac{\text{ArcSec[u]}}{\pi}}\right]$$

In[144]:=
$$N\left[\{\text{MacDonald`Hc1[u, 1], MacDonald`Hc1[u, 1], MacDonald`Hc1[u, 1]}\}\ /.\ u \to \frac{1}{3}\right]$$

Out[144]= {1.55799, 1.55799, 1.55799}

special case μ = 1:

[d'Eon and Williams 2018]

In[145]:=
$$\text{MacDonald`Hu1[1, c\_]} := \sqrt{\frac{2}{1+\sqrt{1-c^2}}}\ \text{Exp}\left[\frac{c}{\text{Pi}}\ \text{HypergeometricPFQ}\left[\left\{\frac{1}{2}, 1, 1\right\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, c^2\right]\right]$$

Benchmark values

Validated against independent implementation by Barry Ganapol, Dec 2019.

In[53]:=
```
TableForm[
 Table[NumberForm[Chop[N[MacDonald`H[u, c], 12]], 12],
  {u, 1/10, 1, 1/10}, {c, 1/10, 1, 1/10}]
 , TableHeadings → {{"μ=0.1", "μ=0.2", "μ=0.3", "μ=0.4", "μ=0.5", "μ=0.6",
    "μ=0.7", "μ=0.8", "μ=0.9", "μ=1.0"}, {"c=0.1", "c=0.2", "c=0.3",
    "c=0.4", "c=0.5", "c=0.6", "c=0.7", "c=0.8", "c=0.9", "c=1.0"}}]
```

Out[53]//TableForm=

	c=0.1	c=0.2	c=0.3	c=0.4	c=0.5
μ=0.1	1.00986237220	1.02035969089	1.03160591899	1.04375534182	1.057
μ=0.2	1.01545117571	1.03214541676	1.05032602070	1.07032520184	1.092
μ=0.3	1.01955400755	1.04090070058	1.06441474026	1.09061229185	1.120
μ=0.4	1.02277520414	1.04783494385	1.07568183575	1.10701407777	1.142
μ=0.5	1.02539994920	1.05352430029	1.08499779016	1.12069475183	1.161
μ=0.6	1.02759251403	1.05830376879	1.09287380232	1.13234524502	1.178
μ=0.7	1.02945790873	1.06238935388	1.09964260728	1.14241993772	1.192
μ=0.8	1.03106780654	1.06592963874	1.10553506623	1.15123714379	1.205
μ=0.9	1.03247341547	1.06903152309	1.11071857647	1.15902972215	1.216
μ=1.0	1.03371259147	1.07177452765	1.11531854174	1.16597352752	1.226

special values

[d’Eon and Williams 2018, Eq.(A.18)]

In[28]:=
```
MacDonald`H[1, 1]
```

Out[28]= $\sqrt{2}\, e^{\frac{2\,\mathrm{Catalan}}{\pi}}$

[d’Eon and McCormick 2019, Eq.(B.13)]

$$H_{2D}(1, c = 1/2) = \frac{2e^{\frac{4C}{3\pi}}}{(2+\sqrt{3})^{2/3}}.$$

In[57]:=
```
FullSimplify[
 Log[MacDonald`H[1, 1/2]] == Log[2 Exp[4 Catalan / (3 Pi)] / (2 + Sqrt[3])^(2/3)]]
```

Out[57]= True

Additional representations

Form 2

[d’Eon and McCormick 2019 - Eq.(B.8)]

In[80]:=
```
MacDonald`H2[u_, c_] := Sqrt[(1 + u)/(1 + u Sqrt[1 - c^2])]
  Exp[1/(2 Pi) (IFock[ArcSec[u] + ArcSin[c]] - IFock[ArcSec[u] - ArcSin[c]])]
```

In[108]:=
```
TableForm[
 Table[NumberForm[Chop[N[MacDonald`H2[u, c], 12]], 12],
  {u, 1/10, 1, 1/10}, {c, 1/10, 1, 1/10}]
 , TableHeadings → {{"μ=0.1", "μ=0.2", "μ=0.3", "μ=0.4", "μ=0.5", "μ=0.6",
    "μ=0.7", "μ=0.8", "μ=0.9", "μ=1.0"}, {"c=0.1", "c=0.2", "c=0.3",
    "c=0.4", "c=0.5", "c=0.6", "c=0.7", "c=0.8", "c=0.9", "c=1.0"}}]
```

Out[108]//TableForm=

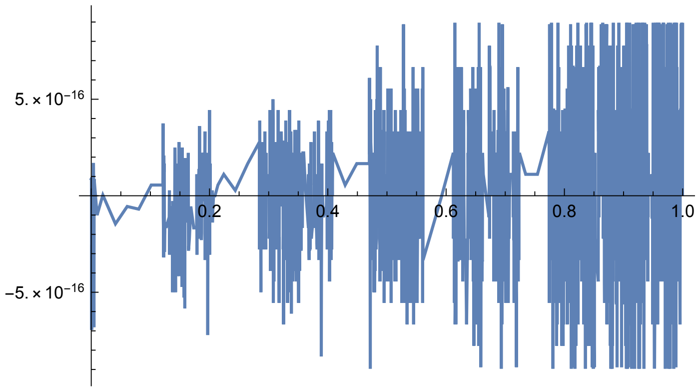
	c=0.1	c=0.2	c=0.3	c=0.4	c=0.5
μ=0.1	1.00986237220	1.02035969089	1.03160591899	1.04375534182	1.057
μ=0.2	1.01545117571	1.03214541676	1.05032602070	1.07032520184	1.092
μ=0.3	1.01955400755	1.04090070058	1.06441474026	1.09061229185	1.120
μ=0.4	1.02277520414	1.04783494385	1.07568183575	1.10701407777	1.142
μ=0.5	1.02539994920	1.05352430029	1.08499779016	1.12069475183	1.161
μ=0.6	1.02759251403	1.05830376879	1.09287380232	1.13234524502	1.178
μ=0.7	1.02945790873	1.06238935388	1.09964260728	1.14241993772	1.192
μ=0.8	1.03106780654	1.06592963874	1.10553506623	1.15123714379	1.205
μ=0.9	1.03247341547	1.06903152309	1.11071857647	1.15902972215	1.216
μ=1.0	1.03371259147	1.07177452765	1.11531854174	1.16597352752	1.226

Form 3

In[81]:=
```
MacDonald`X[y_] := Re[HypergeometricPFQ[{1/2, 1, 1}, {3/2, 3/2}, Sin[y]^2] Sin[y]]
```

In[96]:=
```
Plot[Re[IFock[x]] - MacDonald`X[x], {x, 0, 1}]
```

Out[96]=



In[109]:=
```
MacDonald`H3[u_, c_] := Sqrt[(1 + u)/(1 + u Sqrt[1 - c^2])] Exp[
  1/(2 Pi) (MacDonald`X[ArcSec[u] + ArcSin[c]] - MacDonald`X[ArcSec[u] - ArcSin[c]])]
```

In[105]:=
```
MacDonald`H3[1, 1/2] // FullSimplify
```

Out[105]=

$$\frac{2\, e^{\frac{4\,\text{Catalan}}{3\pi}}}{\left(2+\sqrt{3}\right)^{2/3}}$$

In[110]:=
```
TableForm[
 Table[NumberForm[Chop[N[MacDonald`H3[u, c], 12]], 12],
  {u, 1/10, 1, 1/10}, {c, 1/10, 1, 1/10}]
 , TableHeadings → {{"μ=0.1", "μ=0.2", "μ=0.3", "μ=0.4", "μ=0.5", "μ=0.6",
    "μ=0.7", "μ=0.8", "μ=0.9", "μ=1.0"}, {"c=0.1", "c=0.2", "c=0.3",
    "c=0.4", "c=0.5", "c=0.6", "c=0.7", "c=0.8", "c=0.9", "c=1.0"}}]
```

Out[110]//TableForm=

	c=0.1	c=0.2	c=0.3	c=0.4	c=0.5
μ=0.1	1.00986237220	1.02035969089	1.03160591899	1.04375534182	1.057
μ=0.2	1.01545117571	1.03214541676	1.05032602070	1.07032520184	1.092
μ=0.3	1.01955400755	1.04090070058	1.06441474026	1.09061229185	1.120
μ=0.4	1.02277520414	1.04783494385	1.07568183575	1.10701407777	1.142
μ=0.5	1.02539994920	1.05352430029	1.08499779016	1.12069475183	1.161
μ=0.6	1.02759251403	1.05830376879	1.09287380232	1.13234524502	1.178
μ=0.7	1.02945790873	1.06238935388	1.09964260728	1.14241993772	1.192
μ=0.8	1.03106780654	1.06592963874	1.10553506623	1.15123714379	1.205
μ=0.9	1.03247341547	1.06903152309	1.11071857647	1.15902972215	1.216
μ=1.0	1.03371259147	1.07177452765	1.11531854174	1.16597352752	1.226

Form 4

```
MacDonald`H4[u_, c_] :=
 Sqrt[(1 + u)/(1 + u Sqrt[1 - c^2])] Exp[1/Pi Re[HypergeometricPFQ[{1/2, 1, 1}, {3/2, 3/2}, Sin[#]^2] Sin[#] &[
     ArcSec[u] + ArcSin[c]]]]
```

Form 5

In[111]:=
```
Leftover[x_] := +ⅈ (PolyLog[2, -ⅇ^(ⅈ x)] - PolyLog[2, ⅇ^(ⅈ x)])
```

In[112]:=
```
MacDonald`H5[u_, c_] := Sqrt[(1 + u)/(1 + u Sqrt[1 - c^2])] Abs[
  ((1 - ⅇ^(ⅈ (ArcSec[u]+ArcSin[c])))^(ArcSec[u]+ArcSin[c])
     (1 + ⅇ^(ⅈ (ArcSec[u]+ArcSin[c])))^(-ArcSec[u]-ArcSin[c]))^(1/Pi) ×
   Exp[1/Pi Leftover[ArcSec[u] + ArcSin[c]]]
  ]
```

In[115]:=
```
MacDonald`H5[1, 1/2] == MacDonald`H[1, 1/2] // FullSimplify
```

Out[115]= True

Expansion of Fock’s integral

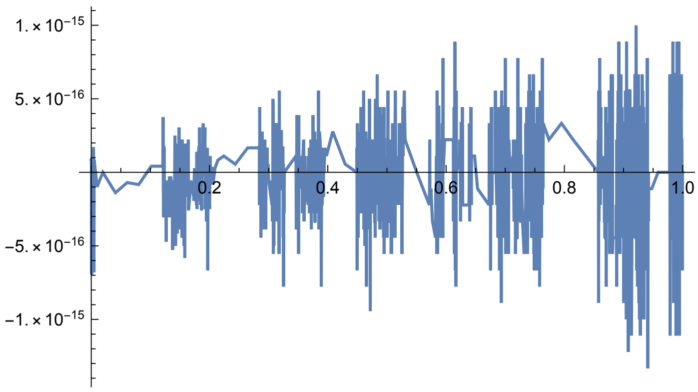
In[58]:=
```
Integrate[x/Sin[x], x]
```

Out[58]=
```
x (Log[1 - ⅇ^(ⅈ x)] - Log[1 + ⅇ^(ⅈ x)]) + ⅈ (PolyLog[2, -ⅇ^(ⅈ x)] - PolyLog[2, ⅇ^(ⅈ x)])
```

In[62]:=
```
IFocksum[x_, J_] := Sum[-((ⅈ^j (-2 + 2^j) BernoulliB[j]) x^(j+1))/(j! (j + 1)), {j, 0, J, 2}]
```

In[74]:=
```
Plot[{Re[IFock[x]] - Re[IFocksum[x, 50]]}, {x, 0, 1}, PlotRange → All]
```

Out[74]=



In[77]:= `IFocksum[x, 15]`

Out[77]= $$x+\frac{x^3}{18}+\frac{7\,x^5}{1800}+\frac{31\,x^7}{105\,840}+\frac{127\,x^9}{5\,443\,200}+\frac{73\,x^{11}}{37\,635\,840}+\frac{1\,414\,477\,x^{13}}{8\,499\,883\,392\,000}+\frac{8191\,x^{15}}{560\,431\,872\,000}$$

In[79]:= $$\texttt{Series}\left[x\left(\texttt{Log}\left[1-e^{i\,x}\right]-\texttt{Log}\left[1+e^{i\,x}\right]\right)+i\left(\texttt{PolyLog}\left[2,-e^{i\,x}\right]-\texttt{PolyLog}\left[2,e^{i\,x}\right]\right),\ \{x,0,15\},\ \texttt{Assumptions}\to 0<x<1\right]$$

Out[79]= $$-\frac{i\,\pi^2}{4}+x+\frac{x^3}{18}+\frac{7\,x^5}{1800}+\frac{31\,x^7}{105\,840}+\frac{127\,x^9}{5\,443\,200}+\frac{73\,x^{11}}{37\,635\,840}+\frac{1\,414\,477\,x^{13}}{8\,499\,883\,392\,000}+\frac{8191\,x^{15}}{560\,431\,872\,000}+O[x]^{16}$$

Numerical Integration - Form 1

In[148]:= $$\texttt{MacDonald`NH1}[u\_,c\_]:=\frac{(1+u)}{1+\sqrt{1-c^2}\,u}\,\texttt{Exp}\left[\texttt{NIntegrate}\left[\frac{c}{\texttt{Pi}}\,\frac{t\,\texttt{ArcTan}[u\,t]}{\left(t^2+1\right)\left(c+\sqrt{t^2+1}\right)},\ \{t,0,\texttt{Infinity}\}\right]\right]$$

In[149]:= $$\texttt{MacDonald`NH1b}[u\_,c\_]:=\texttt{Exp}\left[\texttt{NIntegrate}\left[\frac{c}{\texttt{Pi}}\,\frac{t\,\texttt{ArcTan}[u\,t]}{\left(t^2+1\right)\left(-c+\sqrt{t^2+1}\right)},\ \{t,0,\texttt{Infinity}\}\right]\right]$$

In[150]:= $$\texttt{MacDonald`NH1c}[u\_,c\_]:=\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\,\texttt{Exp}\left[\texttt{NIntegrate}\left[\frac{c}{\texttt{Pi}}\,\frac{t\,\texttt{ArcTan}[t\,u]}{\sqrt{1+t^2}\left(1-c^2+t^2\right)},\ \{t,0,\texttt{Infinity}\}\right]\right]$$

In[151]:= $$\texttt{MacDonald`NH1c}[u\_,c\_,J\_]:=\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\,\texttt{Exp}\left[\texttt{Sum}\left[\frac{c^j\,\texttt{Gamma}\left[\frac{1+j}{2}\right]\texttt{Hypergeometric2F1Regularized}\left[1,\frac{1+j}{2},\frac{2+j}{2},1-\frac{1}{u^2}\right]}{2\,j\,\sqrt{\pi}\,u},\ \{j,1,J-1,2\}\right]+\texttt{NIntegrate}\left[\frac{c^J\,t\,\texttt{ArcTan}[t\,u]}{\left(1+t^2\right)^{J/2}\left(\pi-c^2\,\pi+\pi\,t^2\right)},\ \{t,0,\texttt{Infinity}\}\right]\right]$$

In[152]:= $$\texttt{MacDonald`NH1c}[u\_,c\_,3]:=\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\,\texttt{Exp}\left[\texttt{Chop}\left[\frac{c\,\texttt{ArcSin}\left[\sqrt{\#1}\right]}{\pi\,u\,\sqrt{-(-1+\#1)\,\#1}}\right]\&\left[1-\frac{1}{u^2}\right]+\texttt{NIntegrate}\left[\frac{c^3\,t\,\texttt{ArcTan}[t\,u]}{\left(1+t^2\right)^{3/2}\left(\pi-c^2\,\pi+\pi\,t^2\right)},\ \{t,0,\texttt{Infinity}\}\right]\right]$$

In[153]:= $$\texttt{MacDonald`NH1c}[u\_,c\_,5]:=\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\,\texttt{Exp}\left[\texttt{Chop}\left[\frac{-c^3\sqrt{-(-1+\#1)\,\#1}+c\,\texttt{ArcSin}\left[\sqrt{\#1}\right]\left(c^2+3\,\#1\right)}{3\,\pi\,u\,\sqrt{1-\#1}\,\#1^{3/2}}\right]\&\left[1-\frac{1}{u^2}\right]+\texttt{NIntegrate}\left[\frac{c^5\,t\,\texttt{ArcTan}[t\,u]}{\left(1+t^2\right)^{5/2}\left(\pi-c^2\,\pi+\pi\,t^2\right)},\ \{t,0,\texttt{Infinity}\}\right]\right]$$

In[154]:= MacDonald`NH1c[u_, c_, 7] :=

$$\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\ \text{Exp}\Big[\text{Chop}\Big[\frac{1}{15\,\pi\,u\,\sqrt{1-\#1}\ \#1^{5/2}}\Big(c\,\text{ArcSin}\big[\sqrt{\#1}\big]\ \big(3\,c^4+5\,c^2\,\#1+15\,\#1^2\big)-c^3\,\Big(\big(5+2\,c^2\big)\sqrt{1-\#1}\ \#1^{3/2}+3\,c^2\sqrt{-(-1+\#1)\ \#1}\Big)\Big)\Big]\ \&\Big[1-\frac{1}{u^2}\Big]+\text{NIntegrate}\Big[\frac{c^7\,t\,\text{ArcTan}[t\,u]}{\big(1+t^2\big)^{7/2}\ \big(\pi-c^2\,\pi+\pi\,t^2\big)},\ \{t,0,\text{Infinity}\}\Big]\Big]$$

In[155]:= MacDonald`NH1d[u_, c_, J_] :=

$$\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\ \text{Exp}\Big[\text{Sum}\Big[\frac{c^j\,\text{Gamma}\big[\frac{1+j}{2}\big]\ \text{Hypergeometric2F1Regularized}\big[1,\frac{1+j}{2},\frac{2+j}{2},1-\frac{1}{u^2}\big]}{2\,j\,\sqrt{\pi}\,u},\ \{j,1,J-1,2\}\Big]+\text{NIntegrate}\Big[\frac{c^J\,t\,\text{ArcTan}[t\,u]}{\big(1+t^2\big)^{J/2}\ \big(\pi-c^2\,\pi+\pi\,t^2\big)},\ \{t,0,\text{Infinity}\}\Big]\Big]$$

In[161]:=

$$\text{N}\Big[\Big\{\text{MacDonald`H}\Big[\frac{7}{10},\frac{9}{10}\Big],\ \text{MacDonald`NH1}\Big[\frac{7}{10},\frac{9}{10}\Big],\ \text{MacDonald`NH1b}\Big[\frac{7}{10},\frac{9}{10}\Big],\ \text{MacDonald`NH1c}\Big[\frac{7}{10},\frac{9}{10}\Big],\ \text{MacDonald`NH1c}\Big[\frac{7}{10},\frac{9}{10},10\Big]\Big\}\Big]$$

Out[161]= {1.58199, 1.58199, 1.58199, 1.58199, 1.5848}

Numerical Integration - Form 2

In[162]:= MacDonald`NH2f[u_, c_] :=

$$\frac{(1+u)}{1+\sqrt{1-c^2}\,u}\ \text{Exp}\Big[\frac{-u}{\text{Pi}}\ \text{NIntegrate}\Big[\frac{\text{Log}\Big[\Big(1-\frac{c}{\sqrt{1+t^2}}\Big)\ \frac{t^2+1}{t^2+1-c^2}\Big]}{1+t^2\,u^2},\ \{t,0,\text{Infinity}\}\Big]\Big]$$

In[163]:= MacDonald`NH2[u_, c_] :=

$$\frac{(1+u)}{1+\sqrt{1-c^2}\,u}\ \text{Exp}\Big[\text{NIntegrate}\Big[\frac{u}{\text{Pi}}\ \frac{\text{Log}\Big[1+\frac{c}{\sqrt{1+t^2}}\Big]}{1+t^2\,u^2},\ \{t,0,\text{Infinity}\}\Big]\Big]$$

In[164]:= MacDonald`NH2b[u_, c_] :=

$$\text{Exp}\Big[\text{NIntegrate}\Big[\frac{-u}{\text{Pi}}\ \frac{\text{Log}\Big[1-\frac{c}{\sqrt{1+t^2}}\Big]}{1+t^2\,u^2},\ \{t,0,\text{Infinity}\}\Big]\Big]$$

In[165]:= MacDonald`NH2c[u_, c_, J_] :=

$$\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\ \text{Exp}\Big[\text{Sum}\Big[\frac{c^j\,\text{Gamma}\big[\frac{1+j}{2}\big]\ \text{Hypergeometric2F1Regularized}\big[1,\frac{1+j}{2},\frac{2+j}{2},1-\frac{1}{u^2}\big]}{2\,j\,\sqrt{\pi}\,u},\ \{j,1,J-1,2\}\Big]+\text{NIntegrate}\Big[\frac{u}{\text{Pi}}\ \frac{\frac{(-1)^{1+J}\,c^J\,\big(1+t^2\big)^{-J/2}\ \text{Hypergeometric2F1}\big[1,\frac{J}{2},1+\frac{J}{2},\frac{c^2}{1+t^2}\big]}{J}}{1+t^2\,u^2},\ \{t,0,\text{Infinity}\}\Big]\Big]$$

In[170]:=
```
N[{MacDonald`H[7/10, 9/10], MacDonald`NH2f[7/10, 9/10], MacDonald`NH2[7/10, 9/10],
  MacDonald`NH2b[7/10, 9/10], MacDonald`NH2c[7/10, 9/10, 100]}]
```
Out[170]= {1.58199, 1.58199, 1.58199, 1.58199, 1.58199}

Numerical Integration - Form 3

In[172]:= MacDonald`NH3[u_, c_] :=

$$\frac{(1+u)}{1+\sqrt{1-c^2}\,u}\,\mathrm{Exp}\left[\mathrm{NIntegrate}\left[\frac{1}{\mathrm{Pi}}\,\frac{u\,y\,\mathrm{Log}\left[1+\frac{c}{y}\right]}{\sqrt{-1+y^2}\,\left(1+u^2\,(-1+y^2)\right)},\,\{y,1,\mathrm{Infinity}\}\right]\right]$$

In[173]:= MacDonald`NH3b[u_, c_] :=

$$\mathrm{Exp}\left[\mathrm{NIntegrate}\left[\frac{-u}{\mathrm{Pi}}\,\frac{y\,\mathrm{Log}\left[1-\frac{c}{y}\right]}{\sqrt{-1+y^2}\,\left(1+u^2\,(-1+y^2)\right)},\,\{y,1,\mathrm{Infinity}\}\right]\right]$$

In[174]:= MacDonald`NH3c[u_, c_] :=

$$\sqrt{\frac{1+u}{1+\sqrt{1-c^2}\,u}}\,\mathrm{Exp}\left[\mathrm{NIntegrate}\left[\frac{u\,y\,\mathrm{ArcTanh}\left[\frac{c}{y}\right]}{\pi\,\sqrt{-1+y^2}\,\left(1-u^2+u^2\,y^2\right)},\,\{y,1,\mathrm{Infinity}\}\right]\right]$$

In[178]:=
```
N[{MacDonald`H[7/10, 9/10], MacDonald`NH3[7/10, 9/10],
  MacDonald`NH3b[7/10, 9/10], MacDonald`NH3c[7/10, 9/10]}]
```
Out[178]= {1.58199, 1.58199, 1.58199, 1.58199}

Numerical Integration - Form 4

In[179]:= MacDonald`NH4[u_, c_] :=

$$\frac{(1+u)}{1+\sqrt{1-c^2}\,u}\,\mathrm{Exp}\left[\mathrm{NIntegrate}\left[\frac{u\,\mathrm{Csc}[x]^2\,\mathrm{Log}[1+c\,\mathrm{Sin}[x]]}{\pi+\pi\,u^2\,\mathrm{Cot}[x]^2},\,\{x,0,\mathrm{Pi}/2\}\right]\right]$$

In[180]:= MacDonald`NH4o[u_, c_] :=

$$\frac{(1+u)}{1+\sqrt{1-c^2}\,u}\,\mathrm{Exp}\left[\mathrm{NIntegrate}\left[\frac{u}{\mathrm{Pi}}\,\frac{\mathrm{Log}[1+c\,\mathrm{Sin}[x]]}{u^2\,\mathrm{Cos}[x]^2+\mathrm{Sin}[x]^2},\,\{x,0,\mathrm{Pi}/2\}\right]\right]$$

In[181]:= MacDonald`NH4oo[u_, c_] :=

$$\frac{(1+u)}{1+\sqrt{1-c^2}\,u}\,\mathrm{Exp}\left[\frac{u}{\mathrm{Pi}}\left(\frac{1}{2}\,\pi\,\sqrt{\frac{1}{u^2}}\,\mathrm{Log}[1+c]-\mathrm{NIntegrate}\left[\frac{c\,\mathrm{ArcCot}[u\,\mathrm{Cot}[x]]\,\mathrm{Cos}[x]}{u+c\,u\,\mathrm{Sin}[x]},\,\{x,0,\mathrm{Pi}/2\}\right]\right)\right];$$

In[182]:= MacDonald`NH4oo2[u_, c_] :=

$$\frac{(1+u)\,\sqrt{1+c}}{1+\sqrt{1-c^2}\,u}\,\mathrm{Exp}\left[\frac{-u}{\mathrm{Pi}}\left(\mathrm{NIntegrate}\left[\frac{c\,\mathrm{ArcCot}[u\,\mathrm{Cot}[x]]\,\mathrm{Cos}[x]}{u+c\,u\,\mathrm{Sin}[x]},\,\{x,0,\mathrm{Pi}/2\}\right]\right)\right];$$

In[183]:=
```
MacDonald`NH4oo3[u_, c_] := Exp[(-u)/Pi
  ((1/2) π Sqrt[1/u^2] Log[1 - c] - NIntegrate[(-c ArcCot[u Cot[x]] Cos[x])/(u - c u Sin[x]), {x, 0, Pi/2}])];
```

In[184]:=
```
MacDonald`NH4oo4[u_, c_] :=
  (Sqrt[1 - c])^-1 Exp[u/Pi (NIntegrate[(-c ArcCot[u Cot[x]] Cos[x])/(u - c u Sin[x]), {x, 0, Pi/2}])];
```

In[193]:=
```
MacDonald`NH4oo5[u_, c_] := Chop[
  ((1 + u) Sqrt[1 + c])/(1 + Sqrt[1 - c^2] u) Exp[(-u)/Pi (NIntegrate[-(c^2 ArcCot[u Cot[x]] Cos[x])/(c u + u Csc[x]), {x, 0, Pi/2}]) -
    (c (π + (2 ⅈ u ArcSec[u])/Sqrt[1-u^2]))/(2 π)]];
```

In[186]:=
```
MacDonald`NH4b[u_, c_] :=
  Exp[NIntegrate[(-u Csc[x]^2 Log[1 - c Sin[x]])/(π + π u^2 Cot[x]^2), {x, 0, Pi/2}]]
```

In[187]:=
```
MacDonald`NH4c[u_, c_] :=
  Sqrt[(1 + u)/(1 + Sqrt[1 - c^2] u)] Exp[NIntegrate[(u ArcTanh[c Sin[x]] Csc[x]^2)/(π (1 + u^2 Cot[x]^2)), {x, 0, Pi/2}]]
```

In[188]:=
```
MacDonald`NH4c[u_, c_, J_] := Sqrt[(1 + u)/(1 + Sqrt[1 - c^2] u)]
  Exp[Sum[(c^j Gamma[(1+j)/2] Hypergeometric2F1Regularized[1, (1+j)/2, (2+j)/2, 1 - 1/u^2])/(2 j Sqrt[π] u),
    {j, 1, J - 1, 2}] + NIntegrate[
    ((-1)^J c^J u Csc[x]^2 Hypergeometric2F1[1, J/2, 1 + J/2, c^2 Sin[x]^2] (-Sin[x])^J)/
    (J π (1 + u^2 Cot[x]^2)), {x, 0, Pi/2}]]
```

In[205]:=
```
N[{MacDonald`H[7/10, 9/10], MacDonald`NH4[7/10, 9/10], MacDonald`NH4o[7/10, 9/10],
  MacDonald`NH4oo[7/10, 9/10], MacDonald`NH4oo2[7/10, 9/10],
  MacDonald`NH4oo2[7/10, 9/10], MacDonald`NH4oo3[7/10, 9/10],
  MacDonald`NH4oo4[7/10, 9/10], MacDonald`NH4oo5[7/10, 9/10], MacDonald`NH4b[7/10, 9/10],
  MacDonald`NH4c[7/10, 9/10], MacDonald`NH4c[7/10, 9/10, 11]}]
```

Out[205]=
```
{1.58199, 1.58199, 1.58199, 1.58199, 1.58199,
 1.58199, 1.58199, 1.58199, 1.58199, 1.58199, 1.58199, 1.58199}
```

Numerical Integration - Form 5

In[207]:=
```
MacDonald`NH5c[u_, c_] :=
 Sqrt[(1 + u)/(1 + Sqrt[1 - c^2] u)] Exp[NIntegrate[(c^2 u ArcTanh[y])/(π Sqrt[(c - y) (c + y)] (y^2 + u^2 (c - y) (c + y))), {y, 0, c}]]
```

In[208]:=
```
N[{MacDonald`H[7/10, 9/10], MacDonald`NH5c[7/10, 9/10]}]
```

Out[208]= {1.58199, 1.58199}

Numerical Integration - Form 6

In[209]:=
```
MacDonald`NH6c[u_, c_] :=
 Sqrt[(1 + u)/(1 + Sqrt[1 - c^2] u)] Exp[NIntegrate[(c^2 u ArcTanh[Sqrt[c^2 - Y^2]])/(π Y (c^2 + (-1 + u^2) Y^2)) (Y/Sqrt[c^2 - Y^2]), {Y, 0, c}]]
```

In[210]:=
```
N[{MacDonald`H[7/10, 9/10], MacDonald`NH6c[7/10, 9/10]}]
```

Out[210]= {1.58199, 1.58199}

Numerical Integration - Fox / Mullikin / Case & Zweifel forms

In[211]:=
```
MacDonald`HFox[u_, c_] :=
 (Sqrt[1 + c] (1 + u))/(1 + Sqrt[1 - c^2] u) Exp[-1/Pi NIntegrate[ArcTan[(c t)/Sqrt[1 - t^2]]/(t + u), {t, 0, 1}]]
```

In[212]:=
```
MacDonald`HMullikin54[u_, c_] :=
 (1 + u)/(1 + Sqrt[1 - c^2] u) Exp[u/Pi NIntegrate[ArcTan[(c t)/Sqrt[1 - t^2]]/(t (t + u)), {t, 0, 1}]]
```

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In[213]:=
```
MacDonald`HCZ1[u_, c_] := (Sqrt[1 + c] (1 + u))/(1 + Sqrt[1 - c^2] u)
 Exp[-1/Pi (Log[1 + u] Pi/2 - NIntegrate[Log[t + u] c/(Sqrt[1 - t^2] (1 + (-1 + c^2) t^2)), {t, 0, 1}])]
```

In[214]:=
```
MacDonald`HCZ2[u_, c_] := (Sqrt[1 + c] Sqrt[(1 + u)])/(1 + Sqrt[1 - c^2] u)
 Exp[1/Pi (NIntegrate[Log[t + u] c/(Sqrt[1 - t^2] (1 + (-1 + c^2) t^2)), {t, 0, 1}])]
```

In[218]:=
```
N[{MacDonald`H[7/10, 9/10], MacDonald`HFox[7/10, 9/10], MacDonald`HMullikin54[7/10, 9/10],
  MacDonald`HCZ1[7/10, 9/10], MacDonald`HCZ2[7/10, 9/10]}]
```

Out[218]=
```
{1.58199, 1.58199, 1.58199, 1.58199, 1.58199}
```