Research

ENERGY CHANGE IN A HARD BINARY DUE TO DISTANT ENCOUNTERS

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(Received: 4 March 2003; accepted: 3 June 2003)

Abstract. An approximate analytical technique for computing the change in the binding energy of a binary due to an incoming third star moving in a distant parabolic orbit is presented. This is an example of a tidal encounter since we assume that the distance of the third star always considerably exceeds the size of the binary. The perturbation is also adiabatic, varying on a time scale much exceeding the binary period, and the change has an exponential form. Different cases arise depending on the choice of the masses and the angle of inclination of the plane in which the star moves. Some numerical experiments are performed as a means of checking the analytical theory.

Key words: stellar dynamics, binaries:general

1. Introduction

Three-body scattering plays an essential role in stellar dynamics. An important dynamical process is the interaction between binary stars and single stars in globular clusters. This process involves an energy exchange and provides support against core collapse (Spitzer, 1986; Hut et al., 1992). A comprehensive theoretical picture of the behaviour of binaries in *N*-body systems can be found in Heggie (1975). The present paper which deals with the case of wide encounters is an extension of Heggie's work.

Suppose that a third body approaches a binary, whose initial binding energy is x, with a relative velocity of magnitude V_0 when the separation is still very large. During the ensuing encounter, let the binding energy of the pair change by an amount y, and, if the third body escapes to infinity afterwards, let its final velocity be V_1. If all the masses are equal to m, conservation of energy in the rest frame of the centre of mass of the three bodies implies that

$$\tfrac{1}{3}mV_1^2 = \tfrac{1}{3}mV_0^2 + y. \tag{1}$$

If $y > 0$, the third body will escape, and the encounter results in a 'hardening'. If $-(1/3)mV_0^2 < y < 0$, escape of the third body still occurs, and if in addition $-x < y$ then the binary survives the encounter, which is referred to as a 'softening'. On the other hand, if $-(1/3)mV_0^2 < y < -x$, the binary is destroyed, the process known (by analogy with atomic physics) as 'ionisation'. It may possibly happen in this case that the third body forms a new binary with one component of the old. Such an event, called an 'exchange' encounter, will occur in general when $y < -x$ and $y < -(1/3)mV_0^2$, for these conditions imply the original binary is disrupted and yet, by Equation (1), the third body is unable to escape to infinity from the centre of mass of the old binary. If finally, $-x < y < -(1/3)mV_0^2$, no particle escapes to infinity, at least not immediately, and the outcome of such a 'resonance' encounter is decided by further interactions.

The particular problem which is investigated in this paper is concerned with the effect of a passing third star on the energy of a hard binary. The binary is *hard* since its binding energy far exceeds the kinetic energy of the relative motion of the third body and the centre of mass of the binary, that is, $x \gg mV_0^2/3$ (cf. Spitzer, 1986). Theoretical and numerical techniques are used to compute the energy change when the third body remains *outside* the binary and moves on a nearly parabolic path. This study is motivated by a need to understand the release of energy in three-body encounters in star clusters.

In this adiabatic regime, where the time scale of the passage of the third body much exceeds the period of the binary, the change in energy of the binary depends exponentially on the ratio of the time scales. This regime has also been explored by Percival and Richards (1967) who gave formulae for the case of distant, very hyperbolic encounters (when the orbit of the third body is approximately rectilinear). Heggie (1975) computed a formula for the energy change in the regime we consider here. His formula, however, suffers from certain deficiencies.

(i) Certain coefficients which depend on the eccentricity, e, and the orientation of the orbits are not worked out explicitly; this is put right with a full rederivation in Section 3.
(ii) It fails when the binary is circular which is an important special case, because tidal effects in close binary systems lead to nearly circular orbits (Eggleton et al., 1998). This is rectified in Section 4.1.
(iii) This formula itself fails when the masses of the components of the binary are equal and Section 4.2 deals with this special case.
(iv) Finally, for circular orbits and equal masses it was found that the lowest order approximation in Section 4.2 failed to account for the non-zero mean value of the change in energy of the binary observed in the numerical data. The theory is corrected in Section 4.4 using second order perturbation theory.

Sections 3 and 4.2 largely follow an unpublished manuscript by Haddow and Heggie (1996) but with more complete numerical results. Sections 4.1 and 4.4 are entirely new.

2. Preliminaries

Let m_1 and m_2 be the masses of the components of the binary, and m_3 be the mass of the third star approaching the binary from infinity. (We now consider general masses.) Let $\mathbf{r}$ be the position of star 2 relative to star 1 and let $\mathbf{R}$ be the position of star 3 relative to the barycentre of the binary. The situation is illustrated in Figure 1.

Notation: G is the universal constant of gravitation, $M_{12} = m_1 + m_2$, $M_{123} = m_1 + m_2 + m_3$, and $\mu_i = m_i/M_{12} (i = 1, 2)$ so that $\mu_1 + \mu_2 = 1$.

The equations governing the relative motion of these three stars are easily derived from the equations of motion of the three-body problem, and may be written in the form

$$\ddot{\mathbf{r}} = -GM_{12}\frac{\mathbf{r}}{r^3} + Gm_3\frac{\partial\mathcal{R}}{\partial\mathbf{r}}, \tag{2}$$

$$\ddot{\mathbf{R}} = GM_{123}\mu_1\mu_2\frac{\partial\mathcal{R}}{\partial\mathbf{R}}, \tag{3}$$

where $\mathcal{R}$ is the *perturbing function* given by

$$\mathcal{R} = \frac{1}{\mu_2|\mathbf{R} + \mu_2\mathbf{r}|} + \frac{1}{\mu_1|\mathbf{R} - \mu_1\mathbf{r}|}.$$

Strictly, $\mathcal{R}$ is *proportional* to the perturbing function, as this term is usually defined in celestial mechanics. (Since the third body moves on a *distant* parabolic orbit we may assume that $|\mathbf{R}| \gg |\mathbf{r}|$.) This can be expanded in terms of Legendre polynomials (Plummer, 1960; Heggie, 1975) to give[1]

$$\mathcal{R} = \frac{1}{R}\left(\frac{1}{\mu_2} + \frac{1}{\mu_1}\right) + \frac{1}{2}\frac{r^2}{R^3}\left(3\left(\frac{\mathbf{r}\cdot\mathbf{R}}{rR}\right)^2 - 1\right) + \mathrm{O}\left(\frac{r^3}{R^4}\right). \tag{4}$$

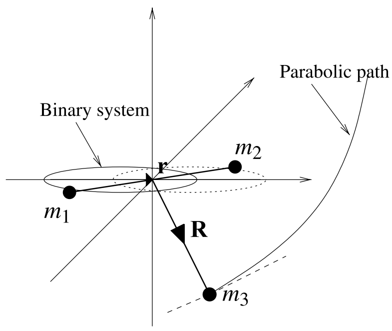

Figure 1. The frame of the barycentre of the binary showing the approximate elliptical motion of bodies m_1, m_2 and the approximate parabolic path of the third star.

[1] In many applications $\mathcal{R}$ would include a term $-1/\mu_1\mu_2 R$ (so that Eq. (3) takes the form of perturbed Kepler motion) and different mass factors.

The energy of the binary, ε, is given by

$$\varepsilon = -\frac{Gm_1m_2}{r} + \frac{m_1m_2}{2M_{12}}\dot{\mathbf{r}}^2.$$

Its derivative with respect to time is

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{m_1m_2}{M_{12}}\left(GM_{12}\frac{\mathbf{r}}{r^3} + \ddot{\mathbf{r}}\right)\cdot\dot{\mathbf{r}} = \frac{Gm_1m_2m_3}{M_{12}}\dot{\mathbf{r}}\cdot\frac{\partial\mathcal{R}}{\partial\mathbf{r}} \quad \text{(by (2))}. \tag{5}$$

The total change in the energy of the binary over the duration of the encounter is found by integrating Equation 5 from $-\infty$ to ∞ with respect to time t. This means that

$$\delta\varepsilon = \int_{-\infty}^{\infty}\frac{Gm_1m_2m_3}{M_{12}}\dot{\mathbf{r}}\cdot\frac{\partial\mathcal{R}}{\partial\mathbf{r}}\,\mathrm{d}t. \tag{6}$$

3. Binaries with Non-circular Orbits

3.1. THEORY

The integration in Equation (6) may be transformed by noting that

$$\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\cdot\frac{\partial\mathcal{R}}{\partial\mathbf{r}} + \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\cdot\frac{\partial\mathcal{R}}{\partial\mathbf{R}}.$$

Equation (6) is therefore equivalent to

$$\delta\varepsilon = \frac{Gm_1m_2m_3}{M_{12}}\int_{-\infty}^{\infty}\left[\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}t} - \dot{\mathbf{R}}\cdot\frac{\partial\mathcal{R}}{\partial\mathbf{R}}\right]\mathrm{d}t.$$

But $\mathcal{R} \to 0$ as $t \to \pm\infty$ and so

$$\delta\varepsilon = -\frac{Gm_1m_2m_3}{M_{12}}\int_{-\infty}^{\infty}\dot{\mathbf{R}}\cdot\frac{\partial\mathcal{R}}{\partial\mathbf{R}}\,\mathrm{d}t. \tag{7}$$

In the first instance we use a zero-order approximation to $\mathcal{R}$ to evaluate the above integral. If the perturbation is ignored in Equation (2), this has the solution

$$\mathbf{r} = a(\cos E - e)\hat{\mathbf{a}} + b\sin E\,\hat{\mathbf{b}} \tag{8}$$

(cf. Plummer, 1960), where a, b are the lengths of the semi-axes of the orbit, $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors parallel to them, the eccentricity $e \equiv (a^2 - b^2)^{1/2}/a$, and E is the eccentric anomaly. It is related to the time t by Kepler's equation which is

$$n(t - t_0) = E - e\sin E, \tag{9}$$

where t_0 is constant and n (the mean motion) is given by

$$n^2a^3 = GM_{12}. \tag{10}$$

Similarly, substitution of the first term of Equation (4) into Equation (3) gives

$$\ddot{\mathbf{R}} = -GM_{123}\frac{\mathbf{R}}{R^3},$$

which (assuming a parabolic orbit for the third body) has the solution

$$\mathbf{R} = q(1 - \sigma^2)\hat{\mathbf{A}} + 2q\sigma\hat{\mathbf{B}}, \tag{11}$$

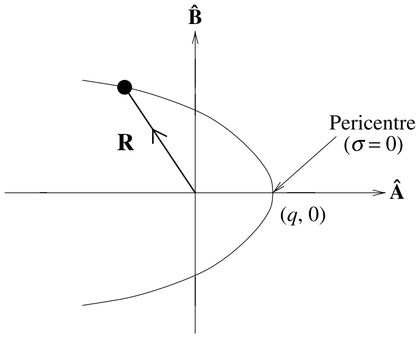

Figure 2. The orbital plane of motion of m_3 relative to the barycentre of the binary showing the orientation of the parabolic path of the third star with respect to the coordinate axes ($\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$).

where q is the periastron distance of the third star to the barycentre of the binary, and $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ are unit vectors in the plane of relative motion of the third body, as shown in Figure 2. The orientation of the parabola with respect to the axes is illustrated in Figure 2. σ, which is a dimensionless parameter specifying position on the parabolic orbit, is related to t by

$$\sqrt{\frac{GM_{123}}{2q^3}}t \approx \sigma + \frac{1}{3}\sigma^3. \tag{12}$$

(cf. Plummer, 1960). We shall assume that the masses are comparable and that $q \gg a$, that is, the distance of closest approach of the third body much exceed the semi-major axis of the binary. This also justifies neglect of higher-order terms in Equation (4).

The expressions for $\mathbf{r}$ and $\mathbf{R}$ in Equations (8) and (11) are now substituted into the expansion for

$$\frac{\partial \mathcal{R}}{\partial \mathbf{R}}$$

obtained from Equation (4).

Our goal is to compute the integral in Equation (7). It is easy to see that the first term in Equation (4) will vanish when inserted in Equation (7) and so we consider the second term in Equation (4).

Since $r = a(1 - e\cos E)$ the expression for $\mathcal{R}$ is given by

$$\begin{aligned}\mathcal{R} = \frac{1}{2R^5}\{&3[a^2(\cos E - e)^2(\hat{\mathbf{a}}\cdot\mathbf{R})^2 + 2ab(\cos E - e)\sin E(\hat{\mathbf{a}}\cdot\mathbf{R})(\hat{\mathbf{b}}\cdot\mathbf{R}) + \\ &+ b^2(\hat{\mathbf{b}}\cdot\mathbf{R})^2\sin^2 E] - R^2a^2(1 - e\cos E)^2\}.\end{aligned} \tag{13}$$

An infinite expansion for any term of the form $\sin mE$ or $\cos mE$ as a Fourier series in $M = n(t - t_0)$ may be obtained from Kepler's equation (9). For our purpose the following approximations will be used (cf. Plummer, 1960):

$$\begin{aligned} \sin E &\simeq \sin M(J_0(e) + J_2(e)), & \cos E &\simeq -\tfrac{1}{2}e + \cos M(J_0(e) - J_2(e)), \\ \sin 2E &\simeq 2\sin M(J_{-1}(e) + J_3(e)), & \cos 2E &\simeq 2\cos M(J_{-1}(e) - J_3(e)), \end{aligned} \tag{14}$$

where $J_p(e)$ is the Bessel function of order p and argument e. As mentioned by Heggie (1975) the additional terms of the expansions may be neglected since they oscillate more rapidly than $\sin M$ or $\cos M$ and it will be seen from what follows that their contribution to $\delta\varepsilon$ would be insignificant.

Equations (14) are now substituted into the expression for $\mathcal{R}$ in Equation (13). Terms which are independent of M may be ignored since they are perfect differentials of functions which vanish as $R \to \infty$. This yields the following expression for $\mathcal{R}$:

$$\begin{aligned} \mathcal{R} = \frac{1}{R^5}\Bigg\{&\left[\frac{3}{2}e_1 a^2(\hat{\mathbf{a}}\cdot\mathbf{R})^2 - \frac{3}{2}e_2 b^2(\hat{\mathbf{b}}\cdot\mathbf{R})^2 - \frac{1}{2}ee_3 a^2 R^2\right]\cos M + \\ &+ 3e_4 ab\,\hat{\mathbf{a}}\cdot\mathbf{R}\hat{\mathbf{b}}\cdot\mathbf{R}\sin M\Bigg\}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} e_1 &= J_{-1}(e) - 2eJ_0(e) + 2eJ_2(e) - J_3(e), \qquad e_2 = J_{-1}(e) - J_3(e), \\ e_3 &= eJ_{-1}(e) - 2J_0(e) + 2J_2(e) - eJ_3(e), \\ e_4 &= J_{-1}(e) - eJ_0(e) - eJ_2(e) + J_3(e). \end{aligned}$$

In order to compute the integral in Equation (7) it is necessary to evaluate

$$\dot{\mathbf{R}}\cdot\frac{\partial\mathcal{R}}{\partial\mathbf{R}}.$$

It can easily be shown from Equation (11) that

$$\mathbf{R}\cdot\dot{\mathbf{R}} = 2q^2(1+\sigma^2)\sigma\dot{\sigma} = 2qR\sigma\dot{\sigma}.$$

This gives

$$\begin{aligned} \dot{\mathbf{R}}\cdot\frac{\partial}{\partial\mathbf{R}}\left(\frac{(\hat{\mathbf{a}}\cdot\mathbf{R})^2}{R^5}\right) &= \frac{2\,\hat{\mathbf{a}}\cdot\mathbf{R}\,\hat{\mathbf{a}}\cdot\dot{\mathbf{R}}}{R^5} - \frac{5(\hat{\mathbf{a}}\cdot\mathbf{R})^2\,\mathbf{R}\cdot\dot{\mathbf{R}}}{R^7} \\ &= \frac{2\,\hat{\mathbf{a}}\cdot\mathbf{R}\,\hat{\mathbf{a}}\cdot\mathbf{R}'\dot{\sigma}}{R^5} - \frac{10(\hat{\mathbf{a}}\cdot\mathbf{R})^2 q\sigma\dot{\sigma}}{R^6} \\ &= \frac{(2\,\hat{\mathbf{a}}\cdot\mathbf{R}\,\hat{\mathbf{a}}\cdot\mathbf{R}'R - 10(\hat{\mathbf{a}}\cdot\mathbf{R})^2 q\sigma)\dot{\sigma}}{R^6}, \end{aligned} \tag{16}$$

where

$$R' = \frac{\mathrm{d}R}{\mathrm{d}\sigma}.$$

The other terms in Equation (15) are handled in similar fashion. Writing $\cos M = \mathrm{Re}(\mathrm{e}^{\mathrm{i}M}) = \mathrm{Re}(\mathrm{e}^{-\mathrm{i}nt_0}\,\mathrm{e}^{\mathrm{i}nt})$ and $\sin M = \mathrm{Re}(-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}nt_0}\,\mathrm{e}^{\mathrm{i}nt})$ we see that it is necessary to evaluate integrals of the form

$$I_\alpha = \int_{-\infty}^{\infty} \frac{e^{int}\sigma^\alpha}{(1+\sigma^2)^6}\dot{\sigma}\,dt,$$

where $\sigma(t)$ is given by Equation (12) and $\alpha = 0, 1, 2, \ldots, 5$. To make explicit the dependence on σ it is necessary to expand

$$\hat{\mathbf{a}}\cdot\mathbf{R} \text{ as } q(1-\sigma^2)\hat{\mathbf{a}}\cdot\hat{\mathbf{A}} + 2q\sigma\hat{\mathbf{a}}\cdot\hat{\mathbf{B}}, \qquad \hat{\mathbf{a}}\cdot\mathbf{R}' \text{ as } -2q\sigma\hat{\mathbf{a}}\cdot\hat{\mathbf{A}} + 2q\hat{\mathbf{a}}\cdot\hat{\mathbf{B}}$$

and similarly for $\hat{\mathbf{b}}\cdot\mathbf{R}$ and $\hat{\mathbf{b}}\cdot\mathbf{R}'$.

From Equations (12) and (10) it is clear that

$$nt = K\left(\sigma + \frac{1}{3}\sigma^3\right), \quad \text{where } K = \sqrt{\frac{2M_{12}q^3}{M_{123}a^3}}.$$

The integral, I_α, is then given by

$$I_\alpha = \int_{-\infty}^{\infty} \frac{e^{iK(\sigma+(1/3)\sigma^3)}\sigma^\alpha}{(1+\sigma^2)^6}\,d\sigma.$$

Three integrations by parts, bearing in mind that K is very large ($q \gg a$ implies that $K \gg 1$), permits the following approximation:

$$I_\alpha \simeq \frac{(iK)^3}{10\times 6\times 2}\int_{-\infty}^{\infty} \sigma^{\alpha-3}\,e^{iK(\sigma+(1/3)\sigma^3)}\,d\sigma.$$

This integral can be evaluated using the method of steepest descents (cf. Heggie, 1975). The relevant saddle point is at $\sigma = i$, and if we write $\sigma = i + U$ we find that

$$I_\alpha \simeq \frac{(iK)^3}{120}\int_{-\infty}^{\infty} e^{iKi(2/3+U^2)}\,i^{\alpha-3}\,dU.$$

This then yields

$$I_\alpha \simeq \frac{i^\alpha\sqrt{\pi}}{120}K^{5/2}\,e^{-(2/3)K} \quad (K\to\infty). \tag{17}$$

Substituting Equation (15) into Equation (7) and using Equation (17) gives

$$\begin{aligned}\delta\varepsilon \simeq & -\frac{Gm_1m_2m_3}{M_{12}q^3}\frac{\sqrt{\pi}}{120}K^{5/2}\,e^{-(2/3)K}\{60\,a^2e_1(\sin nt_0(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})^2 + \\ & + 2\cos nt_0\,\hat{\mathbf{a}}\cdot\hat{\mathbf{A}}\hat{\mathbf{a}}\cdot\hat{\mathbf{B}} - \sin nt_0(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})^2) + \\ & + 120\,ab\,e_4(-\sin nt_0\,\hat{\mathbf{b}}\cdot\hat{\mathbf{B}}\,\hat{\mathbf{a}}\cdot\hat{\mathbf{A}} - \sin nt_0\,\hat{\mathbf{a}}\cdot\hat{\mathbf{B}}\,\hat{\mathbf{b}}\cdot\hat{\mathbf{A}} + \\ & + \cos nt_0\,\hat{\mathbf{a}}\cdot\hat{\mathbf{B}}\,\hat{\mathbf{b}}\cdot\hat{\mathbf{B}} - \\ & - \cos nt_0\,\hat{\mathbf{a}}\cdot\hat{\mathbf{A}}\,\hat{\mathbf{b}}\cdot\hat{\mathbf{A}}) + 60\,b^2e_2(\sin nt_0(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^2 - \\ & - 2\cos nt_0\,\hat{\mathbf{b}}\cdot\hat{\mathbf{B}}\,\hat{\mathbf{b}}\cdot\hat{\mathbf{A}} - \sin nt_0(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^2)\}\end{aligned}$$

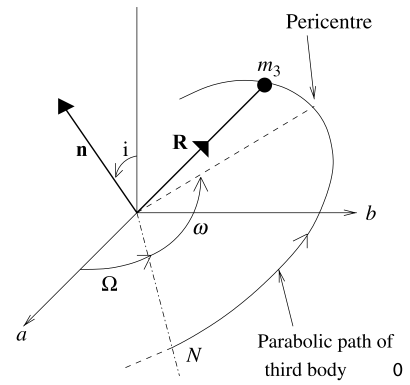

Figure 3. The angles describing the orientation of the parabolic orbit. Ω, ω, and i are the longitude of the ascending node, the argument of pericentre and the inclination, respectively; *ON* shows the line in which the plane of motion of the third body intersects the plane of motion of the binary; **n** is the vector perpendicular to the plane of motion of the third body.

(A simple way to arrive at this expression from Equation (16) is to replace the denominator in Equation (16) by q^6, to replace $\mathbf{R}$ and $\mathbf{R}'$ by their values at the saddle point $\sigma = \mathrm{i}$, that is, $\mathbf{R} = 2q(\hat{\mathbf{A}} + \mathrm{i}\hat{\mathbf{B}})$ and $\mathbf{R}' = 2q(-\mathrm{i}\hat{\mathbf{A}} + \hat{\mathbf{B}})$, and to multiply by $(\sqrt{\pi}/120)K^{5/2}\,\mathrm{e}^{-2K/3}$.)

It is convenient to introduce the angles ω, Ω and i to describe the parabolic path of the third body. This is illustrated in Figure 3. If a frame of reference is chosen such that $\hat{\mathbf{a}} = (1, 0, 0)$ and $\hat{\mathbf{b}} = (0, 1, 0)$ then (cf. Plummer, 1960)

$$\begin{aligned}\hat{\mathbf{A}} &= (\cos\Omega\cos\omega - \cos i\sin\Omega\sin\omega,\ \sin\Omega\cos\omega + \\ &\quad + \cos i\cos\Omega\sin\omega,\ \sin i\sin\omega),\\ \hat{\mathbf{B}} &= (-\cos\Omega\sin\omega - \cos i\sin\Omega\cos\omega, -\sin\Omega\sin\omega + \\ &\quad + \cos\omega\cos i\cos\Omega, \sin i\cos\omega).\end{aligned} \tag{18}$$

Thus,

$$\begin{aligned}\delta\varepsilon \simeq\ & -\frac{Gm_1m_2m_3}{M_{12}q^3}\frac{\sqrt{\pi}}{8}K^{5/2}\,\mathrm{e}^{-(2/3)K}\{e_1a^2[\sin(2\omega + nt_0)(\cos 2i - 1) - \\ & - \sin(2\omega + nt_0)\cos(2i)\cos(2\Omega) - 3\sin(nt_0 + 2\omega)\cos(2\Omega) - \\ & - 4\sin(2\Omega)\cos(2\omega + nt_0)\cos i] + e_2b^2[\sin(2\omega + nt_0)\times \\ & \times (1 - \cos 2i) - \sin(2\omega + nt_0)\cos(2i)\cos(2\Omega) - 3\sin(nt_0 + 2\omega)\times \\ & \times \cos(2\Omega) - 4\cos(nt_0 + 2\omega)\sin(2\Omega)\cos i] + \\ & + e_4\,ab[-2\cos(2i)\cos(2\omega + nt_0)\sin(2\Omega) - \\ & -6\cos(2\omega + nt_0)\sin(2\Omega) - 8\cos(2\Omega)\sin(2\omega + nt_0)\cos i]\}.\end{aligned} \tag{19}$$

In the coplanar case when $i = 0$, Equation (5) reduces to

$$\delta\varepsilon \simeq \frac{Gm_1m_2m_3}{M_{12}q^3}\frac{\sqrt{\pi}}{2}K^{5/2}\,\mathrm{e}^{-(2K/3)}(e_1a^2 + 2e_4ab + e_2b^2) \times \\ \times \sin(2\omega + 2\Omega + nt_0).$$

The following comments may be made about Equation (5):

(i) The exponential term involves the ratio of the mean motion of the third body to the mean motion of the binary. This is characteristic of the behaviour of adiabatic invariants (cf. Goldstein, 1980).

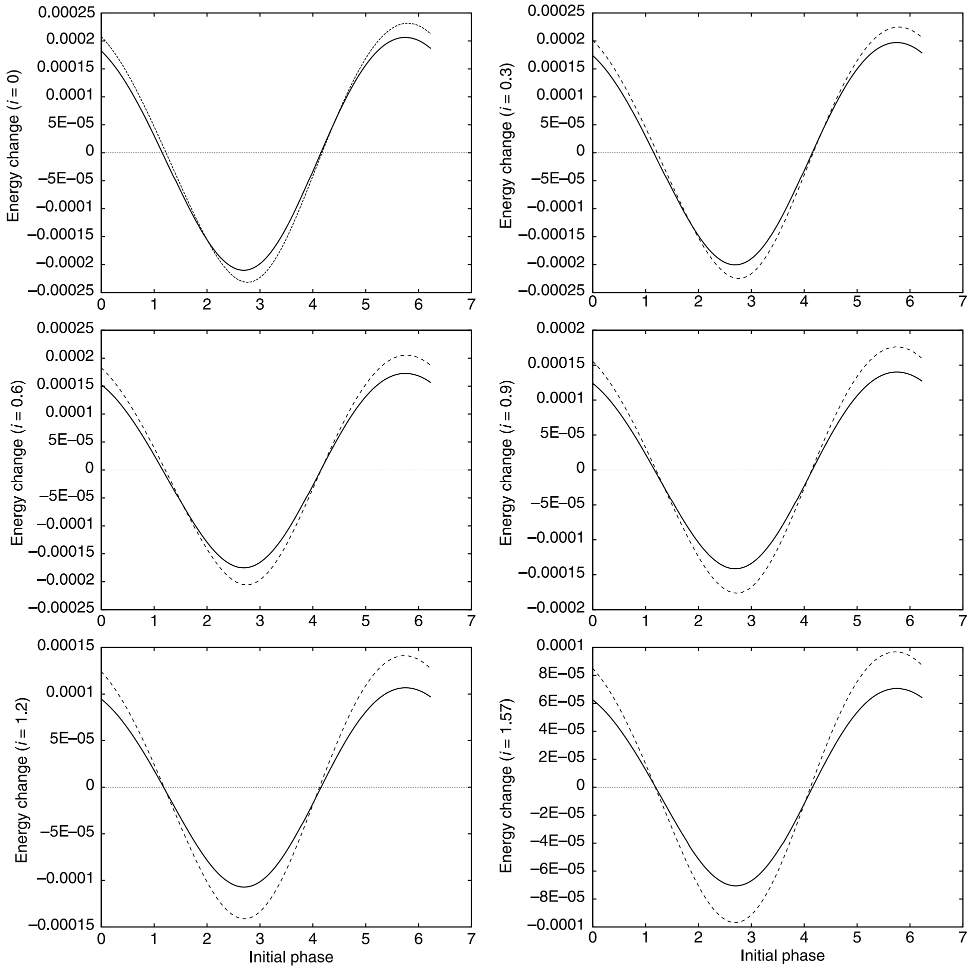

Figure 4. $\delta\varepsilon$ against the initial phase of the binary for various angles of inclination. For each value of i (given in radians on the ordinate axis) a comparison is made between the numerical results (continuous curves) and the theoretical prediction (dotted curves), obtained from Equation (5); $q = 5$ and initially $R = 80$, $e = 0.1$.

(ii) The influence of higher frequency terms such as sin $2M$ and cos $2M$ has been neglected. These give exponentials which decay even more rapidly with increasing q (i.e., increasing K).

(iii) The derivation of exponentially small terms lacks rigour, because many terms in the expansion (e.g., higher-order perturbations), which could dominate the exponential result, have been neglected. Our justification for this is based on analogous problems where the results can be justified rigorously (cf. Goldstein), and numerical results (see below).

3.2. COMPARISON WITH NUMERICAL RESULTS

Equation (5) has been tested against the results obtained from numerical scattering experiments in the case of equal masses $m_1 = m_2 = m_3 = 1$. We use units such

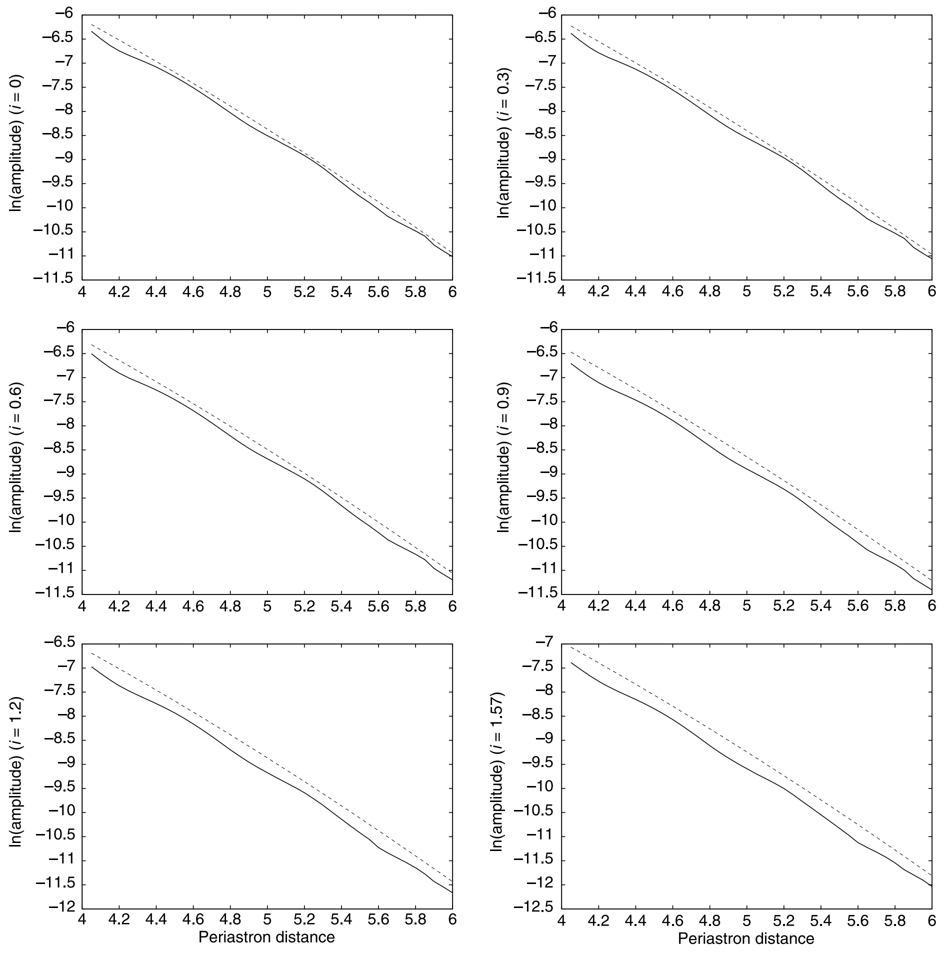

Figure 5. $\ln(\delta\varepsilon)$ against the periastron distance, q, for various angles of inclination. The numerical results (continuous curves), are obtained using, initially, $e = 0.1$ and $R = 80$. The dotted curves are obtained from the analytical theory.

that $G = 1$. Initially $a = 1$ and $\Omega = 0$, and the third body lies at a sufficiently large distance (80 units) from the barycentre of the binary so that its influence on the binary is insignificant. The input parameters are then i, q and the initial values of e, E. An appropriate impact parameter is calculated using a Keplerian approximation. The integration was performed using the subroutine ODEINT from Numerical Recipes (Press et al., 1992). The integration was stopped when the distance of m_3 from the barycentre of the binary exceeds 80. At this point the new semi-major axis is computed and hence the energy change. The program also computes the argument of pericentre, ω and the mean anomaly of the binary, nt_0 when the third body is at pericentre. These quantities are required so that the theoretical formula may be applied.

The results shown in Figure 4 show the change in energy as a function of the initial phase, E_{init}, of the binary for various inclinations. For any q, the amplitude of the energy change can be calculated by fitting a sinusoid through the data. Figure 5 shows the natural log of the amplitude of $\delta\varepsilon$ against the periastron distance q, for various angles of inclination. The difference between theoretical and numerical results may be attributed to neglect of higher-order terms in Equation (4) and other sources.

4. Binaries with Circular Orbits

Circular binaries are particularly important since they occur frequently in globular clusters. The foregoing theory gives a null result in this special case, and we now give a rederivation including the dominant terms.

4.1. CIRCULAR ORBITS/NON-EQUAL MASSES

In order to find the dominant contribution to the energy we are forced to look at the third term in the expansion of the perturbing function $\mathcal{R}$ in Equation (4). (We shall see in Section 4.2 that the second term in the expansion of the perturbing function gives a less significant contribution due to integrations involving $\sin 2M$, $\cos 2M$.) This term is given by

$$-\frac{r^3}{R^4}\frac{1}{2}\left[5\left(\frac{\mathbf{r}\cdot\mathbf{R}}{rR}\right)^3 - 3\left(\frac{\mathbf{r}\cdot\mathbf{R}}{rR}\right)\right](\mu_2-\mu_1).$$

For a circular orbit the eccentricity $e = 0$ and the expression for $\mathbf{r}$ is given by

$$\mathbf{r} = a\cos E\,\hat{\mathbf{a}} + a\sin E\,\hat{\mathbf{b}},$$

where $E = M$. Substituting this into the expression for $\mathcal{R}$ gives

$$\begin{aligned}\mathcal{R} \;=\; \frac{a^3}{2R^4}\Bigg[&\frac{5}{R^3}(\cos^3 E(\hat{\mathbf{a}}\cdot\mathbf{R})^3 + 3\cos^2 E\sin E(\hat{\mathbf{a}}\cdot\mathbf{R})^2(\hat{\mathbf{b}}\cdot\mathbf{R}) + \\ &+ 3\sin^2 E\cos E(\hat{\mathbf{a}}\cdot\mathbf{R})(\hat{\mathbf{b}}\cdot\mathbf{R})^2 + \sin^3 E(\hat{\mathbf{b}}\cdot\mathbf{R})^3) - \\ &- \frac{3}{R}(\cos E(\hat{\mathbf{a}}\cdot\mathbf{R}) + \sin E(\hat{\mathbf{b}}\cdot\mathbf{R}))\Bigg](\mu_2-\mu_1).\end{aligned}$$

This gives

$$\mathcal{R} = \Bigg[\left(\frac{15a^3(\hat{\mathbf{a}}\cdot\mathbf{R})^3}{8R^7} + \frac{15a^3(\hat{\mathbf{a}}\cdot\mathbf{R})(\hat{\mathbf{b}}\cdot\mathbf{R})^2}{8R^7} - \frac{3a^3(\hat{\mathbf{a}}\cdot\mathbf{R}}{2R^5}\right)\cos E + \\ + \left(\frac{15a^3(\hat{\mathbf{a}}\cdot\mathbf{R})^2(\hat{\mathbf{b}}\cdot\mathbf{R})}{8R^7} + \frac{15a^3(\hat{\mathbf{b}}\cdot\mathbf{R})^3}{8R^7} - \frac{3a^3(\hat{\mathbf{b}}\cdot\mathbf{R})}{2R^5}\right)\sin E\Bigg] \times \\ \times (\mu_2 - \mu_1) + \text{ terms of higher frequency.} \tag{20}$$

The first integral is evaluated asymptotically by applying Equation (A.8) in Appendix A:

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}}\cdot\frac{\partial}{\partial\mathbf{R}}\left(\frac{15a^3(\hat{\mathbf{a}}\cdot\mathbf{R})^3\cos E}{8R^7}\right)\mathrm{d}t \\ = -\frac{15a^3}{8q^4}\frac{\sqrt{\pi}}{120}K^{7/2}\,\mathrm{e}^{-(2/3)K}(8(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})^3\sin nt_0 - 24(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})^2(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})\cos nt_0 - \\ - 24\sin nt_0(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})^2 + 8\cos nt_0(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})^3). \tag{21}$$

The next integral is obtained by applying Equation (A.12):

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}}\cdot\frac{\partial}{\partial\mathbf{R}}\left(\frac{15a^3(\hat{\mathbf{a}}\cdot\mathbf{R})(\hat{\mathbf{b}}\cdot\mathbf{R})^2\cos E}{8R^7}\right)\mathrm{d}t \\ = -\frac{-a^3}{8q^4}\sqrt{\pi}K^{7/2}\,\mathrm{e}^{-(2K/3)}(\sin nt_0(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^2 - \\ - 2\cos nt_0(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}}) - \cos nt_0(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^2 - \\ - 2\sin nt_0(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}}) - \sin nt_0(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^2 + \\ + \cos nt_0(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^2(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})). \tag{22}$$

By Equation (A.1), where the distinction between E and M is immaterial, the integral

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}}\cdot\frac{\partial}{\partial\mathbf{R}}\left(\frac{-3a^3(\mathbf{a}\cdot\mathbf{R})\cos E}{2R^5}\right)\mathrm{d}t \\ = \frac{-a^3\sqrt{\pi}}{4q^4}K^{5/2}\,\mathrm{e}^{-2K/3}(-\sin nt_0(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}}) + \cos nt_0(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})). \tag{23}$$

Notice that this is one order smaller in K, and will be neglected.

The contribution from the terms $\sim\sin E$ is given as follows. From Equation (A.11) it is found that

$$\int_{-\infty}^{\infty} \frac{15a^3}{8}\dot{\mathbf{R}}\cdot\frac{\partial}{\partial\mathbf{R}}\left(\frac{(\hat{\mathbf{b}}\cdot\mathbf{R})^3}{R^7}\right)\sin E\,\mathrm{d}t \\ = \frac{-a^3}{q^4}\frac{\sqrt{\pi}}{8}K^{7/2}\,\mathrm{e}^{-2K/3}(\cos nt_0(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^3 + 3\sin nt_0(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^2(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}}) - \\ - 3\cos nt_0(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^2 - \sin nt_0(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^3). \tag{24}$$

From Equation (A.13) the integral

$$\int_{-\infty}^{\infty} \frac{15a^3}{8} \dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}} \left(\frac{(\hat{\mathbf{a}} \cdot \mathbf{R})^2 (\hat{\mathbf{b}} \cdot \mathbf{R})}{R^7} \right) \sin E \, \mathrm{d}t$$
$$= \frac{-a^3}{8q^4} \sqrt{\pi} K^{7/2} \, \mathrm{e}^{-2K/3} (\cos nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})^2 +$$
$$+ 2 \sin nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{B}}) + \sin nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{B}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})^2 -$$
$$- 2 \cos nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{B}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{B}}) - \cos nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{B}})^2 -$$
$$- \sin nt_0 (\hat{\mathbf{a}} \cdot \hat{\mathbf{B}})^2 (\hat{\mathbf{b}} \cdot \hat{\mathbf{B}})). \tag{25}$$

Applying Equation (A.4) the integral

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}} \left(\frac{-3a^3 (\hat{\mathbf{b}} \cdot \mathbf{R}) \sin E}{2R^5} \right) \mathrm{d}t$$

is of lower order in K. The faster oscillating terms $\sim \sin 3E$, $\cos 3E$ may be ignored since their contribution to the energy change is exponentially smaller. This means that $\delta\varepsilon$ is given by (combining the results for Equations (21), (22), (24) and (25))

$$\delta\varepsilon \simeq \frac{Gm_1 m_2 m_3}{M_{12}} \frac{\sqrt{\pi} a^3}{8q^4} K^{7/2} \, \mathrm{e}^{-2K/3} (\mu_2 - \mu_1) [\sin nt_0 (\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})^3 -$$
$$- 3 \cos nt_0 (\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})^2 (\hat{\mathbf{a}} \cdot \hat{\mathbf{B}}) - 3 \sin nt_0 (\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{B}})^2 + \cos nt_0 (\hat{\mathbf{a}} \cdot \hat{\mathbf{B}})^3 +$$
$$+ \sin nt_0 (\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})^2 - 2 \cos nt_0 (\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{b}} \cdot \hat{\mathbf{B}}) -$$
$$- \cos nt_0 (\hat{\mathbf{a}} \cdot \hat{\mathbf{B}})(\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})^2 - 2 \sin nt_0 (\hat{\mathbf{a}} \cdot \hat{\mathbf{B}})(\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{b}} \cdot \hat{\mathbf{B}}) -$$
$$- \sin nt_0 (\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{b}} \cdot \hat{\mathbf{B}})^2 + \cos nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{B}})^2 (\hat{\mathbf{a}} \cdot \hat{\mathbf{B}}) +$$
$$+ \cos nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})^3 + 3 \sin nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})^2 (\hat{\mathbf{b}} \cdot \hat{\mathbf{B}}) - 3 \cos nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{b}} \cdot \hat{\mathbf{B}})^2 -$$
$$- \sin nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{B}})^3 + \cos nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})^2 +$$
$$+ 2 \sin nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{B}}) + \sin nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{B}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})^2 -$$
$$- 2 \cos nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{B}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{B}}) - \cos nt_0 (\hat{\mathbf{b}} \cdot \hat{\mathbf{A}})(\hat{\mathbf{a}} \cdot \hat{\mathbf{B}})^2 -$$
$$- \sin nt_0 (\hat{\mathbf{a}} \cdot \hat{\mathbf{B}})^2 (\hat{\mathbf{b}} \cdot \hat{\mathbf{B}})] + \text{terms smaller by a factor of order } 1/\mathrm{K}.$$

We choose axes such that the longitude of the ascending node of the orbit of the third body, relative to the plane of motion of the binary, is $\Omega = 0$. Since the binary is initially circular we may also take $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ anywhere in its plane of motion, and therefore take

$$\hat{\mathbf{a}} = (1, 0, 0), \qquad \hat{\mathbf{b}} = (0, 1, 0), \qquad \hat{\mathbf{A}} = (\cos\omega, \ \cos i \sin\omega, \ \sin i \sin\omega),$$
$$\hat{\mathbf{B}} = (-\sin\omega, \ \cos i \cos\omega, \sin i \cos\omega).$$

The expression for $\delta\varepsilon$ may be simplified to give

$$\delta\varepsilon \simeq \frac{Gm_1 m_2 m_3}{M_{12}} \frac{\sqrt{\pi} a^3}{8q^4} K^{7/2} \, \mathrm{e}^{-2K/3} (\mu_2 - \mu_1)(1 + \cos i) \sin^2 i \times$$
$$\times [(\cos^3\omega - 3 \sin^2\omega \cos\omega) \sin nt_0 + (3 \cos^2\omega \sin\omega - \sin^3\omega) \cos nt_0].$$

4.2. CIRCULAR ORBITS/EQUAL MASSES

Unfortunately, the theory must be rederived in the case when the masses of the components of the binary are equal ($\mu_1 = \mu_2$). As before, lowest order perturbation theory is used and in the case of a circular binary $E = M$ from Equation (10). Then all the 'odd' terms in the expansion of Equation (4) vanish, that is, those with factors $(1/R)(r/R)^n$ with n odd. The dominant contribution is obtained when $n = 2$ from terms in $\cos 2M$ and $\sin 2M$. This is because all the terms involving $\sin M$ and $\cos M$ in the expansion of Equation (13) disappear when $e = 0$.

In performing the integral in Equation (7) integrals of the following form require to be evaluated:

$$I_\alpha = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{2\mathrm{i}K(\sigma+(1/3)\sigma^3)}\sigma^\alpha}{(1+\sigma^2)^6}\,\mathrm{d}\sigma,$$

where $\alpha = 0, 1, 2, \ldots, 5$.

By the methods of Section 3.1 it is found that

$$\begin{aligned} I_\alpha &\simeq \frac{(2\mathrm{i}K)^3}{10\times 6\times 2}\mathrm{e}^{-(4/3)K}\sqrt{\frac{\pi}{2K}}i^{\alpha-3} \\ &= \frac{i^\alpha}{15}\sqrt{\frac{\pi}{2}}\,\mathrm{e}^{-(4/3)K}\,K^{5/2} \quad (K\to\infty). \end{aligned}$$

Hence, using Equations (18)

$$\delta\varepsilon \simeq \frac{8Gm_1m_2m_3\sqrt{2\pi}}{aM_{123}}\,\mathrm{e}^{-(4/3)K}\,K^{1/2}\sin(2\omega+2\Omega+2nt_0)\cos^4\frac{i}{2}. \qquad (26)$$

4.3. COMPARISON WITH NUMERICAL RESULTS

Equation (26) has been tested numerically in the case when $i = 0$. The same numerical procedure is used as before in Section 3.2 but in this case $e_{\text{init}} = 0$ since the binary is initially circular. Figure 6 compares the numerical results with the analytical theory in Equation (26). The sinusoidal component of Equation (26) is in phase with the numerical results and has nearly the correct amplitude but it fails to account for the non-zero mean which is observed.

4.4. SECOND-ORDER PERTURBATION CALCULATION

We show in this section that the non-zero mean can be explained in terms of second-order perturbation theory, at least in the coplanar case. (A similar non-zero mean arises in the non-coplanar case but the following theory is restricted to the case $i = 0$ for simplicity.) The following argument explains why this might be expected.

We have already seen, by comparing Equations (5) and (26), that the change of energy in an eccentric binary is much greater than in a circular binary, in the sense

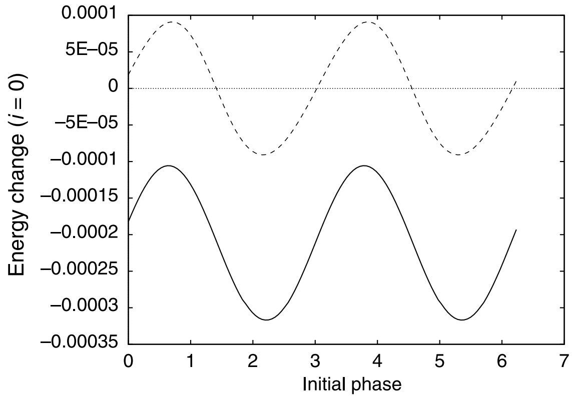

Figure 6. The numerical result (continuous curve) compared with the analytical theory (dotted curve) in Equation (26) when the binary motion is circular ($q = 4$).

that $e^{-2K/3} \gg e^{-4K/3}$ for $K \gg 1$. But these results are based on first-order perturbation theory. At second order we have to allow for the fact that the passing star induces a tiny eccentricity on the binary. This may induce a much larger mean energy change than was obtained from first-order theory.

The Hamiltonian for the binary is a periodic function of the mean anomaly of the binary and is given by

$$\mathcal{H} = -\frac{\mu^2 m^3}{2L^2} + H',$$

where

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

and we use Delaunay's variables defined as follows (cf. Plummer, 1960):

$$\begin{aligned} L &= m\sqrt{G(m_1+m_2)a}, \quad l = nt + M_0, \\ G &= m\sqrt{G(m_1+m_2)a(1-e^2)}, \quad g = \omega, \\ H &= m\sqrt{G(m_1+m_2)a(1-e^2)}\cos i, \quad h = \Omega. \end{aligned} \tag{27}$$

We concentrate attention on the Delaunay variable, L, which is directly related to the energy of the binary. Its conjugate variable l is the only 'fast' variable in the problem. In the coplanar problem (since we are examining the case $i = 0$) the Delaunay variables H and h may be ignored. Without changing L, the variables (L, G; l, g) may be transformed to a new set (L, ρ_1; λ, ω_1), where

$$\rho_1 = L - G, \qquad \omega_1 = -g, \qquad \lambda = l + g.$$

These are still canonical, L and ρ_1 being conjugate to λ and ω_1, respectively. Since g is ill-defined when e becomes very small, it is better to use the eccentric variables x, y instead of ρ_1 and ω_1. The eccentric variables are particularly well suited to the study of motions with small eccentricity. Both the eccentric and Delaunay variables are useful in celestial mechanics because they form a set of canonical variables which incorporate the orbital elements of the binary. The canonical momenta have physical meaning: L is associated with the energy of the binary and G gives the angular momentum; l is the mean motion and g gives the argument of pericentre. The eccentric variables are related to the Delaunay variables by

$$x = \sqrt{2\rho_1}\cos\omega_1 \simeq \sqrt{L}e\cos g, \qquad y = -\sqrt{2\rho_1}\sin\omega_1 \simeq \sqrt{L}e\sin g$$

and, being canonical, they satisfy

$$\dot{x} = \frac{\partial H'}{\partial y}, \qquad \dot{y} = -\frac{\partial H'}{\partial x}. \tag{28}$$

H' is then a periodic function of the mean longitude, λ, and may be expanded as follows:

$$H' = \sum_{k=-\infty}^{\infty} c_k(L, x, y)\,\mathrm{e}^{\mathrm{i}k\lambda},$$

where x and y are the eccentric variables.

To explain the non-zero mean a contribution to the energy change is sought which is independent of the phase. This is the only result we shall seek. Roughly speaking, this contribution is found (in second-order perturbation theory) by combining the terms in $\delta\varepsilon$ corresponding to the coefficients c_{-1} and c_1 in the expansion of H', so that the oscillatory components cancel. Because the calculation is complicated enough, and because it is narrowly focussed on a single result, we retain only the most significant relevant terms and ignore those which contribute only oscillatory corrections and those which are of relatively negligible order in the small expansion parameters r/R (equivalently, K^{-1}) and e.

From Equation (28)

$$\dot{x} \simeq \frac{\partial c_{-1}}{\partial y}\,\mathrm{e}^{-\mathrm{i}\lambda} + \frac{\partial c_1}{\partial y}\,\mathrm{e}^{\mathrm{i}\lambda} \quad \text{and} \quad \dot{y} \simeq -\frac{\partial c_{-1}}{\partial x}\,\mathrm{e}^{-\mathrm{i}\lambda} - \frac{\partial c_1}{\partial x}\,\mathrm{e}^{\mathrm{i}\lambda},$$

the approximate equality implying that we ignore all irrelevant terms.

Using first order perturbation theory (cf. Born, 1960; Goldstein, 1980)

$$x_1 \simeq \int_{-\infty}^{t} \frac{\partial c_{-1}}{\partial y}\,\mathrm{e}^{-\mathrm{i}\lambda_0}\,\mathrm{d}t + \int_{-\infty}^{t} \frac{\partial c_1}{\partial y}\,\mathrm{e}^{\mathrm{i}\lambda_0}\,\mathrm{d}t$$

and

$$y_1 \simeq -\int_{-\infty}^{t} \frac{\partial c_{-1}}{\partial x}\,\mathrm{e}^{-\mathrm{i}\lambda_0}\,\mathrm{d}t - \int_{-\infty}^{t} \frac{\partial c_1}{\partial x}\,\mathrm{e}^{\mathrm{i}\lambda_0}\,\mathrm{d}t, \tag{29}$$

where subscripts 0 and 1 on the variables x, y and λ denote terms at zeroth and first-order in a perturbation expansion and the partial derivatives

$$\frac{\partial c_{-1}}{\partial y}, \qquad \frac{\partial c_1}{\partial y}, \qquad \frac{\partial c_{-1}}{\partial x}, \qquad \frac{\partial c_1}{\partial x}$$

are evaluated for the unperturbed motion $L = L_0$, $x = x_0 = 0$, $y = y_0 = 0$.

The rate of change of L to second order is,

$$\dot{L}_2 = -\frac{\partial H'}{\partial \lambda},$$

where the partial derivative is evaluated at

$$L = L_0 + L_1, \qquad \lambda = \lambda_0 + \lambda_1, \qquad x = x_0 + x_1, \qquad y = y_0 + y_1.$$

This gives

$$\begin{aligned}
\dot{L}_2 &= \sum_{k=\pm 1} -\mathrm{i}kc_k(L_0 + L_1, x_0 + x_1, y_0 + y_1)\,\mathrm{e}^{\mathrm{i}k(\lambda_0+\lambda_1)} \\
&= \sum_{k=\pm 1} -\mathrm{i}kc_k(L_0, x_0, y_0)\,\mathrm{e}^{\mathrm{i}k\lambda_0} - \sum_{k=\pm 1} \mathrm{i}k\left(L_1\frac{\partial c_k}{\partial L} + \mathrm{i}k\lambda_1 c_k\right)\mathrm{e}^{\mathrm{i}k\lambda_0} + \\
&\quad + \sum_{k=\pm 1} -\mathrm{i}k\left(x_1\frac{\partial c_k}{\partial x} + y_1\frac{\partial c_k}{\partial y}\right)\mathrm{e}^{\mathrm{i}k\lambda_0}.
\end{aligned} \tag{30}$$

The partial derivatives in Equation (30) are evaluated at the zero-order solution. On integrating the first term, the first-order contribution to L and hence the first-order approximation for the change in energy of the binary is obtained. This has been previously calculated in Equation (26). We shall see that the second term vanishes. Finally, integration of the last term leads to a phase independent contribution which is calculated as follows:

$$\begin{aligned}
\dot{L}_2 &\simeq \sum_{k=\pm 1} -\mathrm{i}k\left(x_1\frac{\partial c_k}{\partial x} + y_1\frac{\partial c_k}{\partial y}\right)\mathrm{e}^{\mathrm{i}k\lambda_0} \\
&= \mathrm{i}\left(x_1\frac{\partial c_{-1}}{\partial x} + y_1\frac{\partial c_{-1}}{\partial y}\right)\mathrm{e}^{-\mathrm{i}\lambda_0} - \mathrm{i}\left(x_1\frac{\partial c_1}{\partial x} + y_1\frac{\partial c_1}{\partial y}\right)\mathrm{e}^{\mathrm{i}\lambda_0} \quad \text{(by Eq. (29))} \\
&= \mathrm{i}\left(\frac{\partial c_{-1}}{\partial x}\mathrm{e}^{-\mathrm{i}\lambda_0}\int_{-\infty}^{t}\frac{\partial c_1}{\partial y}\mathrm{e}^{\mathrm{i}\lambda_0}\,\mathrm{d}t - \mathrm{e}^{-\mathrm{i}\lambda_0}\frac{\partial c_{-1}}{\partial y}\int_{-\infty}^{t}\frac{\partial c_1}{\partial x}\mathrm{e}^{\mathrm{i}\lambda_0}\,\mathrm{d}t\right) - \\
&\quad - \mathrm{i}\left(\frac{\partial c_1}{\partial x}\mathrm{e}^{\mathrm{i}\lambda_0}\int_{-\infty}^{t}\frac{\partial c_{-1}}{\partial y}\mathrm{e}^{-\mathrm{i}\lambda_0}\,\mathrm{d}t - \mathrm{e}^{\mathrm{i}\lambda_0}\frac{\partial c_1}{\partial y}\int_{-\infty}^{t}\frac{\partial c_{-1}}{\partial x}\mathrm{e}^{-\mathrm{i}\lambda_0}\,\mathrm{d}t\right) \\
&= \mathrm{i}\frac{\partial}{\partial t}\left(\int_{-\infty}^{t}\frac{\partial c_{-1}}{\partial x}\mathrm{e}^{-\mathrm{i}\lambda_0}\,\mathrm{d}t\int_{-\infty}^{t}\frac{\partial c_1}{\partial y}\mathrm{e}^{\mathrm{i}\lambda_0}\,\mathrm{d}t - \right. \\
&\quad \left. - \int_{-\infty}^{t}\mathrm{e}^{-\mathrm{i}\lambda_0}\frac{\partial c_{-1}}{\partial y}\,\mathrm{d}t\int_{-\infty}^{t}\frac{\partial c_1}{\partial x}\mathrm{e}^{\mathrm{i}\lambda_0}\,\mathrm{d}t\right),
\end{aligned}$$

retaining phase independent terms only. This implies that, at the end of the encounter,

$$\begin{aligned}
L_2 &= \mathrm{i}\left(\int_{-\infty}^{\infty}\frac{\partial c_{-1}}{\partial x}\mathrm{e}^{-\mathrm{i}\lambda_0}\,\mathrm{d}t\int_{-\infty}^{\infty}\frac{\partial c_1}{\partial y}\mathrm{e}^{\mathrm{i}\lambda_0}\,\mathrm{d}t - \right. \\
&\quad \left. - \int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\lambda_0}\frac{\partial c_{-1}}{\partial y}\,\mathrm{d}t\int_{-\infty}^{\infty}\frac{\partial c_1}{\partial x}\mathrm{e}^{\mathrm{i}\lambda_0}\,\mathrm{d}t\right).
\end{aligned} \tag{31}$$

In the coplanar case the perturbing Hamiltonian is given in terms of the Legendre polynomial P_2 (cf. Eq. (4)). In terms of $\mathcal{R}$, an explicit expression is given in Equation (13) but it must be transformed to the new variables. Suppose that $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are orthogonal unit vectors fixed in an inertial frame and in the plane of the binary. If g is measured with respect to this frame and $\hat{\mathbf{x}}_1$, $\hat{\mathbf{y}}_1$ are orthogonal unit vectors in the direction of the major and minor axes of the relative orbit then

$$\hat{\mathbf{x}}_1 = \cos g\,\hat{\mathbf{a}} + \sin g\,\hat{\mathbf{b}}, \qquad \hat{\mathbf{x}}_2 = -\sin g\,\hat{\mathbf{a}} + \cos g\,\hat{\mathbf{b}}.$$

Then we expand Equation (9) to first order in e noting that $l = M$. From the relation between H' and $\mathcal{R}$ it is eventually found that, to first order in e,

$$\begin{aligned} H' = &-\frac{G_1 m_1 m_2 m_3}{M_{12} R^5}\Bigg\{\frac{3}{2}\Bigg(a\Bigg[\cos\lambda - \frac{3}{2}\frac{x}{\sqrt{L}} + \frac{1}{2}\cos 2\lambda\left(\frac{x}{\sqrt{L}}\right) + \\ &+ \frac{1}{2}\sin 2\lambda\left(\frac{y}{\sqrt{L}}\right)\Bigg]\hat{\mathbf{a}}\cdot\mathbf{R} + a\Bigg[\sin\lambda + \frac{1}{2}\sin 2\lambda\left(\frac{x}{\sqrt{L}}\right) - \\ &- \frac{1}{2}\cos 2\lambda\left(\frac{y}{\sqrt{L}}\right) - \frac{3}{2}\frac{y}{\sqrt{L}}\Bigg]\hat{\mathbf{b}}\cdot\mathbf{R}\Bigg)^2 - \\ &- \frac{1}{2}\frac{a^2}{R^3}\left(1 - \cos\lambda\left(\frac{x}{\sqrt{L}}\right) - \sin\lambda\left(\frac{y}{\sqrt{L}}\right)\right)^2\Bigg\}. \end{aligned}$$

This may be expanded to lowest order in x, y to give

$$\begin{aligned} H' = &-\frac{G m_1 m_2 m_3}{M_{12}}\Bigg(\Bigg[\frac{-15a^2}{4R^5}\frac{x}{\sqrt{L}}(\hat{\mathbf{a}}\cdot\mathbf{R})^2 - \frac{9a^2}{2R^5}\frac{y}{\sqrt{L}}(\hat{\mathbf{a}}\cdot\mathbf{R})(\hat{\mathbf{b}}\cdot\mathbf{R}) + \\ &+ \frac{3a^2 x}{4\sqrt{L}R^5}(\hat{\mathbf{b}}\cdot\mathbf{R})^2 + \frac{a^2 x}{\sqrt{L}R^3}\Bigg]\cos\lambda + \Bigg[\frac{3a^2 y}{4\sqrt{L}R^5}(\hat{\mathbf{a}}\cdot\mathbf{R})^2 - \\ &- \frac{9a^2 x}{2\sqrt{L}R^5}(\hat{\mathbf{a}}\cdot\mathbf{R})(\hat{\mathbf{b}}\cdot\mathbf{R}) - \frac{15a^2 y}{4\sqrt{L}R^5}(\hat{\mathbf{b}}\cdot\mathbf{R})^2 + \frac{a^2 y}{\sqrt{L}R^3}\Bigg]\sin\lambda\Bigg), \end{aligned}$$

where we have retained only the terms in $\sin\lambda$ and $\cos\lambda$.

Let us denote the coefficients of $\cos\lambda$ and $\sin\lambda$ by a_1 and b_1. The complex coefficients c_1, c_{-1} are then given by

$$c_1 = \tfrac{1}{2}(a_1 - \mathrm{i}b_1), \qquad c_{-1} = \tfrac{1}{2}(a_1 + \mathrm{i}b_1).$$

Since the orbit of the binary system is assumed to be initially circular the zero-order solution may be written as

$$x = x_0 = 0, \qquad y = y_0 = 0, \qquad L = L_0, \qquad a = a_0.$$

The derivative $\partial c_k/\partial L$ and c_k, $k = 0, 1$ are zero when $x = y = 0$ which explains why the second term in Equation (30) vanishes. Two of the partial derivatives, evaluated at the zero-order solution, are given by

$$\begin{aligned}\frac{\partial c_1}{\partial x} &= -\frac{Gm_1m_2m_3}{M_{12}}\left[\frac{-15a_0^2}{8\sqrt{L_0}R^5}(\hat{\mathbf{a}}\cdot\mathbf{R})^2+\frac{3a_0^2}{8\sqrt{L_0}R^5}(\hat{\mathbf{b}}\cdot\mathbf{R})^2+\right.\\ &\quad\left.+\frac{a_0^2}{2\sqrt{L_0}R^3}+\frac{9\mathrm{i}a_0^2}{4\sqrt{L_0}R^5}(\hat{\mathbf{a}}\cdot\mathbf{R})(\hat{\mathbf{b}}\cdot\mathbf{R})\right],\\ \frac{\partial c_1}{\partial y} &= -\frac{Gm_1m_2m_3}{M_{12}}\left[\frac{-9a_0^2}{4\sqrt{L_0}R^5}(\hat{\mathbf{a}}\cdot\mathbf{R})(\hat{\mathbf{b}}\cdot\mathbf{R})-\frac{3\mathrm{i}a_0^2}{8\sqrt{L_0}R^5}(\hat{\mathbf{a}}\cdot\mathbf{R})^2+\right.\\ &\quad\left.+\frac{15\mathrm{i}a_0^2}{8\sqrt{L_0}R^5}(\hat{\mathbf{b}}\cdot\mathbf{R})^2-\frac{\mathrm{i}a_0^2}{2\sqrt{L_0}R^3}\right].\end{aligned}$$

If * denotes the complex conjugate then the remaining pairs are easily obtained from

$$\left(\frac{\partial c_{-1}}{\partial x}\right)^* = \left(\frac{\partial c_1}{\partial x}\right) \quad\text{and}\quad \left(\frac{\partial c_{-1}}{\partial y}\right)^* = \left(\frac{\partial c_1}{\partial y}\right).$$

Having obtained the expressions for the partial derivatives the next step is to perform the integrations in Equation (31). Thus

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\partial c_{-1}}{\partial x}\,\mathrm{e}^{-\mathrm{i}\lambda_0}\,\mathrm{d}t &= -\frac{Gm_1m_2m_3}{M_{12}}\left(\frac{-15a_0^2}{8\sqrt{L_0}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}(n_0t+\lambda_{00})}(\hat{\mathbf{a}}\cdot\mathbf{R})^2\,\mathrm{d}t}{R^5}+\right.\\ &\quad+\frac{3a_0^2}{8\sqrt{L_0}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}(n_0t+\lambda_{00})}(\hat{\mathbf{b}}\cdot\mathbf{R})^2\,\mathrm{d}t}{R^5}+\\ &\quad+\frac{a_0^2}{2\sqrt{L_0}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}(n_0t+\lambda_{00})}\,\mathrm{d}t}{R^3}-\\ &\quad\left.-\frac{9a_0^2\mathrm{i}}{4\sqrt{L_0}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}(n_0t+\lambda_{00})}(\hat{\mathbf{a}}\cdot\mathbf{R})(\hat{\mathbf{b}}\cdot\mathbf{R})\,\mathrm{d}t}{R^5}\right),\end{aligned}$$

where we have written $\lambda_0 = \lambda_{00} + n_0t$. The appendix contains formulae which enable these integrals to be computed approximately. From Equation (A.5) the integral

$$\begin{aligned}&-\frac{15a_0^2}{8\sqrt{L_0}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}(n_0t+\lambda_{00})}(\hat{\mathbf{a}}\cdot\mathbf{R})^2\,\mathrm{d}t}{R^5}\\ &= -\frac{a_0^2}{\sqrt{L_0}}\frac{\mathrm{e}^{-\mathrm{i}\lambda_{00}}\,K^{5/2}\sqrt{\pi}\,\mathrm{e}^{-2/3K}}{nq^3}\left[\frac{5}{8}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})^2-\frac{5\mathrm{i}}{4}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})-\frac{5}{8}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})^2\right].\end{aligned}$$

Similarly, from Equation (A.6),

$$\begin{aligned}&\frac{3a_0^2}{8\sqrt{L_0}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}(n_0t+\lambda_{00})}(\hat{\mathbf{b}}\cdot\mathbf{R})^2\,\mathrm{d}t}{R^5}\\ &= -\frac{a_0^2}{\sqrt{L_0}}\frac{\mathrm{e}^{-\mathrm{i}\lambda_{00}}\,K^{5/2}\sqrt{\pi}\,\mathrm{e}^{-2/3K}}{nq^3}\left[\frac{1}{8}(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^2-\frac{\mathrm{i}}{4}(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})-\frac{1}{8}(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^2\right].\end{aligned}$$

Next, we evaluate

$$\begin{aligned}\frac{\mathrm{e}^{-\mathrm{i}\lambda_{00}}a_0^2}{2\sqrt{L_0}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}n_0t}\,\mathrm{d}t}{R^3} &= \frac{\mathrm{e}^{-\mathrm{i}\lambda_{00}}a_0^2K}{2\sqrt{L_0}nq^3}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}n_0t}\,\mathrm{d}\sigma}{(1+\sigma^2)^2}\\ &= \frac{\mathrm{e}^{-\mathrm{i}\lambda_{00}}a_0^2K^{3/2}\mathrm{e}^{-(2/3)K}\sqrt{\pi}}{4\sqrt{L_0}nq^3},\end{aligned}$$

which is smaller by a factor $\mathrm{O}(K)$. Finally from Equation (A.7) the integral

$$\begin{aligned}&-\frac{9a_0^2\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\lambda_{00}}}{4\sqrt{L_0}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}n_0t}(\hat{\mathbf{a}}\cdot\mathbf{R})(\hat{\mathbf{b}}\cdot\mathbf{R})\,\mathrm{d}t}{R^5}\\ &\quad= -\frac{3a_0^2\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\lambda_{00}}\,K^{5/2}\sqrt{\pi}\,\mathrm{e}^{-(2/3)K}}{4\sqrt{L_0}nq^3}[(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})-\mathrm{i}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})-\\ &\qquad-\mathrm{i}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})-(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})].\end{aligned}$$

Hence, to leading order (neglecting the term $\sim K^{3/2}$)

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\partial c_{-1}}{\partial x}\,\mathrm{e}^{-\mathrm{i}\lambda_0}\,\mathrm{d}t\\ &\quad= -\frac{Gm_1m_2m_3}{M_{12}}\frac{a_0^2\,\mathrm{e}^{-\mathrm{i}\lambda_{00}}\,K^{5/2}\sqrt{\pi}\,\mathrm{e}^{-(2K/3)}}{\sqrt{L_0}nq^3}\times\\ &\qquad\times\bigg[-\frac{5}{8}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})^2+\frac{5\mathrm{i}}{4}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})+\frac{5}{8}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})^2+\frac{1}{8}(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^2-\\ &\qquad-\frac{\mathrm{i}}{4}(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})-\frac{1}{8}(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^2-\frac{3\mathrm{i}}{4}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})-\frac{3}{4}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})-\\ &\qquad-\frac{3}{4}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})+\frac{3\mathrm{i}}{4}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})\bigg].\end{aligned}\tag{32}$$

Similarly, the next integral may be shown to be approximately equal to

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\partial c_{1}}{\partial y}\,\mathrm{e}^{-\mathrm{i}\lambda_0}\,\mathrm{d}t\\ &\quad= -\frac{Gm_1m_2m_3}{M_{12}}\frac{a_0^2\,\mathrm{e}^{\mathrm{i}\lambda_{00}}\,K^{5/2}\sqrt{\pi}\,\mathrm{e}^{-(2/3)K}}{\sqrt{L_0}nq^3}\times\\ &\qquad\times\bigg[-\frac{3}{4}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})-\frac{3\mathrm{i}}{4}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})-\frac{3\mathrm{i}}{4}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})+\\ &\qquad+\frac{3}{4}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})-\frac{\mathrm{i}}{8}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})^2+\frac{1}{4}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})+\frac{\mathrm{i}}{8}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})^2+\\ &\qquad+\frac{5\mathrm{i}}{8}(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^2-\frac{5}{4}(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})-\frac{5\mathrm{i}}{8}(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^2\bigg].\end{aligned}\tag{33}$$

To compute L_2 the results from Equations (32), (33) and their complex conjugates are substituted into (31). The terms proportional to K^4 and lower powers of

K may be neglected since it is assumed that K is large. It is then found that

$$\begin{aligned}
L_2 = &-\frac{G^2 m_1^2 m_2^2 m_3^2}{M_{12}^2}\frac{K^5\pi\,\mathrm{e}^{-(4/3)K}a_0^4}{L_0 n^2 q^6}\bigg[\frac{5}{32}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})^4+\frac{5}{32}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})^4+\\
&+\frac{5}{32}(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^4+\frac{5}{32}(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^4-\frac{9}{8}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})^2(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})+\\
&+\frac{9}{8}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^2(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})+\frac{5}{16}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})^2(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^2-\\
&-\frac{13}{4}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})+\frac{9}{8}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})^3(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})+\\
&+\frac{9}{8}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})^2(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})+\frac{9}{8}(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^3(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})+\frac{31}{16}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})^2(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^2-\\
&-\frac{9}{8}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})^3(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})-\frac{9}{8}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^3-\frac{9}{8}(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^2(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})+\\
&+\frac{31}{16}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})^2(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^2+\frac{5}{16}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})^2(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^2+\\
&+\frac{5}{16}(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}})^2(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})^2+\frac{5}{16}(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}})^2(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})^2\bigg].
\end{aligned} \tag{34}$$

Since the orbit of the binary is initially circular this result should be independent of the orientation of the $(\hat{\mathbf{A}},\hat{\mathbf{B}})$ axis and the $(\hat{\mathbf{a}},\hat{\mathbf{b}})$ axis. This may be checked by replacing

$$\begin{aligned}
(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}}) &= \cos\theta, \qquad (\hat{\mathbf{b}}\cdot\hat{\mathbf{A}}) = \sin\theta,\\
(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}}) &= -\sin\theta, \qquad (\hat{\mathbf{b}}\cdot\hat{\mathbf{B}}) = \cos\theta.
\end{aligned}$$

The result in Equation (34) simplifies to give

$$L_2 = -\frac{G^2 m_1^2 m_2^2 m_3^2}{M_{12}^2}\frac{K^5\pi\,\mathrm{e}^{-(4K/3)}a_0^4}{L_0 n^2 q^6}\left(\frac{9}{2}\right).$$

To second order this means that

$$\begin{aligned}
\delta\varepsilon_{\mathrm{mean}} &= \frac{\mu^2 m^3}{L_0^3}L_2\\
&= -\frac{\mu^2 m^3}{L_0^3}\frac{K^5\pi\,\mathrm{e}^{-(4K/3)}}{a_0^2 L_0 n^2}\left(\frac{18M_{12}^2}{M_{123}^2K^4}\right)\frac{G^2 m_1^2 m_2^2 m_3^2}{M_{12}^2}
\end{aligned}$$

using the fact that

$$\frac{a_0^6}{q^6} = \frac{4M_{12}^2}{M_{123}^2K^4}.$$

We also have

$$L_0 = m\sqrt{\mu a_0} \quad \text{and} \quad \frac{\mu^2 m^3}{L_0^4 n^2} = \frac{a_0}{m\mu} \quad \text{and} \quad \frac{a_0^6}{q^6} = \frac{4M_{12}^2}{M_{123}^2K^4},$$

where we have used the definition of L in Equations (27) and (10). Therefore

$$\delta\varepsilon_{\text{mean}} = -\frac{18Gm_1m_2m_3^2K\pi\,\mathrm{e}^{-(4K/3)}}{a_0M_{123}^2}.$$

When $q = 4$, $a_0 = 1$, $G = m_1 = m_2 = m_3 = 1$ we find that $\delta\varepsilon_{\text{mean}} = -0.000259\ldots$. If this contribution is added to the analytical result plotted in Figure 6 there is much better agreement with the numerical results.

It might be of interest to extend this result by calculating $\delta\varepsilon_{\text{mean}}$ at higher angles of inclination. We would then have to consider the Delaunay variables h and H which were neglected in the coplanar case.

5. Conclusions

In this paper we have considered the change in energy in a hard binary resulting from a nearly parabolic encounter with a third star whose distance at closest approach considerably exceeds the semi-major axis of the binary. The asymptotic result is

$$\delta\varepsilon \simeq -\frac{Gm_1m_2m_3}{M_{12}q^3}\frac{\sqrt{\pi}}{2}K^{5/2}\,\mathrm{e}^{-(2/3)K}(e_1a^2 + 2e_4ab + e_2b^2),$$

where $K = \sqrt{2M_{12}q^3/M_{123}a^3}$. We have found that this gives a null result in the case of a circular binary, for which the dominant asymptotic result is instead

$$\delta\varepsilon \simeq \frac{Gm_1m_2m_3}{M_{12}}\frac{\sqrt{\pi}a^3}{8q^4}K^{7/2}\,\mathrm{e}^{-(2K/3)}(\mu_2 - \mu_1).$$

This itself gives a null result when the components of the binary have equal masses. In this highly degenerate case of a circular binary with equal masses we have given a complete result only for the case of coplanar motion of the third body, as second-order perturbation theory is required. We find in this case that

$$\delta\varepsilon \simeq \frac{8Gm_1m_2m_3\sqrt{2\pi}}{aM_{123}}\,\mathrm{e}^{-(4/3)K}K^{1/2} - \frac{18Gm_1m_2m_3^2K\pi\,\mathrm{e}^{-(4K/3)}}{a_0M_{123}^2}$$

(where the oscillatory result has been combined with the mean energy change into one formula).

Acknowledgements

AR gratefully acknowledges the award of an EPSRC scholarship. The authors would like to thank Professor Douglas Heggie, University of Edinburgh, for many useful discussions.

Appendix A. Integral formulas

These approximate formulae are used in Sections 4.1 and 4.4 and are derived in Section 3.1.

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}} \cdot \frac{\partial}{\partial R}\left(\frac{(\hat{\mathbf{a}}\cdot\mathbf{R})\cos M}{R^5}\right) \mathrm{d}t = \frac{\sqrt{\pi} K^{5/2}\,\mathrm{e}^{-2K/3}}{12q^5}\mathfrak{F} \times \times [(2q\mathrm{i}\hat{\mathbf{B}} + 2q\hat{\mathbf{A}})\cdot\hat{\mathbf{a}}\,\mathrm{e}^{-\mathrm{i}nt_0}], \tag{A.1}$$

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}} \cdot \frac{\partial}{\partial R}\left(\frac{(\hat{\mathbf{b}}\cdot\mathbf{R})\cos M}{R^5}\right) \mathrm{d}t = \frac{\sqrt{\pi} K^{5/2}\,\mathrm{e}^{-2K/3}}{12q^5}\mathfrak{F} \times \times [(2q\mathrm{i}\hat{\mathbf{B}} + 2q\hat{\mathbf{A}})\cdot\hat{\mathbf{b}}\,\mathrm{e}^{-\mathrm{i}nt_0}], \tag{A.2}$$

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}} \cdot \frac{\partial}{\partial R}\left(\frac{(\hat{\mathbf{a}}\cdot\mathbf{R})\sin M}{R^5}\right) \mathrm{d}t = \frac{\sqrt{\pi} K^{5/2}\,\mathrm{e}^{-2K/3}}{12q^5}\mathfrak{F} \times \times [-\mathrm{i}(2q\mathrm{i}\hat{\mathbf{B}} + 2q\hat{\mathbf{A}})\cdot\hat{\mathbf{a}}\,\mathrm{e}^{-\mathrm{i}nt_0}], \tag{A.3}$$

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}} \cdot \frac{\partial}{\partial R}\left(\frac{(\hat{\mathbf{b}}\cdot\mathbf{R})\sin M}{R^5}\right) \mathrm{d}t = \frac{\sqrt{\pi} K^{5/2}\,\mathrm{e}^{-2K/3}}{12q^5}\mathfrak{F} \times \times [-\mathrm{i}(2q\mathrm{i}\hat{\mathbf{B}} + 2q\hat{\mathbf{A}})\cdot\hat{\mathbf{b}}\,\mathrm{e}^{-\mathrm{i}nt_0}], \tag{A.4}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}n_0 t}(\hat{\mathbf{a}}\cdot\mathbf{R})^p\,\mathrm{d}t}{R^5} = \frac{K^{3/2}\sqrt{\pi}\,\mathrm{e}^{-2K/3}}{12nq^4}[2(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}}) - 2q\mathrm{i}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})]^p, \tag{A.5}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}n_0 t}(\hat{\mathbf{b}}\cdot\mathbf{R})^p\,\mathrm{d}t}{R^5} = \frac{K^{5/2}\sqrt{\pi}\,\mathrm{e}^{-2K/3}}{12nq^5}[2q(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}}) - 2q\mathrm{i}(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})]^p, \tag{A.6}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}n_0 t}(\hat{\mathbf{a}}\cdot\mathbf{R})(\hat{\mathbf{b}}\cdot\mathbf{R})\,\mathrm{d}t}{R^5} = \frac{K^{5/2}\sqrt{\pi}\,\mathrm{e}^{-2K/3}}{12nq^5}[2q(\hat{\mathbf{a}}\cdot\hat{\mathbf{A}}) - 2q\mathrm{i}(\hat{\mathbf{a}}\cdot\hat{\mathbf{B}})] \times \times [2q(\hat{\mathbf{b}}\cdot\hat{\mathbf{A}}) - 2q\mathrm{i}(\hat{\mathbf{b}}\cdot\hat{\mathbf{B}})], \tag{A.7}$$

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}}\left(\frac{(\hat{\mathbf{a}}\cdot\mathbf{R})^p\cos E}{R^7}\right) \mathrm{d}t = \frac{K^{7/2}\sqrt{\pi}\,\mathrm{e}^{-2K/3}}{120q^7}\mathfrak{F}([(2q\mathrm{i}(\hat{\mathbf{B}} + 2q\hat{\mathbf{A}})\cdot\hat{\mathbf{a}}]^p\,\mathrm{e}^{-\mathrm{i}nt_0}), \tag{A.8}$$

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}} \left(\frac{(\hat{\mathbf{a}} \cdot \mathbf{R})^p \sin E}{R^7} \right) \mathrm{d}t = \frac{K^{7/2}\sqrt{\pi}\,\mathrm{e}^{-2K/3}}{120} \mathfrak{F}(-\mathrm{i}[(2q\mathrm{i}\hat{\mathbf{B}} + 2q\hat{\mathbf{A}}) \cdot \hat{\mathbf{a}}]^p\,\mathrm{e}^{-\mathrm{i}nt_0}), \tag{A.9}$$

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}} \left(\frac{(\hat{\mathbf{b}} \cdot \mathbf{R})^p \cos E}{R^7} \right) \mathrm{d}t = \frac{K^{7/2}\sqrt{\pi}\,\mathrm{e}^{-2K/3}}{120} \mathfrak{F}([(2q\mathrm{i}\hat{\mathbf{B}} + 2q\hat{\mathbf{A}}) \cdot \hat{\mathbf{b}}]^p\,\mathrm{e}^{-\mathrm{i}nt_0}), \tag{A.10}$$

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}} \left(\frac{(\hat{\mathbf{b}} \cdot \mathbf{R})^p \sin E}{R^7} \right) \mathrm{d}t = \frac{K^{7/2}\sqrt{\pi}\,\mathrm{e}^{-2K/3}}{120q^7} \mathfrak{F}(-\mathrm{i}[(2q\mathrm{i}\hat{\mathbf{B}} + 2q\hat{\mathbf{A}}) \cdot \hat{\mathbf{b}}]^p\,\mathrm{e}^{-\mathrm{i}nt_0}), \tag{A.11}$$

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}} \left(\frac{(\hat{\mathbf{a}} \cdot \mathbf{R})^p (\hat{\mathbf{b}} \cdot \mathbf{R})^q \cos E}{R^7} \right) \mathrm{d}t = \frac{K^{7/2}\sqrt{\pi}\,\mathrm{e}^{-2K/3}}{120q^7} \mathfrak{F}([(2q\mathrm{i}\hat{\mathbf{B}} + 2q\hat{\mathbf{A}}) \cdot \hat{\mathbf{a}}]^p[(2q\mathrm{i}\hat{\mathbf{B}} + 2q\hat{\mathbf{A}}) \cdot \hat{\mathbf{a}}]^q\,\mathrm{e}^{-\mathrm{i}nt_0}), \tag{A.12}$$

$$\int_{-\infty}^{\infty} \dot{\mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{R}} \left(\frac{(\hat{\mathbf{a}} \cdot \mathbf{R})^p (\hat{\mathbf{b}} \cdot \mathbf{R})^q \sin E}{R^7} \right) \mathrm{d}t = \frac{K^{7/2}\sqrt{\pi}\,\mathrm{e}^{-2K/3}}{120q^7} \mathfrak{F}(-\mathrm{i}[(2q\mathrm{i}\hat{\mathbf{B}} + 2q\hat{\mathbf{A}}) \cdot \hat{\mathbf{a}}]^p[(2q\mathrm{i}\hat{\mathbf{B}} + 2q\hat{\mathbf{A}}) \cdot \hat{\mathbf{b}}]^q\,\mathrm{e}^{-\mathrm{i}nt_0}). \tag{A.13}$$

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