1 Torque criterion

We start from the assumption that the thermal diffusivity is proportional to $H^2\Omega : \chi = KH^2\Omega$. K can be constant, or variable. The torque is

$$\Gamma_h = 1.61\frac{\gamma-1}{\gamma}\frac{x_p}{\lambda_c}\frac{L}{L_c}\Gamma_0$$

where

$$\begin{aligned}\frac{x_p}{\lambda} &= \frac{c_s^2}{\gamma r\Omega^2}\sqrt{\frac{3\gamma\Omega}{2\chi}} \\ &= \frac{H}{r}\sqrt{\frac{3}{2\gamma K}}\end{aligned}$$

and

$$\frac{L}{L_c} = \epsilon\frac{4\pi Gmc/\kappa}{4\pi Gm\rho\chi/\gamma} = \epsilon\frac{\gamma c}{\kappa\rho\chi} = \epsilon\frac{\gamma c}{K\tau H\Omega}$$

So the rthermal torue is

$$\Gamma_h = 1.61\frac{\gamma-1}{\gamma}\frac{H}{r}\sqrt{\frac{3}{2\gamma}}\epsilon\frac{\gamma c}{K^{3/2}\tau H\Omega}\Gamma_0$$

this thermal torque need to overcome the type I torque:

$$\Gamma_h > \Gamma_I = C_I\frac{H}{r}\Gamma_0$$

or

$$1.61\frac{\gamma-1}{\gamma C_I}\sqrt{\frac{3}{2\gamma}}\epsilon\frac{\gamma c}{K^{3/2}\tau H\Omega} > 1$$

or

$$\tau < 1.61\frac{\gamma-1}{C_I}\sqrt{\frac{3}{2\gamma}}\frac{\epsilon c}{K^{3/2}c_s} \approx 10^3\left(\frac{c_s}{5\times10^6\mathrm{cm/s}}\right)^{-1}\frac{\epsilon}{K^{3/2}}$$

For the previous case, $K \approx \alpha \sim 0.01$ and we have $\tau < 10^6$, with accordance to our results, and also with armitage (but a larger value due to additional division by H/r).

However, the latter was derived using Eq. 11, which assumes the temperature is related to the accretion rate. Instead, in the graviataionally unstable regime, the temperature must be optained from the pressure equation. Assuming only radiation pressure is valid:

$$p_{\mathrm{rad}} = \frac{4\sigma}{3c}T^4 = c_s^2\rho.$$

For the thermal diffusivity we have

$$\chi = \frac{16\gamma(\gamma-1)\sigma T^4}{3\kappa\rho_0^2H^2\Omega^2} = \frac{4\gamma(\gamma-1)c}{\tau c_s}H^2\Omega \approx \frac{10^5}{\tau}H^2\Omega$$

so in this case K is varying and uqual to $10^5/\tau$ and we have

$$\begin{aligned}\tau &< 1.61\frac{\gamma-1}{C_I}\sqrt{\frac{3}{2\gamma}}\frac{\epsilon c c_s^{3/2}\tau^{3/2}}{8\gamma^{3/2}(\gamma-1)^{3/2}c^{3/2}c_s} \\ &= 1.61\frac{\gamma-1}{C_I 8\gamma^2(\gamma-1)^{3/2}}\sqrt{\frac{3}{2}}\frac{\epsilon c_s^{1/2}\tau^{3/2}}{c^{1/2}} \\ &\approx 0.013 c_{0.53}^{1/2}\epsilon\tau^{3/2}\end{aligned}$$

or

$$\tau^{1/2} > \frac{77}{c_{0.53}^{1/2}\epsilon} \implies \tau > \frac{6\times 10^3}{c_{0.53}\epsilon}$$

Here we note that τ must be **larger** than some value, not smaller, so thermal torque may operate only in the very optically thick disc, and probably must heat the disc due to Eddington luminosity.

1.1 Validity of the hierarchy of length scales

Let's check the hierarchy:

$$\frac{x}{\lambda} = \frac{H}{r}\sqrt{\frac{3}{2\gamma K}} = \frac{H}{r}\sqrt{\frac{3\tau c_s}{8(\gamma-1)c}} \approx \frac{H}{r}0.014\tau^{1/2}$$

in order for $x/\lambda < 1/3$, we need

$$\frac{H}{r}0.014\tau^{1/2} < \frac{1}{5}$$

$$\tau < 450\frac{r^2}{H^2} \sim 5\cdot 10^6 h_{0.01}^{-2}$$

which is generalyl satisfied throughout the disc.

For

$$\frac{\lambda}{H} = \sqrt{\frac{2K}{3\gamma}} = \sqrt{\frac{8(\gamma-1)c}{3\tau c_s}} \approx 71\tau^{-1/2}$$

so $\lambda < H$ for $\tau > 5000$, which is roughly thermal torques should operate.

2 Conclusion

The assumptions for thermal torques valid for $5\cdot 10^3 < \tau < 5\cdot 10^6$, and thermal torques will dominate for $\tau > 6000/\epsilon$.

2.1 Application to discs of different masses:

The conditions for $Q_T = 1$ are usually met outward. In the Sirko and Goodman case it occurs at $r_s = 800 r_g$. For more general Shakura-sunyaev discs, it occurs at $r_s \sim 10^4 r_g$ and at larger value for smaller $M_\bullet$. The former analysis (where $K = \alpha$) should be valid for low $M_\bullet$ cases, while the latter should be valid for large $M_\bullet$ cases.