# Computational searches for fractional Calabi-Yau algebras

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September 1, 2023

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**Example:**  $\alpha_2\alpha_1$  is a zero path in  $Q_3$ , whereas  $\alpha_1\alpha_2$  is non-zero.



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**Example:**  $kQ_3$  has basis  $\{e_1, e_2, e_3, \alpha_1, \alpha_2, \alpha_1\alpha_2\}$ , where  $e_i$  are the **lazy paths**. We write  $e_i \cdot \alpha_i = \alpha_i$ .



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We can embed relations into an algebra by forming the quotient algebra  $kQ_n/I$ , where  $kQ_n \supseteq I = (r_1, \ldots, r_m)$  is the ideal generated by the relations  $r_1, \ldots, r_m$ .

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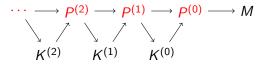
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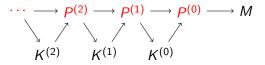


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We can represent this projective resolution using a matrix. Then, if the matrix doesn't have finite order, then  $kQ_n/I$  is not fCY.



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**Theorem:**  $kQ_n$  (i.e. l=0) has Calabi-Yau dimension  $\frac{n-1}{n+1}$ .

However, less is known about  $kQ_n/I$  for other ideals I. This is the primary motivation behind performing these computational searches.

 $Programming\ Interlude...$ 

Some data...

**Question 1:** Does there exist an I and an n such that  $kQ_n/I$  is not fCY?

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**Question 2:** For  $n \ge 3$ , is  $kQ_n/I$  fCY whenever I is generated by length 2 relations?

**Question 3:** Does the number of algebras passing the matrix test tend to zero as n tends to infinity?

Thank you for listening.