Computational searches for fractional Calabi-Yau algebras

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Quivers and Paths

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Example: $\alpha_2\alpha_1$ is a zero path in Q_3, whereas $\alpha_1\alpha_2$ is non-zero.

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A **path algebra** $A = kQ_n$ of a quiver Q_n is a vector space over a field (e.g. $\mathbb{R}$ or $\mathbb{C}$) k with basis containing all paths in Q_n together with multiplication

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If b_1 and b_2 are basis vectors, then

$$b_1 \cdot b_2 = \begin{cases} b_1 b_2, & \text{if } t(b_1) = s(b_2); \\ 0, & \text{otherwise.} \end{cases}$$

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Example: kQ_3 has basis $\{e_1, e_2, e_3, \alpha_1, \alpha_2, \alpha_1\alpha_2\}$, where e_i are the **lazy paths**. We write $e_i \cdot \alpha_i = \alpha_i$.

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We can embed relations into an algebra by forming the quotient algebra kQ_n/I, where $kQ_n \supseteq I = (r_1, \ldots, r_m)$ is the ideal generated by the relations $r_1, \ldots, r_m$.

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A **projective resolution** of a left A-module M is given by

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\begin{tikzcd}
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& K^{(2)} \arrow[ur] & & K^{(1)} \arrow[ur] & & K^{(0)} \arrow[ur] & &
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& K^{(2)} \arrow[u] & K^{(1)} \arrow[u] & K^{(0)} \arrow[u] &
\end{tikzcd}$$

We can represent this projective resolution using a matrix. Then, if the matrix doesn't have finite order, then kQ_n/I is not fCY.

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Theorem: kQ_n (i.e. $I = 0$) has Calabi-Yau dimension $\dfrac{n-1}{n+1}$.

However, less is known about kQ_n/I for other ideals I. This is the primary motivation behind performing these computational searches.

Programming Interlude...

Some data...

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Question 2: For $n \geq 3$, is kQ_n/I fCY whenever I is generated by length 2 relations?

Question 3: Does the number of algebras passing the matrix test tend to zero as n tends to infinity?

Thank you for listening.