

# The Dodecagram Polygonal Harmony Uniqueness Theorem

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## Abstract

We characterize a critical family of regular star polygons described by the Schläfli symbol  $\{n/k\}$ . Under the Diophantine relation  $k = n/2 - 1$  with  $k - 2 > 1$ , we establish The Dodecagram Polygonal Harmony Uniqueness Theorem, proving that the regular dodecagram  $\{12/5\}$  is the unique polygon for which all non-convex internal stellation rings are composite in the sense of  $\gcd(n, s) > 1$ . This establishes  $\{12/5\}$  as a sharp arithmetic and combinatorial threshold in the theory of regular stellations.

## 1 Introduction

Regular star polygons constitute a classical topic in Euclidean geometry and are denoted by the Schläfli symbol  $\{n/k\}$ . Their stellations generate internal rings whose components may be simple polygons, star polygons, or unions of multiple regular polygons.

In this paper we introduce a critical arithmetic relation between  $n$  and  $k$  and show that the dodecagram  $\{12/5\}$  occupies a unique structural position: it is the only polygon in this family whose non-convex internal rings are all composite. This property follows from elementary number-theoretic constraints rather than geometric coincidence.

## 2 Preliminaries

**Definition 1.** A regular star polygon  $\{n/k\}$  consists of  $n$  equally spaced vertices on a circle, with edges joining each vertex to the  $k$ -th subsequent vertex, where  $\gcd(n, k) = 1$  and  $1 < k < n/2$ .

**Definition 2.** An internal ring of  $\{n/k\}$  is the orbit, under the action of the dihedral group  $D_n$ , of intersection points of pairs of non-adjacent edges that lie on a common concentric circle. Rings are indexed from the exterior to the interior.

**Definition 3.** A ring with step size  $s$  is composite if  $\gcd(n, s) > 1$ , in which case it consists of  $\gcd(n, s)$  congruent regular polygons. A ring is simple if  $\gcd(n, s) = 1$ .

## 3 Critical Diophantine Relation

We impose the Diophantine condition

$$k = \frac{n}{2} - 1, \quad n = 2(k + 1). \tag{1}$$

## 4 Main Result

**Theorem 1** (Dodecagram Polygonal Harmony Uniqueness Theorem). *Let  $\{n/k\}$  be a regular star polygon satisfying  $k = n/2 - 1$  and  $k - 2 > 1$ . Then:*

1. *The polygon has exactly  $k - 1$  internal rings with step sizes  $s = k - 1, \dots, 2, 1$ .*
2. *The only integer solution  $(n, k)$  for which all rings with  $s \geq 2$  are composite is  $(n, k) = (12, 5)$ .*
3. *The three composite rings of  $\{12/5\}$  decompose respectively into four equilateral triangles, three squares, and two regular hexagons; the innermost ring is the convex dodecagon.*

## 5 Proofs

**Lemma 1.** *If  $k = n/2 - 1$  and  $\gcd(n, k) = 1$ , then  $k$  is odd.*

*Proof.* Since  $n = 2(k + 1)$ , we have  $\gcd(n, k) = \gcd(2(k + 1), k) = \gcd(2, k)$ . Hence  $\gcd(n, k) = 1$  if and only if  $k$  is odd.  $\square$

**Lemma 2.** *A ring of step size  $s$  in  $\{n/k\}$  is composite if and only if  $\gcd(n, s) > 1$ .*

*Proof.* If  $d = \gcd(n, s) > 1$ , the traversal closes after  $n/d$  steps, yielding  $d$  congruent regular polygons. If  $d = 1$ , all vertices are visited before closure, yielding a single polygon.  $\square$

**Lemma 3** (Ring Structure). *Let  $P_i^{(m)}$  denote the intersection of edge  $i$  with edge  $i + m$  in  $\{n/k\}$ , for  $1 \leq m \leq k - 1$  and indices taken mod  $n$ . Then:*

- (a) *The  $n$  points  $\{P_i^{(m)}\}_{i=0}^{n-1}$  lie equally spaced on a circle of radius  $r_m$ , with  $r_1 < r_2 < \dots < r_{k-1}$ .*
- (b) *The ring  $m$  has step  $s = m$ : the points  $P_i^{(m)}$  and  $P_{i+m}^{(m)}$  are connected by a segment of edge  $i + m$ .*
- (c) *The  $j$ -th ring from the exterior has step  $s = k - j$ . Thus  $\{n/k\}$  has exactly  $k - 1$  internal rings.*

*Proof.* (a) The  $n$ -fold rotational symmetry  $R$  (rotation by  $2\pi/n$ ) satisfies  $R(P_i^{(m)}) = P_{i+1}^{(m)}$ , so the  $n$  points are equally spaced on a circle of radius  $r_m = |P_0^{(m)}|$ . Parameterizing the edges and solving the resulting  $2 \times 2$  linear system shows  $r_m$  is strictly increasing in  $m$ .

(b) Both  $P_0^{(m)}$  and  $P_m^{(m)}$  lie on edge  $m$ :  $P_0^{(m)}$  is the intersection of edge 0 with edge  $m$ , and  $P_m^{(m)}$  is the intersection of edge  $m$  with edge  $2m$ . These are the only two intersections of edge  $m$  with the ring of radius  $r_m$ , and the segment of edge  $m$  between them is an edge of ring  $m$ . Since  $P_i^{(m)}$  occupies angular position  $i$  (in units of  $2\pi/n$ ), the step from  $P_0^{(m)}$  to  $P_m^{(m)}$  is  $s = m$ .

(c) By (a), the outermost ring is  $m = k - 1$  and the innermost is  $m = 1$ . The  $j$ -th ring from the exterior is  $m = k - j$ , with step  $s = k - j$ .  $\square$

**Lemma 4** (Uniqueness). *Let  $k \geq 5$  be odd and  $n = 2(k + 1)$ . Then  $\gcd(n, s) > 1$  for all  $s \in \{2, \dots, k - 1\}$  if and only if  $(n, k) = (12, 5)$ .*

*Proof.* The condition requires that every integer in  $\{2, \dots, k - 1\}$  shares a prime factor with  $n = 2(k + 1)$ . Equivalently, letting  $p_0$  be the smallest prime not dividing  $n$ , we need  $p_0 \geq k$ .

**Case  $(n, k) = (12, 5)$ :** The prime factors of  $12 = 2^2 \cdot 3$  are  $\{2, 3\}$ . Each element of  $\{2, 3, 4\}$  shares a factor with 12, and  $p_0 = 5 = k$ .  $\checkmark$

**Case  $k \geq 7$ :** We show  $p_0 < k$  by exhaustive sub-case analysis.

- If  $3 \nmid n$ : then  $p_0 = 3$ . Since  $k \geq 7 > 3$ , we have  $p_0 < k$ .
- If  $3 \mid n$  but  $5 \nmid n$ : then  $p_0 = 5$ . Since  $k \geq 7 > 5$ , we have  $p_0 < k$ .
- If  $3 \mid n$  and  $5 \mid n$  but  $7 \nmid n$ : then  $p_0 = 7$ . Since  $15 \mid (k + 1)$ , we get  $k \geq 29 > 7 = p_0$ .
- In general: suppose the primes  $3, 5, \dots, q$  all divide  $n$ , and  $q' \nmid n$  where  $q'$  is the next prime after  $q$ . Then  $2 \cdot 3 \cdot 5 \cdots q \mid (k + 1)$ , so  $k \geq 2 \cdot 3 \cdot 5 \cdots q - 1$ . For  $q \geq 5$ , this bound exceeds  $q'$  (since  $q' > q$  and  $2 \cdot 3 \cdot 5 > 5$ ), giving  $p_0 = q' < k$ .

In every sub-case, the smallest prime  $p_0$  not dividing  $n$  satisfies  $p_0 < k$ , so  $s = p_0 \in \{2, \dots, k - 1\}$  gives  $\gcd(n, p_0) = 1$ , a simple ring. ✓ □

## 6 Geometric Structure and Figures

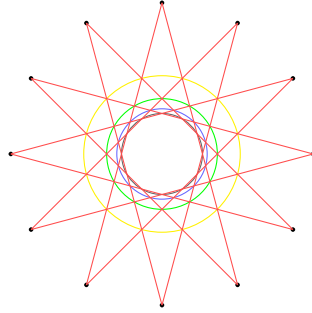


Figure 1: Regular dodecagram  $\{12/5\}$  with schematic internal rings.

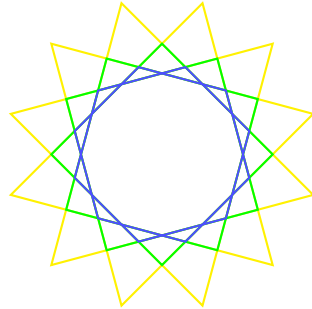


Figure 2: Composite rings of  $\{12/5\}$ : four equilateral triangles (yellow,  $r_3$ ), three squares (green,  $r_2$ ), and two regular hexagons (blue,  $r_1$ ). Vertex positions match the exact intersection radii and angular offsets.

## 7 Discussion

The dodecagram appears as a minimal arithmetic configuration in which the prime factors of  $n$  cover the full interval of ring step sizes. This property does not persist for larger  $n$ , where simple stellations inevitably appear. Thus  $\{12/5\}$  constitutes a sharp transition point between fully composite and mixed stellation regimes.

## 8 Conclusion

We have proven that  $\{12/5\}$  is the unique regular star polygon satisfying  $k = n/2 - 1$  with  $k - 2 > 1$  whose non-convex internal rings are all composite. The result follows from elementary number theory and symmetry arguments and suggests further connections between stellations theory and arithmetic geometry.

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