

The Dodecagram Polygonal Harmony Uniqueness Theorem

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Abstract

We characterize a critical family of regular star polygons described by the Schläfli symbol $\{n/k\}$. Under the Diophantine relation $k = n/2 - 1$ with $k - 2 > 1$, we establish The Dodecagram Polygonal Harmony Uniqueness Theorem, proving that the regular dodecagram $\{12/5\}$ is the unique polygon for which all non-convex internal stellation rings are composite in the sense of $\gcd(n, s) > 1$. This establishes $\{12/5\}$ as a sharp arithmetic and combinatorial threshold in the theory of regular stellations.

1 Introduction

Regular star polygons constitute a classical topic in Euclidean geometry and are denoted by the Schläfli symbol $\{n/k\}$. Their stellations generate internal rings whose components may be simple polygons, star polygons, or unions of multiple regular polygons.

In this paper we introduce a critical arithmetic relation between n and k and show that the dodecagram $\{12/5\}$ occupies a unique structural position: it is the only polygon in this family whose non-convex internal rings are all composite. This property follows from elementary number-theoretic constraints rather than geometric coincidence.

2 Preliminaries

Definition 1. *A regular star polygon $\{n/k\}$ consists of n equally spaced vertices on a circle, with edges joining each vertex to the k -th subsequent vertex, where $\gcd(n, k) = 1$ and $1 < k < n/2$.*

Definition 2. *An internal ring of $\{n/k\}$ is the orbit, under the action of the dihedral group D_n , of intersection points of pairs of non-adjacent edges that lie on a common concentric circle. Rings are indexed from the exterior to the interior.*

Definition 3. *A ring with step size s is composite if $\gcd(n, s) > 1$, in which case it consists of $\gcd(n, s)$ congruent regular polygons. A ring is simple if $\gcd(n, s) = 1$.*

3 Critical Diophantine Relation

We impose the Diophantine condition

$$k = \frac{n}{2} - 1, \quad n = 2(k + 1). \tag{1}$$

4 Main Result

Theorem 1 (Dodecagram Polygonal Harmony Uniqueness Theorem). *Let $\{n/k\}$ be a regular star polygon satisfying $k = n/2 - 1$ and $k - 2 > 1$. Then:*

1. *The polygon has exactly $k - 1$ internal rings with step sizes $s = k - 1, \dots, 2, 1$.*
2. *The only integer solution (n, k) for which all rings with $s \geq 2$ are composite is $(n, k) = (12, 5)$.*
3. *The three composite rings of $\{12/5\}$ decompose respectively into four equilateral triangles, three squares, and two regular hexagons; the innermost ring is the convex dodecagon.*

5 Proofs

Lemma 1. *If $k = n/2 - 1$ and $\gcd(n, k) = 1$, then k is odd.*

Proof. Since $n = 2(k + 1)$, we have $\gcd(n, k) = \gcd(2(k + 1), k) = \gcd(2, k)$. Hence $\gcd(n, k) = 1$ if and only if k is odd. \square

Lemma 2. *A ring of step size s in $\{n/k\}$ is composite if and only if $\gcd(n, s) > 1$.*

Proof. If $d = \gcd(n, s) > 1$, the traversal closes after n/d steps, yielding d congruent regular polygons. If $d = 1$, all vertices are visited before closure, yielding a single polygon. \square

Lemma 3 (Ring Structure). *Let $P_i^{(m)}$ denote the intersection of edge i with edge $i + m$ in $\{n/k\}$, for $1 \leq m \leq k - 1$ and indices taken mod n . Then:*

- (a) *The n points $\{P_i^{(m)}\}_{i=0}^{n-1}$ lie equally spaced on a circle of radius r_m , with $r_1 < r_2 < \dots < r_{k-1}$.*
- (b) *The ring m has step $s = m$: the points $P_i^{(m)}$ and $P_{i+m}^{(m)}$ are connected by a segment of edge $i + m$.*
- (c) *The j -th ring from the exterior has step $s = k - j$. Thus $\{n/k\}$ has exactly $k - 1$ internal rings.*

Proof. (a) The n -fold rotational symmetry R (rotation by $2\pi/n$) satisfies $R(P_i^{(m)}) = P_{i+1}^{(m)}$, so the n points are equally spaced on a circle of radius $r_m = |P_0^{(m)}|$. Parameterizing the edges and solving the resulting 2×2 linear system shows r_m is strictly increasing in m .

(b) Both $P_0^{(m)}$ and $P_m^{(m)}$ lie on edge m : $P_0^{(m)}$ is the intersection of edge 0 with edge m , and $P_m^{(m)}$ is the intersection of edge m with edge $2m$. These are the only two intersections of edge m with the ring of radius r_m , and the segment of edge m between them is an edge of ring m . Since $P_i^{(m)}$ occupies angular position i (in units of $2\pi/n$), the step from $P_0^{(m)}$ to $P_m^{(m)}$ is $s = m$.

(c) By (a), the outermost ring is $m = k - 1$ and the innermost is $m = 1$. The j -th ring from the exterior is $m = k - j$, with step $s = k - j$. \square

Lemma 4 (Uniqueness). *Let $k \geq 5$ be odd and $n = 2(k + 1)$. Then $\gcd(n, s) > 1$ for all $s \in \{2, \dots, k - 1\}$ if and only if $(n, k) = (12, 5)$.*

Proof. The condition requires that every integer in $\{2, \dots, k - 1\}$ shares a prime factor with $n = 2(k + 1)$. Equivalently, letting p_0 be the smallest prime not dividing n , we need $p_0 \geq k$.

Case $(n, k) = (12, 5)$: The prime factors of $12 = 2^2 \cdot 3$ are $\{2, 3\}$. Each element of $\{2, 3, 4\}$ shares a factor with 12, and $p_0 = 5 = k$. \checkmark

Case $k \geq 7$: We show $p_0 < k$ by exhaustive sub-case analysis.

- If $3 \nmid n$: then $p_0 = 3$. Since $k \geq 7 > 3$, we have $p_0 < k$.
- If $3 \mid n$ but $5 \nmid n$: then $p_0 = 5$. Since $k \geq 7 > 5$, we have $p_0 < k$.
- If $3 \mid n$ and $5 \mid n$ but $7 \nmid n$: then $p_0 = 7$. Since $15 \mid (k+1)$, we get $k \geq 29 > 7 = p_0$.
- In general: suppose the primes $3, 5, \dots, q$ all divide n , and $q' \nmid n$ where q' is the next prime after q . Then $2 \cdot 3 \cdot 5 \cdots q \mid (k+1)$, so $k \geq 2 \cdot 3 \cdot 5 \cdots q - 1$. For $q \geq 5$, this bound exceeds q' (since $q' > q$ and $2 \cdot 3 \cdot 5 > 5$), giving $p_0 = q' < k$.

In every sub-case, the smallest prime p_0 not dividing n satisfies $p_0 < k$, so $s = p_0 \in \{2, \dots, k-1\}$ gives $\gcd(n, p_0) = 1$, a simple ring. ✓ \square

6 Geometric Structure and Figures

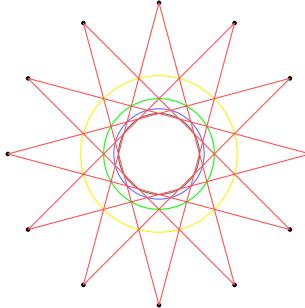


Figure 1: Regular dodecagram $\{12/5\}$ with schematic internal rings.

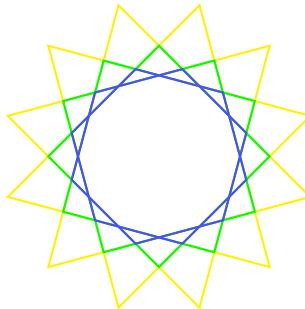


Figure 2: Composite rings of $\{12/5\}$: four equilateral triangles (yellow, r_3), three squares (green, r_2), and two regular hexagons (blue, r_1). Vertex positions match the exact intersection radii and angular offsets.

7 Discussion

The dodecagram appears as a minimal arithmetic configuration in which the prime factors of n cover the full interval of ring step sizes. This property does not persist for larger n , where simple stellations inevitably appear. Thus $\{12/5\}$ constitutes a sharp transition point between fully composite and mixed stellation regimes.

8 Conclusion

We have proven that $\{12/5\}$ is the unique regular star polygon satisfying $k = n/2 - 1$ with $k - 2 > 1$ whose non-convex internal rings are all composite. The result follows from elementary number theory and symmetry arguments and suggests further connections between stellation theory and arithmetic geometry.

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