① a) $a\cdot x + b\cdot y + c = 0$ $\qquad l = (a\ \ b\ \ c)^T$

Show that $x^T \cdot l = 0$

Let's say $X = (x,y) \Rightarrow X = (x,y,1) \Rightarrow$

$\Rightarrow (x,y,1)\cdot(a,b,c)^T = (x,y,1)\cdot l = 0 \Rightarrow \boxed{x^T\cdot l = 0}$

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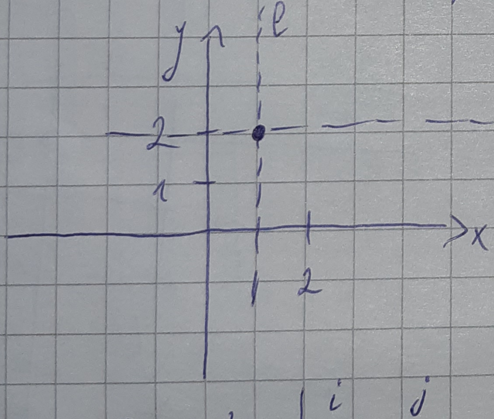
b) $l = (a\ \ b\ \ c)^T$, $\ l' = (a'\ \ b'\ \ c')^T$

$X = l \times l'$, From triple scalar product identity $l\cdot(l\times l') = l'\cdot(l\times l') = 0 \Rightarrow$

$\Rightarrow l^T\cdot X = l'^T \cdot X = 0$, point X belong to both lines.

Example: $l_1 = a\cdot x + by + c = 0$.

We have a point $[x,y] = [1,2]$



$l = (-1\ \ 0\ \ 1)^T = (a\ \ b\ \ c)^T$.

$-1\cdot 1 + 0\cdot y + 1 = 0$.

$l' = (0\ \ -1\ \ 2)^T$

$0\cdot x - 1\cdot 2 + 2 = 0$.

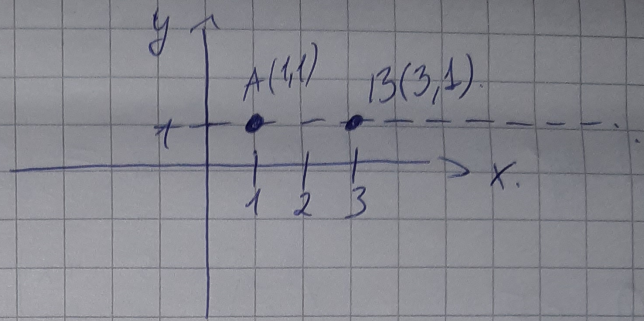
$x = l\times l' = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -1 & 2\end{vmatrix} = k + i + 2j = +1\cdot i + 2j + k$

x, y

(1,2)

1) c) $\boxed{l = x \times x'}$

$l = A \times B = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} \Leftrightarrow$



$\Leftrightarrow i + k + 3j - 3k - i - j = 0 \cdot i + 2j - 2k$

$l = (a\ \ b\ \ c)^T = (0\ \ 2\ \ -2)^T$

check that $x^T \cdot l = 0$. $\begin{cases} A^T \cdot l = (1\ \ 1\ \ 1) \cdot (0\ \ 2\ \ -2)^T = 0 \\ B^T \cdot l = (3\ \ 1\ \ 1) \cdot (0\ \ 2\ \ -2)^T = 0 \end{cases}$

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d) $y = \alpha x + (1-\alpha) \cdot x'$

assume: $x : (x_1, y_1)$, $x' : (x_2, y_2)$

$$\alpha x + (1-\alpha) x' = \alpha \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-\alpha) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + (1-\alpha) x_2 \\ \alpha \cdot y_1 + (1-\alpha) \cdot y_2 \end{pmatrix}$$

gradient $x - x'$: $\dfrac{y_2 - y_1}{x_2 - x_1}$

gradient $x - x''$: $\dfrac{\alpha \cdot y_1 + (1-\alpha) y_2 - y_1}{\alpha \cdot x_1 + (1-\alpha) x_2 - x_1} = \dfrac{(\alpha - 1) y_1 + (1-\alpha) y_2}{(\alpha - 1) \cdot x_1 + (1-\alpha) x_2} =$

$$= \frac{\cancel{(1-\alpha)}(y_2 - y_1)}{\cancel{(1-\alpha)}(x_2 - x_1)} = \frac{y_2 - y_1}{x_2 - x_1}$$

② a)/b) • Translation $x' = \begin{bmatrix} I & t \\ 0 & 1 \end{bmatrix} \cdot x$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$t = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

2 DOF

$$x' = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

• Euclidean transformation

3 DOF

$$x' = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \cdot x = \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

• Similarity transformation or scaled rotation

4 DOF

$$x' = \begin{bmatrix} s \cdot R & t \\ 0 & 1 \end{bmatrix} \cdot x = \begin{bmatrix} s \cdot r_{11} & s \cdot r_{12} & t_x \\ s \cdot r_{21} & s \cdot r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

• Affine transformation

6 DOF

$$x' = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

• Projective transformation.

$x' = H \cdot x$, where $H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$

8 DOF

2.c). The projective transformation H, has 8 DOF, because matrix H contains only 8 independent ratios.

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3 a). $ax + by + c = 0$, $l^T \cdot x = 0$, $l = [a \; b \; c]^T$

$x' = H \cdot x$ if x lie on l, $\Rightarrow \boxed{l' = H^{-T} \cdot l}$

or $\underline{l'^T = l^T \cdot H^{-1}}$

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b). $$I = \frac{(l_1^T \cdot x_1) \cdot (l_2^T \cdot x_2)}{(l_1^T \cdot x_2) \cdot (l_2^T \cdot x_1)}$$

$$l^T \cdot x = \overbrace{l^T \cdot H^{-1}}^{l'} \cdot \overbrace{H \cdot x}^{x'} = l'^T \cdot x'$$

$$\underbrace{H^{-1} \cdot H}_{I}$$