Aalto University
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ELEC-E8740 — Discretization of Continuous-Time Dynamic Models

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Contents

Intended Learning Outcomes

After this lecture, you will be able to:

- explain why continuous-time dynamic models need to be discretized in practice
- construct discrete-time dynamic models from linear ODE and SDE state-space models
- construct approximate discrete-time dynamic models from non-linear ODE and SDE models

Recap

- Nonlinear continuous-time state-space model:

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t) \\
\mathbf{y}_n &= \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n
\end{aligned}
$$

- Linear discrete-time state-space model:

$$
\begin{aligned}
\mathbf{x}_n &= \mathbf{F}\mathbf{x}_{n-1} + \mathbf{B}_q\mathbf{q}_n \\
\mathbf{y}_n &= \mathbf{G}\mathbf{x}_n + \mathbf{r}_n
\end{aligned}
$$

- Nonlinear discrete-time state-space model:

$$
\begin{aligned}
\mathbf{x}_n &= \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{B}_q(\mathbf{x}_{n-1})\mathbf{q}_n \\
\mathbf{y}_n &= \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n
\end{aligned}
$$

Discretization of Continuous-Time Models: Why?

- Sensor fusion is implemented in digital computers
- Data is only processed at $t_1, t_2, \ldots, t_n$
- Discretized continuous-time models are closely related to discrete-time models
- Example: Vehicle tracking

Discretization of continuous-time models is equivalent to solving the ODE/SDE model between t_{n-1} and t_n

Solving Linear First Order ODEs (1/2)

- Goal: Solve the first order ODE
$$\dot{x}(t) = a x(t) + b u(t),$$
on the interval $(t_{n-1}, t_n]$.
- Ansatz: Multiply by the integrating factor e^{-at}
$$e^{-at} \dot{x}(t) = e^{-at} a x(t) + e^{-at} b u(t)$$
i.e.
$$e^{-at} \dot{x}(t) - e^{-at} a x(t) = e^{-at} b u(t)$$
- We can then identify the derivative on the left hand side:
$$\frac{\mathrm{d}}{\mathrm{d}t} \left[e^{-at} x(t) \right] = e^{-at} \dot{x}(t) - e^{-at} a x(t).$$
- Thus we have
$$\frac{\mathrm{d}}{\mathrm{d}t} \left[e^{-at} x(t) \right] = e^{-at} b u(t).$$

Solving Linear First Order ODEs (2/2)

- We can now integrate the both sides:

$$\int_{t_{n-1}}^{t_n} \frac{\mathrm{d}}{\mathrm{d}t} \left[e^{-at} x(t) \right] \mathrm{d}t = \int_{t_{n-1}}^{t_n} e^{-at} b u(t) \mathrm{d}t$$

- Solution:

$$e^{-at_n} x(t_n) - e^{-at_{n-1}} x(t_{n-1}) = \int_{t_{n-1}}^{t_n} e^{-at} b u(t) \mathrm{d}t$$

- Rearranged:

$$x(t_n) = e^{a(t_n - t_{n-1})} x(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{a(t_n - t)} b u(t) \mathrm{d}t$$

- Defining $\Delta t = t_n - t_{n-1}$ this is

$$x(t_n) = e^{a\Delta t} x(t_{n-1}) + \int_{t_{n-1}}^{t_{n-1}+\Delta t} e^{a(t_{n-1}+\Delta t - t)} b u(t) \mathrm{d}t$$

Vector-Valued Linear First Order ODE

- General linear dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$

- This is a vector-valued first order ODE

What is the integrating factor for vector-valued first order ODEs?

Matrix Exponential

- Definition of the exponential:

$$e^a = \sum_{k=0}^{\infty} \frac{1}{k} a^k$$

- Definition of the matrix exponential:

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k$$

- Derivative of matrix exponential w.r.t. scalar t:

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\mathbf{A} t} = e^{\mathbf{A} t} \mathbf{A}$$

- Matrix exponential of $\mathbf{A}^\mathsf{T}$:

$$(e^{\mathbf{A}})^\mathsf{T} = e^{\mathbf{A}^\mathsf{T}}$$

Solving Linear First Order Vector ODEs (1/2)

- General linear dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}(t)$$

- Multiplication by the integrating factor $e^{-\mathbf{A}t}$:

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) = e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) + e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

- Rearranging:

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) = e^{-\mathbf{A}t}\mathbf{B}_u\mathbf{u}(t)$$

- Substituting $\frac{\mathrm{d}}{\mathrm{d}t}e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}}\mathbf{A}\mathbf{x}(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t}\mathbf{B}_u\mathbf{u}(t)$$

Solving Linear First Order Vector ODEs (2/2)

- We now have the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t}\mathbf{B}_u\mathbf{u}(t)$$

- Integration w.r.t. t:

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} \mathrm{d}\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right] &= \int_{t_{n-1}}^{t_n} e^{-\mathbf{A}t}\mathbf{B}_u\mathbf{u}(t)\mathrm{d}t \\
\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right]_{t=t_{n-1}}^{t_n} &= \int_{t_{n-1}}^{t_n} e^{-\mathbf{A}t}\mathbf{B}_u\mathbf{u}(t)\mathrm{d}t \\
e^{-\mathbf{A}t_n}\mathbf{x}(t_n) - e^{-\mathbf{A}t_{n-1}}\mathbf{x}(t_{n-1}) &= \int_{t_{n-1}}^{t_n} e^{-\mathbf{A}t}\mathbf{B}_u\mathbf{u}(t)\mathrm{d}t
\end{aligned}$$

- Rearranging:

$$\mathbf{x}(t_n) = e^{\mathbf{A}(t_n - t_{n-1})}\mathbf{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)}\mathbf{B}_u\mathbf{u}(t)\mathrm{d}t$$

Zero-Order-Hold Inputs

- Solution:

$$\mathbf{x}(t_n) = e^{\mathbf{A}(t_n - t_{n-1})} \mathbf{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_u \mathbf{u}(t) \mathrm{d}t,$$

- The input $\mathbf{u}(t)$ can be often assumed to be constant between sampling instants (zero-order-hold; ZOH)
- Then:

$$\int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_u \mathbf{u}(t) \mathrm{d}t = \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_u \mathrm{d}t \, \mathbf{u}(t_{n-1})$$

Discretized Deterministic Linear Dynamic Model

- Linear continuous-time dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u \mathbf{u}(t)$$

- Discretized dynamic model:

$$\mathbf{x}_n = \mathbf{F}_n \mathbf{x}_{n-1} + \mathbf{L}_n \mathbf{u}_{n-1},$$

where

$$\mathbf{F}_n \triangleq e^{\mathbf{A}(t_n - t_{n-1})}$$

$$\mathbf{L}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_u \, \mathrm{d}t$$

The discretized dynamic model is completely equivalent to the continuous-time model

Example: Deterministic 1D Motion Model (1/4)

- Dynamic model:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

- Recall:

$$\mathbf{F}_n = e^{\mathbf{A} \Delta t} = \sum_{j=0}^{\infty} \frac{1}{j!} \mathbf{A}^j (\Delta t)^j$$

Example: Deterministic 1D Motion Model (2/4)

- Powers of $\mathbf{A}$:

$$\mathbf{A}^0 = \mathbf{I}$$
$$\mathbf{A}^1 = \mathbf{A}$$
$$\mathbf{A}^j = \mathbf{0}\ j \geq 2$$

- Hence:

$$\mathbf{F}_n = \sum_{j=0}^{\infty} \frac{1}{j!} \mathbf{A}^j (\Delta t)^j = \frac{1}{0!} \mathbf{I} (\Delta t)^0 + \frac{1}{1!} \mathbf{A} \Delta t$$
$$= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

Example: Deterministic 1D Motion Model (3/4)

- Input matrix:

$$\mathbf{L}_n = \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_u \mathrm{d}t$$

where:

$$e^{\mathbf{A}(t_n - t)} = \begin{bmatrix} 1 & t_n - t \\ 0 & 1 \end{bmatrix}, \; \mathbf{B}_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example: Deterministic 1D Motion Model (4/4)

- Continuous-time model:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

- Discretized model:

$$\mathbf{x}_n = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \mathbf{x}_{n-1} + \begin{bmatrix} \frac{(\Delta t)^2}{2} \\ \Delta t \end{bmatrix} \mathbf{u}_{n-1}$$

Stochastic Linear Dynamic Model

- Stochastic linear dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_w \mathbf{w}(t),$$

- The only difference to the deterministic model is the input $\mathbf{u}(t)$ ($\mathbf{w}(t)$)
- $\mathbf{w}(t)$ is a zero-mean white stochastic process
- Auto-correlation function:

$$R_{ww}(\tau) = \mathrm{E}\{\mathbf{w}(t+\tau)\mathbf{w}(t)\} = \mathbf{\Sigma}_w \delta(\tau)$$

- Here $\mathbf{\Sigma}_w$ is the spectral density of the while noise.
- Hence:

$$\mathbf{x}(t_n) = e^{\mathbf{A}(t_n - t_{n-1})}\mathbf{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)}\mathbf{B}_w \mathbf{w}(t)\mathrm{d}t$$

Integration of the Stochastic Process

- Model:

$$\mathbf{x}(t_n) = e^{\mathbf{A}(t_n - t_{n-1})}\mathbf{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)}\mathbf{B}_w \mathbf{w}(t)\mathrm{d}t$$

- $\mathbf{w}(t)$ is stochastic; not ZOH and not even integrable (with standard tools)
- Define a random variable as the process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)}\mathbf{B}_w \mathbf{w}(t)\mathrm{d}t$$

- Then:

$$\mathbf{x}_n = \mathbf{F}_n \mathbf{x}_{n-1} + \mathbf{q}_n$$

Mean of the Process Noise

- Process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \mathbf{w}(t) \mathrm{d}t.$$

- We get

$$\begin{aligned}
\mathrm{E}\{\mathbf{q}_n\} &= \mathrm{E}\left\{ \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \mathbf{w}(t) \mathrm{d}t \right\} \\
&= \int_{t_{n-1}}^{t_n} \mathrm{E}\left\{ e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \mathbf{w}(t) \right\} \mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \, \mathrm{E}\left\{ \mathbf{w}(t) \right\} \mathrm{d}t \\
&= 0.
\end{aligned}$$

Covariance of the Process Noise (1/2)

- Process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \mathbf{w}(t) \mathrm{d}t$$

- Covariance:

$$\begin{aligned}
&\mathrm{Cov}\{\mathbf{q}_n\} \\
&= \mathrm{E}\{(\mathbf{q}_n - \mathrm{E}\{\mathbf{q}_n\})(\mathbf{q}_n - \mathrm{E}\{\mathbf{q}_n\})^\mathsf{T}\} \\
&= \mathrm{E}\{\mathbf{q}_n \mathbf{q}_n^\mathsf{T}\} \\
&= \mathrm{E}\left\{ \left(\int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \mathbf{w}(t) \mathrm{d}t \right) \left(\int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - \tau)} \mathbf{B}_w \mathbf{w}(\tau) \mathrm{d}\tau \right)^\mathsf{T} \right\} \\
&= \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \, \mathrm{E}\left\{ \mathbf{w}(t) \mathbf{w}(\tau)^\mathsf{T} \right\} \mathbf{B}_w^\mathsf{T} (e^{\mathbf{A}(t_n - \tau)})^\mathsf{T} \mathrm{d}\tau \mathrm{d}t \ldots
\end{aligned}$$

Covariance of the Process Noise (2/2)

- Process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \mathbf{w}(t) \mathrm{d}t$$

- Covariance:

$$\begin{aligned}
\mathrm{Cov}\{\mathbf{q}_n\} &= \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \, \mathrm{E}\left\{ \mathbf{w}(t) \mathbf{w}(\tau)^\mathsf{T} \right\} \mathbf{B}_w^\mathsf{T} (e^{\mathbf{A}(t_n - \tau)})^\mathsf{T} \mathrm{d}\tau \mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w R_{ww}(t - \tau) \mathbf{B}_w^\mathsf{T} (e^{\mathbf{A}(t_n - \tau)})^\mathsf{T} \mathrm{d}\tau \mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \boldsymbol{\Sigma}_w \delta(t - \tau) \mathbf{B}_w^\mathsf{T} (e^{\mathbf{A}(t_n - \tau)})^\mathsf{T} \mathrm{d}\tau \mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - \tau)} \mathbf{B}_w \boldsymbol{\Sigma}_w \mathbf{B}_w^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_n - \tau)} \mathrm{d}\tau
\end{aligned}$$

Properties of the Process Noise

- Process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_w \mathbf{w}(t) \mathrm{d}t$$

- Mean and covariance:

$$\begin{aligned}
\mathrm{E}\{\mathbf{q}_n\} &= 0 \\
\mathrm{Cov}\{\mathbf{q}_n\} &= \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - \tau)} \mathbf{B}_w \boldsymbol{\Sigma}_w \mathbf{B}_w^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_n - \tau)} \mathrm{d}\tau \triangleq \mathbf{Q}_n
\end{aligned}$$

- Distribution:

$$\mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n)$$

Discretized Stochastic Linear Dynamic Model

- Discretized stochastic dynamic model:

$$\mathbf{x}_n = \mathbf{F}_n \mathbf{x}_{n-1} + \mathbf{q}_n$$

where:

$$\begin{aligned}
\mathbf{F}_n &\triangleq e^{\mathbf{A}(t_n - t_{n-1})} \\
\mathbf{q}_n &\sim \mathcal{N}(0, \mathbf{Q}_n) \\
\mathbf{Q}_n &= \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n-\tau)} \mathbf{B}_w \boldsymbol{\Sigma}_w \mathbf{B}_w^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_n-\tau)} \mathrm{d}\tau
\end{aligned}$$

The discretized stochastic dynamic model is completely equivalent to the continuous-time model

Example: 1D Wiener Velocity Model (1/3)

- Dynamic model:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

  with white noise process $w(t)$ and $R_{ww}(\tau) = \sigma_w^2 \delta(\tau)$

- Process noise covariance:

$$\mathbf{Q}_n = \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - \tau)} \mathbf{B}_w \mathbf{\Sigma}_w \mathbf{B}_w^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_n - \tau)} \mathrm{d}\tau$$

- Recall:

$$e^{\mathbf{A}(t_n - \tau)} = \begin{bmatrix} 1 & t_n - \tau \\ 0 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}(t_n - \tau)} \mathbf{B}_w = \begin{bmatrix} 1 & t_n - \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_n - \tau \\ 1 \end{bmatrix}$$

Example: 1D Wiener Velocity Model (2/3)

- Process noise covariance:

$$
\begin{aligned}
\mathbf{Q}_n &= \int_{t_{n-1}}^{t_n} \begin{bmatrix} t_n - \tau \\ 1 \end{bmatrix} \sigma_w^2 \begin{bmatrix} t_n - \tau \\ 1 \end{bmatrix}^\mathsf{T} \mathrm{d}\tau \\
&= \sigma_w^2 \int_{t_{n-1}}^{t_n} \begin{bmatrix} t_n - \tau \\ 1 \end{bmatrix} \begin{bmatrix} t_n - \tau & 1 \end{bmatrix} \mathrm{d}\tau \\
&= \sigma_w^2 \int_{t_{n-1}}^{t_n} \begin{bmatrix} (t_n - \tau)^2 & t_n - \tau \\ t_n - \tau & 1 \end{bmatrix} \mathrm{d}\tau \\
&= \sigma_w^2 \begin{bmatrix} -\frac{1}{3}(t_n - \tau)^3 & -\frac{1}{2}(t_n - \tau)^2 \\ -\frac{1}{2}(t_n - \tau)^2 & \tau \end{bmatrix}_{\tau = t_{n-1}}^{t_n} \\
&= \sigma_w^2 \begin{bmatrix} \frac{(\Delta t)^3}{3} & \frac{(\Delta t)^2}{2} \\ \frac{(\Delta t)^2}{2} & \Delta t \end{bmatrix}
\end{aligned}
$$

Example: 1D Wiener Velocity Model (3/3)

- Dynamic model:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

- Discretized model:

$$\begin{bmatrix} p_n \\ v_n \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{n-1} \\ v_{n-1} \end{bmatrix} + \mathbf{q}_n$$

with $\mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n)$ and

$$\mathbf{Q}_n = \sigma_w^2 \begin{bmatrix} \frac{(\Delta t)^3}{3} & \frac{(\Delta t)^2}{2} \\ \frac{(\Delta t)^2}{2} & \Delta t \end{bmatrix}$$

Discretization of Nonlinear Dynamic Models

- Objective: Discretization of nonlinear models

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_u(\mathbf{x}(t))\mathbf{u}(t)$$

and

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Problem: In most cases, no exact approach exists
- A few possible approaches:
  - Linearization of the nonlinear model followed by discretization
  - Approximation of the derivative (integral)
  - Exact integration (of at least the dynamics)
  - & many more...

Linearization of Nonlinear Models

- Nonlinear dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_u(\mathbf{x}(t))\mathbf{u}(t)$$

- 1st order Taylor series approximation of $\mathbf{f}(\mathbf{x}(t))$ around $\mathbf{x}(t) = \mathbf{x}(t_{n-1})$:

$$\mathbf{f}(\mathbf{x}(t)) \approx \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{A}_x(\mathbf{x}(t) - \mathbf{x}_{n-1})$$

- Approximation of the ODE:

$$\dot{\mathbf{x}}(t) \approx \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{A}_x(\mathbf{x}(t) - \mathbf{x}_{n-1}) + \mathbf{B}_u\mathbf{u}(t)$$

Discretization of Linearized Models (1/2)

- Approximation of the ODE:

$$\dot{\mathbf{x}}(t) \approx \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{A}_x(\mathbf{x}(t) - \mathbf{x}_{n-1}) + \mathbf{B}_u \mathbf{u}(t)$$

- Rewritten approximation of the ODE

$$\dot{\mathbf{x}}(t) \approx \mathbf{A}_x \mathbf{x}(t) + \mathbf{f}(\mathbf{x}_{n-1}) - \mathbf{A}_x \mathbf{x}_{n-1} + \mathbf{B}_u \mathbf{u}(t)$$

- Solution of the approximation:

$$\begin{aligned}\mathbf{x}_n &\approx e^{\mathbf{A}_x \Delta t} \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathrm{d}t \, \mathbf{f}(\mathbf{x}_{n-1}) \\ &\quad - \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathrm{d}t \, \mathbf{A}_x \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathbf{B}_u \mathbf{u}(t) \mathrm{d}t\end{aligned}$$

Discretization of Linearized Models (2/2)

- Solution of the approximation:

$$
\begin{aligned}
\mathbf{x}_n \approx e^{\mathbf{A}_x \Delta t} \mathbf{x}_{n-1} &+ \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathrm{d}t \, \mathbf{f}(\mathbf{x}_{n-1}) \\
&- \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathrm{d}t \, \mathbf{A}_x \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathbf{B}_u \mathbf{u}(t) \mathrm{d}t
\end{aligned}
$$

- Simplified solution:

$$
\mathbf{x}_n \approx \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathrm{d}t \, \mathbf{f}(\mathbf{x}_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathbf{B}_u \mathbf{u}(t) \mathrm{d}t
$$

Discretization of Linearized Models (Stochastic)

- Stochastic nonlinear model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Discretization is the same as for the ODE model:

$$\begin{aligned}
\mathbf{x}_n &\approx \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathrm{d}t \, \mathbf{f}(\mathbf{x}_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathbf{B}_w \mathbf{w}(t) \mathrm{d}t \\
&= \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathrm{d}t \, \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{q}_n &\sim \mathcal{N}(0, \mathbf{Q}_n), \\
\mathbf{Q}_n &\approx \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - \tau)} \mathbf{B}_w \mathbf{\Sigma}_w \mathbf{B}_w^\mathsf{T} e^{\mathbf{A}_x^\mathsf{T} (t_n - \tau)} \mathrm{d}\tau
\end{aligned}$$

Properties of the Discretization

- Stochastic nonlinear model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Linearized model:

$$\dot{\mathbf{x}}(t) \approx \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{A}_x(x(t) - x_{n-1}) + \mathbf{B}_w\mathbf{w}(t)$$

- Discretized model:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x(t_n - t)} \mathrm{d}t \, \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

- Integration is exact, model is not
- Discretization is not exact
- Linearization is local, may cause problems

Example: Quasi-Constant Turn Model (1/5)

- Model:

$$\begin{bmatrix} \dot{p}^x(t) \\ \dot{p}^y(t) \\ \dot{v}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} v(t)\cos(\varphi(t)) \\ v(t)\sin(\varphi(t)) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{w}(t)$$

- Jacobian of $\mathbf{f}(\mathbf{x}(t))$:

$$\mathbf{A}_x = \begin{bmatrix} 0 & 0 & \cos(\varphi(t)) & -v(t)\sin(\varphi(t)) \\ 0 & 0 & \sin(\varphi(t)) & v(t)\cos(\varphi(t)) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cos(\varphi_{n-1}) & -v_{n-1}\sin(\varphi_{n-1}) \\ 0 & 0 & \sin(\varphi_{n-1}) & v_{n-1}\cos(\varphi_{n-1}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example: Quasi-Constant Turn Model (2/5)

- Powers of $\mathbf{A}_x$:

$$
\begin{aligned}
\mathbf{A}_x^0 &= \mathbf{I} \\
\mathbf{A}_x^1 &= \mathbf{A}_x \\
\mathbf{A}_x^2 &= \mathbf{0}
\end{aligned}
$$

- Matrix exponential:

$$
\begin{aligned}
e^{\mathbf{A}_x (t_n - t)} &= \mathbf{I} + \mathbf{A}_x (t_n - t) \\
&= \begin{bmatrix}
1 & 0 & \cos(\varphi_{n-1})(t_n - t) & -v_{n-1} \sin(\varphi_{n-1})(t_n - t) \\
0 & 1 & \sin(\varphi_{n-1})(t_n - t) & v_{n-1} \cos(\varphi_{n-1})(t_n - t) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{aligned}
$$

Example: Quasi-Constant Turn Model (3/5)

- Integral:

$$\int_{t_{n-1}}^{t_n} \begin{bmatrix} 1 & 0 & \cos(\varphi_{n-1})(t_n - t) & -v_{n-1}\sin(\varphi_{n-1})(t_n - t) \\ 0 & 1 & \sin(\varphi_{n-1})(t_n - t) & v_{n-1}\cos(\varphi_{n-1})(t_n - t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{d}t$$

$$= \begin{bmatrix} t & 0 & -\frac{(t_n-t)^2}{2}\cos(\varphi_{n-1}) & \frac{(t_n-t)^2}{2} v_{n-1}\sin(\varphi_{n-1}) \\ 0 & t & -\frac{(t_n-t)^2}{2}\sin(\varphi_{n-1}) & -\frac{(t_n-t)^2}{2} v_{n-1}\cos(\varphi_{n-1}) \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{bmatrix}_{t=t_{n-1}}^{t_n}$$

$$= \begin{bmatrix} \Delta t & 0 & \frac{(\Delta t)^2}{2}\cos(\varphi_{n-1}) & -\frac{(\Delta t)^2}{2} v_{n-1}\sin(\varphi_{n-1}) \\ 0 & \Delta t & \frac{(\Delta t)^2}{2}\sin(\varphi_{n-1}) & \frac{(\Delta t)^2}{2} v_{n-1}\cos(\varphi_{n-1}) \\ 0 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & \Delta t \end{bmatrix}$$

Example: Quasi-Constant Turn Model (4/5)

- Discretized model:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathrm{d}t \, \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

- Second term:

$$\begin{bmatrix} \Delta t & 0 & \frac{(\Delta t)^2}{2} \cos(\varphi_{n-1}) & -\frac{(\Delta t)^2}{2} v_{n-1} \sin(\varphi_{n-1}) \\ 0 & \Delta t & \frac{(\Delta t)^2}{2} \sin(\varphi_{n-1}) & \frac{(\Delta t)^2}{2} v_{n-1} \cos(\varphi_{n-1}) \\ 0 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & \Delta t \end{bmatrix} \begin{bmatrix} v_{n-1} \cos(\varphi_{n-1}) \\ v_{n-1} \sin(\varphi_{n-1}) \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \Delta t v_{n-1} \cos(\varphi_{n-1}) \\ \Delta t v_{n-1} \sin(\varphi_{n-1}) \\ 0 \\ 0 \end{bmatrix}$$

Example: Quasi-Constant Turn Model (5/5)

- Discretized model:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathrm{d}t \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

- Discretization of Linearized Model:

$$\begin{bmatrix} p_n^x \\ p_n^y \\ v_n \\ \varphi_n \end{bmatrix} = \begin{bmatrix} p_{n-1}^x \\ p_{n-1}^y \\ v_{n-1} \\ \varphi_{n-1} \end{bmatrix} + \begin{bmatrix} \Delta t v_{n-1} \cos(\varphi_{n-1}) \\ \Delta t v_{n-1} \sin(\varphi_{n-1}) \\ 0 \\ 0 \end{bmatrix} + \mathbf{q}_n$$

- What about $\mathbf{Q}_n$?

$$\mathbf{Q}_n \approx \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - \tau)} \mathbf{B}_w \boldsymbol{\Sigma}_w \mathbf{B}_w^\mathsf{T} e^{\mathbf{A}_x^\mathsf{T} (t_n - \tau)} \mathrm{d}\tau$$

Euler Approximation

- Dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_u(\mathbf{x}(t))\mathbf{u}(t)$$

- Integral equation:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} \mathbf{f}(\mathbf{x}(t))\mathrm{d}t + \int_{t_{n-1}}^{t_n} \mathbf{B}_u(\mathbf{x}(t))\mathbf{u}(t)\mathrm{d}t$$

- Idea: Approximate the integral rather than the model
- Euler approximation:

$$\mathbf{x}_n \approx \mathbf{x}_{n-1} + \Delta t \mathbf{f}(\mathbf{x}_{n-1}) + \Delta t \mathbf{B}_u(\mathbf{x}_{n-1})\mathbf{u}_{n-1}.$$

Euler–Maruyama Discretization (1)

- Stochastic dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Integral representation:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} \mathbf{f}(\mathbf{x}(t))\mathrm{d}t + \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)\mathrm{d}t$$

- Process noise definition:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)\mathrm{d}t$$

Mean of the Process Noise

- Process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)\mathrm{d}t$$

- Mean:

$$\begin{aligned}
\mathrm{E}\{\mathbf{q}_n\} &= \mathrm{E}\left\{\int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)\mathrm{d}t\right\} \\
&= \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t))\,\mathrm{E}\left\{\mathbf{w}(t)\right\}\mathrm{d}t \\
&= 0
\end{aligned}$$

Covariance of the Process Noise (1/2)

- Process noise:

$$\mathbf{q}_n \triangleq \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)\mathrm{d}t$$

- Covariance:

$$\begin{aligned}
\mathrm{Cov}\{\mathbf{q}_n\} &= \mathrm{E}\left\{ \left(\int_{t_{n-1}}^{t_n} \mathbf{B}_w \mathbf{w}(t) \mathrm{d}t \right) \left(\int_{t_{n-1}}^{t_n} \mathbf{B}_w \mathbf{w}(\tau) \mathrm{d}\tau \right)^\mathsf{T} \right\} \\
&= \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t))\, \mathrm{E}\{\mathbf{w}(t)\mathbf{w}(\tau)^\mathsf{T}\} \mathbf{B}_w(\mathbf{x}(t))^\mathsf{T} \mathrm{d}\tau \mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t)) \boldsymbol{\Sigma}_w \delta(t-\tau) \mathbf{B}_w(\mathbf{x}(t)))^\mathsf{T} \mathrm{d}\tau \mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t))) \boldsymbol{\Sigma}_w \mathbf{B}_w^\mathsf{T}(\mathbf{x}(t)) \mathrm{d}\tau
\end{aligned}$$

Covariance of the Process Noise (2/2)

- Covariance:

$$\mathrm{Cov}\{\mathbf{q}_n\} = \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t))\boldsymbol{\Sigma}_w \mathbf{B}_w(\mathbf{x}(t))^\mathsf{T} \mathrm{d}\tau$$

- Rectangle approximation of the integral:

$$\begin{aligned}
\mathrm{Cov}\{\mathbf{q}_n\} &= \int_{t_{n-1}}^{t_n} \mathbf{B}_w(\mathbf{x}(t)))\boldsymbol{\Sigma}_w \mathbf{B}_w^\mathsf{T}(\mathbf{x}(t)) \mathrm{d}\tau \\
&\approx \mathbf{B}_w(\mathbf{x}_{n-1})\boldsymbol{\Sigma}_w \mathbf{B}_w(\mathbf{x}_{n-1})^\mathsf{T}(t_n - t_{n-1}) \\
&= \Delta t \mathbf{B}_w(\mathbf{x}_{n-1})\boldsymbol{\Sigma}_w \mathbf{B}_w(\mathbf{x}_{n-1})^\mathsf{T} \\
&\triangleq \mathbf{Q}_n
\end{aligned}$$

Euler–Maruyama Discretization (2)

- Dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Euler–Maruyama discretization:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \Delta t \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

  with $\mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n),\ \mathbf{Q}_n \approx \Delta t \mathbf{B}_w(\mathbf{x}_{n-1}) \mathbf{\Sigma}_w \mathbf{B}_w(\mathbf{x}_{n-1})^\mathsf{T}$

- ... or equivalently:

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \Delta t \mathbf{f}(\mathbf{x}_{n-1}) + \sqrt{\Delta t} \mathbf{B}_w(\mathbf{x}_{n-1}) \mathbf{q}_n$$

  with $\mathbf{q}_n \sim \mathcal{N}(0, \mathbf{\Sigma}_w)$

- Discretization is not exact

Summary (1/3)

- The discretization of the linear ODE model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_u \mathbf{u}(t)$$

is

$$\mathbf{x}_n = \mathbf{F}_n \mathbf{x}_{n-1} + \mathbf{L}_n \mathbf{u}_{n-1}$$

$$\mathbf{F}_n \triangleq e^{\mathbf{A}(t_n - t_{n-1})}, \; \mathbf{L}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - t)} \mathbf{B}_u \mathrm{d}t$$

- The discretization of the linear SDE model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_w \mathbf{w}(t)$$

is

$$\mathbf{x}_n = \mathbf{F}_n \mathbf{x}_{n-1} + \mathbf{q}_n, \; \mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n)$$

$$\mathbf{Q}_n = \int_{t_{n-1}}^{t_n} e^{\mathbf{A}(t_n - \tau)} \mathbf{B}_w \mathbf{\Sigma}_w \mathbf{B}_w^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_n - \tau)} \mathrm{d}\tau$$

Summary (2/3)

- Nonlinear dynamic model:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$

- Discretization of the linearized model:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w \mathbf{w}(t) \\ &\approx \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{A}_x(\mathbf{x}(t) - \mathbf{x}_{n-1}) + \mathbf{B}_w \mathbf{w}(t)\end{aligned}$$

$$\Downarrow$$

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - t)} \mathrm{d}t \, \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

with

$$\mathbf{q}_n \sim \mathcal{N}(0, \mathbf{Q}_n), \;\; \mathbf{Q}_n \approx \int_{t_{n-1}}^{t_n} e^{\mathbf{A}_x (t_n - \tau)} \mathbf{B}_w \boldsymbol{\Sigma}_w \mathbf{B}_w^\mathsf{T} e^{\mathbf{A}_x^\mathsf{T} (t_n - \tau)} \mathrm{d}\tau$$

Summary (3/3)

- Euler–Maruyama discretization:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}_w(\mathbf{x}(t))\mathbf{w}(t)$$
$$\Downarrow$$
$$\mathbf{x}_n = \mathbf{x}_{n-1} + \Delta t \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

with

$$\begin{aligned} \mathbf{q}_n &\sim \mathcal{N}(0, \mathbf{Q}_n), \\ \mathbf{Q}_n &\approx \Delta t \mathbf{B}_w(\mathbf{x}_{n-1}) \mathbf{\Sigma}_w \mathbf{B}_w(\mathbf{x}_{n-1})^\mathsf{T}. \end{aligned}$$