Session 2

Tuesday 22. September 2020 11.53

1 $\mathbb{E}[X] = \int_{-\infty}^{\infty} x \, p(x) \, dx$ (1)

$\text{var}[X] = \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 \, p(x) \, dx$ (2).

$\int_{-\infty}^{\infty} p(x) \, dx = 1.$

$\int_{-\infty}^{\infty} \mu^2 \, p(x) \, dx = \mu^2 \int_{-\infty}^{\infty} p(x) \, dx$

$= \mu^2$

a). Show that $\underbrace{\text{var}[X]}_{(2)} = \mathbb{E}[X^2] - (\mathbb{E}(X)^2).$

$$\text{var}[X] = \int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 \, p(x) \, dx$$

$$= \int_{-\infty}^{\infty} \left(x^2 - 2\mathbb{E}(X)\, x + \mathbb{E}(X)^2\right) p(x) \, dx.$$

$$= \int_{-\infty}^{\infty} x^2 \, p(x) \, dx - 2\int_{-\infty}^{\infty} \mathbb{E}(X)\, x \, p(x) \, dx$$

$$+ \int_{-\infty}^{\infty} \mathbb{E}(X)^2 \, p(x) \, dx.$$

$$= \int_{-\infty}^{\infty} x^2 p(x) \, dx - 2\,\mathbb{E}(X) \int_{-\infty}^{\infty} x \, p(x) \, dx$$

$$+ \mathbb{E}(X)^2$$

$$= \mathbb{E}[X^2] - 2\mathbb{E}(X)\,\mathbb{E}[X] + \mathbb{E}[X]^2$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

X is random variable, x is just an integration variable.

b). $p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right).$

$X \sim N(\mu, \sigma^2)$

a&b cnst

$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \, p(x) \, dx$

$\mathbb{E}[aX+b] = a\,\mathbb{E}[X] + b.$

we introduce a change of variable:

define: $z = \underbrace{\frac{(x-\mu)}{\sigma}}$

$\int p(x)\, dx$

$x = \sigma z + \mu.$

$= \int \underbrace{p(z)} \, dz$

$= \int_{-\infty}^{\infty} (\sigma z + \mu) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$

$= \int_{-\infty}^{\infty} \mu \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz\right)$

$+ \sigma \int_{-\infty}^{\infty} z \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz}$

$= \mu + \sigma \underbrace{\int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz}_{\mathrm{I} = 0.}$

generally, $f(z) = k \exp(g(z))$ $\qquad g(z) = \frac{-z^2}{2}$

$\frac{df}{dz} = k \exp(g(z)) \cdot \frac{dg(z)}{dz}$

$= f(z) \cdot dg(z)/dz.$

$p(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$

$$\frac{dp_z}{dz} = \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)}\left(-z\right) = \underline{p(z).(-z)}$$

$$\mathrm{II} = \sigma \int_{-n}^{n} \overbrace{z \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)} dz$$

$$= \sigma \int_{-n}^{n} -\frac{d}{dz}\left[\overbrace{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)}\right] dz$$

$$= \sigma \left(\frac{1}{2\pi} \exp\left(\frac{-z^2}{2}\right)\right)\Bigg|_{-\infty}^{\infty}$$

$$= \sigma . 0 //$$

$$E[x] = \mu //$$

c). what about the variance?

$$var[X] = E[(X - E(X))^2]$$

$$= E[X^2] - E[X]^2$$

$$= \underbrace{E[X^2]} - \mu^2 //$$

$$E[X^2] = \int_{-n}^{n} x^2 \; p(x) \; dx.$$

$$\frac{dx}{dz} = \sigma$$

we use 'again' $x = (\sigma z + \mu)$

$$= \int_{-n}^{n} (\sigma z + \mu)(\sigma z + \mu) \; \underline{p(x) \, dx}$$

$$= \int_{-n} (\sigma z + \mu)(\sigma z + \mu) \; \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz$$

$$= \int^{n} (\sigma^2 z^2 + 2\sigma\mu z + \mu^2) \overbrace{\frac{1}{\quad} \exp\left(\frac{-z^2}{2}\right)}^{P_z} dz$$

$$= \int_{-\infty}^{\infty} (\sigma^2 z^2 + 2\sigma\mu z + \mu^2) \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)} dz$$

$$= \sigma^2 \underbrace{\int_{-\infty}^{\infty} z^2 p_z \, dz}_{\text{II}} + 2\sigma\mu \underbrace{\int_{-\infty}^{\infty} z \, p_z \, dz}_{0} + \mu^2$$

$$\text{II} = \int_{-\infty}^{\infty} z^2 p_z \, dz = \int_{-\infty}^{\infty} \underbrace{(-z)}_{u} \cdot \underbrace{[-z \cdot p_z \, dz]}_{dv} \qquad v = p_z$$

$$= \int_{-\infty}^{\infty} u \, dv = uv \Big|_{-\infty}^{\infty} - \int v \, du$$

$$= -z \cdot p_z(z) \Big|_{-\infty}^{\infty} + \int p_z \, dz$$

$$\frac{z}{\sqrt{2\pi}} \cdot \exp\left(\frac{-z^2}{2}\right) \Big|_{\infty}^{-\infty}$$

$$= 0 + 1$$

$$= 1$$

$$E[X^2] = \sigma^2 \cdot 1 + \mu^2$$

hence $\text{var}[X] = \sigma^2$ //

Exercise 2:

$Z = LX$, $E[X] = m$ and $\text{cov}\{X\} = P$

then $E[Z] = Lm$, $\text{cov}[Z] = LPL^T$.

Let say $X \in \mathbb{R}^{q \times 1}$, $L \in \mathbb{R}^{p \times q}$.

$Z \in \mathbb{R}^{p \times 1}$

$$E[X] = \int x \, p(x) \, dx = m \in \mathbb{R}^{q \times 1}$$

$$x_1, \ldots, x_q = \int \cdots \int x \, p(x) \, dx_1 \, dx_2 \ldots dx_q.$$

$$\operatorname{cov}[X] = \mathbb{E}[\underbrace{(X-\mathbb{E}(X))}_{\mathbb{R}^{q\times 1}} \underbrace{(X-\mathbb{E}(X))^T}_{\mathbb{R}^{1\times q}}]$$

asked: $\operatorname{cov}[Z]$?

$$= \mathbb{E}[XX^T] - \mathbb{E}[X]\mathbb{E}[X^T] = \mathbb{E}[XX^T] - mm^T$$

$$\operatorname{cov}[Z] = \mathbb{E}[[Z-\mathbb{E}[Z]][Z-\mathbb{E}[Z]]^T]$$

$$= \mathbb{E}[[LX - \mathbb{E}[LX]][LX - \mathbb{E}[LX]]^T]$$

$$\mathbb{E}[Z] = \mathbb{E}[LX] = L\,\mathbb{E}[X]$$

$$= Lm$$

$$\operatorname{cov}[Z] = \mathbb{E}[[LX-Lm][LX-Lm]^T]$$

$$= \mathbb{E}[LXX^TL^T - LXm^TL^T - LmX^TL^T$$

$$+ \underbrace{Lmm^TL^T}]$$

$$= L\,\mathbb{E}[XX^T]L^T - L\overbrace{\mathbb{E}[X]}^{m}m^TL^T$$

$$- Lm\,\mathbb{E}[X]L^T$$

$$+ Lmm^TL^T$$

$$= L\,\mathbb{E}[XX^T]L^T - Lmm^TL^T \cancel{- Lmm^TL^T}$$

$$\cancel{+ Lmm^TL^T}$$

$$= L\,\mathbb{E}[XX^T]L^T - Lmm^TL^T$$

$$= L(\mathbb{E}[XX^T] - mm^T)L^T$$

$$:= L \operatorname{cov}[X] L^T$$

$$= L P L^T.$$

Exercise 3.

(a,b)

measument model: $y_n = g_n(x) + r_n$

$$\underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{bmatrix} + \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

$$\underline{y} \quad \underline{g}(x) + \underline{r}$$

non Weighted LS: $J = (\underline{y} - \underline{g}(x))^T \, I \, (\underline{y} - \underline{g}(x))$

$$= (\underline{y} - \underline{g}(x))^T (\underline{y} - \underline{g}(x))$$

if we have now $\sigma_n \neq 1$

we define

Weighted LS = $\underline{J = (y - g(x))^T W (y - g(x))}$

$\underline{\text{where } W = R^{-1}}$

$$= \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_m^2 \end{bmatrix}^{-1}$$

$$= (y - g(x))^T \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}^{-1} (y - g(x))$$

we need to show that

$$\hat{y}_n = \frac{1}{\sigma_n} y_n \, , \quad \hat{g}_n(x) = \frac{1}{\sigma_n} g_n(x)$$

$$\hat{r}_n = \frac{1}{\sigma_n} r_n$$

$$\hat{\underline{y}} = \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \end{bmatrix} y$$

$\text{var}\{\hat{r}_n\} = 1$

$\text{var}\{a x\} = a^2 \text{var}\{x\}$

$$\begin{bmatrix} \ddots & & \\ & \ddots & \\ & & \frac{1}{\sigma_m} \end{bmatrix}$$

$$\triangleq M \cdot y$$

$\therefore \operatorname{var}\{ax\} = a^2 \operatorname{var}\{x\}$

where $MM = R^{-1}$

then $\hat{y} = M\,y = M\,g(x) + M.r$

then if we use $W = I$ then

$$\hat{J} = (\hat{y} - \hat{g}(x))^T \cdot (\hat{y} - g(x))$$

$$= (M(y - g(x)))^T (M(y - g(x)))$$

$$= (y - g(x))^T M^T M (y - g(x))$$

$$= (y - g(x))^T \underbrace{R^{-1}}_{W} (y - g(x)) = J_{WLS.}$$

$E \neq$ Expectation

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$p(\underline{X})$ in eq 8.

$$\underbrace{\cancel{var}\ \text{cov}[X]}_{\text{is } \mathbb{R}^{2\times 2}} = E[(X - E(X))(X - E(X))^T]$$