Aalto University
School of Electrical Engineering

Exercise and Homework Round 11

These exercises (except for the last) will be gone through on **Tuesday December 1st 12:15–14:00** in the exercise session. The last exercise is a homework which you should return via mycourses by **Tuesday December 8th at 12:00**.

Exercise 1. (Particle filter for Gaussian random walk)

Consider the model

$$\begin{aligned} x_n &= x_{n-1} + q_{n-1}, \\ y_n &= x_n + r_n, \end{aligned} \tag{1}$$

where $x_0 \sim \mathcal{N}(0,1)$, $q_{n-1} \sim \mathcal{N}(0,1)$, and $r_n \sim \mathcal{N}(0,1)$.

(a) Simulate data from the model and implement a particle filter for the model. Compare its performance against Kalman filter.

(b) As Kalman filter is exact for this model, particle filter should converge to it when the number of particles goes to infinity. Check numerically that the mean and variance sequences have that property.

Exercise 2. (Particle filter for robot model)

Recall the robot tracking models from Exercises 10.2 and 10.3.

(a) Implement a particle filter to the model with direct position measurements as in Exercise 10.2. Using the simulated data from 10.2, evaluate its performance against the EKF.

(b) Modify the particle filter to work with distance and bearing measurements. Evaluate its performance against the EKF using the data from Exercise 10.3.

Exercise 3. (Particle filter convergence)

Recall that so called Bayesian filter equations (e.g. Särkkä, Bayesian Filtering and Smoothing, CUP, 2013) are given as:

- Prediction step:

$$p(x_n \mid y_1, \ldots, y_{n-1}) = \int p(x_n \mid x_{n-1})\, p(x_{n-1} \mid y_1, \ldots, y_{n-1})\, dx_{n-1}. \quad (2)$$

- Update step:

$$p(x_n \mid y_1, \ldots, y_n) = \frac{p(y_n \mid x_n)\, p(x_n|y_1, \ldots, y_{n-1})}{\int p(y_n \mid x_n)\, p(x_n \mid y_1, \ldots, y_{n-1}) dx_n}. \quad (3)$$

Let us now informally "prove" the convergence of particle filter (for an example of proper proof see, e.g., https://arxiv.org/abs/1403.6585).

(a) Assume that we have a set of samples $\{x_{n-1}^j\}$ for $j = 1, \ldots, J$, which represents the filtering distribution $p(x_{n-1} \mid y_1, \ldots, y_{n-1})$ in the sense that

$$\frac{1}{J} \sum_{j=1}^{J} \phi(x_{n-1}^j) \approx \int \phi(x_{n-1})\, p(x_{n-1} \mid y_1, \ldots, y_{n-1})\, dx_{n-1} \quad (4)$$

where $\phi(x)$ is an arbitrary function. What selection of $\phi(x)$ would correspond to the case that the mean is correct? How about what corresponds to variance?

(b) Convince yourself that simulating the dynamics starting from each x_{n-1}^j will result in samples x_n^j which approximate

$$\int p(x_n \mid x_{n-1})\, p(x_{n-1} \mid y_1, \ldots, y_{n-1})\, dx_{n-1}. \quad (5)$$

Conclude that we then have

$$\frac{1}{J} \sum_{j=1}^{J} \phi(x_n^j) \approx \int \phi(x_n)\, p(x_n \mid y_1, \ldots, y_{n-1})\, dx_n \quad (6)$$

which implies that we have an approximation of the prediction step.

(c) Let us now evaluate $\tilde{w}_n^j = p(y_n \mid x_n^j)$ and use these as the importance weights. Convince yourself that we should now have

$$\sum_{j=1}^{J} \tilde{w}_n^j \, \phi(x_n^j) \approx \int p(y_n \mid x_n) \, \phi(x_n) \, p(x_n \mid y_1, \ldots, y_{n-1}) \, dx_n. \tag{7}$$

Similarly by setting $\phi(x) = 1$ we get

$$\sum_{j=1}^{J} \tilde{w}_n^j \approx \int p(y_n \mid x_n) \, p(x_n \mid y_1, \ldots, y_{n-1}) \, dx_n. \tag{8}$$

From this conclude that we should have

$$\frac{\sum_{j=1}^{J} \tilde{w}_n^j \, \phi(x_n^j)}{\sum_{j=1}^{J} \tilde{w}_n^j} = \sum_{j=1}^{J} w_n^j \, \phi(x_n^j) \approx \int \phi(x_n) \, p(x_n \mid y_1, \ldots, y_n) \, dx_n \tag{9}$$

where

$$w_n^j = \frac{\tilde{w}_n^j}{\sum_{j=1}^{J} \tilde{w}_n^j}. \tag{10}$$

(d) Resampling converts the weighed samples (x_n^j, w_n^j) into set of samples with uniform weights of the form $(x_n^j, 1/J)$. Convince yourself that after the resampling we should have

$$\frac{1}{J} \sum_{j=1}^{J} \phi(x_n^j) \approx \int \phi(x_n) \, p(x_n \mid y_1, \ldots, y_n) \, dx_n, \tag{11}$$

which closes the recursion of the Bayesian filter.

Homework 11 (DL Tuesday December 8th at 12:00)

A classical demonstration of Monte Carlo methods (which a particle filter is) is the approximation of π by simulation. Your task is to implement this demonstration. Recall that the area of a circle of radius r is $A_c = \pi r^2$ and area of a square with side length $2r$ is $A_s = 4r^2$. Hence the ratio of these is $A_c/A_s = \pi/4$ which allows us to approximate the value $\pi/4$ and hence π. Now do the following:

(a) Select suitable radius and simulate N uniform random samples on the square $[-r, r] \times [-r, r]$. Check how many of the samples fall inside a circle of radius r centered on the origin. Let this number be M. The ratio can now be approximated as $A_c/A_s \approx M/N$. Use this result to approximate the value of π.

(b) Check how the accuracy of the approximation increases with increasing N. How much samples would you use to determine the first 2 decimals of π correctly?