Exercise 8

Tuesday 10. November 2020 12.06

Question 1

$$\dot{\underline{x}}_t = A \, \underline{x}_t \quad \ldots (1)$$

assume that $u_t = 0$.

b) The solution of (1) is given by : $\underline{x}_t = \exp(At) \cdot \underline{x}_0 \quad \ldots (2)$

In Ex 6, we compute $\underline{\exp(At)}$ via the Laplace transform.
so if we assume that (expm) routine gives a very close approximation to the anlyfical solution

$t = 0, t_1, \ldots\ldots, t_n$

$t_0 = 0$

$$\underline{x}_t = [x_0, \exp(At_1)x_0, \exp(At_2)x_0 \ldots\ldots] \quad \ldots (3)$$

c) If we discretize the model with $\Delta t = 0.1$?

$$\underline{x}_n = F \, \underline{x}_{n-1}$$

here $\underline{x}_n := \underline{x}_{t_n}$, but since δt is constant
then $\underline{x}_n := \underline{x}_{n.\Delta t}$.

$$\underline{x}_{n\Delta t} = \exp(A\Delta t) \, \underline{x}_{(n-1)\Delta t}$$

(4). $$\left(\underline{x}_n = F \, \underline{x}_{n-1}\right) \; ; \; F = \exp(A\Delta t).$$

we should expect that the dizeretization of x_t at t_0, t_1 t_n, gives exact value ; that is solution obtained from (3) is equivalent to solution of (4).

let say $t_1 = 1.\delta t$

$$\underline{x}_1 = F \, x_0$$
$$= \exp(A\Delta t) \, x_0$$
$$= \exp(At_1) \, x_0 = x_{t_1}$$

$$\underline{x}_2 = F\ \underline{x}_1$$
$$= \underbrace{\exp(A\Delta t)}\cdot\underbrace{\exp(A\Delta t)}\ x_0$$
$$= \exp(A\, 2\cdot\Delta t)\ x_0$$
$$= \exp(A t_2)\ x_0 = x_{t_2}\ !$$

$$\vdots$$

$$\underline{x}_3 = F\cdot x_2 = x_{t_3}.$$

Question 2 :

The Wiener velocity model:

assume that $\mathcal{N}(0, \sigma_w^2 I)$

$$\underline{\dot{x}}_t = A\ \underline{x}_t + B\ \underline{W}_t \quad \ldots (2.1)$$

where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

a) with fixed Δt, write down the discretization model.

agin as before we use $\underset{n\Delta t}{\underline{x}_n} := \underline{x}_{t_n} = x_{\Delta t \cdot n}$.

$$\underline{x}_n = e^{(A\Delta t)}\ \underline{x}_{n-1} + \int_{(n-1)\Delta t}^{n\Delta t} e^{A(n\Delta t - \tau)}\ B\, W_\tau\, d\tau \quad .(2.2)$$

since Δt is fixed, then

$$\underline{x}_n = F\ \underline{x}_{n-1} + \underline{q}_n$$

where
$$\underline{q}_n = \int_{(n-1)\Delta t}^{n\Delta t} e^{A(n\Delta t - \tau)}\ B\, W_\tau\, d\tau \quad \ldots (2.3)$$

First examination : A matrix is special, it is "degenerate"
so its easy to compute $\exp(A\Delta t)$ directly:

$$\exp(A\Delta t) := \sum_{i=0}^{\infty} \frac{(A\Delta t)^i}{i!}$$

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$$= \underbrace{I + (A\Delta t)} + \underbrace{\frac{(A\Delta t)^2}{2!}}_{=0} + \dots$$

but we see that $(A\Delta t)^2 = \Delta t^2 \, A . A$

$$= \Delta t^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \Delta t^2 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

hence $\exp(A\Delta t) = I + A\Delta t$

$$= \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} . \quad \dots\dots \quad (2.4)$$

There fore we can see that :

with respect to the density of w

$$\underbrace{\mathbb{E}[q_n]} = \mathbb{E}\left[\int_{(n-1)\Delta t}^{n\Delta t} e^{(n\Delta t - \tau)} B \, w_\tau \, d\tau\right]$$

Fubini theorem

$$= \int_{(n-1)\Delta t}^{n\Delta t} e^{(n\Delta t - \tau)} B \, \mathbb{E}[w_\tau] \, d\tau = 0. \quad (2.5)$$

so q_n also zero mean.

what about q_n variance?

$$\mathbb{E}\left[(q_n - \underbrace{\mathbb{E}[q_n]}_{=0})(q_n - \underbrace{\mathbb{E}[q_n]}_{=0})^T\right] = \mathbb{E}[q_n \, q_n^T]$$

$$\mathbb{E}[q_n \, q_n^T] = \mathbb{E}\left[\int_{(n-1)\Delta t}^{n\Delta t}\int_{(n-1)\Delta t}^{n\Delta t} e^{A(n\Delta t - \tau)} B \, w_\tau \, w_{\tau'}^T \, B^T e^{A^T(n\Delta t - \tau')} d\tau d\tau'\right]$$

now

$$\int_{(n-1)\Delta t}^{n\Delta t}\int_{(n-1)\Delta t}^{n\Delta t} e^{A(n\Delta t - \tau)} B \, \underbrace{\mathbb{E}[w_\tau w_{\tau'}^T]} B^T e^{A^T(n\Delta t - \tau')} d\tau d\tau'$$

$$\int\int_{(n-1)\ (n-1)\Delta t} e \qquad B\ \underbrace{\mathbb{E}[w_\tau w_{\tau'}]}\ B^{\cdot}\ e \qquad d\tau\, d\tau'$$

$$\underbrace{\mathbb{E}[w_\tau w_{\tau'}]} = \begin{cases} \sigma_w^2 I & , \text{if } \tau = \tau' \\ 0 & , \tau \neq \tau' \end{cases}$$

$$\mathbb{E}[q_n q_n^T] = \sigma_w^2 \int_{(n-1)\Delta t}^{n\Delta t} e^{A(n\Delta t - \tau)} B\, B^T e^{A^T(n\Delta t - \tau)} d\tau \quad \dots (2.6)$$

first : $e^{A(n\Delta t - \tau)} B$ where $(n-1)\Delta t \leq \tau \leq n\Delta t$

change variable 'let say $\tau = (n-1)\Delta t + \tilde{\tau}$

hence $n\Delta t - \tilde{\tau} = n\Delta t - ((n-1)\Delta t + \tilde{\tau})$

$= \Delta t - \tilde{\tau}$

$$e^{A(n\Delta t - \tau)} B = e^{A(\Delta t - \tilde{\tau})} B$$

$$\begin{bmatrix} 1 & \Delta t - \tilde{\tau} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \Delta t - \tilde{\tau} \\ 1 \end{bmatrix} \quad \dots (2.7)$$

$d\tau = d\tilde{\tau}$

substituting (2.7) into (2.6)

$$\mathbb{E}[q_n q_n^T] = \sigma_w^2 \int_0^{\Delta t} \begin{bmatrix} \Delta t - \tilde{\tau} \\ 1 \end{bmatrix} \begin{bmatrix} \Delta t - \tilde{\tau} \\ 1 \end{bmatrix}^T d\tilde{\tau}$$

$$= \sigma_w^2 \underbrace{\begin{bmatrix} \Delta t^3/3 & \Delta t^2/2 \\ \Delta t^2/2 & \Delta t \end{bmatrix}} \quad // \quad \dots (2.8)$$

Question 3

$$\dot{\underline{x}}_t = f(\underline{x}_t, \underline{u}_t) + \underline{w}_t \quad \dots (3.1)$$

a) Euler - Maruyama. here $\underline{w}_t \sim \mathcal{N}(0, Q)$, $Q = \begin{bmatrix} q_1 & & \\ & \ddots & \\ & & q_3 \end{bmatrix}$

a) Euler - Maruyama. here $\underline{w}_t \sim \mathcal{N}(0, Q)$, $Q = \begin{bmatrix} q_1 & & \\ & \ddots & \\ & & q_3 \end{bmatrix}$

as before : $\underline{x}_n := \underline{x}_{t_n} = \underline{x}_{n\Delta t}$

$$\frac{d\underline{x}_t}{dt} = f(\underline{x}_t, \underline{u}_t) + \underline{w}_t$$

$$\int_{(n-1)\delta t}^{n\delta t} d\underline{x}_t = \int_{\boxed{(n-1)\delta t}}^{n\delta t} f(\underline{x}_t, u_t)\, dt + \underbrace{\int_{(n-1)\delta t}^{n\delta t} \underline{w}_t \, dt}$$

$$\underline{x}_n - \underline{x}_{n-1} \approx f(x_{n-1}, u_{n-1})\Delta t$$

$$+ \; q_n$$

$$q_n \sim \mathcal{N}(0, \Delta t\, Q).$$

you can set $q_n = \sqrt{\delta t}\; \tilde{w}_n$, $\tilde{w}_n \sim (0, Q)$.

b). $$\frac{d\underline{x}_t}{dt} = f(\underline{x}_t, \underline{u}_t) + \underline{w}_t \quad \ldots (3.2)$$

now we will expand $f(\underline{x}_t, \underline{u}_t)$ in Taylor series where the point of evaluation is $\underline{x}_{n-1}$, $t_{n-1} \leq t \leq t_n$
or $(n-1)\Delta t \leq t \leq n\Delta t$.

We also keep u_t, constant between $[t_{n-1}, t_n]$ at u_{n-1}.
zero order hold. (ZOH).

$$(3.3) \ldots \; f(\underline{x}_t, \underline{u}_t) = f(\underline{x}_{n-1}, \underline{u}_{n-1}) + \left.\frac{\partial f}{\partial x}\right|_{(x_{n-1}, u_{n-1})} \cdot \tilde{x}_t + \mathcal{O}(\tilde{x}^2)$$

$$\tilde{x}_t = \underline{x}_t - \underline{x}_{n-1} \quad \ldots \; 3.4)$$

then let denote

$$A_x := \frac{\partial f}{\partial x}\Big|_{\underbrace{x_{n-1},\, u_{n-1}}}$$

hence when we neglect the higher order $\mathcal{O}(\tilde{x}^2)$ we can write:

$$\frac{dx_t}{dt} \approx \left[A_x \tilde{x}_t + f(x_{n-1}, u_{n-1}) \right] + \underline{w}_t$$

$$\frac{dx_t}{dt} \approx \left[A_x [x_t - x_{n-1}] + f(x_{n-1}, u_{n-1}) \right] + w_t$$

$$\approx \left[\underline{A_x} \, x_t + \left[f(x_{n-1}, u_{n-1}) - A_x x_{n-1} \right] \right]$$

$$+ \underline{w_t} \; . \qquad \ldots\ldots \quad (3.5)$$

The linearized ODE (3.5) has solution: at $t = t_n$

$$\underline{x}_n = e^{A_x (\underbrace{n\Delta t - (n-1)\Delta t})} \underline{x}_{n-1} + \int_{(n-1)\Delta t}^{n\Delta t} e^{A_x [n\Delta t - \tau]} f(x_{n-1}, u_{n-1}) \, d\tau$$

$$(3.6) \ldots \quad - \int_{(n-1)\Delta t}^{n\Delta t} e^{A_x (n\Delta t - \tau)} A_x x_{n-1} \, d\tau + \underbrace{\int_{(n-1)\Delta t}^{n\Delta t} e^{A_x (n\Delta t - \tau)} w_\tau \, d\tau}_{=\, q_n}$$

now we see that:

$$\left[e^{A_x (\Delta t)} - \left[\int_{(n-1)\Delta t}^{n\Delta t} e^{A_x (n\Delta t - \tau)} d\tau \right] A_x \right]$$

$$= e^{A_x \Delta t} - \left[- A_x^{-1} \left[I - e^{A_x \Delta t} \right] A_x \right] \quad \ldots (3.7)$$

so A_x always commute with e^{A_x}; that is

$$A_x \cdot e^{A_x} = e^{A_x} \cdot A_x .$$

$$A_x \cdot e^{A_x} = e^{A_x} \cdot A_x.$$

hence (3-7) equals to

$$= e^{A_x \Delta t} + \left[I - e^{A_x \Delta t} \right] = I \quad \ldots\ldots (3.8)$$

$$x_n = x_{n-1} + \underbrace{\int_{(n-1)\Delta t}^{n\Delta t} e^{A_x (n\Delta t - \tau)} f(x_{n-1}, u_{n-1})\, d\tau} + q_n \quad \ldots (3.9)$$

$$q_n := \int_{(n-1)\Delta t}^{n\Delta t} e^{A_x (n\Delta t - \tau)} w_\tau \, d\tau \quad \ldots (3.10)$$