Exercise 6

Tuesday 27. October 2020 11.57

Question 1

$$m a_t + k p_t + \eta v_t = 0 \quad \ldots \quad (1).$$

a). $p_t = \underline{C \exp(st)}$ .... (2)

$$a_t = \frac{d}{dt} v_t \quad , \quad v_t = \frac{d}{dt} p_t \qquad (3)$$

$$d^2/dt^2 \left(C \exp(st) \right) = s^2 C \exp(st) = s^2 p_t$$

Substitute (2) & (3) into (1)

$$m \, d^2/dt^2 \, p_t + \eta \, d/dt \, p_t + k p_t = 0$$

$$(m s^2 + \eta s + k) p_t = 0 \quad \ldots \quad (4)$$

Either $p_t = 0$, $\forall t$, or $m s^2 + \eta s + k = 0$ ...(5)

$P_0 \neq 0$

$$(6) \ldots \quad s_{1,2} = \frac{-\eta}{2m} \pm \frac{1}{2m} \sqrt{\eta^2 - 4mk}$$

From eq (6)

\# if $\eta^2 - 4mk \geq 0 \rightarrow$ no oscillation, why? $\sqrt{-4} = i2$

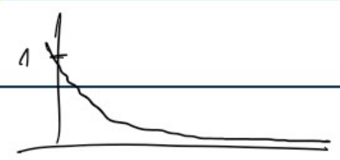
$\eta^2 - 4mk < 0$

$$e^{s_i} = e^{\sigma + i\omega} \qquad e^{i\omega} = \cos(\omega) + i\sin(\omega)$$

\# if $\eta^2 - 4mk < 0 \rightarrow$ there is oscillation

ω freq equivalent to $\frac{1}{2m}\sqrt{\eta^2 - 4mk}$

e^{-1t}



\# if $\underbrace{Re(s_i) < 0}$, $i = 1, 2$, then $P_t \rightarrow 0$.

$$P_t = C . \exp(s_i t) \ldots \qquad \sigma < 0$$

$$= C . \exp(\sigma t + i\omega t) \longrightarrow 0 \quad \text{as } t \to \infty$$

Full solution can be obtained using Laplace transform.

Let $P_t = x_{1,t}$; $V_t = x_{2,t}$ if there is

$\underbrace{\frac{d}{dt} x_{1,t} = \dot{x}_{1,t} = x_{2,t}}$; $\frac{d}{dt} x_{2,t} = \underbrace{\frac{-\eta}{m} x_{2,t} - \frac{k}{m} x_{1,t} + u_t}_{\text{according to eq(1)}}$

The Laplace transform: $x_{1,t} \rightarrow X_1(s)$

$$\mathcal{L}(x_{1,t}) = X_1(s) := \int_0^\infty e^{-st} x_{1,t} \, dt \quad \ldots\ldots (7)$$

$$\mathcal{L}\left(\frac{d}{dt} x_{1,t}\right) = s X_1(s) - x_{1,0} \quad \ldots\ldots (8)$$

$$\frac{d}{dt}\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\eta}{m} \end{bmatrix}\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t$$

$$\dot{\underline{x}} = A \underline{x} + B u \quad \ldots (9)$$

$$\mathcal{L}(\dot{\underline{x}}) = \mathcal{L}(A \underline{x} + B u)$$

We will solve for $u = 0$ | let say $\underline{X}(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$

$$s \underline{X}(s) - \underline{x}_0 = A \underline{X}(s) \quad \ldots (10)$$

$$(sI - A)\underline{X}(s) = \underline{x}_0$$

$$\underbrace{\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{\eta}{m} \end{bmatrix}}_{G(s)} \underline{X}(s) = \underline{x}_0$$

$$\underline{X}(s) = G(s)^{-1} \underline{x}_0 \quad \ldots\ldots (11)$$

$$\underline{X}(s) = \frac{1}{\det(G(s))}\begin{bmatrix} s + \eta/m & 1 \\ -\frac{k}{m} & s \end{bmatrix}\begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix}$$

$$\det(G(s)) = s^2 + \frac{\eta}{m} s + \frac{k}{m}.$$

$$= (s - s_1)(s - s_2) \quad \cdots \quad (12)$$

we can write

$$\underline{X}(s) = \underbrace{\frac{1}{(s-s_1)(s-s_2)} \begin{bmatrix} s+\eta/m & 1 \\ -k/m & s \end{bmatrix}}\begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} \quad \cdots (13)$$

$$\Downarrow \quad X_1(s) = \frac{1}{(s-s_1)(s-s_2)} \left[(s+\eta/m)\, x_{1,0} + x_{2,0} \right]. \quad (14)$$

factorize

$$= a \frac{1}{(s-s_1)} + b \frac{1}{(s-s_2)} \quad \cdots (15)$$

$$\Downarrow \quad x_{1,t} = \mathcal{L}^{-1}(X_1(s))$$

$$= a . e^{s_1 t} + b . e^{s_2 t} \quad \cdots (16)$$

same goes with $x_{2,t}$

Question 2.

a). Is given by eq (9)

b). With $u_t = 0$. solve the equation; i.e eq (9), with $u = 0$

$$\underline{\dot{x}} = A\, \underline{x} \qquad (9)$$

for given $\underline{x}_0$, what is $\underline{x}_t$? at any $t \geq 0$.

if there is $\underline{z}_t \in \mathbb{R}^{2\times 1}$ that satisfy (9) and $\underline{z}_0 = \underline{x}_0$

then $\underline{z}_t$ is the solution to (9) ODE with initial condition $\underline{x}_0$.

suppose I choose $z_t = e^{At}\, \underline{x}_0 \quad \cdots \quad (17)$

then $\underline{z}_0 = e^{A0} \cdot x_0 = \underline{x_0}$

and $\underline{\frac{d}{dt}} \, \underline{z}_t = \underline{A \, e^{At} \, x_0} = \underline{A \, z_t}$

then z_t is the solution to (9).

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c). $\underline{Z_t} = e^{At} \cdot \underline{x_0}$ $\exp(A)$

$\exp(A)$ is not equivalent to taking exp on the element.

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $\exp(A) \neq \begin{pmatrix} \exp(a_{11}) & \exp(a_{12}) \\ \exp(a_{21}) & \exp(a_{22}) \end{pmatrix}$

to obtain e^{At} we can use:

1. Laplace transform (as in Question 1)

$$\underline{e^{At}} = \underbrace{\mathcal{L}^{-1}\left(G(s)^{-1}\right)}..$$

2. or Cayley-Hamilton theorem.

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d). for the case of $u \neq 0$.

$$\underline{\dot{x}}_t = A \, \underline{x}_t + B \, u_t \quad \ldots. (18)$$

as before, if we can find $\underline{z_t}$, such that $\underline{z_t}$ satisfies (18)

and $\underline{z}_0 = \underline{x_0}$. then $\underline{z_t}$ is the solution to initial value problem (18).

now $\underline{z}_t = e^{At} \, \underline{x}_0 + \int_0^t e^{A(t-\tau)} \, B \, u_\tau \, d\tau$ ..(19)

check: $\underline{z}_0 = \underline{x}_0$ ...

check

$$\frac{dz_t}{dt} = \underbrace{\frac{d}{dt}\left(e^{At} x_0\right)}_{\text{I}} + \underbrace{\frac{d}{dt}\left[\int_0^t e^{A(t-\tau)} B u_\tau \, d\tau\right]}_{\text{II}} \quad (20)$$

$$\text{I} = A e^{At} x_0$$

we use Leibniz rule :

$$\frac{d}{dt}\left[\int_0^t f(t,\tau)\, d\tau\right] = f(t,t) + \int_0^t \frac{d}{dt} f(t,\tau)\, d\tau$$

using Leibniz rule to II , with $f(t,\tau) = e^{A(t-\tau)} B u_\tau$

$$\frac{d}{dt}\left[\int_0^t e^{A(t-\tau)} B u_\tau \, d\tau\right] = B u_t + \int_0^t A e^{A(t-\tau)} B u_\tau \, d\tau \quad (21)$$

there fore combining I & II , we have

$$\frac{d}{dt} z_t = A e^{At} x_0 + B u_t + \int_0^t A e^{A(t-\tau)} B u_\tau \, d\tau$$

$$= A\left[e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u_\tau \, d\tau\right] + B u_t$$

$$= A z_t + B u_t$$

which means that z_t is the solution to the initial value problem (18).

$$(sI - A) X(s) = x_0 + B u(s)$$

$$X(s) = \underbrace{(sI - A)^{-1} x_0}_{} + \underbrace{(sI - A)^{-1} B u(s)}_{}$$