Exercise 3

Tuesday 29. September 2020 11.54

1. a)

$$\begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y_6 \end{pmatrix} = \underline{\underline{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ -1 & & \\ & -1 & \\ & & -1 \end{bmatrix}}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ L \\ L \\ L \end{bmatrix} + \begin{bmatrix} r_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ r_6 \end{bmatrix}$$

p^x, p^y, p^z

$$\underline{y} = G \underline{x} + \underline{b} + \underline{r}$$

$$\underline{\tilde{y}} = \underline{y} - \underline{b} = G \underline{x} + \underline{r} \leftarrow \text{form of (3.14).}$$

b). $$J = \sum_i (\tilde{y}_i - G_i x)^2$$

$$= (\underline{\tilde{y}} - G\underline{x})^T (\underline{\tilde{y}} - G\underline{x}) \leftarrow$$

$$\frac{\partial}{\partial x}(\tilde{y}^T\tilde{y})=0, \qquad = \underbrace{\underline{\tilde{y}}^T\underline{\tilde{y}} - 2\underline{x}^T G^T \underline{\tilde{y}} + \underline{x}^T G^T G \underline{x}}$$

$$0 = \left.\frac{\partial J}{\partial x}\right|_{x=\hat{x}} = -2G^T\underline{\tilde{y}} + 2G^T G \underline{\hat{x}}$$

$$= \qquad \underline{\hat{x}} = (\underline{\underline{G}}^T G)^{-1} G^T \tilde{y} \;//$$

c). $G = \begin{bmatrix} I \\ -I \end{bmatrix}$ where I is identity matrix 3X3.

$$\hat{x} = \left([I \;\; -I] \begin{bmatrix} I \\ -I \end{bmatrix} \right)^{-1} [I \;\; -I]\, \tilde{y}$$

$$= \frac{1}{2} \begin{bmatrix} y_1 + (L - y_4) \\ y_2 + (L - y_5) \\ y_3 + (L - y_6) \end{bmatrix} //$$

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2. (3.40).

$$\hat{x}_{ReLS} = (G^T R^{-1} G + P^{-1})^{-1} (G^T R^{-1} y + P^{-1} m).$$

what is the expected value of $\hat{x}_{ReLS}$?

that is, what is $\mathbb{E}[\hat{x}_{ReLS}]$?

probability density

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \, p(x) \, dx$$

covariance matrix

we use the assumption that $\underline{r} \sim \mathcal{N}(0, \underline{R})$.

$$\mathbb{E}[\hat{x}_{ReLS}] = \mathbb{E}[(G^TR^{-1}G + P^{-1})^{-1}(G^TR^{-1}y + P^{-1}m)]$$

$$= (G^TR^{-1}G + P^{-1})^{-1}(G^TR^{-1}\mathbb{E}[y] + P^{-1}m).$$

$$y = Gx + r$$

$$\mathbb{E}[y] = Gx + \mathbb{E}[r] = Gx$$

$$= [G^TR^{-1}G + P^{-1})^{-1}[G^TR^{-1}[Gx] + P^{-1}m]$$

$$G^TR^{-1}Gx + P^{-1}m + \underbrace{P^{-1}(x - x)}_{=0}$$

$$= [G^TR^{-1}G + P^{-1}]^{-1} [[G^TR^{-1}G + P^{-1}]x + P^{-1}[m - x]]$$

$$\approx (x) + (G^TR^{-1}G + P^{-1})^{-1}P^{-1}(m - x)$$

$[m \neq x]$ $\quad$ $[m = x]$

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3. a) $\underline{\tilde{y}} = G\underline{x} + \underline{r}$, $\quad \underline{r} \sim \mathcal{N}(0, \underline{R})$

$$W = R^{-1}$$

$$J_{ReLS} = (\underline{\tilde{y}} - G\underline{x})^T R^{-1} (\tilde{y} - Gx) + (\underline{x} - \underline{m})^T P^{-1}(\underline{x} - \underline{m}).$$

b). Sequential form of Regularized weighted Least square.

$$\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_6$$

r_1, r_2 is independent, $E[r_i r_j] = 0$, $i \neq j$

In lecture notes (3.47) $\longrightarrow$ (3.52) $\leftarrow$ non regularized version.

$$J = \boxed{(\underline{x}-\underline{m})^T P^{-1} (\underline{x}-\underline{m})} + (\tilde{y}_1 - G_1\underline{x})^T R_1^{-1} (\tilde{y}_1 - G\underline{x})$$

$$0 = \frac{\partial J}{\partial x}\Big|_{x=\hat{x}_1} = 2P^{-1}(\hat{\underline{x}} - \underline{m}) - 2G_1^T R_1^{-1}\tilde{y}_1 + 2G_1^T R_1^{-1} G\hat{x} = 0$$

$$\hat{\underline{x}}_1 = [G_1^T R_1^{-1} G_1 + P^{-1}]^{-1} [G_1^T R_1^{-1} \tilde{y}_1 + P^{-1}\underline{m}] \quad ..(0)$$

we use woodbury formula

$$(G_1^T R_1^{-1} G + P^{-1})^{-1} = P - PG_1^T (G_1 P G_1^T + R_1)^{-1} G_1 P \quad (1)$$

substitute (1) to (0), we will have

$$\hat{x}_1 = [P - PG_1^T (G_1 P G_1^T + R_1)^{-1} G_1 P][G_1^T R_1^{-1} \tilde{y} + P^{-1} m]$$

$$= [m - PG_1^T (G_1 P G_1^T + R_1)^{-1} G_1 m + PG_1^T R_1^{-1} \tilde{y}$$

$$- PG_1^T (G_1 P G_1^T + R_1)^{-1} G_1 P G_1^T R_1^{-1} \tilde{y}] \quad ...(2)$$

let say: $K_1 = PG_1^T (G_1 P G_1^T + R_1)^{-1} \quad ...(3)$

substitute (3) to (2)

$$\hat{\underline{x}}_1 = [m - K_1 G_1 m + PG_1^T R^{-1} \tilde{y}_1$$

$$- PG_1^T (G_1 P_1 G_1^T + R_1)^{-1} G_1 P G_1^T R_1^{-1} \tilde{y}_1]$$

$$= [m - K_1 G_1 m + K_1 \tilde{y}_1]$$

$$[PG_1^T R_1^{-1} \tilde{y}_1 - PG_1^T [G_1 P_1 G_1^T + R_1)^{-1} G_1 P G_1^T R_1^{-1} \tilde{y}_1$$

$$= PG^T [-(G P_1 G_1^T + R_1)^{-1} G_1 P G_1^T + I] R_1^{-1} \tilde{y}_1$$

$$= P G_1^T \left[-(G_1 P_1 G_1^T + R_1)^{-1} G_1 P G_1^T + I \right] R_1^{-1} \tilde{y}$$

$$= P G_1^T \left[-(G_1 P_1 G_1^T + R_1)^{-1} (G_1 P G_1^T + R_1) + (G_1 P_1 G_1^T + R_1)^{-1} R_1 + I \right] R_1^{-1} \tilde{y}$$

$$= P G_1^T \left[(G_1 P G_1^T + R_1)^{-1} \overset{I}{\cancel{R_1 R_1^{-1}}} \tilde{y}_1 \right]$$

$$= K_1 \tilde{y}_1$$

R is invertible. also P.