MA 779 ~ Probability Theory I.

Unit 1 – Sets and Probability space

Introduction to set theory. Goal is to measure probabilities of sets of possible outcomes.

Set is collection of objects.

e.g. $A = \{3, 7, 8\}$

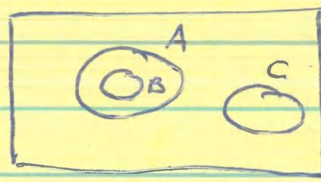
$A = \{\text{blue}, \text{green}, \text{red}\}$

$A = \{\text{even integers}\}$, $\mathbb{N} = \{1, 2, 3, \dots\}$

$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$

<u>Notation</u> :
- Capital letters for sets, lowercase letters for elements of sets
- $3 \notin \{5, 7, 8\}$, $5 \in \{5, 7, 8\}$

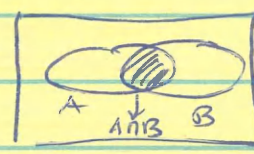
Venn diagram



$B \subseteq A$, $A \cap C = \emptyset$ ↳ empty set , $A \cap B = B$, $A \cup B = A$

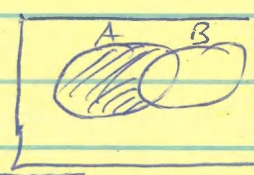
$A \cup B = \{x : x \in A \text{ or } x \in B\}$

$A \cap B = \{x : x \in A \text{ and } x \in B\}$



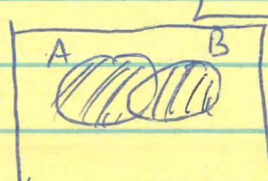
$A^c = \{x : x \notin A\}$

$A \setminus B = \{x \in A : x \notin B\} = A \cap B^c$



$A \triangle B = (A \setminus B) \cup (B \setminus A)$

↳ symmetric difference



$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

i.e. if $x \in A$ then $x \in B$ $\boxed{\text{AND}}$ if $x \in B$ then $x \in A$

Finite union: Given sets $A_1, \dots, A_N$

$$\bigcup_{i=1}^{N} A_i = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_N$$

Finite intersection $\bigcap_{i=1}^{N} A_i = A_1 \cap A_2 \cap \dots \cap A_N$

Countable union: $\bigcup_{i=1}^{\infty} A_i = \{x : x \in A_i \text{ for at least one } i\}$

Countable intersection: $\bigcap_{i=1}^{\infty} A_i = \{x : x \in A_i \text{ for all } i\}$

example: • if $A_n = [\frac{1}{n}, 7 - \frac{1}{n}]$ $n \in \mathbb{N}$

then $\bigcup_{n=1}^{\infty} A_n = (0, 7)$

• if $B_n = (-\frac{1}{n}, 7 - \frac{1}{n})$

then $\bigcap_{n=1}^{\infty} B_n = [0, 3]$

Increasing sets: A collection of sets $\{A_i\}$ is said to be increasing if $A_i \subseteq A_{i+1}$ $\forall i$

Define $\lim A_i = \bigcup_{i=1}^{\infty} A_i$

Decreasing sets: A collection of sets $\{A_i\}$ is said to be decreasing if $A_i \supseteq A_{i+1}$ $\forall i$

Define then $\lim A_i = \bigcap_{i=1}^{\infty} A_i$

- $\liminf A_i = \bigcup_{n=1}^{\infty}\left(\bigcap_{m=n}^{\infty} A_m\right) = \{x : \exists n \text{ s.t. } \forall m \geq n \; x \in A_m\}$

  = "belongs in all sets eventually"

  = "belongs to all but a finite number of n"

  Note that $\bigcap_{m=n}^{\infty} A_m$ is increasing set in n.

- $\limsup A_i = \bigcap_{n=1}^{\infty}\left(\bigcup_{m=n}^{\infty} A_m\right) = \{x : x \in A_i \text{ infinitely many times}\}$

  Note that $\bigcup_{m=n}^{\infty} A_m$ is decreasing set in n.

Remark $\liminf A_i \subseteq \limsup A_i$.

Definition If $\liminf A_i = \limsup A_i$ then set
$\lim A_i = \liminf A_i = \limsup A_i$

Example: $A_{2n} = [\frac{1}{n}, 7 - \frac{1}{n}]$
$A_{2n+1} = (-\frac{1}{n}, 7 + \frac{1}{n})$

Then $\liminf A_n = (0, 3)$
$\limsup A_n = [0, 3]$

Remark liminf and limsup operations for sets is not the same as corresponding liminf and limsup operation for numbers.

$\boxed{\text{DeMorgan's laws}}$ $$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$$

$$\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c$$

$\boxed{\text{Proof}}$ ($\Rightarrow$) Assume $x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c \Rightarrow x \notin \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow x \notin A_n$ for any n

$\Rightarrow x \in A_n^c$ for any n

$\Rightarrow x \in \bigcap_{n=1}^{\infty} A_n^c$

($\Leftarrow$) Assume $x \in \bigcap_{n=1}^{\infty} A_n^c \Rightarrow x \in A_n^c \ \forall n$

$\Rightarrow x \notin A_n \ \forall n$

$\Rightarrow x \notin \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow x \in \left(\bigcup_{n=1}^{\infty} A_n\right)^c$

Similarly for the other statement.

$\boxed{\text{Countable sets}}$: A set A is countable if $\exists$ exists a one-to-one mapping $f: N \to A$

Example: positive even numbers are countable

$f(n) = 2n$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 6 & 8 & 10 \end{array}$$

Example: $\mathbb{Z}$ integers $\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$

Example: $\mathbb{Q}$ rationals

$$\begin{array}{lllll} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1}, & \ldots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2}, & \ldots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3}, & \ldots \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4}, & \ldots \end{array}$$

Example of uncountable set : $\{ \{a_n\}_{n=1}^{\infty} : a_n \in \{0,1\} \}$

Sequences of 0's and 1's

We can prove this by diagonalization. In particular if I assume I have a list, let's say

1 0 0 1 0 1 1 0 _ _
0 1 0 0 0 1 0 1 - -
1 0 1 0 1 0 0 1 . . .
:

we can always find something missing from the list

Another example of uncountable set : $(0,1)$

$\boxed{\text{Definition}}$ Power set of Ω is denoted by 2^{Ω} and is the set of all subsets of Ω

(e.g.) if $\Omega = \{1,2\}$ then $2^{\Omega} = \{\emptyset, \Omega, \{1\}, \{2\}\}$
if $\Omega = \{4,5,6\}$ then
$2^{\Omega} = \{\emptyset, \Omega, \{4\}, \{5\}, \{6\}, \{4,5\}, \{4,6\}, \{5,6\}\}$

$\boxed{\text{Definition}}$ An algebra is a set of sets $\mathcal{A} \subset 2^{\Omega}$ such that :

(α) $\Omega \in \mathcal{A}$
(β) if $B \in \mathcal{A}$ then $B^c \in \mathcal{A}$
(γ) if $B \in \mathcal{A}$ and $C \in \mathcal{A}$ then $(B \cup C) \in \mathcal{A}$

$\boxed{\text{Examele}}$: ~~If $A \in \Omega$ and $B \in A$~~

If $\Omega = \{1,2,3\}$ then ~~the~~ an algebra $\mathcal{A}$ is

$A = \{ \Omega, \emptyset, \{1\}, \{2,3\} \}$

$\boxed{\text{Definition}}$ $A \subset 2^{\Omega}$ is called a <u>σ-algebra</u> if

(a) $\Omega \in A$

(b) if $B \in \mathcal{A}$ then $B^c \in \mathcal{A}$

(γ) If $B_n \in A$ $\forall n$ then $\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$

$\boxed{\text{Examele}}$ Let $\Omega = (0,1)$, $A = \{(a,b) ; 0 \le a < b \le 1\}$

This is an examele of something that is an algebra but not σ-algebra. This is closed under finite unions and intersection.

$\boxed{\text{Definition}}$ A <u>cofinite</u> subset of a set is a subset whose comelement is a finite set

$\boxed{\text{Examele}}$ $\Omega = \mathbb{N}$, $A = \{\text{finite or cofinite sets}\}$

Check: (a) $\Omega \in A$ ✓

(b) if $B \in A$ then $B^c \in A$ by definition

(γ) if $B, C \in A$ then:

- if B, C finite $B \cup C$ finite
- if B, C cofinite $B \cup C$ cofinite
- if B finite, C-cofinite $\Rightarrow B \cup C$ cofinite

$(B \cup C)^c = B^c \cap C^c \subset C^c$ finite

Thus A is algebra.

However it is not σ-algebra : $A = \{\text{even numbers}\}$

$A = \bigcup_{n=1}^{\infty} \{2n\}$

$\boxed{\text{Definition}}$ $\mathcal{A}$ is π-system if $B, C \in \mathcal{A}$ implies $B \cap C \in \mathcal{A}$

$\boxed{\text{Definition}}$ $\mathcal{A}$ is a monotone class if for any increasing family $A_n \subseteq A_{n+1}$ where $A_n \in \mathcal{A}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

and if $A_n \supseteq A_{n+1}$ $A_n \in \mathcal{A}$ then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$.

$\boxed{\text{Definition}}$ $\mathcal{A}$ is called a Dynkin system if $\Omega \in \mathcal{A}$ and

- $B, C \in \mathcal{A}$, $B \subset C \Rightarrow C \setminus B \in \mathcal{A}$
- $B_n \in \mathcal{A}$ $n \geq 1$ $B_n \uparrow \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$.

$\boxed{\text{Example}}$ A Dynkin system that is a π-system is also a σ-algebra.

Indeed, the last two conditions in the definition of a Dynkin system together with closure under finite intersections imply closure under finite unions. This then implies closure under countable unions

$\boxed{\text{Definition}}$ Let A collection of subsets of Ω. Then we have that $\sigma(A) =$ smallest σ-algebra that contains A

$$= \bigcap_{\substack{A \subset B \\ B\ \sigma\text{-algebra}}} B$$

$\boxed{\text{Example}}$ $\Omega = \{1, 2, 3, 4\}$

$A = \{\{1\}\}$ (not a σ-algebra)

$\sigma(A) = \{\{1\}, \{2, 3, 4\}, \Omega, \emptyset\}$

$\boxed{\text{Example}}$ $\Omega = \{1,2,3,4\}$, $A = \{\{1\}, \{3,4\}\}$

$\hookrightarrow \sigma(A) = \{\{1\}, \{3,4\}, \{2,3,4\}, \{1,2\}, \{1,3,4\}, \{2\},$
$\Omega, \emptyset\}$

$\boxed{\text{Example}}$ Borel σ-algebra $\Omega = \mathbb{R}$, $A = \{(a,b) : a < b\}$

$\sigma(A)$ is called Borel σ-algebra

$\boxed{\text{Example}}$ (filtrations)

$\Omega = \{x = (x_1, x_2, x_3, \ldots) : x_i \in \{0,1\}\}$

infinite sequence of 0's and 1's

$\mathcal{F}_n$ - information after n flips of a coin

$\mathcal{F}_1 = \{\{x : x_1 = 1\}, \{x : x_1 = 0\}, \Omega, \emptyset\}$

$\mathcal{F}_2 = \sigma\{\{x : x_1 = 1, x_2 = 1\}, \{x : x_1 = 1, x_2 = 0\}, \{x : x_1 = 0, x_2 = 1\}$
$\{x : x_1 = 0, x_2 = 0\}, \Omega, \emptyset\}$

$\boxed{\text{Definition}}$ A probability space is a triple

$$(\Omega, \mathcal{F}, P)$$

set — σ-algebra of subsets of Ω. — Probability measure

$P : \mathcal{F} \longmapsto [0,1]$ satisfying Kolmogorov axioms:

(i) $P(A) \in [0,1]$ $\forall A \in \mathcal{F}$.

(ii) $P(\Omega) = 1$

(iii) Countable additive: If $\{A_i\}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

$\boxed{\text{Remark}}$

(i) If $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$

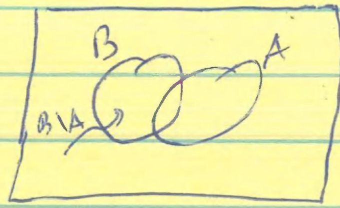
(ii) $P(\emptyset) = 0$

(iii)

(iii) $P(A^c) = 1 - P(A)$

$\left(1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c) \right)$

(iv) Exclusion $\quad P(B \setminus A) = P(B) - P(A \cap B)$



$B = (B \setminus A) \cup (A \cap B)$ (disjoint union)

$P(B) = P(B \setminus A) + P(A \cap B)$

(v) Let $A_n \in \mathcal{F}$. Then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} P(A_i)$$

Indeed let $B_1 = A_1$

$B_2 = A_2 \setminus B_1$

$B_i = A_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right)$, B_j - disjoint

Note that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

So $$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P(B_n) = \sum_{n=1}^{\infty} \left(P(A_n) - P\left(\bigcup_{j=1}^{i-1} B_j\right)\right)$$

$$\le \sum_{n=1}^{\infty} P(A_n).$$

(vi) $P(A \cup B) \le P(A) + P(B)$

(vii) If $A \subseteq B$ then $P(A) \le P(B)$

Indeed, note that $B = A \cup (B \setminus A)$ disjoint.

So $P(B) = P(A) + P(B \setminus A) \ge 0$

So $P(B) \ge P(A)$

(vii) (Continuity): If $A_n \subseteq A_{n+1}$ then we have that
$$\lim_{n\to\infty} P(A_n) = P\left(\bigcup_{n=1}^{\infty} A_n\right)$$

If $A_n \supseteq A_{n+1}$ then we have that
$$\lim_{n\to\infty} P(A_n) = P\left(\bigcap_{n=1}^{\infty} A_n\right) = P\left(\lim_{n\to\infty} A_n\right)$$
limit of numbers — limits of sets

$\boxed{\text{Proof}}$ Assume $A_n \subseteq A_{n+1}$. Consider the following disjoint collection of sets B_n
$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

$$B_1 = A_1$$
$$B_2 = A_2 \setminus A_1 \qquad B_i \cap B_j = \emptyset \text{ if } i \neq j$$
$$B_3 = A_3 \setminus A_2$$
$$\vdots$$
$$B_{n+1} = A_{n+1} \setminus A_n$$

Note that $\bigcup_{i=1}^{n} B_i = A_n \Rightarrow \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$

$$\Rightarrow P\left(\bigcup_{i=1}^{\infty} B_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right)$$
$$\sum_{n=1}^{\infty} P(B_n) = \lim_{N\to\infty} \sum_{n=1}^{N} P(B_n)$$
$$= \lim_{N\to\infty} P\left(\bigcup_{n=1}^{N} B_n\right) \quad \text{(finite additivity)}$$
$$= \lim_{N\to\infty} P(A_N)$$

Let us now assume that $A_n \supseteq A_{n+1}$. Then we note that

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \Rightarrow P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1 - P\left(\bigcup_{n=1}^{\infty} A_n^c\right)$$
$$= 1 - \lim_{n\to\infty} P(A_n^c)$$
$$= \lim_{n\to\infty} P(A_n)$$

$\boxtimes$

(viii) Let $A_n \in \mathcal{F}$, then we have

$$P(\underset{\text{set}}{\liminf A_n}) \le \underset{\text{number}}{\liminf P(A_n)} \le \underset{\text{number}}{\limsup P(A_n)} \le P(\underset{\text{set}}{\limsup A_n})$$

$\boxed{\text{Proof}}$ Let $B_m = \bigcup_{n=m}^{\infty} A_n$ decreasing sets. So we get that

$$\limsup_{n\to\infty} A_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n = \bigcap_{m=1}^{\infty} B_m$$

So by continuity we have that

$$\lim_{n\to\infty} P(B_n) = P\left(\bigcap_{n=1}^{\infty} B_n\right) = P\left(\limsup_{n\to\infty} A_n\right)$$

By monotonicity property $A_n \subseteq B_n$, so we get that

$$\limsup P(A_n) \le \limsup P(B_n) = P(\limsup A_n)$$

(ix) If $\lim A_n$ exists then $\limsup A_n = \liminf A_n$

So we get

$$P(\liminf A_n) = \liminf P(A_n) = \limsup P(A_n) = P(\limsup A_n)$$

$$\Rightarrow \lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right)$$

$\boxtimes$

$\boxed{\text{Example}}$ (Countable probability space.)

$\Omega = \mathbb{N}$, $\mathcal{F} = 2^{\Omega}$, $P(\{k\}) = 2^{-k}$ $\quad \sum_{k=1}^{\infty} 2^{-k} = 1$

$$P(\{\text{even}\}) = \sum_{n=1}^{\infty} P(\{2k\}) = \sum_{k=1}^{\infty} 2^{-2k} = \frac{1}{4}\frac{1}{1-1/4} = \frac{1}{3}$$

$\boxed{\text{Example}}$ $\quad \Omega = [0,1] \quad P((a,b)) = (b-a)$, $0<a<b$.

$\mathcal{F}$: Borel σ-algebra smallest σ-algebra that contains all open intervals

$$P(\{1/3\}) = \lim_{n\to\infty} P\left(\left(\frac{1}{3}-\frac{1}{n}, \frac{1}{3}+\frac{1}{n}\right)\right) = \lim_{n\to\infty} \frac{2}{n} = 0$$

$$1 = P([0,1]) = P\left(\bigcup_{x\in[0,1]} \{x\}\right) \neq \sum_{x} P(\{x\}) = 0$$

If $A = \{a_n\}_{n=1}^{\infty} \subset [a,b]$ then $P(A) = 0$, which follows by $P(A) = \sum_{n=1}^{\infty} P(\{a_n\}) = 0$

$\boxed{\text{Remark}}$ We have that $P(\mathbb{Q} \cap [0,1]) = 0$ rational numbers

~~Conditionin~~

Conditioning and Independence

Definition (Conditional probability)
If $A, B \in \mathcal{F}$ and $P(B) > 0$ then $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Definition (Independence)
Two events A, B are independent if $P(A|B) = P(A)$
$P(B|A) = P(B)$

Putting these two definitions together independence also says $P(A \cap B) = P(A) P(B)$

Remark If $P(A) = 0$ or $P(A) = 1$, then A is independent of any $B \in \mathcal{F}$.

Properties If A and B are independent, then
- A and B^c are independent
- A^c and B^c —"—
- A^c and B —"—

Proof Let us assume that $P(A \cap B) = P(A) P(B)$
Then we have that

$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B) \\ &= P(A) - P(A) P(B) \\ &= P(A)(1 - P(B)) \\ &= P(A) P(B^c) \end{aligned}$$

$\blacksquare$

Definition The events $\{A_k, 1 \le k \le n\}$ are independent if and only if $P\left(\bigcap_{k=1}^{n} A_k\right) = \prod_{k=1}^{n} P(A_k)$

$\boxed{\text{Definition}}$ (Pair-wise independence)

The events $\{A_k, 1 \le k \le n\}$ are pair-wise independent if and only if

$$P(A_i \cap A_j) = P(A_i)\,P(A_j) \quad \forall\ 1 \le i \ne j \le n.$$

These two last definitions do not go along together very well!

$\boxed{\text{Example}}$ Flip two coins $\Omega = \{(0,0), (0,1), (1,0), (1,1)\}$

For $x \in \Omega$ $P(\{x\}) = 1/4$

Define $A = \{(0,1), (1,1)\}$ = "2nd coin tail"

$B = \{(1,0), (1,1)\}$ = "1st coin tail"

$C = \{(0,0), (1,1)\}$ = "pairs"

Let us first check pairwise independence. Indeed

$P(A \cap B) = P(\{1,1\}) = 1/4 = P(A)\cdot P(B)$ ✓

$P(B \cap C) = P(\{1,1\}) = 1/4 = P(B)\cdot P(C)$ ✓

$P(A \cap C) = P(\{1,1\}) = 1/4 = P(A)\cdot P(C)$ ✓

But $P((A \cap B) \cap C) = P(\{1,1\}) = 1/4$

whereas $P(A \cap B) = 1/4$, $P(C) = 1/2 \Rightarrow P(A \cap B)\,P(C) = 1/8$

Hence $P((A \cap B) \cap C) \ne P(A \cap B)\cdot P(C)$

So C is not independent of $A \cap B$

This means that A, B, C are pairwise independent but not independent.

$\boxed{\text{Definition}}$ For a countable family $\{A_k\}_{k=1}^{\infty}$ is independent if every finite subfamily is independent.

If $\{A_k\}_{k=1}^{\infty}$ are independent then

$$P\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{N\to\infty} P\left(\bigcap_{k=1}^{N} A_k\right) = \lim_{N\to\infty} \prod_{k=1}^{N} P(A_k)$$

$$= \prod_{k=1}^{\infty} P(A_k)$$

$\boxed{\text{Definition}}$ (~~Law of total Probability~~) (Partition)

A partition $\{H_i\}_{i=1}^{N}$ or $\{H_i\}_{i=1}^{\infty}$ is a ~~partition~~ family such that $H_i \cap H_j = \emptyset$ if $i \neq j$

i.e. if H_i and H_j do not overlap

$\boxed{\text{Theorem}}$ (Law of total probability)

Assume $\{H_i\}_{i=1}^{\infty}$ is a partition. Then $\forall A \in \mathcal{F}$

$$P(A) = \sum_{i=1}^{\infty} P(A \mid H_i) P(H_i)$$

$\boxed{\text{Proof}}$ We can write $A = \bigcup_{i=1}^{\infty} (A \cap H_i)$ (disjoint union)

Recall that $P(A \mid H_i) = \dfrac{P(A \cap H_i)}{P(H_i)}$

$$\Rightarrow P(A \cap H_i) = P(A \mid H_i) P(H_i)$$

So $P(A) = \underbrace{\sum_{i=1}^{\infty} P(A \cap H_i)}_{\text{countable additive}} = \sum_{i=1}^{\infty} P(A \mid H_i) P(H_i)$ ☒

$\boxed{\text{Example}}$ Roll a six side die. Then flip a coin the number of times shown on die

$A = \{\text{total of 3 heads}\} \Rightarrow P(A) = ?$

Consider the partition $H_i = \{\text{dice was } i\}$

$$\cancel{P(A \mid H_i) = \begin{cases} 0 & \text{if } i < 2 \\ \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^{i}, & \text{if } i \geq 3 \end{cases}}$$

Note that $P(A \mid H_i) = \begin{cases} 0 & , \text{if } i \le 2 \\ \binom{i}{3}\left(\frac{1}{2}\right)^i & , \text{if } i \geq 3 \end{cases}$

$\hookrightarrow$ combinations

So we conclude that

$$P(A) = \sum_{i=3}^{6} \binom{i}{3}\left(\frac{1}{2}\right)^i \frac{1}{6} = \sum_{i=1}^{6} P(A \mid H_i)\, P(H_i)$$