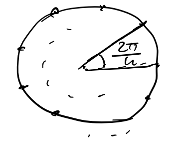
Last time: n-th root.

EX: Find n-th root of unity $= \{w \in \mathbb{C} : w^n = 1\}$.



Suppose $w = \cos\theta + i\sin\theta$, then $w^n = \cos n\theta + i \sin n\theta$.

$\Rightarrow$ $\cos n\theta = 1$, $\sin n\theta = 0$ $\Rightarrow$ $n\theta = 0 \pmod{2\pi}$

$\Leftrightarrow$ $n\theta = 2k\pi$, $k \in \mathbb{Z}$, $\theta = \frac{2k\pi}{n}$.

But as k takes value in $\mathbb{Z}$, $\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ gives n different values when $k = 0, \cdots, n-1$.

Elementary complex functions.

- Polynomials, rational functions.

$$P : \mathbb{C} \to \mathbb{C}, \qquad P(z) = a_n z^n + \cdots + a_1 z + a_0, \quad a_j \in \mathbb{C}, \; a_n \neq 0.$$

Recall (from last time): if z is a zero of a polynomial P with $a_0, \cdots, a_n \in \mathbb{R}$, then $\bar{z}$ is also a zero of P.

Rational functions: $f : \mathbb{C} \setminus \{z_1, \cdots, z_n\} \to \mathbb{C}$, $f(z) = \frac{P(z)}{Q(z)}$,

where $P(z)$ and $Q(z)$ are polynomials without common zeros, $z_1, \cdots, z_n$ are zeros of Q.

Call $z_1, \cdots, z_n$ the poles of f. Order of zero and pole.

EX: For $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$, consider

$$f(z) = \frac{az + b}{cz + d}. \qquad (\text{Note: } ad - bc \neq 0 \Leftrightarrow \text{no common zeros}).$$

zero of f: $-\frac{b}{a}$. Pole of f: $-\frac{d}{c}$.

Such f is called a Möbius transform.

- Exponential, trigonometric.

Recall (from real analysis):

$$\exp : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto e^x \left(= \exp(x)\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The function e^x is the unique function that satisfies:

- $e^0 = 1$, and $(e^x)' = e^x$.

Moreover, the Taylor expansion

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ is convergent for all } x \in \mathbb{R}, \text{ and } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Now: for $y \in \mathbb{R}$, formally insert iy into the power series:

$$\exp(iy) = 1 + \frac{iy}{1} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots$$

$$= \left(1 - \frac{y^2}{2} + \frac{y^2}{4} \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$$

$$= \cos y + i \sin y.$$

Def: We define the complex exponential function

$$\exp: \mathbb{C} \to \mathbb{C}, \qquad \exp(x+iy) = \exp(x) \cdot (\cos y + i \sin y)$$

as motivated by the formal eq. $\exp(z+w) = \exp z \cdot \exp w$.

Prop: The following properties hold:

(i) $\forall z, w \in \mathbb{C}$, $\exp(z+w) = \exp z \cdot \exp w$

(ii) $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$

(iii) for $x, y \in \mathbb{R}$, $\exp(iy) = \cos y + i \sin y$, and $|\exp(x+iy)| = \exp(x)$.

(iv) $\exp(z + 2\pi k i) = \exp(z)$, $\forall z \in \mathbb{C}$, $k \in \mathbb{Z}$, and $\exp(z) = 1 \Leftrightarrow z \in 2\pi i \mathbb{Z}$.

(v) $\overline{\exp(x+iy)} = \exp(x - iy)$, $\forall x, y \in \mathbb{R}$. $\Leftrightarrow$ $\exp(\bar{z}) = \overline{\exp(z)}$, $\forall z \in \mathbb{C}$.

Proof: (i) let $z = x+iy$, $w = a+bi$. Then

$$\exp(z)\exp(w) = e^x (\cos y + i \sin y) \cdot e^a(\cos b + i \sin b)$$

$$= e^{x+a}\left((\cos y \cos b - \sin y \sin b) + i(\sin y \cos b + \cos y \sin b)\right)$$

$$= e^{x+a}\left(\cos(y+b) + i \sin(y+b)\right)$$

$$= \exp\left((x+a) + i(y+b)\right) = \exp(z+w)$$

(ii) Note: $e^{x+iy} = e^x (\cos y + i \sin y)$. For $w = \rho(\cos\theta + i \sin\theta)$, take $x = \log \rho$, $y = \theta$, then $e^{x+iy} = w$.

$$\Rightarrow \exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$$

(iii) trivial; $|\exp(x+iy)| = |e^x(\cos y + i \sin y)|$

$$= e^x \cdot |\cos y + i \sin y|$$

$$= e^x \cdot \sqrt{\cos^2 y + \sin^2 y} = e^x.$$

(iv) · cos and sin are 2π-periodic.

· $\exp(x+iy) = e^x (\cos y + i \sin y) = 1$, Compare modulus $\Rightarrow$ $x = 0$

Compare argument $\Rightarrow$ $y = 0 \mod 2\pi$.

(v) $\overline{\cos y + i \sin y} = \cos y - i \sin y$

$$= \cos(-y) + i \sin(-y).$$

$\square$

· Trigonometric functions.

when $y \in \mathbb{R}$, $e^{iy} = \cos y + i \sin y$, $\quad e^{-iy} = \cos y - i\sin y$. We extend this to $\gamma \in \mathbb{C}$.

Def: We define the complex sine and cosine as:

$$\sin : \mathbb{C} \to \mathbb{C}, \qquad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$\cos : \mathbb{C} \to \mathbb{C}, \qquad \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

Properties: (i) $\sin^2(z) + \cos^2(z) = 1$

(ii) (Euler formula) $\exp(iz) = \cos z + i \sin z$

(iii) $\sin(z+w) = \sin z \cos w + \cos z \sin w$

$\cos(z+w) = \cos z \cos w - \sin z \sin w$, $\quad z, w \in \mathbb{C}$.

Proof: (i)
$$\sin^2 z + \cos^2 z = \left(\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\right)^2 + \left(\frac{1}{2}\left(e^{iz} + e^{-iz}\right)\right)^2$$
$$= -\frac{1}{4}\left(e^{2iz} - 2 + e^{-2iz}\right) + \frac{1}{4}\left(e^{2iz} + 2 + e^{-2iz}\right)$$
$$= 1$$

(ii) $$\cos z + i \sin z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) + i \cdot \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = e^{iz}$$

(iii)
$$\cos z \cos w - \sin z \sin w = \frac{e^{iz} + e^{-iz}}{2} \cdot \frac{e^{iw} + e^{-iw}}{2} - \frac{e^{iz} - e^{-iz}}{2i} \cdot \frac{e^{iw} - e^{-iw}}{2i}$$
$$= \frac{e^{i(z+w)} + e^{i(-z+w)} + e^{i(z-w)} + e^{-i(z+w)}}{4} + \frac{e^{i(z+w)} - e^{i(-z+w)} - e^{i(z-w)} + e^{-i(z+w)}}{4}$$
$$= \frac{1}{2}\left(e^{i(z+w)} + e^{-i(z+w)}\right) = \cos(z+w)$$

- Logarithms

Motivation: For $w \in \mathbb{C}\setminus\{0\}$, a solution z to the equation $\exp(z) = w$ is called a logarithm of w, denoted $z = \log(w)$

Issue: $\log(w)$ is multivalued. EX: $e^0 = 1$. and $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$.
More generally:

Prop: Every $\mathbb{C}\setminus\{0\}$ has countably many logarithms, given by

$$\log w = \left\{ \log(|w|) + i\left(\arg(w) + 2\pi n\right) : n \in \mathbb{Z} \right\}$$

Proof: First, if $z = \log(|w|) + i(\arg(w) + 2\pi n)$, then

$$e^z = e^{\log|w|} \cdot \left(\cos\left(\arg w + 2\pi n\right) + i \sin\left(\arg w + 2\pi n\right)\right)$$
$$= |w| \cdot \left(\cos(\arg w) + \sin(\arg w)\right) = w.$$

Conversely, if $\underset{=x+iy}{z}$ is a logarithm of $w = \rho(\cos\theta + i\sin\theta)$, then

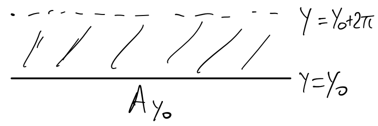
$$e^z = e^x(\cos y + i \sin y) \Rightarrow e^x = \rho, \begin{cases} \cos y = \cos\theta \\ \sin y = \sin\theta \end{cases}$$

$\Rightarrow \quad y = \theta + 2\pi n, \quad n \in \mathbb{Z}.$ $\square$

To obtain a single valued function, we need to restrict the range of log.

<u>Def</u>: For $Y_0 \in \mathbb{R}$, define

$$A_{Y_0} = \{ x + iy \,;\, x \in \mathbb{R},\ Y_0 \le Y < Y_0 + 2\pi \}$$



Observe: $\exp: A_{Y_0} \to \mathbb{C}\setminus\{0\}$ is one-to-one and onto.

<u>Def</u>: The function $\log: \mathbb{C}\setminus\{0\} \to \mathbb{C}$, with range in A_{Y_0}, is called the branch of logarithm lying in A_{Y_0}.

Warning: · when applying log, must specify either: · log is a multivalued function, or:
· restrict it to a branch.

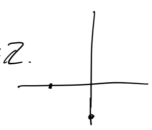
· However, when $x \in \mathbb{R}$, we let $\log w$ be the unique real log.

<u>EX</u>: $z_1, z_2 \in \mathbb{C}\setminus\{0\}$. Then: as multivalued functions.

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

· if restricted in a branch, $\log(z_1 z_2) = \log z_1 + \log z_2 \pmod{2\pi i}$

<u>EX</u>: · Regarded as a multivalued map, $\log(-2i) = \log 2 + i\left(\frac{3}{2}\pi + 2\pi n\right), n \in \mathbb{Z}.$



· If we restrict to the branch $A_{-2\pi} = \{z : -2\pi \le \operatorname{Im} z < 0\}$,

$$\log(-2i) = \log 2 - \frac{\pi}{2} i$$

- Complex powers.

If $(a, b) \in \mathbb{C}\setminus\{0\} \times \mathbb{C}$, define $a^b = e^{b \log a}$

With our previous convention. if $a \in \mathbb{R}_+$, a^b is unique.

But if $a \notin \mathbb{R}_+$, then a^b is either:

- multivalued (differ by $e^{2\pi i n b}$, $n \in \mathbb{Z}$)
- or need to restrict to a branch A_{Y_0} of log.

Continuity.

<u>Def</u>: A sequence (z_n) of complex numbers is bounded, if $\exists\, C \ge 0$ s.t.

$$|z_n| \le C, \quad \forall n \in \mathbb{Z}_+$$

It's called convergent, if $\exists\, z \in \mathbb{C}$ s.t. for every $\varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{Z}_+$ s.t.

$$|z_n - z| < \varepsilon \text{ for all } n \ge N.$$

In this case, say z the limit of (z_n), denoted $z = \lim_{n\to\infty} z_n$.

Basic properties of limits.

Prop. (i) The limit of a convergent complex seq. (z_n) is unique.

(ii) Convergent sequences are bounded

(iii) If $\lim_{n\to\infty} z_n = z$, $\lim_{n\to\infty} w_n = w$, then

$$\lim (z_n + w_n) = z + w, \qquad \lim (z_n \cdot w_n) = zw$$

If $w \neq 0$ then $\lim \frac{z_n}{w_n} = \frac{z}{w}$.

(iv) If $\lim z_n = z$, then $\lim \bar{z}_n = \bar{z}$, $\lim |z_n| = |z|$.

$$\lim \operatorname{Re} z_n = \operatorname{Re} z, \qquad \lim \operatorname{Im} z_n = \operatorname{Im} z$$

(v) If a seq. (z_n) satisfies $\lim \operatorname{Re}(z_n) = \operatorname{Re}(z)$ and $\lim \operatorname{Im}(z_n) = \operatorname{Im} z$ then $\lim z_n = z$.