Problem 1. Consider two points $M_1 = (x_1, y_1, z_1)$, $M_2 = (x_2, y_2, z_2)$ on the Riemann sphere S that are not the north pole, and w_1, w_2 their stereographic projection in the complex plane. Show that $M_1 = -M_2$ iff $w_1 \overline{w_2} = -1$

Sol) ($\Rightarrow$) Let $M_1 = -M_2$. Then we need to show that $w_1 \overline{w_2} = -1$. We have

$$x_1 = -x_2, \quad y_1 = -y_2, \quad z_1 = -z_2 \quad \text{---} \quad (1)$$

Now, let $w_1 = \dfrac{x_1 + iy_1}{1 - z_1}$ and $w_2 = \dfrac{x_2 + iy_2}{1 - z_2}$

Because of (1), w_2 and $\overline{w_2}$ can be written

$$w_2 = \frac{-x_1 - iy_1}{1 + z_1}, \quad \overline{w_2} = \frac{-x_1 + iy_1}{1 + z_1}$$

Then,

$$w_1 \overline{w_2} = \frac{-x_1^2 - y_1^2}{1 - z_1^2}$$

$$= \frac{-(1 - z_1^2)}{1 - z_1^2} \quad (\because M_1, M_2 \text{ are on Riemann sphere and thus } x_1^2 + y_1^2 + z_1^2 = 1 \text{ holds true}$$

$$= -1.$$

($\Leftarrow$) Let $w_1 \overline{w_2} = -1$ for $w_1 = (x_1 + iy_1)/(1 - z_1)$ and $w_2 = (x_2 + iy_2)/(1 - z_2)$

Then we need to show that $M_1 = -M_2$.

$$w_1 \overline{w_2} = \frac{x_1 + iy_1}{1 - z_1} \cdot \frac{x_2 - iy_2}{1 - z_2} = \frac{(x_1x_2 + y_1y_2) + (x_2y_1 - x_1y_2)i}{1 - z_1 - z_2 + z_1z_2} = -1$$

$$\Rightarrow \quad x_1x_2 + y_1y_2 + (x_2y_1 - x_1y_2)i = -1 + z_1 + z_2 - z_1z_2$$

Let $x_1 = cx_2$, $y_1 = cy_2$, $z_1 = cz_2$ for constant c. Then we have

$$x_1x_2 + y_1y_2 = -1 + z_1 + z_2 - z_1z_2 \quad \text{(real values from both sides)}$$

$$cx_2^2 + cy_2^2 = -1 + cz_2 + z_2 - cz_2^2$$

$$c(x_2^2 + y_2^2 + z_2^2) = -1 + z_2(1 + c)$$

$$c = -1 + z_2(1 + c) \quad (\because x_2^2 + y_2^2 + z_2^2 = 1)$$

(Cont. Problem 1)

$$C = -1 + Z_2 + C Z_2$$

$$C(1 - Z_2) = -(1 - Z_2)$$

$$C = -1 \quad \text{for } Z_2 \neq 1$$

Thus, we've shown that

$$X_1 = -X_2, \; Y_1 = -Y_2, \text{ and } Z_1 = -Z_2$$

$$\Rightarrow \quad M_1 = -M_2$$

$\boxed{\checkmark}$

Problem 2. Which of the following functions are holomorphic in their domain of definition?

(1) $f(z) = z^3$.

Sol) Let $z = x+iy$. Then, we have

$$f(z) = (x+iy)^3 = x^3 + 3x^2yi + 3x(iy)^2 + (iy)^3$$
$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

Then let $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$.

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial v}{\partial x} = 6xy \quad , \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

By Cauchy-Riemann equations, we can confirm that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, $f(z) = z^3$ is holomorphic.

(2) $f(z) = |z|^2$.

Sol) $f(z) = |z|^2 = z \cdot \bar{z} = (x+iy)(x-iy) = x^2+y^2$ for $z = x+iy$.

Let $u(x,y) = x^2+y^2$ $v(x,y) = 0$.

$\partial u/\partial x = 2x$, $\partial u/\partial y = 2y$, $\partial v/\partial x = 0$, $\partial v/\partial y = 0$

It doesn't satisfy the Cauchy-Riemann equation because

$\partial u/\partial x \neq \partial v/\partial x$ and $\partial u/\partial y = -\partial v/\partial x$.

Thus, $f(z) = |z|^2$ is NOT holomorphic.

(Cont. Problem 2)

3. $f(z) = \frac{1}{z}$

Sol) Let $x+iy = z$. Then

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

It yields that

$$u(x,y) = \frac{x}{x^2+y^2}, \quad v(x,y) = \frac{-y}{x^2+y^2}$$

Then, we have

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x(x^2+y^2)^{-1}\right) = (x^2+y^2)^{-1} + x(-1)(x^2+y^2)^{-2}(2x)$$

$$= \frac{1}{x^2+y^2} + \frac{-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x(x^2+y^2)^{-1}\right) = x(-1)(x^2+y^2)^{-2}(2y) = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(-y(x^2+y^2)^{-1}\right) = -y(-1)(x^2+y^2)^{-2}(2x) = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(-y(x^2+y^2)^{-1}\right) = -(x^2+y^2)^{-1} - y(-1)(x^2+y^2)^{-2}(2y)$$

$$= \frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

We've shown that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, $f(z) = 1/z$ is holomorphic. $\boxed{\checkmark}$

Problem 3.

Sol) $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$. Then $f(z)$ can be written

$$f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$$

① Find differentiable at z_0 when $r \to r_0$ at $\theta = \theta_0$

$$f'(z_0) = \lim_{r\to r_0} \frac{f(re^{i\theta_0}) - f(r_0e^{i\theta_0})}{re^{i\theta_0} - r_0e^{i\theta_0}} = \frac{1}{e^{i\theta_0}} \lim_{r\to r_0} \frac{f(re^{i\theta_0}) - f(r_0e^{i\theta_0})}{r - r_0}$$

$$= e^{-i\theta_0}\left[\lim_{r\to r_0} \frac{u(r,\theta_0) - u(r_0,\theta_0) + iv(r,\theta_0) - iv(r_0,\theta_0)}{r - r_0}\right]$$

$$= e^{-i\theta_0}\left[\lim_{r\to r_0} \frac{u(r,\theta_0) - u(r_0,\theta_0)}{r - r_0} + i\lim_{r\to r_0} \frac{v(r,\theta_0) - v(r_0,\theta_0)}{r - r_0}\right]$$

$$= e^{-i\theta_0}\left[\frac{\partial u}{\partial r}(r_0,\theta_0) + i\frac{\partial v}{\partial r}(r_0,\theta_0)\right] \quad \text{------} \quad (1)$$

② Find differentiable at z_0 when $\theta \to \theta_0$ where $r = r_0$

$$f'(z_0) = \lim_{\theta\to\theta_0} \frac{f(r_0e^{i\theta}) - f(r_0e^{i\theta_0})}{r_0e^{i\theta} - r_0e^{i\theta_0}} = \frac{1}{r_0} \lim_{\theta\to\theta_0} \frac{f(r_0e^{i\theta}) - f(r_0e^{i\theta_0})}{e^{i\theta} - e^{i\theta_0}}$$

$$= \frac{1}{r_0} \lim_{\theta\to\theta_0}\left[\frac{u(r_0,\theta) - u(r_0,\theta_0) + iv(r_0,\theta) - iv(r_0,\theta_0)}{\theta - \theta_0} \times \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}}\right]$$

$$= \frac{1}{r_0} \lim_{\theta\to\theta_0}\left[\left(\frac{u(r_0,\theta) - u(r_0,\theta_0)}{\theta - \theta_0} + i\frac{v(r_0,\theta) - v(r_0,\theta_0)}{\theta - \theta_0}\right) \times \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}}\right]$$

$$= \frac{1}{ir_0e^{i\theta_0}}\left[\frac{\partial u}{\partial \theta}(r_0,\theta_0) + i\frac{\partial v}{\partial \theta}(r_0,\theta_0)\right] \left(\because \frac{\theta-\theta_0}{e^{i\theta}-e^{i\theta_0}} \doteq \frac{1}{ie^{i\theta_0}} \text{ and details explained next page}\right) (2)$$

$$= \frac{e^{-i\theta_0}}{r_0}\left[\frac{\partial v}{\partial \theta}(r_0,\theta_0) - i\frac{\partial u}{\partial \theta}(r_0,\theta_0)\right] \quad \text{------} \quad (3)$$

(Cont. Problem 3)

Putting all together, (1) = (3), thus, we have

$$e^{-2\theta_0}\frac{\partial u}{\partial r}(r_0,\theta_0) = \frac{e^{-2\theta_0}}{r_0}\frac{\partial v}{\partial \theta}(r_0,\theta_0) \Rightarrow r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$$

$$i\frac{\partial v}{\partial r}(r_0,\theta_0) = \frac{-i}{r_0}\frac{\partial u}{\partial \theta}(r_0,\theta_0) \Rightarrow \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Last, to solve (3), we have

$$\frac{e^{i\theta}-e^{i\theta_0}}{\theta-\theta_0} = \frac{\cos\theta + i\sin\theta - (\cos\theta_0 + i\sin\theta_0)}{\theta-\theta_0}$$

$$= \frac{\cos\theta - \cos\theta_0}{\theta-\theta_0} + i\left(\frac{\sin\theta - \sin\theta_0}{\theta-\theta_0}\right)$$

As $\theta \to \theta_0$,

$$\lim_{\theta\to\theta_0} \frac{\cos\theta - \cos\theta_0}{\theta-\theta_0} = (\cos\theta)'\Big|_{\theta=\theta_0} = -\sin\theta_0$$

$$\lim_{\theta\to\theta_0} \frac{\sin\theta - \sin\theta_0}{\theta-\theta_0} = (\sin\theta)'\Big|_{\theta=\theta_0} = \cos\theta_0$$

Thus, $$\frac{e^{i\theta}-e^{i\theta_0}}{\theta-\theta_0} = -\sin\theta_0 + i\cos\theta_0 = i(\cos\theta_0 + i\sin\theta_0)$$

$$= ie^{i\theta}$$

Our goal is to find $(\theta-\theta_0)/(e^{i\theta}-e^{i\theta_0})$. Thus, we can conclude that

$$\frac{\theta-\theta_0}{e^{i\theta}-e^{i\theta_0}} = \frac{1}{ie^{i\theta}}$$

☑

Problem 4.

Sol) Let Ω be an open set and f be holomorphic.
Then define $\tilde{\Omega} = \{z \in \mathbb{C} : \bar{z} \in \Omega\}$. It follows that $\exists r > 0$ such that $D(\bar{z}, r) \subseteq \Omega$ because $\bar{z} \in \Omega$ and Ω is open. Then for $z \in \mathbb{C}$ there exists $r > 0$ such that $D(z, r) \subseteq \tilde{\Omega}$. Thus, we've shown that $\tilde{\Omega}$ is open.

Now, it's given that $\tilde{f} : z \in \tilde{\Omega} \mapsto \overline{f(\bar{z})} \in \mathbb{C}$.
Then we need to show that $\tilde{f}$ is holomorphic.

$$\tilde{f}(x+iy) = \overline{f(x-iy)} = \overline{u(x,-y) + i v(x,-y)}$$
$$= u(x,-y) - i v(x,-y)$$

f is holomorphic where $f(x+iy) = u(x,y) + i v(x,y)$.
Then it holds $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. (1)

Then, for $\tilde{f}$, let $u = u(x,-y)$, $v = -v(x,-y)$.
Then we have $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y} = +\frac{\partial v}{\partial y}$. (2)

Applying (2) to (1),
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus, we've shown that $\tilde{f}$ is holomorphic. $\boxtimes$

Problem 5.

1. Show that u, v satisfy Laplace's equation

Sol) f is a holomorphic function. Thus it follows that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{— (2)}$$

To show $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$,

differentiate (1) with respect to x

and differentiate (2) with respect to y.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}.$$

This yields $\quad \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2} \Rightarrow \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

To show $\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 = 0$,

differentiate (1) with respect to y

and differentiate (2) with respect to x

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}.$$

This yields $\quad \dfrac{\partial^2 v}{\partial y^2} = -\dfrac{\partial^2 v}{\partial x^2} \Rightarrow \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0.$

(Cont. Problem 5)

5.2. $|f(x+iy)| = \sqrt{u^2+v^2}$. Then we have

$$\phi(x,y) = \log|f(x+iy)| = \log\sqrt{u^2+v^2} = \frac{1}{2}\log(u^2+v^2)$$

$$\frac{\partial \phi}{\partial x} = \frac{1}{2}\,\frac{2u\left(\frac{\partial u}{\partial x}\right) + 2v\left(\frac{\partial v}{\partial x}\right)}{u^2+v^2} = \frac{u\left(\frac{\partial u}{\partial x}\right) + v\left(\frac{\partial v}{\partial x}\right)}{u^2+v^2} = (u^2+v^2)^{-1}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial^2 \phi}{\partial x^2} = -(u^2+v^2)^{-2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)$$

$$+ (u^2+v^2)^{-1}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial x} + v\frac{\partial^2 v}{\partial x^2}\right)$$

$$= -2(u^2+v^2)^{-2}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2$$

$$+ (u^2+v^2)^{-1}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 v}{\partial x^2}\right)$$

$$= -2(u^2+v^2)^{-2}\left(u^2\left(\frac{\partial u}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + v^2\left(\frac{\partial v}{\partial x}\right)^2\right)$$

$$+ (u^2+v^2)^{-1}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 v}{\partial x^2}\right) \quad (1)$$

Likewise, since $\partial^2\phi/\partial y^2$ and $\partial^2\phi/\partial x^2$ are symmetric, $x \leftrightarrow y$, then

$$\frac{\partial^2 \phi}{\partial y^2} = -2(u^2+v^2)^{-2}\left(u^2\left(\frac{\partial u}{\partial y}\right)^2 + 2uv\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + v^2\left(\frac{\partial v}{\partial y}\right)^2\right)$$

$$+ (u^2+v^2)^{-1}\left(\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2} + v\frac{\partial^2 v}{\partial y^2}\right) \quad (2)$$

Putting them together,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2(u^2+v^2)^{-2}\left[u^2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) + 2uv\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right) + v^2\left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)\right]$$

$$+ (u^2+v^2)^{-1}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

(Cont. Problem 5.2)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2(u^2+v^2)^{-2}\left[u^2\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right) + v^2\left(\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right)\right]$$

$$+(u^2+v^2)^{-1}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$\left(\because \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\right.$$
$$\left.\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}=0 \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\right)$$

$$=\frac{1}{(u^2+v^2)^2}\left[-2u^2\left(\frac{\partial u}{\partial x}\right)^2-2u^2\left(\frac{\partial u}{\partial y}\right)^2-2v^2\left(\frac{\partial v}{\partial x}\right)^2-2v^2\left(\frac{\partial v}{\partial y}\right)^2+u^2\left(\frac{\partial u}{\partial x}\right)^2+v^2\left(\frac{\partial u}{\partial x}\right)^2\right.$$

$$\left.+u^2\left(\frac{\partial v}{\partial x}\right)^2+v^2\left(\frac{\partial v}{\partial x}\right)^2+u^2\left(\frac{\partial u}{\partial y}\right)^2+v^2\left(\frac{\partial u}{\partial y}\right)^2+u^2\left(\frac{\partial v}{\partial y}\right)^2+v^2\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$=\frac{1}{(u^2+v^2)^2}\left[-2u^2\left(\frac{\partial u}{\partial x}\right)^2-2u^2\left(\frac{\partial u}{\partial y}\right)^2-2v^2\left(\frac{\partial v}{\partial x}\right)^2-2v^2\left(\frac{\partial v}{\partial y}\right)^2\right.$$

$$\left.+2u^2\left(\frac{\partial u}{\partial x}\right)^2+2u^2\left(\frac{\partial u}{\partial y}\right)^2+2v^2\left(\frac{\partial v}{\partial x}\right)^2+2v^2\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$=0 \quad \left(\because \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\right)$$

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