Complex Variables I – Problem Set 2

Due at 5 pm on Friday, September 22, 2023 via Gradescope

Problem 1

Regard log as a multivalued function. Find all the values of

a) $\log(-i)$

b) $\log(1+i)$

c) $\log 2$.

Solution: a) $\frac{3\pi}{2}i + 2k\pi i,\ k \in \mathbb{Z}$.
b) $\log(\sqrt{2}) + \frac{\pi}{4}i + 2k\pi i,\ k \in \mathbb{Z}$.
c) $\log 2 + 2k\pi i,\ k \in \mathbb{Z}$.

Problem 2

Solve the following equations (make sure to find all solutions):

a) $\cos z = 2$

b) $\sin z = 2$.

Solution:
a) $\cos z = 2 \Leftrightarrow \frac{1}{2}(e^{iz} + e^{-iz}) = 2$. Solve the quadratic equation and obtain:

$$e^{iz} = 2 + \sqrt{3} \quad \text{or} \quad e^{iz} = 2 - \sqrt{3}.$$

We thus get:

$$z = 2k\pi - i\log(2+\sqrt{3}) \text{ or } 2k\pi - i\log(2-\sqrt{3}), k \in \mathbb{Z}.$$

b) $\sin z = 2 \Leftrightarrow \frac{1}{2i}(e^{iz} - e^{-iz}) = 2$. Solve the quadratic equation and obtain:

$$e^{iz} = (2+\sqrt{3})i \text{ or } e^{iz} = (2-\sqrt{3})i.$$

Thus:

$$z = (2k\pi + \frac{\pi}{2}) - i\log(2+\sqrt{3}) \text{ or } (2k\pi + \frac{\pi}{2}) - i\log(2-\sqrt{3}), k \in \mathbb{Z}.$$

Problem 3

Prove that the function $\sin z$ maps the strip $-\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}$ onto the set $\mathbb{C} \setminus \{z : \operatorname{Im} z = 0 \text{ and } |\operatorname{Re} z| \geq 1\}$.

Solution: For $w \in \mathbb{C}$, consider the equation $\sin z = w \Leftrightarrow e^{iz} - e^{-iz} = 2iw$. We show that this equation has a solution in the region $\{z \in \mathbb{C} : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}\}$ if and only if $\operatorname{Im} w \neq 0$ or $|\operatorname{Re} w| < 1$.

Write $z = x + iy$ and $w = a + bi$ for $x, y, a, b \in \mathbb{R}$. Then the equation becomes

$$\frac{1}{2i}(e^{ix-y} + e^{-ix+y}) = a + bi \Leftrightarrow \begin{cases} \sin x \cosh y = a \\ \cos x \sinh y = b. \end{cases}$$

When $b = 0$, since $\cos x \neq 0$ when $x \in (-\pi/2, \pi/2)$, we have that $\sinh y = 0$, and thus $y = 0$, $\cosh y = 1$. Therefore $a = \sin x$ has a solution for $x \in (-\pi/2, \pi/2)$ if and only if $|a| < 1$.

On the other hand, given $b \neq 0$, we can always find a solution (x, y) with $x \in (-\pi/2, \pi/2)$ as follows. From the equation, it follows

$$\frac{a^2}{\cosh^2 y} + \frac{b^2}{\sinh^2 y} = 1.$$

Using that $\cosh^2 y = \sinh^2 y + 1$, we solve the quadratic equation and find

$$\cosh^2 y = \frac{a^2 + b^2 + 1 + \sqrt{(a^2 + b^2 - 1)^2 + 4b^2}}{2} > a^2.$$

Here we have used that $b \neq 0$ in the last inequality. Thus, there exists a unique $x \in (-\pi/2, \pi/2)$ solving $\sin x = a/\cosh y, \cos x = b/\sinh y$.

Problem 4

Find all Möbius transforms $f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$, such that $|f(z)| = 1$ whenever $|z| = 1$.

(Hint: when $a \neq 0$, you can divide a on both the numerator and the denominator. Thus, you can assume $a = 1$ without loss of generality.)

Solution: We distinguish two cases according to whether $a = 0$.

If $a = 0$, we have that $f(z) = \frac{b}{cz+d}$. We have that

$$f(z)\overline{f(z)} = \frac{b\bar{b}}{c\bar{c}z\bar{z} + d\bar{d} + c\bar{d}z + \bar{c}d\bar{z}}.$$

When $|z| = 1$, we should have that $|f(z)| = 1$. Thus, we have

$$1 = \frac{b\bar{b}}{c\bar{c} + d\bar{d} + c\bar{d}z + \bar{c}d\bar{z}} \Leftrightarrow b\bar{b} = c\bar{c} + d\bar{d} + c\bar{d}z + \bar{c}d\bar{z},$$

whenever $|z| = 1$. This tells us that $c\bar{d} = 0$ (otherwise, the equation is the equation for a line in $\mathbb{C}$, not a circle). Note that $c \neq 0$ (otherwise f is a constant). Thus $d = 0$, and hence we have $f(z) = \frac{b}{cz}$ with $|b| = |c| \neq 0$. One may divide b on both the neumerator and the denominator and write this into $f(z) = \frac{1}{cz}$ for some $|c| = 1$.

If $a \neq 0$, we divide a on both the numerator and the denominator, and assume that $f(z) = \frac{z+b}{cz+d}$. As before, we deduce that

$$1 = f(z)\overline{f(z)} = \frac{z\bar{z} + b\bar{b} + \bar{b}z + b\bar{z}}{c\bar{c}z\bar{z} + d\bar{d} + c\bar{d}z + \bar{c}d\bar{z}}$$

whenever $|z| = 1$. This tells us

$$1 + b\bar{b} + \bar{b}z + b\bar{z} = c\bar{c} + d\bar{d} + c\bar{d}z + \bar{c}d\bar{z},$$

whenever $|z| = 1$. Thus, we deduce that

$$1 + |b|^2 = |c|^2 + |d|^2, \quad \bar{b} = c\bar{d}.$$

The second equation tells us $|b| = |c||d|$. Plugging this into the first equation gives $(|c| - 1)(|d| - 1) = 0 \Rightarrow |c| = 1$ or $|d| = 1$. If $|c| = 1$, then $f(z) = \bar{c}\frac{z+b}{c\bar{c}+d\bar{c}} = \bar{c}$ a constant, which cannot happen. Thus, we have $|d| = 1$. Therefore, we may rewrite

$$f(z) = \bar{d}\frac{z + b}{c\bar{d}z + 1} = \bar{d}\frac{z + b}{\bar{b}z + 1}.$$

Remember to justify your answers and acknowledge collaborations and outside help!