Problem 1. Solve the following equations

a) $z^2 + iz + 6 = 0$

Sol) Let $z = x + yi$. Applying it to the original equation, we have

$$(x+yi)^2 + i(x+yi) + 6$$
$$= x^2 + 2xyi - y^2 + xi - y + 6$$
$$= [x^2 - (y^2+y-6)] + (2xy+x)i = 0$$

To make the last equation true, the following must hold true.

$$\begin{cases} x^2 - (y^2+y-6) = 0 & ① \\ 2xy + x = 0 & ② \end{cases}$$

Solve ① : $x^2 = y^2+y-6 = (y+3)(y-2)$

$\Rightarrow$ $x = 0$ and $y = 2$ or -3

Solve ② : $x(2y+1) = 0$

$\Rightarrow$ $x = 0$ or $y = -1/2$

We can conclude that $\underline{z = 2i}$ or $\underline{z = -3i}$ ☑

b) $z^6 - 64 = 0$

Sol) $z^6 - 64 = (z^3-8)(z^3+8)$
$$= (z-2)(z^2+2z+4)(z+2)(z^2-2z+4)$$
$$= (z-2)(z+2)(z^2+2z+4)(z^2-2z+4)$$

$$z = -2,\ 2,\ \frac{-2 \pm \sqrt{-12}}{2},\ \frac{2 \pm \sqrt{-12}}{2} = \underline{-2, 2, -1 \pm i\sqrt{3}, 1 \pm i\sqrt{3}}$$ ☑

c) $z^3 + 1 = 0$

Sol) $z^3 + 1 = (z+1)(z^2 - z + 1) = 0$

By quadratic formula

$$z = -1,\ \frac{1 \pm \sqrt{1-4}}{2} = \underline{-1,\ \frac{1 \pm i\sqrt{3}}{2}}$$ ☑

Problem 2 Let z, w be complex numbers. Prove the parallelogram identity:

$$|z-w|^2 + |z+w|^2 = 2(|z|^2 + |w|^2)$$

sol) Each term of left side of equation can be expressed as follow

$$\begin{aligned} |z-w|^2 &= (z-w)(\bar{z}-\bar{w}) \\ &= z\bar{z} - z\bar{w} - w\bar{z} + w\bar{w} \\ &= |z|^2 - z\bar{w} - w\bar{z} + |w|^2 \quad (1) \end{aligned}$$

$$\begin{aligned} |z+w|^2 &= (z+w)(\bar{z}+\bar{w}) \\ &= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + z\bar{w} + w\bar{z} + |w|^2 \quad (2) \end{aligned}$$

Then, simplify terms by doing $(1)+(2)$,

$$\begin{aligned} (1)+(2) &= 2|z|^2 + 2|w|^2 \\ &= 2(|z|^2 + |w|^2) \end{aligned}$$

$\boxtimes$

Problem 3. Prove that $\left|\frac{z-w}{1-\bar{z}w}\right|=1$ if either $|z|=1$ or $|w|=1$, and $\bar{z}w \neq 1$.

Sol) By modulus property, the given equation can be written as

$$\left|\frac{z-w}{1-\bar{z}w}\right| = \frac{|z-w|}{|1-\bar{z}w|} = 1$$

and it implies that $|z-w| = |1-\bar{z}w|$.

Let $z = a+bi$ and $w = x+yi$ $(a,b,x,y \in \mathbb{R})$. It follows that

$$\begin{aligned} |(a-x)+(b-y)i| &= |1-(a-bi)(x+yi)| \\ &= |1-(ax+ayi-bxi+by)| \\ &= |(1-ax-by)+(bx-ay)i| \end{aligned}$$

Taking modulus square of the equation yields

$$(a-x)^2+(b-y)^2 = (1-ax-by)^2+(bx-ay)^2$$

$$a^2-2ax+x^2+b^2-2by+y^2 = 1+a^2x^2+b^2y^2-2ax-2by+2abxy + b^2x^2+a^2y^2-2abxy$$

After canceling out,

$$a^2+b^2+x^2+y^2 = 1+a^2x^2+a^2y^2+b^2x^2+b^2y^2 \quad \text{—— (1)}$$

By property of fraction, $\bar{z}w \neq 1$ should remain true.

Assume that $|z|=1$. It follows that $a^2+b^2=1$.

Applying it to the equation (1),

$$\begin{aligned} 1+x^2+y^2 &= 1+x^2(a^2+b^2)+y^2(a^2+b^2) \\ &= 1+x^2+y^2. \end{aligned}$$

Now, assume that $|w|=1$. It follows that $x^2+y^2=1$.

Applying it to the equation (1),

$$\begin{aligned} 1+a^2+b^2 &= 1+a^2(x^2+y^2)+b^2(x^2+y^2) \\ &= 1+a^2+b^2. \end{aligned}$$

We've shown that $\left|\frac{z-w}{1-\bar{z}w}\right|=1$ if either $|z|=1$ or $|w|=1$, and $\bar{z}w \neq 1$.

$\square$

Problem 4. Let $c_0, c_1, \ldots, c_n$ be n complex numbers, and consider the following polynomial in z:

$$P(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0$$

Show that there exists a number $R > 0$ such that for all $z \in \mathbb{C}$ with $|z| > R$, we have $\left|\frac{1}{P(z)}\right| < \frac{2}{|c_n| R^n}$

Sol) By Triangle inequality, $|P(z)|$ can be expressed as

$$|P(z)| \geq |c_n z^n| - |c_{n-1} z^{n-1} + \ldots + c_1 z + c_0|$$

$$= \frac{|c_n z^n|}{2} + \left(\frac{|c_n z^n|}{2} - |c_{n-1} z^{n-1} + \ldots + c_1 z + c_0|\right) \quad (1)$$

$$> \frac{|c_n z^n|}{2} \quad (\text{holds true because } (1) \geq 0) \quad (2)$$

Suppose there exists a number $R > 0$ such that for all $z \in \mathbb{C}$ with $|z| > R$.

Continue with the expression (2), since $|z| > R$, we have

$$|P(z)| > \frac{|c_n z^n|}{2} > \frac{|c_n| R^n}{2}$$

Thus, for all $|c_n| \neq 0$, we have

$$\left|\frac{1}{P(z)}\right| < \frac{2}{|c_n| R^n}$$

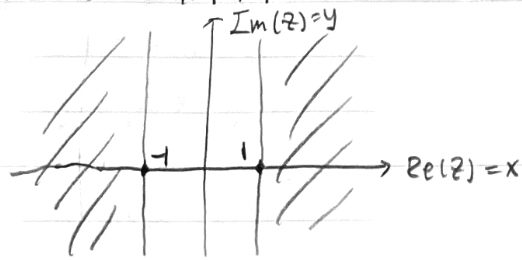
$\boxed{\checkmark}$

Problem 5. Sketch following regions of the complex plane

(a) $\{z \in \mathbb{C} : |Re(z)| \geq 1\}$

ans) Let $z = x + yi$ where $x, y \in \mathbb{R}$. Then $Re(z) = x$ and $Im(z) = y$.

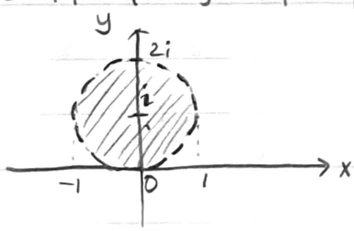
$|Re(z)| \geq 1 \Rightarrow |x| \geq 1 \Rightarrow x \geq 1$ or $x \leq -1$



(b) $\{z \in \mathbb{C} : |z - i| < 1\}$

ans) Let $z = x + yi$. Then we have

$$|z - i| = |x + yi - i| = |x + (y-1)i| = \sqrt{x^2 + (y-1)^2} < 1$$

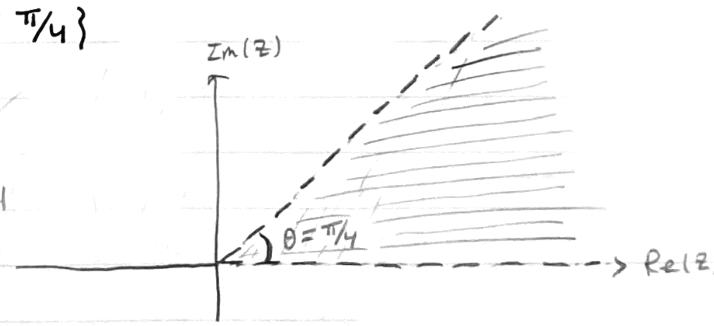


(c) $\{z \in \mathbb{C} : 0 < \arg(z) < \pi/4\}$

ans) Let $z = r(\cos\theta + i\sin\theta)$

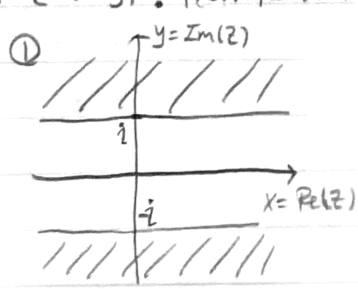
Then $\arg(z) = \theta$.

Thus it can be expressed on graph as picture on the right



(d) $\{z \in \mathbb{C} : |Im(z)| \geq 1\} \cap \{z \in \mathbb{C} : |z| \leq \sqrt{2}\}$

Let $z = x + yi$. ① Then $|Im(z)| = |y| \geq 1 \Rightarrow y \geq 1$ or $y \leq -1$. ② $|z| = \sqrt{x^2 + y^2} \leq \sqrt{2}$



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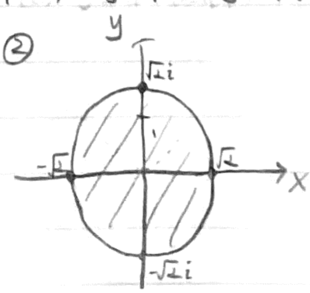

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