Check Prof's Lecture note Page 1.

Lecture 5 (10/5)

$\boxed{\text{Path Independence of contour integral}}$

Thm $A \subseteq \mathbb{C}$ is a ________ . $f: A \to \mathbb{C}$ is continuous.

The following are equivalent.

(1) Integrals of f are path independent

i.e, $\forall\, r_1, r_2$ with the same start and end points

$$\int_{r_1} f = \int_{r_2} f$$

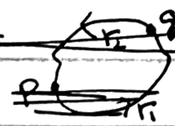
(2) $\int_C f = 0$ for any closed curve C

(3) f has a primitive in A, i.e $\exists\, F: A \to \mathbb{C}$ holomorphic

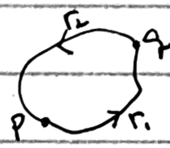
$F'(z) = f(z)$.

~~A~~

________ to path independent in multivariable.

~~Pf (1) $\Leftrightarrow$ (2) (1) $\Rightarrow$ (2) C =~~  ~~r_1 and $= r_2$~~

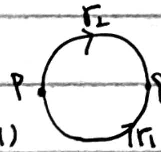
$A \subseteq \mathbb{R}^2$ [ ]

Pf (1) $\Leftrightarrow$ (2). (1) $\Rightarrow$ (2) $C =$  r_1 and $-r_2$ both

________ p to q

$\Rightarrow \int_{r_1} f = \int_{r_2} f$

$\Rightarrow \int_{r_1} f + \int_{r_2} f = 0 \Rightarrow \int_C f = 0$



(2) $\Rightarrow$ (1) $r_1 + (-r_2)$ is closed $\Rightarrow \int_{r_1 + (-r_2)} f = 0 \Rightarrow \int_{r_1} f = \int_{r_2} f$

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Integral w.r.t arclength

Verify that $\lim_{w \to z} \frac{F(w) - F(z)}{w - z}$ exists. Fix $z \in A$. For any $\varepsilon > 0$ find $\delta > 0$ s.t $|f(w) - f(\delta)| < \varepsilon$ for $w \in D(z, \delta)$. Compare the difference $F(w) - F(z)$.

Fix any γ connecting p to z. Let γ_{zw} be the line segment from z to w.

Parameterize : $t \in [0,1] \mapsto z(1-t) + wt$

$$F(w) - F(z) = \int_{\gamma_{zw}} f(t)\,dt = \int_0^1 f(z(1-t)+wt)(w-z)\,dt$$

$$\Rightarrow \frac{F(w) - F(z)}{w-z} = \int_0^1 f(z(1-t) + wt)\,dt$$

$$\frac{F(w) - F(z)}{w-z} - f(z) = \int_0^1 f(z(1-t)+wt) - f(z)\,dt$$

$$\Rightarrow \left| \frac{F(w) - F(z)}{w-z} - f(z) \right| \leq \int_0^1 \left| f(z(1-t)+wt) - f(z) \right| dt$$

$$\lim_{w \to z} \frac{F(w) - F(z)}{w-z} = f(z) < \int_0^1 \varepsilon\,dt = \varepsilon$$

$\square$.

$\boxed{\text{Integral w.r.t arclength}}$ For a piecewise C^1 curve γ,

$$\int_\gamma f(z)\,ds = \int_\gamma f(z)\,|dz| \quad s = \text{arclength parameter}$$
$$= \int_a^b f(z(t))\,|z'(t)|\,dt$$

Complete length of a curve.

$$L(\gamma) = \int_\gamma ds = \int_a^b |z'(t)|\,dt$$

Primitive, Continuous function. Lecture 5 (10/5)

Q. which continuous functions $f: A \to \mathbb{C}$ has a primitive?

Thm (Cauchy's Integral Theorem) $A \subseteq \mathbb{C}$ is a ~~domain~~ disk $f: A \to \mathbb{C}$ is a holomorphic. Then $\int_C f\,dz = 0$ for any closed curve C in A

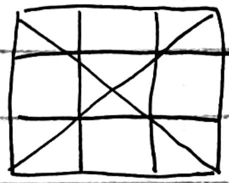
Pf 1. Cauchy's Thm holds for more ________ domains

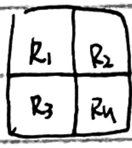
2. $\Rightarrow$ any holomorphic f disk $\to \mathbb{C}$ has a primitive.

3. Thm holds f with minor singularities and for rectifiable curves (curves with a definition of length)

$R =$ rectangle $a \le x \le b$, $c \le y \le d$.

Coursat's proof. want $0 = \int_{\partial R} f\,dz$

$R =$ 



Divide R into 4 equal parts R_1, R_2, R_3, R_4

$$\int_{\partial R_1} f\,dz + \int_{\partial R_2} f\,dz + \int_{\partial R_3} f\,dz + \int_{\partial R_4} f\,dz = \int_{\partial R} f\,dz$$

$$\left| \int_{\partial R} f\,dz \right| = \left| \sum_j \int_{\partial R_j} f\,dz \right| \le \sum_j \left| \int_{\partial R_j} f\,dz \right|$$

$$\Rightarrow \exists j \text{ s.t } \left| \int_{\partial R} f\,dz \right| \le 4 \left| \int_{\partial R_j} f\,dz \right|$$

Call $R_j = R^{(1)}$

Denote $\eta(R) = \int_{\partial R} f \Rightarrow |\eta(R)| \le 4 |\eta(R^{(1)})|$

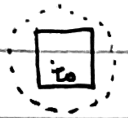
Repeat $R \supseteq R^{(1)} \supseteq R^{(2)} \supseteq \dots \Rightarrow \bigcap_{n \ge 1} R^{(n)} = \{z_0\}$

and $|\eta(R)| \le 4|\eta(R^{(1)})| \le \dots \le 4^n |\eta(R^{(n)})| \le \dots$

f is holomorphic For $\varepsilon > 0$, $\exists \delta$ s.t

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \text{ for } z \in D(z_0, \delta)$$

Primitive.



$R^{(n)}$ $\Rightarrow$ for n large enough $R^{(n)} \subseteq D(z_0, \delta)$

$$\eta(R^{(n)}) = \int_{\partial R^{(n)}} f(z)\,dz = \int_{\partial R^{(n)}} \Big[f(z) - \underbrace{\big(f(z_0) + f'(z_0)(z-z_0) \big)}_{\text{has a primitive} \rightsquigarrow \text{integral}=0} \Big] dz$$

$$\Rightarrow \left| \eta(R^{(n)}) \right| \leq \int_{\partial R^{(n)}} \left| f(z) - f(z_0) - f'(z_0)(z-z_0) \right| \cdot |dz|$$

$$< \int_{\partial R^{(n)}} \varepsilon |z - z_0| \cdot |dz| \leq \text{diameter of } R^{(n)} = d_n$$

<u>Ex</u> Compute the Fourier Transformation of $f(x) = e^{-\pi x^2}$

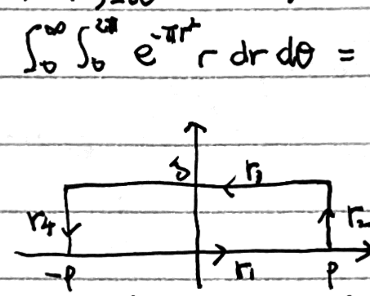
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx = \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x \xi} dx$$

$$= e^{-\pi \xi^2} \int_{-\infty}^{\infty} e^{-\pi (x + i\xi)^2} \, [\ ①\ ]\ dx.$$

To ~~Find~~ ①, recall $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1. \Rightarrow \left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx \right)^2$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-\pi y^2} dx\,dy = \int_0^{\infty} \int_0^{2\pi} e^{-\pi r^2} r\,dr\,d\theta = \dots = 1$$

<u>claim</u> $\int_{-\infty}^{\infty} e^{-\pi (x + i\xi)^2} dx = 1$



<u>Pf</u> $f(z) = e^{-\pi z^2}$ is holomorphic

Cauchy $\Rightarrow 0 = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz$

$\int_{\gamma_1} f(z)\,dz$: γ_1 $t \in [-p, p] \longmapsto z(t) = t$. $\int_{\gamma_1} f(z)\,dz = \int_{-p}^{p} e^{-\pi t^2} dt$

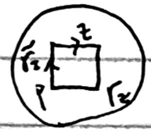
$\int_{\gamma_2} f(z)\,dz$: γ_2 $t \in [0, \xi] \longmapsto z(t) = p + it$ $\int_{\gamma_2} f(z)\,dz = \int_0^{\xi} e^{-\pi (p + it)^2} i\,dt$

$$= \int_0^{\xi} e^{-\pi (p+it)^2} i\,dt = \int_0^{\xi} e^{-\pi p^2 + \pi t^2} \cdot e^{-2\pi p t i}\, i\,dt$$

$$\Rightarrow \left| \int_{\gamma_2} f\,dz \right| \leq \int_0^{\xi} e^{-\pi p^2 + \pi t^2} dt \leq e^{-\pi p^2 + \pi \xi^2} \xi \to 0 \text{ as } p \to \infty$$

(Check lecture note)

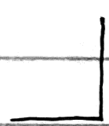
Cauchy for disks: $\int_C f\,dz = 0$ C closed curve in D.



want: find $F: D \to \mathbb{C}$. $F' = f$

Fix $p \in D$ for $z \in D$ γ_z = curve first horizontal then vertical from p to z. $F(z) = \int_{\gamma_z} f(w)\,dw$

Observe Cauchy Thm for rectangular $\int_{\gamma_z} f = \int_{\tilde{\gamma}_z} f$

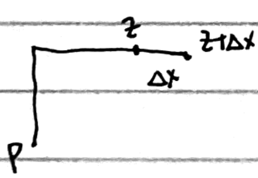


we $\frac{\partial F}{\partial y}(z)$ exists. $F(x, y+\Delta y) - F(x,y) = \int_{\gamma_1} f - \int_{\gamma_2} f = \int_{\gamma_{\Delta y}} f(z)\,dz$

$$\lim_{\Delta y \to 0} \frac{1}{\Delta y}\left(F(x+\Delta y) - F(x)\right) = \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{\gamma_{\Delta y}} f(z)\,dz = i f(z)$$

(why: $\gamma_{\Delta y}$ $t \in [0,1] \longmapsto z(t) = z + i\Delta y \cdot t$

$$\frac{1}{\Delta y} \int_0^1 f(z + i\Delta y t)\, i\Delta y\, dt = i \int_0^1 f(z + i\Delta y t)\, dt$$



$$\frac{\partial F}{\partial x}(z) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\int_{\tilde{\gamma}_{z+\Delta x}} f - \int_{\tilde{\gamma}_z} f\right) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{\gamma_{\Delta x}} f(z)\,dz$$

Lecture 5 (10/5)

$$\Rightarrow \quad \frac{\partial F}{\partial x} = f(z) \qquad \frac{\partial F}{\partial y} = i f(z)$$

write $F(x,y) = u(x,y) + i\,v(x,y)$

$$i(u_x + i v_x) = u_y + i v_y$$

$$\Rightarrow \quad u_x = v_y$$

$$u_y = -v_x$$