Problem 1.

Sol) Let $f(z) = u(x,y) + iv(x,y)$ and suppose $|f(z)|$ is constant in A where A is connected open set in $\mathbb{C}$ and f be holomorphic. Then, we need to show that $f(z)$ is a constant function.

$|f(z)| = \sqrt{u^2+v^2} = c$ for any constant c.

Then we have $u^2+v^2 = c^2$. —— (1)

Since f is holomorphic, we have $u_x = v_y$ and $u_y = -v_x$.

From (1), we have

$$\begin{cases} 2u \cdot u_x + 2v \cdot v_x = 0 \\ 2u \cdot u_y + 2v \cdot v_y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} u \cdot u_x + v \cdot u_y = 0 & \text{—— (2)} \\ u \cdot u_y + v \cdot u_x = 0 & \text{—— (3)} \end{cases}$$

$$u(2) + v(3) = (u^2+v^2)u_x = (u^2+v^2)v_y = 0$$

$$-v(2) + u(3) = (u^2+v^2)u_y = (u^2+v^2)(-v_x) = 0$$

Thus, we have that

$$(u^2+v^2)u_x = 0$$

$$(u^2+v^2)v_y = 0$$

$$(u^2+v^2)u_y = 0$$

$$(u^2+v^2)v_x = 0$$

Since $u^2+v^2 = C$, we conclude that

$$u_x = v_y = u_y = v_x = 0$$

Thus, $f(z)$ is a constant. $\square$

Problem 2.

Sol) Suppose $f'(z) = 1/z$ and $f(1)=0$ where $f(z)$ is a holomorphic in the disk $|z-1|<1$. To show $f(z) = \log z$ in the disk with $A_{-\pi} = \{z : -\pi < \operatorname{Im} z < \pi\}$,

we need to show that

$$(f - \log z)(1) = 0$$
$$(f - \log z)' = 0$$

It's given that $f(1) = 0$. We can naturally Calculate the value of $\log z$ at $z=1$, $\log(1) = 0$.
Thus we've shown that $f(1) - \log(1) = 0$.

Now, we need to show that $(f(z) - \log z)' = 0$.
It's given that $f'(z) = 1/z$. then we need to $(\log z)' = 1/z$

Let g be the holomorphic in the disk $|z-1|<1$ and $g(z) = \log z$ where log is the branch that takes value in $A_{-\pi}$ as defined

Then $\log z = \log|z| + i\arg(z)$

$$\log z = \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x) \text{ where } z = x+iy.$$

Then based on given condition explained above (property of log and holomorphic), g is differentiable, so we have

$$u(x,y) = \log\sqrt{x^2+y^2} \text{ and } v(x,y) = \tan^{-1}(y/x)$$

$$u_x = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$$

$$v_x = \frac{-y}{x^2+y^2}$$

(Cont. 2)

$g(z) = u + iv$ so it follows that $g'(z) = u_x + iv_y$.

$$g'(z) = \frac{x}{x^2+y^2} + i\frac{(-y)}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy}$$

$$= \frac{1}{z}.$$

Thus, we've shown that $(f-g)' = (f-\log z)' = 0$.

$\boxtimes$

Problem 3

1. Verify that the function $u(x,y) = \sin x \cosh y$ is harmonic in $\mathbb{R}^2$.

Sol) We need to show that $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$.

① $\dfrac{\partial u}{\partial x} = \cos x \cosh y \Rightarrow \dfrac{\partial^2 u}{\partial x^2} = -\sin x \cosh y$

② $\dfrac{\partial u}{\partial y} = \sin x \sinh y \Rightarrow \dfrac{\partial^2 u}{\partial y^2} = \sin x \cosh y$

① + ② = 0. Thus $u(x,y)$ is harmonic in $\mathbb{R}^2$

2. Find the harmonic conjugate v of u such that $v(0,0) = 3$

Sol) Let $f(z) = u(x,y) + i v(x,y)$ and v be the harmonic conjugate to u. Thus it holds the following true
$\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ are continuous
AND $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ (CR equations)

$\partial u/\partial x = \partial v/\partial y = \cos x \cdot \cosh y$

$\Rightarrow \quad v(x,y) = \cos x \cdot \sinh y + p(x)$ —— (1)

$\Rightarrow \quad v(0,0) = 1 \cdot 0 + p(0) = p(0) = 3.$ — (2)

$-\partial v/\partial x = \partial u/\partial y = \sin x \cdot \sinh y$ —— (3)

Derivative of (1) w.r.t x, results in

$\partial v/\partial x = -\sin x \cdot \sinh y + p'(x)$

$\Rightarrow -\partial v/\partial x = \sin x \cdot \sinh y - p'(x)$ —— (4)

(3) = (4). Thus, $p'(x) = 0 \Rightarrow p(x) = C$

Apply it to (1) and (2), then we have

$\underline{v(x,y) = \cos x \cdot \sinh y + 3.}$

Problem 4

Sol) Let $f(z) = u(x,y) + iv(x,y)$.

Then let $u(x,y) = \varphi(x^2+y^2)$ by Fundamental thm of Integral.

$$u_x = 2x\varphi'(x^2+y^2)$$

$$u_{xx} = 2\varphi'(x^2+y^2) + 4x^2\varphi''(x^2+y^2)$$

Then we also have

$$u_y = 2y\varphi'(x^2+y^2)$$

$$u_{yy} = 2\varphi'(x^2+y^2) + 4y^2\varphi''(x^2+y^2)$$

Since the real part of a holomorphic function is harmonic, we have that $u_{xx} + u_{yy} = 0$. It yields that

$$0 = 4\varphi'(x^2+y^2) + 4(x^2+y^2)\varphi''(x^2+y^2)$$

Let $t = x^2+y^2$. Then we have

$$0 = 4\varphi'(t) + 4t\varphi''(t)$$

$$\frac{\varphi'(t)}{t} + \varphi''(t) = 0$$

$$\Rightarrow \quad \varphi'(t) = c_1/t$$

$$\Rightarrow \quad \varphi(t) = c_1 \log t + c_2 \quad \text{for any constant } c_1, c_2$$

$$\Rightarrow \quad f(z) = c_1 \log z + c_2$$

$\square$

Problem 5.

1. $\int_C \log z \, dz \quad A_0 = \{z : 0 \leq \text{Im } z < 2\pi\}$

sol) Let $z = e^{i\theta}$, it follows that $dz = ie^{i\theta} d\theta$

$$\int_C \log z \, dz = \int_0^{2\pi} (\log e^{i\theta})(ie^{i\theta}) d\theta$$

$$= \int_0^{2\pi} (i\theta) i e^{i\theta} d\theta = -\int_0^{2\pi} (-\theta) e^{i\theta} d\theta$$

Let $u = -\theta \quad V_\theta = e^{i\theta} d\theta$ and integral becomes

$$-\theta \left. \frac{e^{i\theta}}{i} \right|_0^{2\pi} - \int_0^{2\pi} \frac{e^{i\theta}}{i} (-d\theta)$$

$$= i\theta e^{i\theta} \Big|_0^{2\pi} - i \cdot \frac{1}{i} e^{i\theta} \Big|_0^{2\pi}$$

$$= i 2\pi e^{2\pi i} - e^{i\theta} \Big|_0^{2\pi}$$

$$= \boxed{2\pi i} \qquad (\because e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$$

$$\text{AND } e^{i\theta} \Big|_0^{2\pi} = e^{2\pi i} - e^0 = 1 - 1 = 0)$$

□

2. $\int_C \frac{1}{z} dz$

sol) Let $z = e^{i\theta}$. Then $dz = ie^{i\theta} \cdot d\theta$.

Then, convert $(-\sqrt{3}-i)/2$ and $(1+\sqrt{3}i)/2$ to r, θ coordinate

we have $(-\sqrt{3}-i)/2$ can be translated into $r=1$ $\theta = 7\pi/6$

$(1+\sqrt{3}i)/2$ can be translated into $r=1$ $\theta = \pi/3$.

Putting altogether,

$$\int_C \frac{1}{z} dz = \int_{\pi/3}^{7\pi/6} e^{-i\theta} \cdot ie^{i\theta} \cdot d\theta = \int_{\pi/3}^{\frac{7\pi}{6}} i \, d\theta$$

$$= i\left(\frac{7\pi}{6} - \frac{2\pi}{6}\right) = \frac{5\pi i}{6}$$

□

Problem 6

Sol) It's given that f is holomorphic in the open disk centered at the origin $D(0,1)$ where

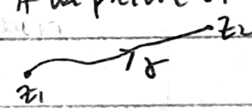
$$\forall z \in D(0,1),\ |f'(z)| \leq M.$$

Now, we need to show that $\forall z_1, z_2 \in D(0,1)$,

$$|f(z_1) - f(z_2)| \leq M|z_1 - z_2|.$$

We have that

$$|f(z_2) - f(z_1)| = \left|\int_\gamma f'(z)\,dz\right| \quad (\because \text{ If we picture } \gamma,$$



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$$\leq \int_\gamma |f'(z)| \cdot |dz| \quad (\text{by triangle inequality})$$

$$\leq M\int_\gamma |dz| \quad (\because |f'(z)| \leq M$$

$$= M|z_2 - z_1|$$

Thus, we've shown that $\forall z_1, z_2 \in D(0,1)$,

$|f(z_1) - f(z_2)| \leq M|z_1 - z_2|$.