Problem 1

1) $\int_{|z+i|=1} [e^z/(z^2+1)]\,dz$

Sol) $\dfrac{e^z}{z^2+1} = e^z\left[\dfrac{A}{z+i} + \dfrac{B}{z-i}\right] = e^z\left[\dfrac{(A+B)z + (-A+B)i}{z^2+1}\right]$

$\Rightarrow \quad A+B = 0 \quad \Rightarrow \quad A = -B$

$\Rightarrow \quad -A+B = -i \quad \Rightarrow \quad A = i/2, \; B = -i/2$

Thus, the integrand can be expressed as

$$e^z\left[\frac{i/2}{z+i} - \frac{i/2}{z-i}\right] = \frac{i}{2}e^z\left[\frac{1}{z+i} - \frac{1}{z-i}\right]$$

Putting it altogether,

$$\frac{i}{2}\int_{|z+i|=1} \frac{e^z}{z+i}\,dz - \frac{i}{2}\int_{|z+i|=1} \frac{e^z}{z-i}\,dz$$

$$\frac{i}{2}\int_{|z+i|=1} \frac{e^z}{z+i}\,dz = \frac{i}{2}\cdot 2\pi i\, e^{-i} = \frac{-\pi}{e^i}$$

$$\frac{i}{2}\int_{|z+i|=1} \frac{e^z}{z-i}\,dz = 0 \quad (w \notin D \text{ where } w=i)$$

Thus, $\displaystyle\int_{|z+i|=1} \frac{e^z}{(z^2+1)}\,dz = \boxed{\frac{-\pi}{e^i}}$

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2) $\displaystyle\int_{|z|=1} \frac{1}{(z-a)^2(z-b)}\,dz$

Sol) First, split the integrand.

$$\frac{1}{(z-a)^2(z-b)} = \frac{A}{(z-a)^2} + \frac{B}{(z-a)} + \frac{C}{z-b}$$

$$= \frac{A(z-b) + B(z-a)(z-b) + C(z-a)^2}{(z-a)^2(z-b)}$$

$$= \frac{Az - Ab + B(z^2-(a+b)z+ab) + Cz^2 - 2Caz + a^2C}{(z-a)^2(z-b)}$$

$$\Rightarrow = \frac{(B+C)z^2 - [B(a+b)+2Ca-A]z + [abB - Ab + a^2C]}{(z-a)^2(z-b)}$$

It follows that

$B+C=0 \qquad \Rightarrow \quad B=-C$

$B(a+b)+2Ca-A=0 \qquad \Rightarrow \quad A = B(a+b)-2Ba = B(b-a)$

$abB - Ab + a^2C = 1$

$abB - B(b-a)b + a^2B = B(ab-b^2+ab-a^2) =$

$= -B(a-b)^2 = 1$

$\Rightarrow \quad B = -(a-b)^{-2}$

Thus, $A = B(b-a) = [-(a-b)^{-2}][-(a-b)] = (a-b)^{-1}$

$B = -(a-b)^{-2}$

$C = (a-b)^{-2}$

Now, we have

$$\int_{|z|=1} \frac{(a-b)^{-1}}{(z-a)^2} - \frac{(a-b)^{-2}}{(z-a)} + \frac{(a-b)^{-2}}{(z-b)}\,dz.$$

and we need to consider following 4 cases.

1) a inside domain, b outside

$$\int_{|z|=1}\left[\underbrace{\frac{(a-b)^{-1}}{(z-a)^2}}_{=0}-\frac{(a-b)^{-2}}{(z-a)}+\underbrace{\frac{(a-b)^{-2}}{(z-b)}}_{=0.}\right]dz = -2\pi i a \cdot (a-b)^{-2}$$

2) a outside domain, b inside

$$\int_{|z|=1}\left[\underbrace{\frac{(a-b)^{-1}}{(z-a)^2}}_{=0}-\underbrace{\frac{(a-b)^{-2}}{(z-a)}}_{=0}+\frac{(a-b)^{-2}}{(z-b)}\right]dz = 2\pi i b (a-b)^{-2}$$

3) a and b inside

$$\int_{|z|=1}\left[\frac{(a-b)^{-1}}{(z-a)^2}-\frac{(a-b)^{-2}}{(z-a)}+\frac{(a-b)^{-2}}{(z-b)}\right]dz = 2\pi i (a-b)^{-2}(a-b)$$

$$= -2\pi i (a-b)^{-1}$$

4) a and b outside

$$\int_{|z|=1}\left[\underbrace{\frac{(a-b)^{-1}}{(z-a)^2}}_{=0}-\underbrace{\frac{(a-b)^{-2}}{(z-a)}}_{=0}+\underbrace{\frac{(a-b)^{-2}}{(z-b)}}_{=0}\right]dz = 0$$

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Problem 2.

Sol) We need to show that $\lim_{h\to 0} \frac{1}{h}[f(z+h)-f(z)]$ does exist.

$$f(z+h)-f(z)=\frac{1}{2\pi i}\int \phi(w)\left[\frac{1}{w-(z+h)}-\frac{1}{w-z}\right]dw$$

$$=\frac{1}{2\pi i}\int \phi(w)\frac{w-z-w+(z+h)}{[w-(z+h)][w-z]}dw$$

$$=\frac{1}{2\pi i}\int \phi(w)\frac{h}{(w-(z+h))(w-z)}dw$$

$$\Rightarrow \quad \frac{f(z+h)-f(z)}{h}=\frac{1}{2\pi i}\int \frac{\phi(w)}{(w-(z+h))(w-z)}dw$$

Now, we have

$$\left|\frac{1}{w-(z+h)}-\frac{1}{w-z}\right| \le \frac{|h|}{|w-(z+h)|\cdot|w-z|} \le \frac{|h|}{(\delta/2)^2} \to 0 \qquad (*)$$

for some δ and $\forall w \in \gamma$.

By Morera's theorem, when a function is integrable, we can consider the function as uniform. Hence, $(*)$ is uniform thus $\lim_{h\to 0}\frac{1}{h}[f(z+h)-f(z)]$ does exist. So f is holomorphic on $\mathbb{C}\setminus\gamma$ $\square$

Problem 3.

Sol) First, we need to show that f is continuous.
It's given that $f_n \to f$ uniformly on all compact subsets of A and f_n is a sequence of holomorphic function. Hence f is continuous.

Second, we need to show that $\int_C f(z)dz = 0$ for every closed curve $C \subseteq A$ where $A \subseteq \mathbb{C}$ open and connected and $f: A \to \mathbb{C}$ is continuous. It's given that $f_n \to f$ uniformly on all compact subsets of A. It follows that $\int f_n dz \to \int f dz$.
Since f_n is a holomorphic, $\int f_n dz = 0$, thus ~~the~~ the sequence f_n converges to 0. thus, $\int_C f(z)dz = 0$.

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Problem 4

1) Sol) Cauchy's integral formula gives us that

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_D \frac{f(z)}{(z-w)^{n+1}}\,dz \qquad \text{where } D(w,r) \text{ disk.}$$

Then, we can parameterize $z = w + re^{i\theta}$ and $0 \le \theta \le 2\pi$

It follows that

$$|f^{(n)}(w)| = \left|\frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(w+re^{i\theta})\, r i e^{i\theta}}{(re^{i\theta})^{n+1}}\,d\theta\right|$$

$$\le \frac{n!}{2\pi}\int_0^{2\pi} \left|\frac{f(w+re^{i\theta})}{(re^{i\theta})^{n}}\right| d\theta$$

$$= \frac{n!}{2\pi}\int_0^{2\pi} \frac{|f(w+re^{i\theta})|}{r^n}\,d\theta \qquad (\because |e^{i\theta}|=1)$$

$$\le \frac{n!}{2\pi}\int_0^{2\pi} \frac{M}{r^n}\,d\theta \qquad (\because |f(z)| \le M)$$

$$= \frac{n!\cdot M}{r^n}$$

$\square$

2) Sol) Let f be entire holomorphic. From previous part of this problem set, we have $|f^{(n)}(w)| \le [n!M]/r^n$, and for $n=1$, we have $|f'(w)| \le M/r$ for all $r>0$.

Since f is entire, we can conclude that $f'(w) = 0 \; \forall r > 0$. $\forall w \in \mathbb{C}$. Hence, $f'(z) = 0 \; \forall z \in \mathbb{C}$. Thus, $f = \deg \le 1$ polynomial. Hence, we're done. $\square$

Problem 5.

Sol) By Cauchy's integral formula, we have

$$A = \int_\gamma \frac{e^z}{z}\,dz = 2\pi i$$

Let $z = e^{i\theta} = \cos\theta + i\sin\theta$. and $dz = i e^{i\theta}$.

Putting altogether, we have

$$A = \int_\gamma \frac{e^{\cos\theta + i\sin\theta}}{e^{i\theta}} (i e^{i\theta})\,d\theta = i\int_\gamma e^{\cos\theta + i\sin\theta}\,d\theta$$

$$= i\int_\gamma e^{\cos\theta}\left[\cos(\sin\theta) + i\sin(\sin\theta)\right] d\theta$$

$$= i\int_\gamma e^{\cos\theta}\cos(\sin\theta)\,d\theta - \int_\gamma e^{\cos\theta}\sin(\sin\theta)\,d\theta$$

The second term of the last equation is 0.

$$\int_{-\pi}^{\pi} e^{\cos\theta}\sin(\sin\theta)\,d\theta = \int_0^{\pi} e^{\cos\theta}\sin(\sin\theta)\,d\theta + \int_{-\pi}^{0} e^{\cos\theta}\sin(\sin\theta)\,d\theta$$

$$= \int_0^{\pi} e^{\cos\theta}\sin(\sin\theta)\,d\theta - \int_0^{\pi} e^{\cos\theta}\sin(\sin\theta)\,d\theta = 0.$$

Thus, now we have

$$A = i\int_{-\pi}^{\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta = 2i\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta = 2\pi i$$

It gives us that

$$\underline{\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta = \pi}$$

Problem 6

Sol) Let f be ~~a~~ nonconstant entire holomorphic function.

Assume that ~~f~~ doesn't map $\mathbb{C}$ onto a dense subset of $\mathbb{C}$. Then suppose $|f(z) - w| > r$ for some $w \in \mathbb{C}$ and r be a radius of disk.

Then it follows that $g(z) = 1/(f(z) - \lambda)$ is entire and bounded (because $|g(z)| \leq 1/r$). Thus, by Liouville's Theorem, g is constant. Hence f is and it contradicts the assumption. $\square$