Complex Variables I – Problem Set 5

Due at 5 pm on Friday, Oct 13, 2023 via Gradescope

Problem 1

Evaluate the following contour integrals.

1. $\int_C \frac{1}{z^n} dz$, where C is the unit circle oriented counterclockwise, and n is an integer. Your answer may depend on n.
2. $\int_C \frac{e^z}{z^2-16} dz$, where C is the circle centered at 1 with radius 1.

Problem 2

Evaluate the following integrals with arc length parameters.

1. $\int_{|z|=1} \frac{|dz|}{z}$.
2. $\int_{|z|=1} |z-1||dz|$.

Problem 3

Let $A \subset \mathbb{C}$ be a domain, $f : A \to \mathbb{C}$ be holomorphic on $A \setminus \{\zeta\}$, where $\zeta \in A$ is an interior point. Suppose that

$$\lim_{z \to \zeta} (z - \zeta) f(z) = 0.$$

Prove that $\int_{\partial R} f(z) dz = 0$, for any rectangle $R \subset A$ containing ζ as an interior point.

Problem 4

Assume that f is holomorphic and satsifies the inequality $|f(z) - 1| < 1$ in a domain Ω. Taking for granted that f' is continuous in Ω, show that

$$\int_\gamma \frac{f'(z)}{f(z)} = 0$$

for every closed curve γ in Ω.

Problem 5

Evaluate

$$\int_0^\infty \frac{1 - \cos x}{x^2} dx$$

by considering the function $f(z) = \frac{1-e^{iz}}{z^2}$ along a suitable choice of contour.

Hint: use the statement of Problem 7 below (without proving it). Note that f is not defined at the origin, so your contour should avoid a tiny neighborhood of the interval - e.g. try an indented semicircle as shown in the figure below.

Problem 6

Prove that

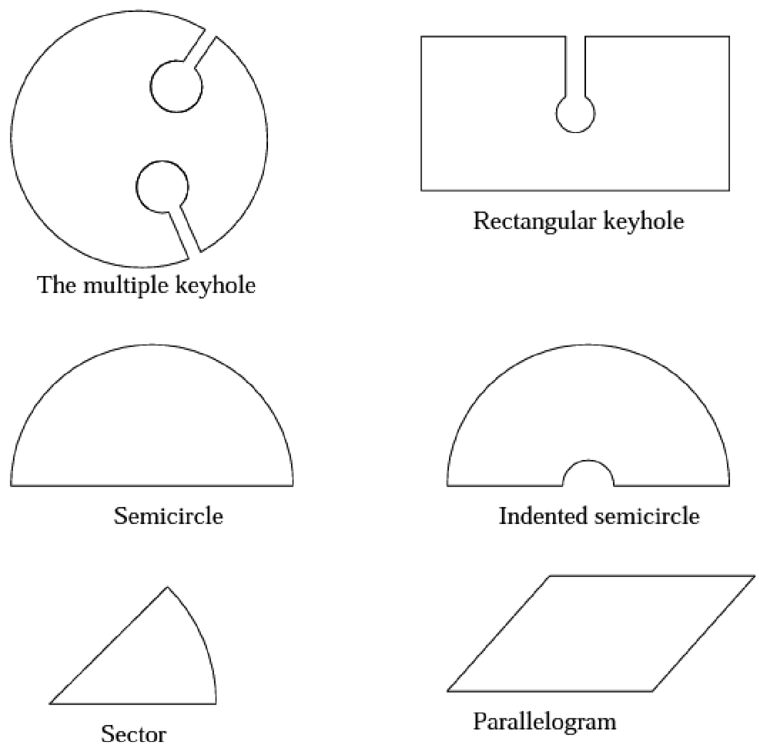
$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2}\pi}{4},$$

by considering the function $f(z) = e^{-z^2}$ over a 'sector' of angle $\pi/4$, as shown in the figure below.

Problem 7 - optional

Extend our proof of Cauchy's theorem on disks to more general contours: suppose C is a connected closed piecewise C^1 curve which bounds a connected open set D of $\mathbb{C}$, and any two points p, q can be connected by a polygonal curve inside D, that is, the union of finitely many horizontal or vertical line segments. Show that if f is holomorphic on a neighborhood of D, then $\int_C f(z)dz = 0$.

Some examples of valid contours (figure taken from Stein-Shakarchi, Complex Analysis):



Remember to justify your answers and acknowledge collaborations and outside help!