9/14/2023

Lecture 2

<u>n-th root of $z \in \mathbb{C}$</u> $= \{ w \in \mathbb{C} \;\; w^n = z \}$ a set ~~of~~ with n elements

e.g) nth root of unity $= \{ w \in \mathbb{C},\; w^n = 1 \}$

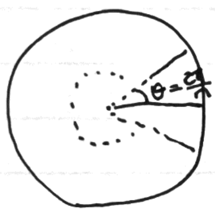
To find such w, $\quad w = \rho(\cos\theta + i\sin\theta)$

$w^n = 1 \iff \rho = 1 \qquad \cos(n\theta) + i\sin(n\theta) = 1$

$\Rightarrow \cos\theta = 1 \qquad \Rightarrow n\theta = 2k\pi \quad k \in \mathbb{Z}$

$\sin n\theta = 1 \qquad \Rightarrow \theta = \frac{2\pi k}{n} \quad k \in \mathbb{Z}$

$\Rightarrow w = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right), \quad k \in \mathbb{Z}$



$\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ $(k=1)$

$(1,0)$ $k=0$

$\cos\frac{2\pi(n-1)}{n} + i\sin\frac{2\pi(n-1)}{n}$ $\quad k=n-1$

unit circle in $\mathbb{R}^2$

$\Rightarrow w = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right), \quad k=0,1,\ldots,n-1$

ex) nth root of 2

$= \frac{1}{2^n}\left(\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right), \quad k=0,\ldots,n-1$

ex) nth root of $z_0 \to$ find some w_0 such that $w_0^n = z_0$

Then nth root of z_0

$= \left\{ w_0\left(\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right),\; k=0,1,\ldots,n-1 \right\}$

<u>Elementary complex functions</u>

Polynomial $P : \mathbb{C} \to \mathbb{C}$, $\quad P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

$a_n, \ldots, a_0 \in \mathbb{C}$ and $a_n \neq 0$.

Call n = degree of P

If $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$ say z_0 is a zero of P

<u>Fact</u> A degree n polynomial has n zeroes (concluded w/ multiplicity)

e.g) $(z-1)^2 = 0 \quad z=1$ is a zero ~~of~~ with mult 2

Rational function

$f: \mathbb{C} \setminus \{z_0, \dots, z_n\} \to \mathbb{C}$, $\quad f(z) = \dfrac{P(z)}{Q(z)}$, $z_0, \dots, z_n$ = zeros of Q

P, Q = Polynomials without common zeros

e.g) $\dfrac{(\cancel{z-1})(z-i)}{(\cancel{z-1})(z+i)}$

Notation Call zero of P = the zero of f
Call zero of Q = the poles of f

ex) $f(z) = \dfrac{az+b}{cz+d}$ $(a, b, c, d \in \mathbb{C}, \ ad-bc \neq 0)$ ** c can be zero

zero of $f = -b/a$ and pole of $f = -d/c$

f is called Möbius Function

(Property: Such f maps a line / circle to a line / circle)

Exponential

Recall $\exp: \mathbb{R} \to \mathbb{R}$

① 1st way to define exp is the unique function s.t
$(e^x)' = e^x$ and $e^0 = 1$

② 2nd definition

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ (Taylor expansion)

Property $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent for $\forall x \in \mathbb{R}$

Motivation $$\sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = 1 + \frac{iy}{1} - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + \dots$$

(number 1)

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right)$$

$$= \cos y + i \sin y$$

Recall

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots$$

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots$$

} absolutely convergence $\forall\, y \in \mathbb{R}$

$$\exp(iy) = \cos y + i \sin y$$

and $\exp(a+b) = \exp(a) \cdot \exp(b)$

Def For $z = x+iy \in \mathbb{C}$ define the complex exponential function

$$\exp(z) = e^x(\cos y + i\sin y) \longleftarrow \exp(x+iy) = \exp(x)(\cos y + i \sin y)$$

Properties

(1) $\exp(z+w) = \exp(z)\cdot\exp(w)$ for all $z, w \in \mathbb{C}$

(2) $\exp : \mathbb{C} \longrightarrow \mathbb{C}\setminus\{0\}$ (real $\exp : \mathbb{R} \to \mathbb{R}_{>0}$)

(3) For $x, y \in \mathbb{R}$, $|\exp(x+iy)| = e^x$, $\arg(\exp(x+iy)) = y$, $\exp(iy) = \cos y + i \sin y$

(4) $\exp(z + 2\pi k i) = \exp(z)$, $\forall z \in \mathbb{C}, k \in \mathbb{Z}$, and $\exp(z) = 1 \Leftrightarrow z \in 2\pi i \mathbb{Z}$

(5) $\overline{\exp(z)} = \exp(\bar{z}) \Rightarrow \overline{\exp(x+iy)} = \exp(x-iy)$, $\forall\, x,y \in \mathbb{R} \Leftrightarrow \exp(\bar{z}) = \overline{\exp(z)}$ $\forall z \in \mathbb{C}$

Pf (1) $z = x+iy$ $\quad w = a+ib$

$$\begin{aligned}
e^z \cdot e^w &= e^{x+iy} \cdot e^{a+ib} \\
&= e^x(\cos y + i\sin y)\cdot e^a(\cos b + i \sin b) \\
&= e^{x+a}\big((\cos y \cos b - \sin y \sin b) + i(\cos y \sin b + \sin y \cos b)\big) \\
&= e^{x+a}(\cos(y+b) + i\sin(b+y)) \\
&= \exp((x+a) + i(y+b)) = e^{z+w}
\end{aligned}$$

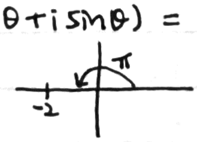
(2) Take any $w \in \mathbb{C}\setminus\{0\}$ write $w = p(\cos\theta + i \sin\theta)$

$p > 0$ and $\theta \in [0, 2\pi)$

Then $z = \log p + i\theta$

$$\exp(z) = e^{\log p}(\cos\theta + i\sin\theta) = p(\cos\theta + i \sin\theta) = w$$

$$= \exp(\underset{\ln 2}{\overset{\|}{\log 2}} + i3\pi)$$



(3) and (4) $\Rightarrow$ Check Lecture Note.

(5) $z = x+iy \quad \bar{z} = x-iy$

$e^z = e^x(\cos y + i\sin y) \quad e^{\bar{z}} = e^x(\cos(-y) + i\sin(-y))$

$= e^x(\cos y - i\sin y)$

$\overline{\cos y + i\sin y} = \cos y - i\sin y = \cos(-y) + i\sin(-y)$

Trigonometric Functions

Recall $e^{iy} = \cos y + i\sin y$ when $y \in \mathbb{R}$

$e^{-iy} = \cos y - i\sin y$ when $y \in \mathbb{C}$

Def For $z \in \mathbb{C}$, define $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ sin, cos

$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$ $\mathbb{C} \to \mathbb{C}$

Properties (1) $\sin^2 z + \cos^2 z = 1$

(2) (Euler Formula) $\exp(iz) = \cos z + i\sin z$, $z \in \mathbb{C}$

(3) $\sin(z+w) = \sin z \cos w + \cos z \cdot \sin w$

$\cos(z+w) = \cos z \cdot \cos w - \sin z \cdot \sin w$

Proof (1) $\left(\frac{1}{2i}(e^{iz} - e^{-iz})\right)^2 + \left(\frac{1}{2}(e^{iz} + e^{-iz})\right)^2$

$= \frac{-1}{4}\left(e^{2iz} - 2 + e^{-2iz}\right) + \frac{1}{4}\left(e^{2iz} + 2 + e^{-2iz}\right) = 1$

(2) $\cos z + i\sin z = \frac{1}{2}(e^{iz} + e^{-iz}) + \frac{1}{2}(e^{iz} - e^{-iz}) = e^{iz}$

(3) $\sin z \cos w + \sin w \cos z = \frac{1}{2i}(e^{iz} - e^{-iz})\frac{1}{2}(e^{iw} + e^{-iw})$

$+ \frac{1}{2i}(e^{iw} - e^{-iw})\frac{1}{2}(e^{iz} + e^{-iz}) = \sin(z+w)$

Hyperbolic cos/sin

$\cosh(z) = (e^z + e^{-z})/2 \qquad \cosh^2(z) - \sinh^2(z) = 1$

$\sinh(z) = (e^z - e^{-z})/2 \qquad z \in \mathbb{C}$

Logarithms

Motivation $z \in \mathbb{C} \setminus \{0\}$ Try to define $\log z$ to be $w \in \mathbb{C}$ such that $\exp(w) = z$

Issue such w is not unique

Recall $\exp(w) = \exp(w + 2\pi k i)$, $k \in \mathbb{Z}$

ex) $\exp(0) = 1$ $\exp(2\pi k i) = 1$, $k \in \mathbb{Z}$

1^{st} way to define log : log is a multi-valued function

$\boxed{1^{st} \text{ way to define log}}$: log is a multi-valued function

Prop. Every $z \in \mathbb{C} \setminus \{0\}$ has countably many logarithms given by

$$\{ \log|z| + i(\arg z + 2\pi n) : n \in \mathbb{Z} \}$$

Pf) For any $w = \log|z| + i(\arg z + 2\pi n)$

$$\exp(w) = \underbrace{e^{\log|z|}}_{=|z|} \big(\underbrace{\cos(\arg z + 2\pi n)}_{= \cos(\arg z)} + i \underbrace{\sin(\arg z + 2\pi n)}_{= \sin(\arg z)} \big)$$

$$= z$$

Conversely, if $\exp(w) = z$ $w = \cancel{r(\cos\theta + i\sin\theta)} = x + iy$

$\Rightarrow$ $e^x(\cos y + i \sin y) = z$

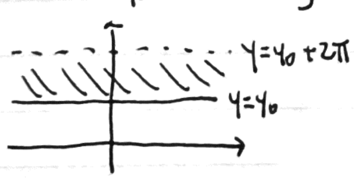
Compute modulus $e^x = |z|$

Compute argument $\cos y + i \sin y = \cos(\arg z) + i \sin(\arg z)$

$\Leftrightarrow$ $y = \arg z + 2\pi n$, $n \in \mathbb{Z}$.

$\boxed{2^{nd} \text{ way to define log}}$ Restrict the range of log to

$A_{y_0} = \{ x + iy : y_0 \le y < y_0 + 2\pi \}$



Check : given any $y_0 \in \mathbb{R}$

$\exp : A_{y_0} \to \mathbb{C} \setminus \{0\}$

is one to one and onto

<u>Def</u> (2^{nd} way) The function $\log : \mathbb{C} \setminus \{0\} \to A_{y_0}$ is the inverse of exp of A_0 called branch of log lying in A_{y_0}

<u>Warning</u> Regarded as multi valued functions $\log(zw) = \log z + \log w$ (as sets) Here: $A, B \in \mathbb{C}$, $A+B = \{ a+b : a \in A, b \in B \}$

Fix $y_0 \in \mathbb{R}$, consider log as a branch in A_{y_0}.
$\log(zw) = \log z + \log w$ no longer holds

ex) $y_0 = 0$, $A_0 = \{ z . 0 \leq \operatorname{Im} z < 2\pi \}$
$\log(-1) = 0 + i\pi$
$\log 1 = 0 + i0 = 0$
$\log((-1)(-1)) = \log 1 = 0$
$\log(-1) + \log(-1) = 2\pi i$

Generally, a branch of log on A_{y_0} satisfies
$\log(zw) = \log z + \log w \mod 2\pi i$
or $\log(zw) = \log z + \log w - 2\pi i \cdot n$

<u>Remark</u> when $x \in \mathbb{R}_+$, $\log x$ = real log of x.

<u>Complex Powers</u> a^b. If $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$, define $a^b = e^{b \log a}$
If $b \in \mathbb{Z}$, $a^2 = a \cdot a$ b^n or $1/b^n$
If $a \in \mathbb{R}_+$, then $\log a$ is the unique real log
$\Rightarrow e^{b \log a}$ is unique
In all other cases, a^b is either $\begin{cases} \text{multi valued} \\ \text{restricted to a branch of log} \end{cases}$