Complex Variables I – Problem Set 6

Due at 5 pm on Friday, Oct 20, 2023 via Gradescope

Problem 1

Evaluate the following integrals (all closed curves are oriented counterclockwise):

1. $\int_{|z+i|=1} \frac{e^z}{z^2+1} dz$.
2. $\int_{|z|=1} \frac{1}{(z-a)^2(z-b)} dz$. Here a, b are not on the circle $|z| = 1$. You answer should depend on a and b.

Solution:

1. Write $\frac{1}{z^2+1} = \frac{1}{2}\left(\frac{1}{z+i} + \frac{1}{z-i}\right)$, and hence $\frac{e^z}{z^2+1} = \frac{1}{2}\left(\frac{e^z}{z+i} + \frac{e^z}{z-i}\right)$. Note that the function $\frac{e^z}{z-i}$ is holomorphic in an open neighborhood of the disk $D = |z+i| < 1$, and hence its contour integral along ∂D is zero. On the other hand, by Cauchy's integral formula,

$$\int_{\partial D} \frac{e^z}{z+i} dz = 2\pi i e^{-i}.$$

Therefore

$$\int_{|z+i|=1} \frac{e^z}{z^2+1} dz = \pi i e^{-i}.$$

2. Write the integrand as partial fractions:

$$\frac{1}{(z-a)^2(z-b)} = \frac{1}{a-b}\frac{1}{(z-a)^2} - \frac{1}{(a-b)^2}\frac{1}{z-a} + \frac{1}{(a-b)^2}\frac{1}{z-b}.$$

We know that $\int_C \frac{1}{(z-a)^2} dz = 0$ regardless of the position of a. On ther other hand, $\int_C \frac{1}{z-a} dz = 2\pi i$ if a is inside the unit disk, and equals 0 if a is not inside the disk. Hence we conclude (letting $D = \{|z| < 1\}$):

$$\int_C \frac{1}{(z-a)^2(z-b)} dz = \begin{cases} -\frac{2\pi i}{(a-b)^2} & a \in D, b \notin D \\ \frac{2\pi i}{(a-b)^2} & a \notin D, b \in D \\ 0 & \text{all other cases.} \end{cases}$$

Problem 2

Let γ be a bounded smooth curve containing its endpoints, and $\phi : \gamma \to \mathbb{R}$ be a continuous function. Use the definition of derivative to prove that the function

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{\phi(w)}{w-z} dw$$

is holomorphic on $\mathbb{C} \setminus \gamma$.

Proof: Fix $z \in \mathbb{C} \setminus \gamma$. For $h \in \mathbb{C}$ such that $D(z, |h|) \in \mathbb{C} \setminus \gamma$, we have that

$$f(z+h) - f(z) = \frac{1}{2\pi i} \int_\gamma \phi(w) \left(\frac{1}{w-z-h} - \frac{1}{w-z}\right) dz = \frac{1}{2\pi i} \int_\gamma \phi(w) \frac{h}{(w-z-h)(w-z)} dz.$$

Hence

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\int_\gamma \frac{\phi(w)}{(z-w-h)(z-w)}$$

As $h \to 0$, the function $\frac{\phi(w)}{(w-z-h)(w-z)}$ converges uniformly to $\frac{\phi(w)}{(w-z)^2}$. Thus, the limit exists and the function is complex differentiable at z.

Problem 3

Let $A \subset \mathbb{C}$ be open, and $f_n : A \to \mathbb{C}$ a sequence of holomorphic functions. Suppose that $f_n \to f$ uniformly on all compact subsets of A. Prove that $f : A \to \mathbb{C}$ is also holomorphic.

Proof: We use Morera's theorem. For any closed C^1 curve $C \subset A$, since C is a compact subset of A, we have that $f_n \to f$ uniformly on C. Since each f_n is holomorphic, we have that

$$\int_C f_n(z)dz = 0.$$

Since $f_n \to f$ uniformly on C, we have that

$$\int_C f(z)dz = \lim_{n\to\infty}\int_C f_n(z)dz = 0.$$

By Morera's theorem, f is holomorphic in A.

Problem 4

1. Prove Cauchy's estimates: suppose that $A \subset \mathbb{C}$ is open, $f : A \to \mathbb{C}$ is holomorphic, and $|f(z)| \leq M$ on A. Then for any disk $D(w,r)$ with $\overline{D(w,r)} \subset A$ and any integer n, we have

$$|f^{(n)}(w)| \leq \frac{n!M}{r^n}.$$

2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire holomorphic, and there exists constant $M > 0$ and integer n such that $|f(z)| \leq M|z|^n$. Show that f is a polynomial of degree $\leq n$.

Proof: 1. By Cauchy's integral formula,

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_{\partial D(w,r)} \frac{f(z)}{(z-w)^{n+1}}dz.$$

Thus, we estimate that

$$|f^{(n)}(w)| \leq \frac{n!}{2\pi}2\pi r\frac{M}{r^{n+1} = \frac{n!M}{r^n}}.$$

2. For any $w \in \mathbb{C}$ and any $r > 0$, apply the previous estimate to $f^{(n+1)}(w)$ in $D(w,r)$ and obtain:

$$f^{(n+1)}(w) \leq \frac{(n+1)!(|w|+r)^n}{r^{n+1}}.$$

As $r \to \infty$, $\frac{(|w|+r)^n}{r^{n+1}} \to 0$. Thus, we have that $f^{(n+1)}(w) = 0$. This holds for every $w \in \mathbb{C}$, and hence f is a polynomial of degree at most n.

Problem 5

Evaluate
$$\int_0^\pi e^{\cos\theta}\cos(\sin\theta)d\theta.$$

Hint: integrate $\int_\gamma \frac{e^z}{z}dz$ along a suitable curve.

Solution: We compute $I = \int_{\partial D(0,1)} \frac{e^z}{z}dz$. On one hand, by Cauchy's integral formula, $I = 2\pi i e^0 = 2\pi i$. On the other hand, parametrize $\partial D(0,1)$ by $z(\theta) = e^{i\theta}$, $0 \le \theta < 2\pi$, we have
$$I = \int_0^{2\pi} e^{e^{i\theta}} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = i\int_0^{2\pi} e^{\cos\theta}(\cos(\sin\theta) + i\sin(\sin\theta))d\theta.$$
Comparing the imaginary parts, we conclude that
$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = 2\pi.$$
On the other hand, observe that
$$\begin{aligned}\int_\pi^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta &= \int_\pi^0 e^{\cos(2\pi-t)}\cos(\sin(2\pi-t))dt \\ &= \int_0^\pi e^{\cos t}\cos(\sin t)dt.\end{aligned}$$
Hence $\int_0^\pi e^{\cos\theta}\cos(\sin\theta)d\theta = \pi$.

Problem 6

Prove that a nonconstant entire holomorphic function $f : \mathbb{C} \to \mathbb{C}$ maps $\mathbb{C}$ onto a dense subset of $\mathbb{C}$.

Remark: a subset A of $\mathbb{C}$ is dense, if its intersection with any disk is nonempty.

Proof: Suppose otherwise, that there exists an open disk $D(w,\delta)$ such that $f(\mathbb{C}) \cap D(w,\delta) = \emptyset$. Then we have that $|f(z) - w| > \delta$ for all $z \in \mathbb{C}$. Consider the function $g(z) = \frac{1}{f(z)-w}$. Then $g(z)$ is a nonconstant entire function with the property that $|g(z)| = \frac{1}{|f(z)-w|} < \frac{1}{\delta}$, contradicting the Liouville theorem.

Remember to justify your answers and acknowledge collaborations and outside help!