Complex Variables I – Problem Set 7

Due at 5 pm on Friday, Nov 3, 2023 via Gradescope

Problem 1

Study for the midterm, and prepare for an one-sided, single sheet reference sheet (standard letter-sized).

Problem 2

Suppose $f(z)$ is holomorphic on the annulus $D = \{1 < |z| < 2\}$ and is continuous on $\overline{D}$, and

$$\max_{|z|=1} |f(z)| \leq 1, \quad \max_{|z|=2} |f(z)| \leq 2.$$

Show that for every $z \in D$, $|f(z)| \leq |z|$.

Problem 3

Let f be a nonconstant holomorphic function in a domain $A \subset \mathbb{C}$. Suppose that $|f|$ has a minimum in A. Show that f has a zero in A.

Remark: by considering $A = B(0, R)$ for sufficiently large R, try to give an alternative proof of the fundamental theorem of algebra.

Problem 4 (optional)

1. Suppose $f : A \to \mathbb{C}$ is holomorphic and $f(x + iy) = u(x, y) + iv(x, y)$ for real valued functions u, v. Prove that for any $z_0 = x_0 + iy_0$ in A and $\overline{D(z_0, r)} \subset A$, we have

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})dt, \quad v(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} v(z_0 + re^{it})dt.$$

2. (Maximum principle for u and v.) Suppose that A is a bounded domain, $f : A \to \mathbb{C}$ is holomorphic and $f : \overline{A} \to \mathbb{C}$ is continuous. Writing $f(z) = u(x, y) + iv(x, y)$, prove that

$$\max_{(x,y)\in\overline{A}} u(x, y) = \max_{(x,y)\in\partial A} u(x, y), \quad \max_{(x,y)\in\overline{A}} v(x, y) = \max_{(x,y)\in\partial A} v(x, y)$$

3. Suppose that f is holomorphic in a neighborhood of the unit disk $D(0, 1)$, and that for every $z \in \partial D(0, 1)$, $f(z) \in \mathbb{R}$. Prove that f is a constant function.

Problem 5 (optional)

Suppose $f(z)$ is holomorphic in the unit disk D, $|f(z)| \leq 1$, and

$$f(0) = f'(0) = \cdots = f^{(k)}(0) = 0.$$

Prove that for every $z \in D$, $|f(z)| \leq |z|^{k+1}$. Moreover, if there exists $z_0 \in D$ with $|f(z_0)| = |z_0|^{k+1}$, then $f(z) = cz^{k+1}$ for some $|c| = 1$.

Remember to justify your answers and acknowledge collaborations and outside help!