GA 2450 Complex analysis I

Textbook: Basic complex analysis, Marsden-Hoffman.
Office hours: Thursday 1:30 – 3:00 pm.

Notations:
- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_+$
- open intervals: (a,b), $a,b \in \mathbb{R}$ — textbook: $]a,b[$.
- closed intervals: $[a,b]$
- mappings (functions): $f: A \to \mathbb{C}$.
- $\exists$: exist, $\forall$: for all.

Complex numbers.

Historically: Solving quadratic equations. $x^2+1=0 \Rightarrow x=\sqrt{-1}$.
— called imaginary, denoted by i.

Generally: a complex number $z = x + iy$

$x = \operatorname{Re} z$, real part of z.

$y = \operatorname{Im} z$, imaginary part of z.

Complex variables are essential to many math problems:

- Solving polynomial equations of any degree.

Thm (Fundamental Thm of algebra): $P(z) = a_n z^n + \cdots + a_1 z + a_0$, $a_j \in \mathbb{C}$.

Then $P(z)=0$ has n roots (counted w/ multiplicity) in $\mathbb{C}$.

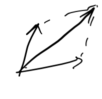
- Computing real integrals.

EX: $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$, $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin(\alpha\pi)}$ $(0<\alpha<1)$

The complex number system: $\mathbb{C}$.

Def: $\mathbb{C}$ is the set $\mathbb{R}^2$ with the following operations: for any $z = x+iy$. identify $\mathbb{C}$ with the pair $(x,y) \in \mathbb{R}^2$. Then for $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$,

- Addition $z_1 + z_2 = (x_1+x_2) + i(y_1+y_2)$

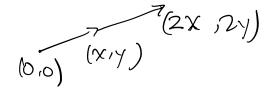


$\Leftrightarrow$ vector addition

$(x_1,y_1) + (x_2,y_2) = (x_1+x_2, y_1+y_2)$

- scalar multiplication: if $a \in \mathbb{R}$, then

$a z = ax + i(by)$



- complex multiplication.

$z_1 z_2 = (x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

Reason: complex linear + distribution law.

<u>Ex</u>: $\cdot\ (\pi + \sqrt{2}i) + (2-i) = (\pi+2) + (\sqrt{2}-1)i$

$\cdot\ (\pi + \sqrt{2}i)\cdot(2-i) = (2\pi + \sqrt{2}) + (2\sqrt{2} - \pi)i$

<u>Note</u>: $z_1, z_2 \in \mathbb{C}$ are equal $\Leftrightarrow$ $\operatorname{Re} z_1 = \operatorname{Re} z_2$, $\operatorname{Im} z_1 = \operatorname{Im} z_2$

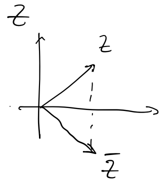
<u>Field structure of $\mathbb{C}$</u>

- $\mathbb{C}$ is closed under addition and multiplication.
- Addition and multiplication are associative and commutative, and multiplication is distributive over addition.
- Identity for $+$: $0 = 0 + 0\cdot i$.
  Identity for multiplication: $1 = 1 + 0\cdot i$
- $z = a + bi$ has additive inverse $-z = -a - bi$
  Multiplicative inverse:
$$\frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$
$\hookrightarrow$ $\frac{1}{z}$ exists whenever $z \neq 0$

<u>EX</u>: $\frac{1}{3-2i} = \frac{3+2i}{(3-2i)(3+2i)} = \frac{3+2i}{3^2+2^2} = \frac{3}{13} + \frac{2}{13}i$

Important operations:

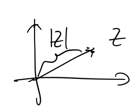
- Conjugation: $z = a + bi$. Say $w = a - bi$ the complex conjugate of z



  - $\bar{\bar{z}} = z$
  - $\operatorname{Re} z = \frac{z+\bar{z}}{2}$, $\operatorname{Im} z = \frac{z-\bar{z}}{2}$
  - $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
    $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

  Application: Consider a polynomial equation
$$a_n z^n + \cdots + a_1 z + a_0 = 0 \qquad (1)$$
  Suppose w is a solution. Then $\bar{w}$ solves
$$\bar{a}_n z^n + \cdots + \bar{a}_1 z + \bar{a}_0 = 0$$
  Thus, if each a_j is real, $a_j = \bar{a}_j$, then $\bar{w}$ is also a solution to (1)
  $\Rightarrow$ nonreal roots of a polynomial of real coefficients occur in complex conjugate pairs

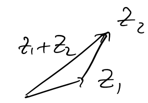
- Modulus
  - For $z = a + bi$, call $|z| = \sqrt{a^2+b^2}$ the modulus of z.



- $|z|^2 = z\bar{z}$. Thus, multiplicative inverse
$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$
- $\forall z \in \mathbb{C}$, $-|z| \le \operatorname{Re} z \le |z|$, $\quad -|z| \le \operatorname{Im} z \le |z|$

- For $z_1, z_2 \in \mathbb{C}$, $\quad |z_1 z_2|^2 = z_1 z_2 \overline{z_1 z_2} = z_1\bar{z}_1 \, z_2\bar{z}_2 = |z_1|^2 \, |z_2|^2$
$$\Rightarrow \quad |z_1 z_2| = |z_1| \cdot |z_2|$$

- For $z_1, z_2 \in \mathbb{C}$,
$$|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2) = z_1\bar{z}_1 + \underbrace{z_1\bar{z}_2 + \bar{z}_1 z_2}_{2\operatorname{Re}(z_1\bar{z}_2)} + z_2\bar{z}_2$$
$$= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2)$$
$$\le |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2$$
$$\Rightarrow \quad |z_1+z_2| \le |z_1| + |z_2| \quad \text{— triangle inequality.}$$



Square roots.

Let $z = a+bi$. Suppose w is a square root of z, $w = x+yi$.

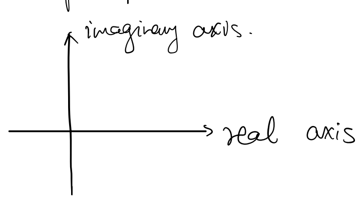
Then:
$$(x+yi)^2 = a+bi$$
$$\Leftrightarrow x^2-y^2+2xyi = a+bi \quad \Leftrightarrow \quad \begin{cases} x^2-y^2 = a & (1) \\ 2xy = b & (2) \end{cases}$$

First, observe $\quad (x^2+y^2)^2 = a^2+b^2 \quad (|w|^2 = |z|)$.
$$\Rightarrow \quad x^2+y^2 = \sqrt{a^2+b^2}.$$
Thus
$$x^2 = \frac{1}{2}\left(a+\sqrt{a^2+b^2}\right), \qquad y^2 = \frac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)$$
(2) tells us how to choose sign of x, y.

if $b \ge 0$, then
$$(x,y) = \pm\left(\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)}, \ \sqrt{\frac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)}\right)$$
if $b < 0$, then
$$(x,y) = \pm\left(\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)}, \ -\sqrt{\frac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)}\right)$$

Geometric representation of complex numbers.

- Complex plane



- Triangle inequality: $\quad |z_1+z_2| \le |z_1| + |z_2|$

Similarly: $\quad |z_1+z_2| \ge \big||z_1| - |z_2|\big|$

- Modulus: distance

e.g. $z_0 \in \mathbb{C}$, $R>0$. Then circle w/ center z_0 and radius R

$$= \{z \in \mathbb{C} : |z - z_0| = R\}.$$

Disk w/ center z_0 and radius $R = \{z \in \mathbb{C} : |z - z_0| \leq R\}$

- ε neighborhoods:

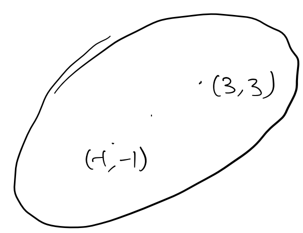
$$\{z : |z - z_0| < \varepsilon\}$$

Sometimes, we also talk about deleted nbhds, or punctured disk:

$$\{z : 0 < |z - z_0| < \varepsilon\}.$$

- <u>EX</u>: $\{z \in \mathbb{C} \mid |z+1+i| + |z-3-3i| = 6\sqrt{2}\}$

represents an ellipse. focci $= (-1,-1)$ and $(3,3)$



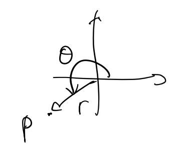
- Polar coordinate representation:

recall: for $p = (x,y) \in \mathbb{R}^2$, polar coord (r,θ) represents:

$r =$ dist from $(0,0)$ to p.

$\theta =$ angle from positive x-axis to $\overrightarrow{OP}$

$\theta \in [0, 2\pi)$.



and: $x = r\cos\theta$, $y = r\sin\theta$. Hence:

$$z = x + yi = r\cos\theta + r\sin\theta\, i = r(\cos\theta + \sin\theta\, i)$$

θ is called the argument of z.

Now if we let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then

$$z_1 z_2 = r_1 r_2 \left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right)$$

$$= r_1 r_2 \left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

<u>Prop</u>: $|z_1 z_2| = |z_1| \cdot |z_2|$, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}$

- In particular, when w is unit, i.e. $|w| = 1$, then $w = \cos\theta + i\sin\theta$ for some θ

Thus, $zw =$ rotating z counterclockwise by θ.

- If $z = r(\cos\theta + i\sin\theta)$, $n \in \mathbb{Z}_+$, then

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Prop (de Moivre): $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

- n-th root: if $z = r(\cos\theta + i\sin\theta)$, then n-th roots of z can be computed:

$w = \rho(\cos\gamma + i\sin\gamma)$ satisfies $w^n = z$

$$\Leftrightarrow \quad \rho^n = r, \qquad n\gamma = \theta \mod 2\pi$$

$$\Longleftrightarrow \quad \begin{cases} \rho = r^{\frac{1}{n}} \\ \gamma = \frac{\theta}{n} + \frac{2\pi k}{n}, \quad k=0,\cdots,n-1 \end{cases}$$

$$\Longleftrightarrow \quad w = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right), \quad k=0,\cdots,n-1$$

Particularly, n-th root of unity: $\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$, $k=0,\cdots,n-1$.

The Riemann sphere.

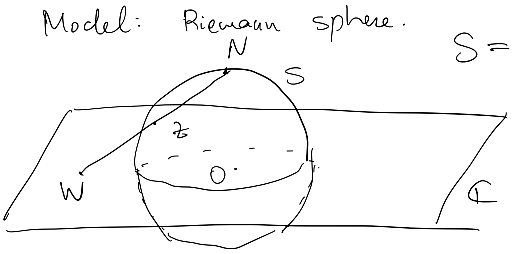
- We will soon see: consider the set $\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$.

∞ = infinity, s.t.:

- $\forall\, z\in\mathbb{C}$, $z+\infty = \infty + z = \infty$.
- $\forall\, z\in\hat{\mathbb{C}}\setminus\{0\}$, $z\cdot\infty = \infty\cdot z = \infty$.
- $\forall\, z\in\hat{\mathbb{C}}\setminus\{0\}$, $\frac{z}{0} = \infty$.
- $\forall\, z\in\mathbb{C}$, $\frac{z}{\infty} = 0$.

Geometrically, $\hat{\mathbb{C}}$ = complex plane + one point at ∞.

Model: Riemann sphere. $S = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1\}$



$\forall\, Z = (x,y,z)\in S\setminus N$,

assigns $w\in\mathbb{C}$ by $w = \frac{x+iy}{1-z}$.

this is an one-to-one and onto map.

$Z\mapsto w$ is a homeomorphism between $S\setminus\{N\}$ and $\mathbb{C}$.

Define $\varphi: S\to\hat{\mathbb{C}}$, $\begin{cases} \varphi(N) = \infty \\ \varphi(\underset{(x,y,z)}{\underset{\shortparallel}{Z}}) = \frac{x+iy}{1-z} \end{cases}$

Geometrically this is called the stereographic projection.