Continuity, Sequence, Cauchy, Bounded

Lecture 3

Continuity $(\mathbb{C}, |\cdot|)$ is a metric space

Def A sequence (z_n) in $\mathbb{C}$ is called bounded if $\exists\, C > 0$ s.t $|z_n| \le C$, $\forall n \in \mathbb{Z}_+$.

(z_n) is convergent if $\exists\, z \in \mathbb{C}$, s.t $\forall\, \varepsilon > 0$ $\exists N$ s.t $n > N$. We have $|z_n - z| < \varepsilon$

In this case $\lim\limits_{n\to\infty} z_n = z$.

Properties (1) (z_n) is convergent $\Rightarrow \lim\limits_{n\to\infty} z_n$ is unique

(2) (z_n) is convergent $\Rightarrow (z_n)$ is bounded

(3) If $\lim\limits_{n\to\infty} z_n = z$ $\lim\limits_{n\to\infty} w_n = w$

Then $\lim\limits_{n\to\infty} (z_n \pm w_n) = z \pm w$. and $\lim\limits_{n\to\infty} z_n \cdot w_n = z \cdot w$

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{z}{w} \quad \text{if } w \ne 0.$$

$$\lim_{n\to\infty} \overline{z_n} = \overline{z_n}, \quad \lim_{n\to\infty} |z_n| = |z|$$

$$\lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} z \quad \text{and} \quad \lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} z$$

Def A sequence (z_n) is Cauchy if $\forall\, \varepsilon > 0$, $\exists N$ s.t if $n, m > N$, we have

$$|z_n - z_m| < \varepsilon$$

Thm (z_n) is convergent $\Leftrightarrow$ (z_n) is Cauchy

Then $(\mathbb{C}, |\cdot|)$ is a complete metric space.

Def An infinite series $\sum_{n=1}^{\infty} z_n$, $z_n \in \mathbb{C}$ is defined to be

the sequence (S_n) of partial sums.

$$S_n = \sum_{j=1}^{n} z_j$$

Say $\sum_{n=1}^{\infty} z_n$ is a convergent if (S_n) is convergent,
and $\sum_{n=1}^{\infty} z_n = \lim_{n\to\infty} S_n$

Say $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |z_n|$ is convergent

Pf Absolute convergent $\Rightarrow$ Convergence $\sum_{n=1}^{\infty} z_n$ is absolutely convergent $S_n = \sum_{j=1}^{n} z_j$, $T_n = \sum_{j=1}^{n} |z_n|$. Then $\{T_n\}$ is convergent

$\Rightarrow$ $\{T_n\}$ is cauchy. For $n > m$

$$|S_n - S_m| = \left| \sum_{j=m+1}^{n} z_j \right| \leq \sum_{j=m+1}^{n} |z_j| = T_n - T_m$$

For any $\varepsilon > 0$, $\exists N$ s.t $|T_n - T_m| < \varepsilon$ where $n, m > N$

Then where $n, m > N$ $|S_n - S_m| < T_n - T_m < \varepsilon$.

$\Rightarrow$ (S_n) is Cauchy $\Rightarrow$ S_n has a limit.

Power Series, open and closed

Lecture 3

<u>Import series</u>. <u>Power Series</u> $\sum_{n=0}^{\infty} a_n z^n$

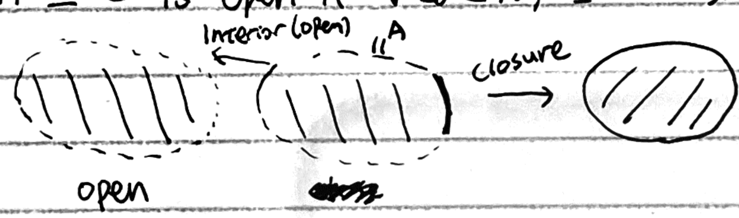
<u>Topology of $\mathbb{C}$ (extended complex plane $\overline{\mathbb{C}}$)</u>

<u>Def</u> (1) for $r>0$, $z_0 \in \mathbb{C}$. Define ~~$D(z_0,r) = \{z \in$~~

$D(z_0, r) = \{ z \in \mathbb{C},\ |z-z_0| < r \}$ open r-disk

$D(z_0, r) = \{ z \in \mathbb{C} : |z-z_0| < r,\ z \neq z_0 \}$ deleted r-disk

(2) $A \subseteq \mathbb{C}$ is open if $\forall z_0 \in A$, $\exists r>0$ s.t $D(z_0,r) \subseteq A$



(3) $A \subseteq \mathbb{C}$ is closed if $\mathbb{C} \setminus A$ is open

(4) $z_0 \in A$ is an interior pt of A if $\exists r>0$ s.t $D(z_0,r) \subseteq A$

$D(z_0, r) =$ 

<u>Property</u> (1) ϕ, $\mathbb{C}$ are open and closed (open-open OR closed-closed)

(2) Union of arbitrarily many open sets is open
(Intersection) (closed) (closed)

(3) Intersection of finitely many open sets is open
(Union) (closed) (closed)

<u>Def</u> $A \subseteq \mathbb{C}$. Interior of A. $\mathring{A} = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U$

Closure of A $\overline{A} = \bigcap_{\substack{C \supseteq A \\ C \text{ closed}}} C$

$\boxed{\text{Ex } \bigcap_{n=1}^{\infty} D(0,\tfrac{2}{n}) = \{0\} \text{ : closed}}$

boundary of A $\partial A = \overline{A} \setminus \mathring{A}$

Connected, Compact, path-connected, domain

Lecture 3

Def $A \subseteq \mathbb{C}$ is compact if every open covering of A contains a finite subcover.

i.e if $\exists \{U_\alpha\}_{\alpha \in I}$ s.t $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ then $\exists\, \alpha, \alpha_n \in I$

s.t $A \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}$

Prop For $A \subseteq \mathbb{C}$ following are equivalent.

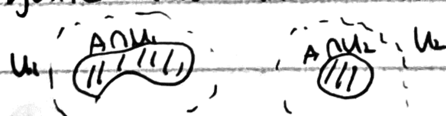
(1) A is compact

(2) A is bounded and closed

(3) Every sequence $\cancel{(z_n) \Rightarrow}$ $(z_n) \subseteq A$ has a converging subsequence whose limit is in A

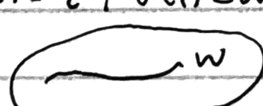
Def $A \subseteq \mathbb{C}$

(1) Say A is connected if it cannot be written as the disjoint union in the form $A = (A \cap U_1) \cup (A \cap U_2)$ U_1, U_2 open



disconnected

(2) A is path-connected if every $z, w \in A$ can be connected if a path i.e $\exists \gamma : [0,1] \to A$ continuous s.t $\gamma(0) = z$, $\gamma(1) = w$

$A =$ 

(3) Say A is a domain, if A is open and connected

Rieman sphere,

Lecture 3

<u>Prop</u> (1) A is path-connected $\Rightarrow$ A is connected (converse if false)

(2) If A is open, then A is connected $\Rightarrow$ A is path-connected

<u>Def</u> $A \subseteq \mathbb{C}$ $f: A \to \mathbb{C}$ is continuous ~~at~~ at $z_0 \in A$

if $\forall \varepsilon > 0$, $\exists r > 0$ s.t whenever $|z - z_0| < r$ $z \in A$

$|f(z) - f(z_0)| < \varepsilon$

<u>Properties</u> Sums, products, quotients, composition of continuous functions are continuous.

Extended complex plance $\overline{\mathbb{C}}$ = Rieman sphere with operations

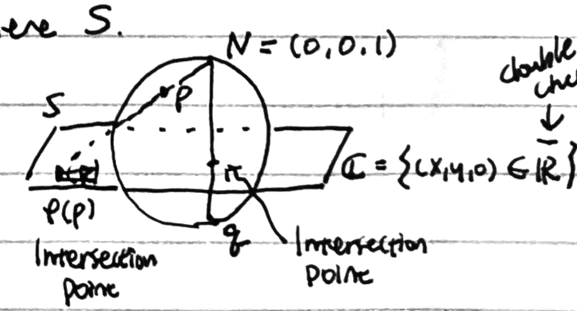
$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$\boxed{\text{Topology of } \overline{\mathbb{C}}}$

<u>Def</u> $A \subseteq \overline{\mathbb{C}}$ is open if $A \cap \mathbb{C}$ is open and if $\infty \in A$

then $\exists K > 0$ s.t $\{z : |z| > K\} \subseteq A$. Geometric

realization of $\overline{\mathbb{C}}$. Rieman Sphere S.



$S = \{(x,y,z) \in \mathbb{R}^3,\ x^2+y^2+z^2 = 1\}$

$\forall P = (x,y,z) \in S$, $P \neq N$,

take the line through N and P

its intersection with the x-y plane is

$P((x,y,z)) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)_S$ zero $= \frac{x+iy}{1-z} \in \mathbb{C}$.

as $p \to N$, it makes a tangent line

Homomorphism. Holomorphic functions

Lecture 3

Then define $\varphi(N) = \infty$ (N: North Pole)

<u>Prop</u> $\varphi : S \to \overline{\mathbb{C}}$ is one-to-one and onto and is a homomorphism

$\boxed{\text{Holomorphic functions}}$

<u>Def</u> $A \subseteq \mathbb{C}$ open a function $f: A \to \mathbb{C}$ is differentiable at $z_0 \in A$ if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists. $\rightsquigarrow f'(z_0)$

And say f is holomorphic / analytic at $z_0 \in A$ if $\exists\, r > 0$ s.t f is differentiable in $D(z_0, r)$

<u>Ex</u> $f(z) = z^n$ $\quad \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{1}{h}\left((z_0+h)^n - z_0^n\right)$

$$= \lim_{h \to \infty} \frac{1}{h}\left(\cancel{z_0^n} + nhz_0^{n-1} + \sum_{k=2}^{n} \binom{n}{k} h^k z_0^{n-k} - \cancel{z_0^n}\right)$$

$$= \lim_{h \to \infty} nz_0^{n-1} + \sum_{k=2}^{n} \binom{n}{k} h^{k-1} z_0^{n-k} = nz_0^{n-1}$$

$f(z) = nz^{n-1}$

<u>Ex</u> $f: \mathbb{C} \to \mathbb{C}$ $\quad f(z) = \mathrm{Re}\, z$

$$\lim_{h \to 0} \frac{1}{h}\left(f(z_0+h) - f(z_0)\right) = \lim_{h \to 0} \frac{1}{h}\left(\mathrm{Re}(z_0+h) - \mathrm{Re}(z_0)\right) = \lim_{h \to 0} \frac{1}{h}\mathrm{Re}(h)$$

When h is real $\lim_{h \to 0} \frac{1}{h} \cdot h = 1$, when h is imaginary $\lim_{h \to 0} \frac{1}{h} \underbrace{\mathrm{Re}(h)}_{=0} = 0$

Cauchy - Riemann equation

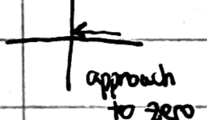
Lecture 3

$\lim_{h\to 0} \frac{1}{h} \mathrm{Re}(h)$ does not exist $\Rightarrow$ f is not differentiable

$\boxed{\text{Cauchy-Riemann equation}}$

f(z) is holomorphic. Write $f(x+iy) = u(x,y) + i v(x,y)$, $u, v : \mathbb{R}^2 \to \mathbb{R}$

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{h\to 0} \frac{1}{h}\Big(f(x_0+h+iy_0) - f(x_0+iy_0) \Big)$$



$$= \lim_{h\to 0} \frac{1}{h}\Big(u(x_0+h, y_0) + i v(x_0+h, y_0) - u(x_0, y_0) - i v(x_0, y_0) \Big) = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)$$

$h \in \mathbb{R}$ $\quad \lim_{h\to 0} \frac{f(z_0 + ih) - f(z_0)}{ih}$

$$= \lim_{h\to 0} \frac{1}{ih}\Big(f(x_0 + i(y_0+h)) - f(x_0+iy_0) \Big)$$

$$= \lim_{h\to 0} \frac{1}{ih}\begin{pmatrix} u(x_0, y_0+h) + i v(x_0, y_0+h) \\ -u(x_0, y_0) - i v(x_0, y_0) \end{pmatrix}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial u}{\partial y}(x_0, y_0) \Rightarrow$$

$$\Rightarrow \begin{cases} \partial u/\partial x = \partial v/\partial y \\ \partial u/\partial y = -\partial v/\partial x \end{cases} \quad (*) \text{ Cauchy-Riemann equations}$$

Thm $f(x_0+iy_0) = u(x_0, y_0) + i v(x_0, y_0)$ is differentiable at z_0

$\Leftrightarrow$ u, v satisfies the C-R equation

Pf $(\Rightarrow)$ ✓

$(\Leftarrow)$ $f(x_0+h+i(y_0+k)) - f(x_0, y_0)$

To do this $u(x_0+h, y_0+k) - u(x_0, y_0)$

$$= \frac{\partial u}{\partial x} \cdot h + \frac{\partial u}{\partial y} \cdot k + o\left((h^2+k^2)^{1/2}\right)$$

$v(x_0+h, y_0+k) - v(x_0, y_0)$

$$= \frac{\partial v}{\partial x} h + \frac{\partial v}{\partial y} k + o\left((h^2+k^2)^{1/2}\right)$$

$$\overset{(C\text{-}R)}{=} -\frac{\partial u}{\partial y} h + \frac{\partial u}{\partial x} k + o\left((h^2+k^2)^{1/2}\right)$$

$$f(z_0+(h+ik)) - f(z_0) = \frac{\partial u}{\partial x} h + \frac{\partial u}{\partial y} \cdot k + i\left(-\frac{\partial u}{\partial y} h + \frac{\partial u}{\partial x} k\right) + o\left((h^2+k^2)^{1/2}\right)$$

$$= \frac{\partial u}{\partial x}(h+ik) + \frac{\partial u}{\partial y}(k-ih) + \underline{o\left((h^2+k^2)^{1/2}\right)}$$

Check Proof not

$$\lim_{h+ik\to 0} \frac{f(z_0+(h+ik)) - f(z_0)}{h+ik} = \frac{\partial u}{\partial x}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0)$$