Complex Variables I – Problem Set 5

Due at 5 pm on Friday, Oct 13, 2023 via Gradescope

Problem 1

Evaluate the following contour integrals.

1. $\int_C \frac{1}{z^n} dz$, where C is the unit circle oriented counterclockwise, and n is an integer. Your answer may depend on n.

2. $\int_C \frac{e^z}{z^2-16} dz$, where C is the circle centered at 1 with radius 1.

Solution:
1. Parametrize C by $t \in [0, 2\pi) \mapsto z(t) = e^{it}$. Thus,

$$\int_C \frac{1}{z^n} dz = \int_0^{2\pi} \frac{1}{e^{int}} ie^{it} dt.$$

If $n \neq 1$, the above integral becomes:

$$\int_0^{2\pi} e^{-i(n-1)t} dt = \frac{-1}{i(n-1)} e^{-i(n-1)t} \bigg|_0^{2\pi} = 0.$$

If $n = 1$, the above integral gives $2\pi i$.

2. We observe that the function $\frac{e^z}{z^2-16}$ is defined and holomorphic everywhere inside the region bounded by C. Thus, Cauchy's integral theorem concludes that its integral along C is zero.

Problem 2

Evaluate the following integrals with arc length parameters.

1. $\int_{|z|=1} \frac{|dz|}{z}$.

2. $\int_{|z|=1} |z-1||dz|$.

Solution:
1. Parametrize the circle by $z = e^{i\theta}$, $0 \leq \theta < 2\pi$. Then

$$\int_{|z|=1} \frac{|dz|}{z} = \int_0^{2\pi} \frac{|z'(\theta)| d\theta}{e^{i\theta}} = \int_0^{2\pi} e^{-i\theta} d\theta = 0.$$

2. Parametrize the circle by $z = e^{i\theta}$. Then $|z-1| = |\cos\theta - 1 + i\sin\theta| = \sqrt{(\cos\theta - 1)^2 + \sin^2\theta} = \sqrt{2 - 2\cos\theta} = 2\sin\frac{\theta}{2}$. Also $|dz| = d\theta$. Thus, we have

$$\int_{|z|=1} |z-1||dz| = \int_0^{2\pi} 2\sin\frac{\theta}{2} d\theta = 4\int_0^{\pi} \sin t dt = 8.$$

Problem 3

Let $A \subset \mathbb{C}$ be a domain, $f : A \to \mathbb{C}$ be holomorphic on $A \setminus \{\zeta\}$, where $\zeta \in A$ is an interior point. Suppose that

$$\lim_{z \to \zeta} (z - \zeta) f(z) = 0.$$

Prove that $\int_{\partial R} f(z)dz = 0$, for any rectangle $R \subset A$ containing ζ as an interior point.

Proof: For any $\epsilon > 0$, let $\delta > 0$ be such that $|(z - \zeta)f(z)| < \epsilon$ when $z \in D(\zeta, \delta)$. Take a square R_δ with side length 2δ contained in $R \cap D(\zeta, \delta)$. Since the region $R \setminus R_\delta$ can be partitioned into rectangles, and by Cauchy's thereom on rectangles, the integral of $f(z)$ along the boundary of each of them is zero, we have that

$$\int_{\partial R} f(z)dz = \int_{\partial R_\delta} f(z)dz.$$

On ∂R_δ, we have $\delta \leq |z - \zeta| < 2\delta$. Thus,

$$\int_{\partial R_\delta} |f(z)||dz| \leq \int_{\partial R_\delta} \frac{\epsilon}{|z - \delta|} |dz| \leq \frac{\epsilon}{\delta} L(\partial R_\delta) = 8\epsilon.$$

Hence $|\int_{\partial R_\delta} f(z)dz| < 8\epsilon$. Sending $\epsilon \to 0$ gives that $\int_{\partial R} f(z)dz = 0$.

Problem 4

Assume that f is holomorphic and satsifies the inequality $|f(z) - 1| < 1$ in a domain Ω. Taking for granted that f' is continuous in Ω, show that

$$\int_\gamma \frac{f'(z)}{f(z)} = 0$$

for every closed curve γ in Ω.

Proof: By assumption, $f(\Omega) \subset D(1, 1)$. In $D(1, 1)$, the branch of log taking values in $A_{-\pi} = \{z : -\pi \leq \operatorname{Im} z \leq \pi\}$ is well-defined and has derivative $\frac{1}{z}$. Therefore, we have that

$$\frac{f'(z)}{f(z)} = (\log f(z))',$$

where log is the branch on $A_{-\pi}$. By the fundamental theorem of contour integral, its contour integral on any closed curve is zero.

Problem 5

Evaluate

$$\int_0^\infty \frac{1 - \cos x}{x^2} dx$$

by considering the function $f(z) = \frac{1 - e^{iz}}{z^2}$ along a suitable choice of contour.

Hint: use the statement of Problem 7 below (without proving it). Note that f is not defined at the origin, so your contour should avoid a tiny neighborhood of the interval - e.g. try an indented semicircle as shown in the figure below.

Solution: Integrate the function in the contour of indented semicircle: for $\epsilon > 0, R > 0$, consider the function $f(z) = \frac{1 - e^{iz}}{z^2}$ on the boundary of the indented semicircle with outer radius R and inner radius ϵ. By Cauchy theorem, the contour integral of f is zero. Thus, we have that:

$$\int_{I_1} f(z)dz + \int_{I_2} f(z)dz + \int_{|z|=\epsilon, \operatorname{Im} z \geq 0} f(z)dz - \int_{|z|=R, \operatorname{Im} z \geq 0} f(z)dz = 0.$$

Here: I_1, I_2 are the two line segments between the indented semicircle, and the minus sign above is to make sure that the orientations match.

Along I_1, I_2, we have:

$$\begin{aligned}
\int_{I_1+I_2} f(z)dz &= \int_{-R}^{\epsilon} f(z)dz + \int_{\epsilon}^{R} f(z)dz \\
&= \int_{-R}^{\epsilon} \frac{1-\cos t - i\sin t}{t^2}dt + \int_{\epsilon}^{R} \frac{1-\cos t - i\sin t}{t^2}dt \\
&= \int_{-R}^{\epsilon} \frac{1-\cos t}{t^2}dt + \int_{\epsilon}^{R} \frac{1-\cos t}{t^2}dt \\
&= 2\int_{\epsilon}^{R} \frac{1-\cos t}{t^2}dt.
\end{aligned}$$

Here we have used the fact that sin is odd and cos is even.

Next we look at the large semicircle. Parametrize $\{|z=R|, \operatorname{Im} z \geq 0\}$ by $z(t) = Re^{it}$, $0 \leq t \leq \pi$. Then $e^{iz} = e^{-R\sin t}e^{iR\cos\theta}$. Thus, we have that

$$|e^{iz}| = e^{-R\sin t} \leq e^0 = 1.$$

Therefore, we can estimate:

$$\left|\frac{1-e^{iz}}{z^2}\right| \leq \frac{1+|e^{iz}|}{|z|^2} = \frac{2}{R^2}.$$

Hence

$$\left|\int_{|z|=R, \operatorname{Im} z\geq 0} \frac{1-e^{iz}}{z^2}dz\right| \leq \frac{2}{R^2}L(\{|z|=1, \operatorname{Im} z \geq 0\}) = \frac{2}{R^2}\pi R = \frac{2\pi}{R} \to 0,$$

as $R \to \infty$.

Next we consider the integral along the inner semicircle. For this, we use the fact that $e^z = 1 + z + O(z^2)$ when $|z| \to 0$ (this can be seen from the Taylor expansions of the real functions $e^x, \cos y, \sin y$ together with the fact that $e^{x+iy} = e^x(\cos y + i\sin y)$). Parametrizing the semicircle by $z(t) = \epsilon e^{it}$, $0 \leq t \leq \pi$, we have:

$$\begin{aligned}
\int_{|z|=1, \operatorname{Im} z\geq 0} f(z)dz &= \int_0^{\pi} \frac{iz + O(z^2)}{z^2}dz \\
&= \int_0^{\pi} \frac{-i\epsilon e^{it} + O(\epsilon^2)}{\epsilon^2 e^{2it}}\epsilon i e^{it}dt \\
&= -\pi + O(\epsilon) \\
&\to -\pi, \text{ as } \epsilon \to 0.
\end{aligned}$$

Finally, putting everything together and sending $\epsilon \to 0$ and $R \to \infty$, we have

$$\int_0^{\infty} \frac{1-\cos x}{x^2}dx = \frac{\pi}{2}.$$

Problem 6

Prove that

$$\int_0^{\infty} \sin(x^2)dx = \int_0^{\infty} \cos(x^2)dx = \frac{\sqrt{2\pi}}{4},$$

by considering the function $f(z) = e^{-z^2}$ over a 'sector' of angle $\pi/4$, as shown in the figure below.

Solution: Consider the contour integral of $f(z) = e^{-z^2}$ in the boundary of a sector of opening angle $\pi/2$ and radius R. By Cauchy, we have that

$$\int_{I_1} f(z)dz - \int_{I_2} f(z)dz + \int_{C_R} f(z)dz = 0,$$

here I_1 is the interval $[0, R]$ on the real axis, I_2 is the ray along the line $x = y$ starting from the origin, and C_R is the circular arc of radius R. Now obviously we have that

$$\int_{I_1} f(z)dz = \int_0^R e^{-x^2} dx = \frac{1}{2} \int_{-R}^R e^{-x^2} dx \to \sqrt{\pi}/2, \text{ as } R \to \infty.$$

Now let's consider the integral over I_2. Parametrize I_2 by $z(t) = \frac{1+i}{\sqrt{2}} t$, $0 \le t \le R$. Then:

$$\int_{I_2} e^{-z^2} dz = \int_0^R e^{-it^2} \frac{1+i}{\sqrt{2}} = \frac{1}{\sqrt{2}} \int_0^R (\cos(t^2) + \sin(t^2)) + i(\cos(t^2) - i\sin(t^2))dt.$$

Finally let's consider the integral over C_R. Parametrize C_R by $z(t) = Re^{it}$, $0 \le t \le \frac{\pi}{4}$, we have that

$$\int_{C_R} e^{-z^2} dz = \int_0^{\frac{\pi}{4}} e^{-R^2(\cos(2t) + i\sin(it))} iRe^{it} dt.$$

Thus, we have that

$$\left| \int_{C_R} e^{-z^2} dz \right| \le \int_0^{\frac{\pi}{4}} e^{-R^2 \cos(2t) Rdt} = \frac{1}{2} \int_0^{\pi/2} e^{-R^2 \sin t} Rdt.$$

Where in the last equation we used a change of variable $t \mapsto \pi/2 - 2t$. Now we use the fact that $\sin t \ge \frac{t}{2}$ on $[0, \pi/2]$ to continue the estimates:

$$\int_0^{\pi/2} e^{-R^2 \sin t} Rdt \le \int_0^{\pi/2} e^{-R^2 t/2} Rdt < \frac{2}{R} \to 0, \text{ as } R \to \infty.$$

Putting everything together and sending $R \to \infty$, we conclude that

$$\frac{1}{\sqrt{2}} \int_0^\infty \sin(t^2) + \cos(t^2) dt = \frac{\sqrt{\pi}}{2}, \quad \int_0^\infty \sin(t^2) - \cos(t^2) dt = 0.$$

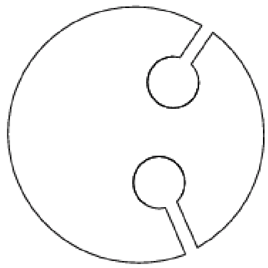
Therefore

$$\int_0^\infty \sin(t^2) dt = \int_0^\infty \cos(t^2) dt = \frac{\sqrt{2\pi}}{4}.$$

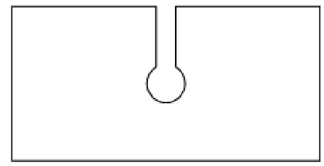
Problem 7 - optional

Extend our proof of Cauchy's theorem on disks to more general contours: suppose C is a connected closed piecewise C^1 curve which bounds a connected open set D of $\mathbb{C}$, and any two points p, q can be connected by a polygonal curve inside D, that is, the union of finitely many horizontal or vertical line segments. Show that if f is holomorphic on a neighborhood of D, then $\int_C f(z)dz = 0$.

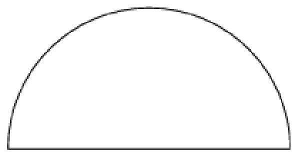
Some examples of valid contours (figure taken from Stein-Shakarchi, Complex Analysis):



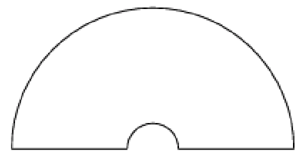
The multiple keyhole



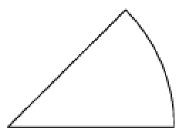
Rectangular keyhole



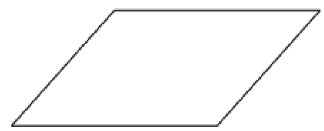
Semicircle



Indented semicircle



Sector



Parallelogram

Remember to justify your answers and acknowledge collaborations and outside help!