Complex Variables I – Problem Set 1

Due at 5 pm on Friday, September 15, 2023 via Gradescope

Problem 1

Solve the following equations

$$\text{a) } z^2 + iz + 6 = 0 \qquad \text{b) } z^6 - 64 = 0 \qquad \text{c) } z^3 + 1 = 0$$

Make sure to find all solutions!

Solution: a) $(z - 2i)(z + 3i) = 0$, hence we have two solutions $z_1 = 2i$ and $z_2 = -3i$.

b) One solution is 2, and the others are 2 multipled with the 6-th roots of unity: $z_k = 2(\cos \frac{2k\pi}{6} + i \sin \frac{2k\pi}{6})$, $k = 0, 1, \cdots, 5$.

c) One solution is -1. Similarly, the three solutions are: $z_k = -(\cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3})$, $k = 0, 1, 2$.

Problem 2

Let z, w be complex numbers. Prove the parallelogram identity:

$$|z - w|^2 + |z + w|^2 = 2(|z|^2 + |w|^2).$$

Proof:

$$\begin{aligned} |z - w|^2 + |z + w|^2 &= (z - w)(\bar{z} - \bar{w}) + (z + w)(\bar{z} + \bar{w}) \\ &= z\bar{z} - w\bar{z} - z\bar{w} + w\bar{w} + z\bar{z} + w\bar{z} + z\bar{w} + w\bar{w} \\ &= 2(z\bar{z} + w\bar{w}) \\ &= 2(|z|^2 + |w|^2). \end{aligned}$$

Problem 3

Prove that

$$\left| \frac{z - w}{1 - \bar{z}w} \right| = 1$$

if either $|z| = 1$ or $|w| = 1$, and $\bar{z}w \neq 1$.

Proof: Note that the denominator satisfies $|1 - \bar{z}w| = |1 - z\bar{w}|$, so the equation is symmetric in z and w. Thus, it suffices to prove the equality when $|w| = 1$. We have:

$$\begin{aligned} \frac{z - w}{1 - \bar{z}w} \cdot \frac{\bar{z} - \bar{w}}{1 - z\bar{w}} &= \frac{z\bar{z} - z\bar{w} - w\bar{z} + w\bar{w}}{1 - \bar{z}w - z\bar{w} + z\bar{z}w\bar{w}} \\ &= \frac{z\bar{z} - z\bar{w} - w\bar{z} + 1}{1 - \bar{z}w - z\bar{w} + z\bar{z}} \\ &= 1. \end{aligned}$$

Problem 4

Let $c_0, c_1, \ldots, c_n$ be n complex numbers, and consider the following polynomial in z:

$$P(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0$$

Show that there exists a number $R > 0$ such that

$$\text{for all } z \in \mathbb{C} \text{ with } |z| > R, \text{ we have } \left|\frac{1}{P(z)}\right| < \frac{2}{|c_n| R^n}$$

Proof:
Let $C > 0$ be large enough such that $|c_{n-1}|, \cdots, |c_0| < C$. Using the triangle inequality, we have:

$$\begin{aligned}
|P(z)| &= |c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0| \\
&\geq |c_n||z|^n - |c_{n-1} z^{n-1} + \cdots + c_0| \\
&\geq |c_n||z|^n - (|c_{n-1}||z|^{n-1} + \cdots + |c_0|) \\
&\geq |c_n||z|^n - C(|z|^{n-1} + \cdots + 1) \\
&= |c_n||z|^n - C\frac{|z|^n - 1}{|z| - 1} \\
&> \left(|c_n| - \frac{C}{|z| - 1}\right)|z|^n.
\end{aligned}$$

Choose R such that $R > \frac{2C}{|c_n|+1}$. Then when $|z| > R$, we have $|c_n| - \frac{C}{|z|-1} > \frac{|c_n|}{2}$, and thus

$$|P(z)| > \frac{|c_n|}{2}|z|^n > \frac{|c_n| R^n}{2}.$$

Problem 5

Sketch the following regions of the complex plane

a) $\{z \in \mathbb{C} : |Re(z)| \geq 1\}$

b) $\{z \in \mathbb{C} : |z - i| < 1\}$

c) $\{z \in \mathbb{C} : 0 < \arg(z) < \frac{\pi}{4}\}$

d) $\{z \in \mathbb{C} : |Im(z)| \geq 1\} \cap \{z \in \mathbb{C} : |z| \leq \sqrt{2}\}$

<u>Note</u>: In the above, arg stands for the argument of z.

Remember to justify your answers and acknowledge collaborations and outside help!