Complex Variables I – Problem Set 4

Due at 5 pm on Friday, Oct 6, 2023 via Gradescope

Problem 1

Let A be a connected open set in $\mathbb{C}$, $f : A \to \mathbb{C}$ is holomorphic. Suppose $|f(z)|$ is constant in A. Prove that $f(z)$ is a constant function.

Problem 2

Let $f(z)$ be holomorphic in the disk $|z-1| < 1$, and suppose that $f'(z) = \frac{1}{z}$, $f(1) = 0$. Prove that $f(z) = \log z$ in the disk, where log is the branch that takes value in $A_{-\pi} = \{z : -\pi < \operatorname{Im} z < \pi\}$.

Problem 3

1. Verify that the function $u(x, y) = \sin x \cosh y$ is harmonic in $\mathbb{R}^2$.
2. Find the harmonic conjugate v of u such that $v(0, 0) = 3$.

Problem 4

Find the general form of a holomorphic function $f(z)$ whose real parts only depend on $|z|$.

Hint: the real part of a holomorphic function is harmonic.

Problem 5

Find the following line integral.

1. $\int_C \log z dz$, here C is the unit circle oriented counterclockwise, and log is the branch which takes value in $A_0 = \{z : 0 \le \operatorname{Im} z < 2\pi\}$.
2. $\int_C \frac{1}{z} dz$, here C is the line segment from $\frac{-\sqrt{3}-i}{2}$ to $\frac{1+\sqrt{3}i}{2}$.

Problem 6

Consider a function f which is holomorphic in the open unit disk centered at the origin $D(0, 1)$, and f satisfies

$$\forall z \in D(0, 1), |f'(z)| \le M,$$

for some $M > 0$. Show that for every $z_1, z_2 \in D(0, 1)$, $|f(z_1) - f(z_2)| \le M|z_1 - z_2|$.

Problem 7 - bonus

Let $f : \mathbb{C} \to \mathbb{R}$ be a continuous real valued function such that $|f(z)| \leq 1$ for all $z \in \mathbb{C}$. Show that

$$\left| \int_C f(z) dz \right| \leq 4,$$

where C is the unit circle oriented counterclockwise.

(Note: this question will not count in the homework grade.)

Remember to justify your answers and acknowledge collaborations and outside help!