Problem 1.

1. $\int_C \frac{1}{z^n} dz$

Sol) Let $z = e^{i\theta}$ ($\because r=1$). Then $dz = ie^{i\theta} \cdot d\theta$.

$$\int_0^{2\pi} \left[\frac{1}{e^{i\theta}}\right]^n \cdot i \cdot e^{i\theta} \cdot d\theta = i\int_0^{2\pi} e^{(1-n)i\theta}\, d\theta = \frac{i}{i(1-n)} e^{(1-n)i\theta}\Big|_0^{2\pi}$$

$$= \frac{1}{(1-n)}\left(e^{-2\pi(n-1)i} - 1\right) \quad (\because e^0 = 1)$$

$$= \frac{1}{1-n}\left(\cos(-2\pi(n-1)) + i\sin(-2\pi(n-1)) - 1\right)$$

$$= \frac{1}{1-n}\left(\cos(2\pi(n-1)) - i\sin(2\pi(n-1)) - 1\right)$$

$$= \frac{1}{1-n}(1 - 0 - 1)$$

$$= \underline{0} \quad \text{for } n \neq 1$$

□

2. $\int_C [e^z/(z^2-16)]\, dz$

Sol) Let $e^z/(z^2-16) = e^z\left(\frac{A}{z-4} + \frac{B}{z+4}\right)$ for constants A and B.

$$\frac{e^z}{z^2-16} = e^z\left[\frac{A(z+4) + B(z-4)}{z^2-16}\right] = \frac{e^z}{z^2-16}\left[(A+B)z + 4(A-B)\right]$$

Thus, it yields that $A = -B$. It follows that $4(2A) = 1 \Rightarrow A = 1/8$.

So, $$\frac{e^z}{z^2-16} = \frac{1}{8}\left[\frac{e^z}{z-4} + \frac{e^z}{z+4}\right].$$

$$\int_C \frac{e^z}{z^2-16} dz = \frac{1}{8}\int_C \left(\frac{e^z}{z-4} + \frac{e^z}{z+4}\right) dz = \frac{1}{8}\left[\int_C \frac{e^z}{z-4} dz + \int_C \frac{e^z}{z+4} dz\right]$$

$$= \underline{0} \quad \left(\because f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w} dz = \begin{cases} 0 & \text{if } w \notin D \\ f(w) & \text{if } w \in D \end{cases}\right)$$

□

Problem 2

1. $\int_{|z|=1} \frac{1}{z} |dz|$

sol) Let $z = e^{i\theta}$ $(\because r=1)$ and $|dz| = d\theta$

Putting altogether, we have

$$\int_0^{2\pi} e^{-i\theta} d\theta = \frac{1}{-i} e^{-i\theta}\Big|_0^{2\pi} = i e^{-i\theta}\Big|_0^{2\pi}$$

$$= i[e^{-2\pi i} - 1]$$

$$= i[\cos 2\pi - i \sin 2\pi - 1]$$

$$= i[1 - 0 - 1]$$

$$= \underline{0}$$

$\square$

2. $\int_{|z|=1} |z-1| \cdot |dz|$

sol) Let $z = e^{i\theta}$

$$\int_0^{2\pi} |e^{i\theta} - 1| \cdot |ie^{i\theta} d\theta| = \int_0^{2\pi} |(\cos\theta - 1) + i\sin\theta| \cdot |ie^{i\theta} d\theta|$$

$$= \int_0^{2\pi} \sqrt{(\cos\theta - 1)^2 + \sin^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\left(1 - 2\sin^2\left(\tfrac{\theta}{2}\right)\right)}\, d\theta \quad \left(\because \cos 2\theta = 1 - 2\sin^2\theta\right)$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\left(\tfrac{\theta}{2}\right)}\, d\theta = 2\int_0^{2\pi} \left|\sin\left(\tfrac{\theta}{2}\right)\right| d\theta$$

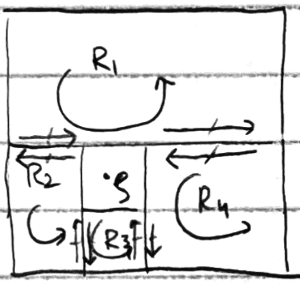
$$= 2 \cdot (-2) \cos\left(\tfrac{\theta}{2}\right)\Big|_0^{2\pi} = -4[\cos\pi - 1] = -4[-2] = +8$$

$\square$

Problem 3.

Sol) Suppose that $\lim_{z\to\xi}(z-\xi)f(z)=0$. Then we need to show that $\int_{\partial R} f(z)dz=0$ for any rectangle $R\subset A$ containing ξ as an interior point.

Let R_1, R_2, R_3, R_4 are rectangles in A.



Start with the rectangle pictured on the left, Sub rectangle's contours are cancelled. Then we can express as

$$\partial R_1+\partial R_2+\partial R_3+\partial R_4=\partial R-\partial R_\xi. \quad (*)$$

It's given that f is holomorphic on $A\setminus\{\xi\}$.
It follows that

$$\int_{\partial R_j} f=0$$

Based on the last equation and $(*)$, we have that

$$\int_{\partial R} f=\int_{\partial R_\xi} f$$

Because $\lim_{z\to\xi}(z-\xi)f(z)=0$, we have that

$$0\le|f|<\frac{\varepsilon}{|z-\xi|} \quad \text{for small number } \varepsilon.$$

on boundary ∂R_ξ. As $\varepsilon\to0$, we can conclude that

$$\int_{\partial R} f(z)dz=0. \qquad \boxtimes$$

Problem 4.

Sol) It's given that f is holomorphic and satisfies $|f(z)-1|<1$ in a domain Ω. In addition, we're given that f' is continuous in Ω.

We need to show that $\int_\gamma f'(z)/f(z) = 0$.

If $f = F'$ holds true, in the given domain Ω, it gives us that F is holomorphic in A.
In addition, it also gives us that $\int_c f\,dz = 0 \ \forall c \subseteq \Omega$
Thus, since f is holomorphic, then f' is also holomorphic.
Therefore, for $f(z) \neq 0$, f'/f is also holomorphic

Let $g(z) = \log z$ and is holomorphic on $|z-1|<1$
Then since f is holomorphic, $g(f(z)) = \log(f(z))$ is also holomorphic on $|f(z)-1|<1$ in a domain Ω.

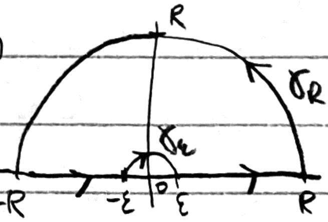
It gives us that $[\log(f(z))]' = f'(z)/f(z)$ and we have shown that f, f', $\log f$, and f'/f are holomorphic.
By Cauchy's integral, we can conclude that

$$\int_\gamma \frac{f'(z)}{f(z)} = 0 \qquad (f(z) \neq 0)$$

for every closed curve γ in Ω. $\boxed{\checkmark}$

Problem 5. Evaluate $\int_0^\infty \frac{1-\cos x}{x^2}dx$

Sol)



By proof of Cauchy's theorem on disks to more general contours (Problem 7), we have that

$$0 = \int_C f(z)dz = \int_{-R}^{-\varepsilon} + \int_{\gamma_\varepsilon} + \int_\varepsilon^R + \int_{\gamma_R}$$

$$= \int_{-R}^{-\varepsilon} + \int_\varepsilon^R + \int_{\gamma_\varepsilon} + \int_{\gamma_R}$$

$$= \int_{\gamma_\varepsilon} + \int_{\gamma_R} + \int_{-\infty}^{\infty} \frac{1-e^{iz}}{z^2}dz \quad \text{as } \varepsilon \to 0 \text{ and } R \to \infty.$$

(*) $\int_{\gamma_R} f dz = 0$ and $\int_{\gamma_\varepsilon} f dz = -\pi$ (details elaborated next page)

Putting all together,

$$0 = -\pi + 0 + \int_{-\infty}^{\infty} \frac{1-\cos z}{z}dz - i\int_{-\infty}^{\infty} \frac{\sin z}{z^2}dz$$

$$\Rightarrow \pi = \int_{-\infty}^{\infty} \frac{1-\cos z}{z^2}dz - i\int_{-\infty}^{\infty} \frac{\sin z}{z^2}dz$$

$$\Rightarrow \pi = \int_{-\infty}^{\infty} \frac{1-\cos z}{z^2}dz \quad (\because \text{Taking only real part})$$

Since $(1-\cos z)/z^2$ is even, we have that

$$2\int_0^\infty \frac{1-\cos z}{z^2}dz = \pi$$

implies that

$$\int_0^\infty \frac{1-\cos x}{x^2}dx = \frac{\pi}{2}.$$

☑

Solve (*)

(Cont 5) We've shown that $\int_C f dz = 0$ by Cauch's theorem.
Then let $R \to \infty$. Then it gives us that

$$|f(z)| = \left| \frac{1-e^{iz}}{z^2} \right| \leq \frac{2}{|z|^2}$$

and this implies that $\displaystyle\int_{\gamma_R} = \underline{\underline{0}}.$

Next, we need to show that $\int_{\gamma_\varepsilon} f dz = -\pi$.
The contour is clockwise and z can be expressed as
$z = \varepsilon e^{i\theta}$ in the given interval. It follows that
$dz = \varepsilon i e^{i\theta}$. Putting altogether,

$$\int_{\gamma_\varepsilon} \frac{1-e^{iz}}{z^2} dz = \int_\pi^0 \frac{1-e^{(i\varepsilon e^{i\theta})}}{\varepsilon^2 \cdot e^{2i\theta}} \cdot \varepsilon i e^{i\theta} d\theta$$

$$= \int_\pi^0 \frac{1-e^{(i\varepsilon e^{i\theta})}}{\varepsilon e^{i\theta}} i \, d\theta$$

$$\overset{\varepsilon \to 0}{=\!=} \int_\pi^0 (-i)(i) \, d\theta = \int_\pi^0 d\theta = \underline{\underline{-\pi}}$$

Problem 6 $\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$

Sol) By Gaussian Integral, we can have that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}$$

Now, let $\alpha = i$, then we have

$$\int_{-\infty}^{\infty} e^{-ix^2}dx = \int_{-\infty}^{\infty}[\cos(x^2) - i\sin(x^2)]dx = \sqrt{\frac{\pi}{i}} \quad \text{——} (*)$$

$\sqrt{1/i} = \sqrt{-i} = \sqrt{e^{i(\frac{-\pi}{2})}} = e^{i(\frac{-\pi}{4})}$. Also, e^{-ix^2} is an even.

Putting altogether,

$$\int_0^\infty e^{-ix^2}dx = \int_0^\infty [\cos(x^2) - i\sin(x^2)]\,dx = \frac{\sqrt{\pi}}{2} e^{i(-\frac{\pi}{4})}$$

$$= \frac{\sqrt{\pi}}{2}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right]$$

$$= \frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4}$$

Thus, we've shown that

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

$\boxed{\checkmark}$

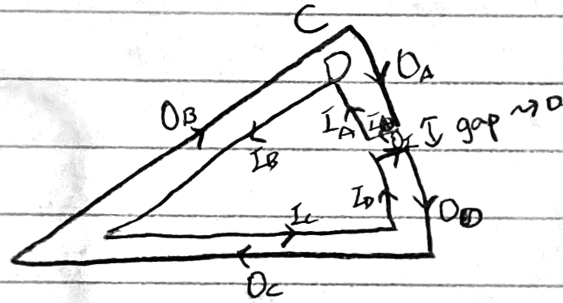
Problem 7.

Sol) Let f be the holomorphic on a neighborhood of D,
then we need to show that $\int_C f(z)dz = 0$.

More specifically, we should show that

$$\int_C f(z) = \int_D f(z) = 0.$$

Let assume contours with a Sector as illustrated below



and D is defined inside of C.

Then since f is holomorphic $\int_C f = 0$.

$$\int_{O_A O_D O_C O_B} f = \int_{I_A I_B I_C I_D} f$$

Thus, ~~if~~ f is holomorphic on neighborhood of D,
then $\int_C f(z)dz = 0$.

☑