Complex Variables I – Problem Set 6

Due at 5 pm on Friday, Oct 20, 2023 via Gradescope

Problem 1

Evaluate the following integrals (all closed curves are oriented counterclockwise):

1. $\int_{|z+i|=1} \frac{e^z}{z^2+1} dz$.
2. $\int_{|z|=1} \frac{1}{(z-a)^2(z-b)} dz$. Here a, b are not on the circle $|z| = 1$. You answer should depend on a and b.

Problem 2

Let γ be a bounded smooth curve containing its endpoints, and $\phi : \gamma \to \mathbb{R}$ be a continuous function. Use the definition of derivative to prove that the function

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{\phi(w)}{w - z} dw$$

is holomorphic on $\mathbb{C} \setminus \gamma$.

Problem 3

Let $A \subset \mathbb{C}$ be open, and $f_n : A \to \mathbb{C}$ a sequence of holomorphic functions. Suppose that $f_n \to f$ uniformly on all compact subsets of A. Prove that $f : A \to \mathbb{C}$ is also holomorphic.

Problem 4

1. Prove Cauchy's estimates: suppose that $A \subset \mathbb{C}$ is open, $f : A \to \mathbb{C}$ is holomorphic, and $|f(z)| \leq M$ on A. Then for any disk $D(w, r)$ with $\overline{D(w, r)} \subset A$ and any integer n, we have
$$|f^{(n)}(w)| \leq \frac{n!M}{r^n}.$$
2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire holomorphic, and there exists constant $M > 0$ and integer n such that $|f(z)| \leq M|z|^n$. Show that f is a polynomial of degree $\leq n$.

Problem 5

Evaluate

$$\int_0^\pi e^{\cos\theta} \cos(\sin\theta) d\theta.$$

Hint: integrate $\int_\gamma \frac{e^z}{z} dz$ along a suitable curve.

Problem 6

Prove that a nonconstant entire holomorphic function $f : \mathbb{C} \to \mathbb{C}$ maps $\mathbb{C}$ onto a dense subset of $\mathbb{C}$.

Remark: a subset A of $\mathbb{C}$ is dense, if its intersection with any disk is nonempty.

Remember to justify your answers and acknowledge collaborations and outside help!