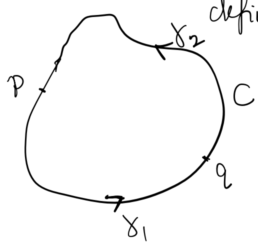
Last time: fundamental Thm of contour integral

Thm: $A \subseteq \mathbb{C}$ open. $I=[a,b]$, $\gamma: I \to \mathbb{C}$ piecewise C^1, $\gamma(I) \subseteq A$. Suppose $F: B \to \mathbb{C}$ s.t. $F'(z)=f(z)$, here B is open and $\gamma(A) \subseteq B$. Then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

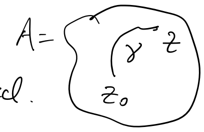
Consequence: If C is closed, and $F'(z)=f(z)$ for holomorphic F, and F is defined in a nbhd of C, then $\int_\gamma f(z) = 0$.



why? Choose $p, q \in C$. Call $\gamma_1 =$ the portion from p to q
$\gamma_2 =$ the portion from q to p.

Then:
$$\int_C f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz$$
$$= \big(F(q)-F(p)\big) + \big(F(p)-F(q)\big) = 0.$$

Cor: If $A \subseteq \mathbb{C}$ is open and connected and $f: A \to \mathbb{C}$ holomorphic, $f'(z)=0$ for every $z \in A$, then f is constant.



Proof: Fix $z_0 \in A$. A is open and connected $\Rightarrow$ A is path connected.
For any other $z \in A$, find a curve γ connecting z_0 to z.
Then $0 = \int_\gamma f'(z)\,dz = f(z) - f(z_0)$ $\square$

Thm: $A \subseteq \mathbb{C}$ is a domain (open and connected), $f: A \to \mathbb{C}$ is continuous. Then the following are equivalent:

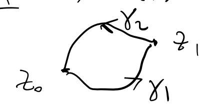
(i) Integrals of f are path independent. i.e. $\forall\, z_0, z_1 \in A$, γ_1, γ_2 two piecewise C^1 curves connecting z_0 to z_1, $\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$.

(ii) $\int_C f\,dz = 0$ for any closed curve C.

(iii) There exists $F: A \to \mathbb{C}$ holomorphic s.t. $F'(z) = f(z)$.

Compare to line integrals in multivariable:
$A \subseteq \mathbb{R}^2$ open-closed. $\int_\gamma P(x,y)\,dx + Q(x,y)\,dy$ is path indep $\Leftrightarrow$ $\int_C P\,dx + Q\,dy = 0$ for any closed C
$\Leftrightarrow$ $\exists\, F: A \to \mathbb{R}$ s.t. $P = \frac{\partial F}{\partial x}$, $Q = \frac{\partial F}{\partial y}$.
(i.e. $(P,Q) = \nabla F$)

Proof: (i) $\Leftrightarrow$ (ii) are equivalent:

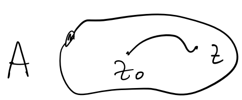


(i) $\Rightarrow$ (ii): we've done it: break C into $C = \gamma_1 + (-\gamma_2)$.
$$\int_{\gamma_1} f\,dz = \int_{-\gamma_2} f\,dz \Rightarrow \int_C f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz = 0.$$

(ii) $\Rightarrow$ (i): For any γ_1, γ_2 connecting z_0, z_1, $\gamma_1 + (-\gamma_2)$ is a closed curve.
$$0 = \int_{\gamma_1 + (-\gamma_2)} f\,dz = \int_{\gamma_1} f\,dz - \int_{\gamma_2} f\,dz.$$

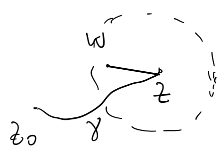
(iii) $\Rightarrow$ (i) : already proven.

(i) $\Rightarrow$ (iii): Fix $z_0 \in A$. Define: $F(z) = \int_\gamma f\,dz$, here γ is a piecewise C^1 curve connecting z_0 to z.



We want : F is holomorphic and $F'(z) = f(z)$.

Fix $z \in A$. For $\varepsilon > 0$, find $\delta > 0$ s.t. $|f(z) - f(w)| < \varepsilon$ in $D(z, \delta)$.



Then for every $w \in D(z, \delta)$, consider the curve

$\gamma_{z,w} : [0,1] \to \mathbb{C}$, $\quad t \mapsto \overset{u(t)}{=} tw + (1-t)z$.

For a curve γ_z from z_0 to z:

Then: $$F(w) - F(z) = \int_{\gamma + \gamma_{z,w}} f(u)\,du - \int_\gamma f(u)\,du$$

$$= \int_{\gamma_{z,w}} f(u)\,du = \int_0^1 f(tw + (1-t)z) \cdot (w - z)\,dt$$

$$\Rightarrow \quad \frac{F(w) - F(z)}{w - z} = \int_0^1 f(tw + (1-t)z)\,dt$$

$$\Rightarrow \quad \left| \frac{F(w) - F(z)}{w - z} - f(z) \right| = \left| \int_0^1 \big(f(tw + (1-t)z) - f(z) \big)\,dt \right|$$

$$\leq \int_0^1 \left| f(tw + (1-t)z) - f(z) \right| dt < \varepsilon$$

$\Rightarrow$ F is holomorphic and $F'(z) = f(z)$. $\square$

Integration w.r.t. arclength.

For a piecewise C^1 curve γ, parametrized by $t \in [a,b] \longmapsto z(t)$, define $\int_\gamma f(z)\,ds := \int_\gamma f(z)\,|dz| = \int_a^b f(z(t))\,|z'(t)|\,dt$.

ds = arclength parameter.

Length of γ : $\quad L(\gamma) = \int_\gamma ds = \int_\gamma |dz|$.

Cauchy's integral Thm.

<u>Thm</u>: $f : A \to \mathbb{C}$ is holomorphic, A is open and connected. Then

$\int_\gamma f(z)\,dz = 0$ for any closed piecewise C^1 curve γ.

<u>Remarks</u>:
- By previous Thm, this $\Rightarrow$ any holomorphic function f has a primitive.
  i.e. $\exists F : A \to \mathbb{C}$ holom, $F'(z) = f(z)$.
- In fact, the Thm holds for curves with less regularity: 'rectifiable' curves.
  (we will never study these curves in this course)

We will prove this step by step.

Thm: R = rectangle given by $a \le x \le b$, $c \le y \le d$, ∂R = boundary curve, counterclock-wise.
Then: $\int_{\partial R} f(z)\,dz = 0$ for all holomorphic f.

Goursat's proof (super elegant): divide R into 4 equal pieces $R_1, \cdots, R_4$, R_j counterclock-wise

$$R = \begin{array}{|c|c|} \hline R_1 & R_2 \\ \hline R_3 & R_4 \\ \hline \end{array}$$

For each rectangle $\tilde{R}$, set
$$\eta(\tilde{R}) = \int_{\partial \tilde{R}} f(z)\,dz$$

Then: $\eta(R) = \eta(R_1) + \eta(R_2) + \eta(R_3) + \eta(R_4)$.

triangle $\Rightarrow$ $|\eta(R)| \le \sum_{j=1}^{4} |\eta(R_j)|$

Thus, $\exists$ some R_j, s.t.
$$|\eta(R)| \le 4\, |\eta(R_j)|.$$

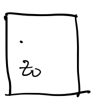
Call this $R_j = R^{(1)}$. Then we repeat this process — divide each $R^{(j)}$ into 4 equal subrectangles, and obtain a sequence
$$R \supseteq R^{(1)} \supseteq \cdots \supseteq R^{(n)} \supseteq \cdots, \quad \text{s.t.}$$
$$|\eta(R^{(n)})| \ge \frac{1}{4} |\eta(R^{(n-1)})| \ge \cdots \ge \frac{1}{4^n} |\eta(R)|.$$

Set $z_0 = \bigcap_{n \ge 1} R^{(n)}$. For $\varepsilon > 0$, find $\delta > 0$ s.t.
$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon, \quad \forall z \in D(z_0, \delta).$$

Fix n large enough s.t. $R^{(n)} \subseteq D(z_0, \delta)$.

Then: $\eta(R^{(n)}) = \int_{\partial R^n} f(z)\,dz$. But the function $f(z_0) + (z - z_0) f'(z_0)$ (degree 1 polynomial in z) has a primitive $\Rightarrow$ integrates to 0 on any closed curve.
$$= \int_{\partial R} f(z) - f(z_0) - (z - z_0) f'(z_0)\,dz$$
$$\Rightarrow \quad |\eta(R^{(n)})| \le \varepsilon \int_{\partial R} |z - z_0|\,|dz|$$

But for $z \in R^{(n)}$, $|z - z_0| \le d_n$, d_n = length of diagonal.

$R^{(n)} =$ 

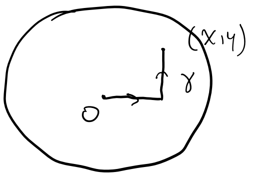
Thus, $|\eta(R^{(n)})| \le \varepsilon \cdot d_n \cdot L_n$, L_n = perimeter of $R^{(n)}$.

On the other hand, $d_n = \frac{d}{2^n}$, $L_n = \frac{L}{2^n}$.
$$\Rightarrow 4^n |\eta(R^{(n)})| \le \varepsilon \cdot dL, \quad \text{Thus } |\eta(R)| \le \varepsilon\, dL.$$
Send $\varepsilon \to 0$, we have that $\eta(R) = 0$. $\square$

Thm (Cauchy-Goursat for a disk). Let D be an open disk, $f: D \to \mathbb{C}$ holomorphic. Then $\int_\gamma f(z)\,dz = 0$ for any closed curve γ in D.

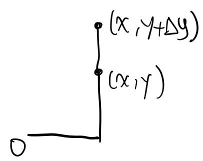
Proof: let's find a primitive.

WLOG suppose D is centered at 0. For every $z = x+iy \in D$, let γ be the arc, horizontal from 0 to $(x,0)$, then vertical from $(x,0)$ to (x,y).



Define $F(z) = \int_\gamma f(z)\, dz$

Let's prove that F is holomorphic and $F' = f$ by checking Cauchy-Riem.

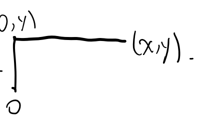


Compute $\frac{\partial F}{\partial y}(z)$.
$$\frac{\partial F}{\partial y} = \lim_{\Delta y \to 0} \frac{F(x,y+\Delta y) - F(x,y)}{\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_{\gamma''} f(z)\, dz$$

Here: γ'' = vertical line from (x,y) to $(x, y+\Delta y)$.

Parametrize γ'' by $z(t) = x + i(y + t\Delta y)$, $0 \le t \le 1$
$$\Rightarrow \quad \frac{1}{\Delta y} \int_{\gamma''} f(z)dz = \frac{1}{\cancel{\Delta y}} \int_0^1 f(x+i(y+\Delta y)) \cdot i\cancel{\Delta y}\, dt \xrightarrow{\text{limit}} i f(z).$$

Next, compute $\frac{\partial F}{\partial x}(z)$. Use the fact that $\int_{\partial R} f dz = 0$,
$$F(z) = \int_{\gamma'} f(z)dz,$$
here γ' = arc, vertical from 0 to $(0,y)$ then horizontal from $(0,y)$ to (x,y).

$\gamma' =$ 

Similarly, $\frac{\partial F}{\partial x}(z) = f(z)$.

Hence, we conclude that $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$. $\left(\text{if } F = u+iv, \text{ this} \iff \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}\right)$

$\Rightarrow$ F is holomorphic, and $\cancel{F'(z)} F' = u_x + iv_x = F_x = f(z)$. □

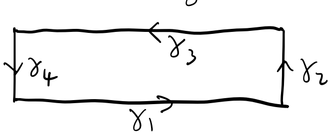
Some initial applications: consider $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-\pi x^2}$. Compute its Fourier transform
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-2\pi i x \xi}\, dx.$$

Note: $e^{-\pi x^2} \cdot e^{-2\pi i x\xi} = e^{-\pi(x+i\xi)^2} \cdot e^{-\pi\xi^2}$.

Recall from real analysis: $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$.

Claim: $\int_{-\infty}^{\infty} \exp(-\pi(x+i\xi)^2)\, dx = 1$, $\xi \in \mathbb{R}$.

Use Cauchy: consider the region $R_\rho := \{z \in \mathbb{C} : |\mathrm{Re}(z)| \le \rho, 0 \le \mathrm{Im}\, z \le \xi\}$.



$f(z) = \exp(-\pi z^2)$.

Cauchy: $\int_{\partial R_\rho} f(z)\, dz = 0$.

Observe: parametrize γ_1 by $z(t) = (t,0)$, $-\rho \le t \le \rho \Rightarrow \int_{\gamma_1} f(z)dz = \int_{-\rho}^{\rho} \exp(-\pi t^2) dt$

Similarly, $\int_{-\gamma_3} f(z)\,dz = \int_{-\rho}^{\rho} \exp(-\pi(t+i\xi)^2)\,dt$

Now parametrize γ_2: $z(t) = \rho + \xi t i, \quad 0 \le t \le 1.$

$$\int_{\gamma_2} f(z)\,dz = \int_0^1 \exp\left(-\pi(\rho+\xi t i)\right)^2 dt$$

$$\Rightarrow \left|\int_{\gamma_2} f(z)\,dz\right| \le \int_0^1 \left|\exp\left(-\pi(\rho+\xi t i)\right)^2\right| dt$$

$$\le \int_0^1 \exp\left(-\pi\rho^2+\xi^2t^2\right)^2 dt \le e^{-\pi\rho^2+\xi^2}$$

$$\Rightarrow \lim_{\rho\to\infty} \int_{\gamma_2} f(z)\,dz = 0. \quad \text{Similarly, } \lim_{\rho\to 0} \int_{\gamma_4} f(z)\,dz = 0$$

Thus, $\lim_{\rho\to\infty} \int_{\gamma_1} f(z)\,dz = \lim_{\rho\to\infty} \int_{-\gamma_3} f(z)\,dz.$

$$\Rightarrow \int_{-\infty}^{\infty} \exp(-\pi(t+i\xi)^2)\,dt = \int_{-\infty}^{\infty} \exp(-\pi t^2)\,dt = 1. \qquad \square$$