Complex Variables I – Problem Set 3

Due at 5 pm on Friday, September 29, 2023 via Gradescope

Problem 1

Consider two points $M_1 = (x_1, y_1, z_1)$, $M_2 = (x_2, y_2, z_2)$ on the Riemann sphere S that are not the north pole, and w_1, w_2 their stereographic projection in the complex plane. Show that $M_1 = -M_2$ (that is, they are diametrically opposite points on S) if and only if $w_1 \bar{w}_2 = -1$.

Proof: We have that $\frac{1}{\bar{w}_2} = \frac{w_2}{|w_2|^2}$. Note that $|w_2|^2 = \frac{x_2^2+y_2^2}{(1-z_2)^2} = \frac{1-z_2^2}{(1-z_2)^2} = \frac{1+z_2}{1-z_2}$. Thus,

$$-\frac{1}{\bar{w}_2} = \frac{-x_2 + iy_2}{1+z_2}.$$

Therefore, $w_1 = -\frac{1}{\bar{w}_2} \Leftrightarrow \frac{x_1 - iy_1}{1-z_1} = \frac{-x_2+iy_2}{1+z_2}$, which is equivalently to

$$\frac{x_1}{1-z_1} = -\frac{x_2}{1+z_2}, \quad \frac{y_1}{1-z_1} = -\frac{y_2}{1-z_2}.$$

Thus, if $M_1 = -M_2$ we have that $w_1\bar{w}_2 = -1$. On the other hand, if $w_1\bar{w}_2 = -1$, since the stereographic projection is a bijection between $S \setminus \{(0,0,1)\}$ and $\mathbb{C}$, there is only one point on $S \setminus \{(0,0,1)\}$ whose stereographic projection satisfies that $w_1\bar{w}_2 = -1$. Therefore, $w_1\bar{w}_2 = -1$ implies that $M_2 = -M_1$.

Problem 2

Which of the following functions are holomorphic in their domain of definition? Justify your answer (you may verify by definition or using the Cauchy-Riemann equation).

1. $f(z) = z^3$
2. $f(z) = |z|^2$
3. $f(z) = \frac{1}{z}$

Solution: 1 and 3 are holomorphic, 2 is not. To see that 2 is not, note that $f(x+iy) = x^2 + y^2$, and hence it does not satisfy the Cauchy-Riemann.

Problem 3

Let $\Omega \subset \mathbb{C}$ be an open set, and $f : \Omega \to \mathbb{C}$ be a function. In polar coordinates $z = r(\cos\theta + i\sin\theta)$, write $f(z) = u(r,\theta) + iv(r,\theta)$ for real-valued functions u, v. Applying appropriate change of variables, deduce the Cauchy-Riemann equation for polar coordinates (r,θ): f is complex differentiable if and only if

$$\begin{cases} r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}. \end{cases} \tag{1}$$

Proof: Since f is holomorphic $\Leftrightarrow$ u, v satisfies the Cauchy-Riemann equations in the Cartesian coordinates (x, y), we show that the Cauchy-Riemann equations in the Cartesian coordinates is equivalent to the form given in the polar coordinates.

We use: $x = r\cos\theta, y = r\sin\theta$. Thus, we have:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta,$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial v}{\partial x} r\sin\theta + \frac{\partial v}{\partial y} r\cos\theta.$$

Therefore, if u, v satisfy the Cauchy-Riemann equation, we have that $u_x = v_y, u_y = -v_x$, and thus $ru_r = v_\theta$. Conversely, if we have $ru_r = v_\theta$ and $u_\theta = -rv_r$, then we can solve from them that $u_x = v_y$ and $u_y = -v_x$.

Problem 4

Let $\Omega \subset \mathbb{C}$ be an open set, and $f : \Omega \to \mathbb{C}$ holomorphic. Consider the set

$$\tilde{\Omega} = \{z \in \mathbb{C} : \bar{z} \in \Omega\}.$$

Show that $\tilde{\Omega}$ is an open set, and that the function

$$\tilde{f} : z \in \tilde{\Omega} \mapsto \overline{f(\bar{z})} \in \mathbb{C}$$

is holomorphic.

Proof: we first prove that $\tilde{\Omega}$ is open. For a point $w \in \tilde{\Omega}$, it is known that $z = \bar{w}$ is in Ω. Since Ω is open, there exists an open neighborhood $D(z, r) \subset \Omega$. Thus, $D(w = \bar{z}, r) \subset \tilde{\Omega}$. Therefore $\tilde{\Omega}$ is open.

Now write f as $f(x + iy) = u(x, y) + iv(x, y)$. Then by definition, $\tilde{f}(x + iy) = u(x, -y) - iv(x, -y)$. We verify:

$$\frac{\partial - v(x, -y)}{\partial y} = \frac{\partial v}{\partial y}(x, -y) = \frac{\partial u}{\partial x}(x, -y),$$

$$\frac{\partial u(x, -y)}{\partial y} = -\frac{\partial u}{\partial y}(x, -y) = \frac{\partial v}{\partial x}(x, -y).$$

Thus $\tilde{f}$ is holomorphic.

Problem 5

1. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function, given as $f(x + iy) = u(x, y) + iv(x, y)$, where u, v are real-valued functions (you may assume that they are smooth). Show that u, v satisfy Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

2. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Show that the function

$$\phi(x, y) = \log|f(x + iy)|$$

satisfies Laplace's equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$. Here log is the real logarithm.

Proof:

1. This is a simple consequence of Cauchy-Riemann.

2. Write $f(x + iy) = u(x, y) + iv(x, y)$. Then by part 1 we know that both u, v are harmonic. Given $\phi(x, y) = \log(u^2 + v^2)^{1/2} = \frac{1}{2}\log(u^2 + v^2)$, we compute:

$$\partial_x \phi = (u^2 + v^2)^{-1}(uu_x + vv_x),$$

$$\partial_{xx}\phi = -2(u^2 + v^2)^{-2}(uu_x + vv_x)^2 + (u^2 + v^2)^{-1}(u_x^2 + v_x^2 + uu_{xx} + vv_{xx}).$$

Similarly (using the symmetry of the expression),

$$\partial_{yy}\phi = -2(u^2+v^2)^{-2}(uu_y+vv_y)^2 + (u^2+v^2)^{-1}(u_y^2+v_y^2+uu_{yy}+vv_{yy}).$$

Using the Cauchy-Riemann equation, we have:

$$u_x^2 = v_y^2, \quad u_y^2 = v_x^2, \quad u_x u_y = -v_x v_y, \quad u_{xx}+u_{yy} = v_{xx}+v_{yy} = 0.$$

Thus we can find that $\phi_{xx} + \phi_{yy} = 0$.

Remember to justify your answers and acknowledge collaborations and outside help!