Last time: Complex differentiability

Recall:

$f(x+iy) = u(x,y) + i v(x,y)$. Say $f: \mathbb{C} \to \mathbb{C}$ <u>entire</u> if it's everywhere holomorphic.

<u>Thm</u>: f is holomorphic with continuous derivative $f'(a)$ at a, if and only if u, v have continuous first order partial derivatives which satisfy the Cauchy-Riem. eq

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Moreover, $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$.

<u>EX</u>: $f(z) = z^2 = x^2 - y^2 + 2i\,xy$. $\quad u = x^2 - y^2$, $v = 2xy$.

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

Harmonic functions.

<u>Def</u>: $A \subseteq \mathbb{R}^2$ open, $f: A \to \mathbb{R}$ twice differentiable. Say f is harmonic, if

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Note: recall that gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, and $\underbrace{\text{divergence}}_{\text{for vector fields } \vec{X} = (g,h)}$ $\operatorname{div} \vec{X} = \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y}$.

Then $\Delta f = \operatorname{div}(\nabla f)$

Now: suppose $f(z) = u(x,y) + i v(x,y)$, $u, v \in C^2(A)$. Then both u, v are harmonic:

$$\frac{\partial^2 u}{\partial x^2} = \partial_x\left(\frac{\partial u}{\partial x}\right) \overset{C\text{-}R}{=} \partial_x\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \partial_y\left(\frac{\partial u}{\partial y}\right) \overset{CR}{=} \partial_y\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x}. \qquad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

<u>Rmk</u>: If $u, v \in C^2(A)$ satisfy the Cauchy-Riem eq, say u, v are conjugate harmonic functions.

Basic properties:

- $A \subseteq \mathbb{C}$ open, f, g holomorphic.
  (1) $f+g$ is holomorphic, and $(f+g)' = f' + g'$.
  (2) fg is holom, $(fg)' = f'g + fg'$.
  (3) $\frac{f}{g}$ is holom. on $\{g \neq 0\}$, and $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
  (4) Chain rule: $h: B \to \mathbb{C}$ holom, $B \supseteq f(A) \Rightarrow h \circ f: A \to \mathbb{C}$ is holom, $(h \circ f)'(z) = h'(f(z)) \cdot f'(z)$.

Derivatives of complex functions.

<u>Prop</u>: $\exp: \mathbb{C} \to \mathbb{C}$ is holomorphic, and $\exp'(z) = \exp z$.

<u>Proof</u>: $\exp(x+iy) = e^x(\cos y + i \sin y)$

$u = e^x \cos y, \quad v = e^x \sin y \quad \Rightarrow \quad \frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial v}{\partial x} = e^x \sin y$

$$\Rightarrow \quad \exp'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x(\cos y + i\sin y) = e^z.$$

Cor: $\sin' z = \cos z, \quad \cos' z = -\sin z.$

Proof: $$\sin' z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)' = \frac{1}{2i}\left(ie^{iz} - (-i)e^{-iz}\right) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$
$$= \cos z$$

$$\cos' z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)' = \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right) = -\frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = -\sin z.$$

Inverse function Thm: $f: A \to \mathbb{C}$ analytic, $f'(z_0) \neq 0$. Then $\exists$ a nbhd U of z_0, V nbhd of $f(z_0)$, s.t. $f: U \to V$ is bijective and $f^{-1}: V \to U$ is analytic, with

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}$$

Warning: f^{-1} exists only in a nbhd. EX: $f(z) = e^z$. $f' \neq 0$ everywhere. But we know $f^{-1}(z) = \log z$ is only defined in a branch.

Proof of existence of f^{-1}: uses the inverse function Thm for real-valued functions in $\mathbb{R}^2$.

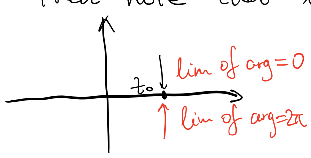
Assume that f^{-1} exists, then use chain rule:

$$(f^{-1})(f(z)) = z \quad \Rightarrow \quad (f^{-1})'(f(z)) \cdot f'(z) = 1$$

□

Derivative of log. Take a branch of log, say $\log: \mathbb{C}\setminus\{0\} \to A_0 = \{z : 0 \le \operatorname{Im} z < 2\pi\}$

Then note that log is not continuous along $\{z \in \mathbb{C}\setminus\{0\} : \arg z = 0\}$:



take any z_0 s.t. $\arg z_0 = 0$.

Approaching z_0 from upper half: $\lim_{\substack{z \downarrow z_0 \\ \text{from above}}} \log z = \lim \big(\log|z| + \underbrace{\arg z}_{\in[0,2\pi)}\big)$
$$= \log|z_0|.$$

approach z_0 from lower half:

$$\lim_{\substack{z \to z_0 \\ \text{from below}}} \log z = \lim\big(\log|z| + \underbrace{\arg z}_{\in[0,2\pi)}\big)$$
$$= \log|z_0| + 2\pi.$$

Thus, log, on branch A_0, is only differentiable on $\mathbb{C}\setminus\{z : \operatorname{Im} z = 0, \operatorname{Re} z \ge 0\}$.

But $$(\log z)' = \frac{1}{\exp'(\log z)} = \frac{1}{\exp(\log z)} = \frac{1}{z}.$$

Remark: On any branch where log is differentiable, we always have

$$(\log z)' = \frac{1}{z}.$$

Chapter 2. Contour integral and Cauchy's Thm.

Construct the integral of a complex function over a curve in $\mathbb{C}$.

Step 1: (Integral over an interval) Consider a complex function $f(t) = u(t) + iv(t)$, $t \in [a,b] \in \mathbb{R}$, u, v real.
If f is continuous, then define
$$\int_a^b f(t)\,dt := \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

Properties:

- domain additive: $\int_a^c f(t)\,dt + \int_c^b f(t)\,dt = \int_a^b f(t)\,dt$
- $\forall \lambda \in \mathbb{C}$, $\int_a^b \lambda f(t)\,dt = \lambda \int_a^b f(t)\,dt$
- $\mathrm{Re}\left(\int_a^b f(t)\,dt\right) = \int_a^b \mathrm{Re}\, f(t)\,dt$, $\quad \mathrm{Im}\left(\int_a^b f(t)\,dt\right) = \int_a^b \mathrm{Im}(f(t))\,dt$.

Triangle inequality:
$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$

Step 2: Let γ be a piecewise differentiable curve in $\mathbb{C}$, parametrized by
$$\gamma: \ z = z(t), \quad a < t < b.$$
For f continuous on γ, $f(z(t))$ is cont. on (a,b). Define:
$$\int_\gamma f(z)\,dz := \int_a^b f(z(t))\,\frac{dz}{dt}\,dt$$

Prop: $\int_\gamma f(z)dz$ is independent of the parametrization of γ.

Proof: Suppose $\gamma: \tau \in (\alpha,\beta) \mapsto z(t(\tau))$ is another parametrization, here $t: \tau \in (\alpha,\beta) \mapsto t(\tau) \in (a,b)$ piecewise differentiable. Then

use change of variable formula $dt = \frac{dt}{d\tau}d\tau$
$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,\frac{dz}{dt}\,dt = \int_\alpha^\beta f(z(t(\tau)))\,\underbrace{\frac{dz}{dt}\,\frac{dt}{d\tau}}_{\frac{d(z(t(\tau)))}{d\tau}}\,d\tau.$$
$\square$

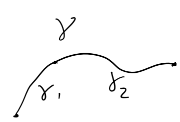
Arclength: $\ell(\gamma) = \int_a^b |z'(t)|\,dt$

Basic properties:

- Let γ: $z = z(t)$, $t \in (a,b)$. Define the opposite curve $-\gamma$, by
$$-\gamma: \quad z = z(-t), \quad t \in (-b, -a)$$
Then:
$$\int_{-\gamma} f(z)\,dz = \int_{-b}^{-a} f(z(-t))\,\frac{d}{dt}(z(-t))\,dt \qquad \text{set } s = -t$$
$$= \int_b^a f(z(s))\cdot -\frac{d}{ds}z(s)\,d(-s) = -\int_a^b f(z(s))\,z'(s)\,ds = -\int_\gamma f(z)dz.$$

- Linearity on f.
$$\int_\gamma (af + bg)\,dz = a\int_\gamma f\,dz + b\int_\gamma g\,dz$$

- Linearity on γ. Suppose γ can be subdivided to γ_1 and γ_2
Then $\int_\gamma f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz$



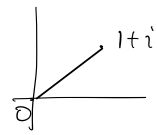
Example: $\cdot$ Simple but fundamental.

$a \in \mathbb{C}$. Consider $\int_\gamma \frac{dz}{z-a}$, $\gamma =$ closed circle w/ radius R centered at a, counterclockwise.

Parametrize γ: $\theta \in [0, 2\pi) \to z(\theta) = a + e^{i\theta}$ $\quad z'(\theta) = ie^{i\theta}$

$$\Rightarrow \int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}}\, d\theta = 2\pi i.$$

$\cdot$ $\int_\gamma z^2\, dz$, $\gamma =$ line segment connecting 0 to $1+i$.



Parametrize γ: $t \mapsto \overset{z(t)}{=} (1+i)t$, $\quad 0 \le t \le 1$.

$z'(t) = 1+i$.

$$\int_0^1 z^2\, dz = \int_0^1 (1+i)^2 t^2 (1+i)\, dt = -2(1+i)\int_0^1 t^2\, dt = -\frac{2}{3}(1+i).$$

Fundamental Thm of calculus for integrals in $\mathbb{C}$.

Thm: Let $A \subseteq \mathbb{C}$ be open, $I = [a,b]$, $\gamma: I \to \mathbb{C}$ piecewise C^1, $\gamma(I) \subseteq A$. Suppose $f: A \to \mathbb{C}$ is continuous, and $F: B \underset{\text{holom.}}{\to} \mathbb{C}$ s.t. $F'(z) = f(z)$ in an open nbhd B containing $\gamma(I)$.

Then:
$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

In particular, if γ is $\underset{(\gamma(a)=\gamma(b))}{\text{closed}}$, then $\int_\gamma f(z)\,dz = 0$.

Proof: Suppose $a = a_0 < \cdots < a_n = b$ s.t. $\gamma|_{[a_i, b_i]}$ is C^1. Then:

$$\int_\gamma f(z)\,dz = \sum_{j=1}^n \int_{\gamma_j} f(z)\,dz = \sum_{j=1}^n \int_{a_{j-1}}^{a_j} f(\gamma(t))\gamma'(t)\,dt$$

$$= \sum_{j=1}^n \int_{a_{j-1}}^{a_j} F'(\gamma(t))\,\gamma'(t)\,dt$$

$$= \sum_{j=1}^n \int_{a_{j-1}}^{a_j} \frac{d}{dt} F(\gamma(t))\,dt$$

$$= \sum_{j=1}^n \Big(F(\gamma(a_j)) - F(\gamma(a_{j-1}))\Big) = F(\gamma(b)) - F(\gamma(a)).$$

Relations to line integral in $\mathbb{R}^2$.

Recall: Suppose $u(x,y)$, $v(x,y): A \to \mathbb{R}$ cont. and $\gamma: [a,b] \to \mathbb{R}^2$ C^1 curve, parametrized by $x(t)$, $y(t)$. Then

$$\int_\gamma u\,dx + v\,dy = \int_a^b \Big(u(x(t), y(t))\, x'(t) + v(x(t), y(t))\, y'(t)\Big)\,dt$$

Thus, for $f(z) = u(x,y) + iv(x,y)$, $\quad z = x + iy$,

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx + i\,dy) = \int_\gamma (u\,dx - v\,dy) + i(u\,dy + v\,dx)$$

(This can be viewed as another definition of $\int_\gamma f(z)\,dz$.)