Continuity. $(\mathbb{C}, |\cdot|)$ is a metric space $\Rightarrow$ define limits and continuity.

<u>Def</u>: A sequence (z_n) of complex numbers is bounded, if $\exists\, C \geq 0$ s.t.
$$|z_n| \leq C, \quad \forall n \in \mathbb{Z}_+$$

It's called convergent, if $\exists\, z \in \mathbb{C}$ s.t. for every $\varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{Z}_+$ s.t.
$$|z_n - z| < \varepsilon \text{ for all } n \geq N.$$

In this case, say z the limit of (z_n), denoted $z = \lim_{n\to\infty} z_n$.

Basic properties of limits.

<u>Prop</u>: (i) The limit of a convergent complex seq. (z_n) is unique.
(ii) Convergent sequences are bounded
(iii) If $\lim_{n\to\infty} z_n = z$, $\lim_{n\to\infty} w_n = w$, then
$$\lim (z_n + w_n) = z + w, \qquad \lim (z_n \cdot w_n) = zw$$
If $w \neq 0$ then $\lim \frac{z_n}{w_n} = \frac{z}{w}$.
(iv) If $\lim z_n = z$, then $\lim \overline{z_n} = \overline{z}$, $\lim |z_n| = |z|$.
$$\lim \mathrm{Re}\, z_n = \mathrm{Re}\, z, \quad \lim \mathrm{Im}\, z_n = \mathrm{Im}\, z$$
(v) If a seq. (z_n) satisfies $\lim \mathrm{Re}(z_n) = \mathrm{Re}(z)$ and $\lim \mathrm{Im}(z_n) = \mathrm{Im}\, z$ then $\lim z_n = z$.

<u>Def</u>: A sequence $(z_n)_{n\in\mathbb{Z}_+}$ is called a Cauchy sequence, if for every $\varepsilon > 0$, $\exists\, N = N(\varepsilon)$ s.t.
$$|z_n - z_m| < \varepsilon \text{ for all } n, m \geq N.$$

<u>Prop</u>: (z_n) is Cauchy $\Leftrightarrow$ it converges. (i.e. $\mathbb{C}$ is complete)

<u>Proof</u>: (z_n) is Cauchy $\Leftrightarrow$ both $(\mathrm{Re}\, z_n)$, $(\mathrm{Im}\, z_n)$ are Cauchy seq. in $\mathbb{R}$
$$\Leftrightarrow (\mathrm{Re}\, z_n), (\mathrm{Im}\, z_n) \text{ converge.}$$
$\square$

<u>Def</u>: An infinite series $\sum_{n=1}^{\infty} z_n$, $z_n \in \mathbb{C}$, is defined to be the seq. $(S_n)_{n\in\mathbb{Z}_+}$ of partial sums, here
$$S_n = \sum_{j=1}^{n} z_j.$$
Say the series $\sum_{n=1}^{\infty} z_n$ converges if and only if (S_n) converges, and write
$$\sum_{n=1}^{\infty} z_n = \lim S_n$$

<u>Def</u>: Say $\sum_{n=1}^{\infty} z_n$ converges absolutely, if $\sum_{n=1}^{\infty} |z_n|$ converges.

Typical examples: power series $\sum_{n=1}^{\infty} a_n z^n$

Topology of $\mathbb{C}$, continuous functions.

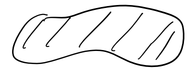
Def: (i) for $r>0$, $z_0\in\mathbb{C}$, call $D(z_0,r)=\{z\in\mathbb{C}: |z-z_0|<r\}$ the open r disk around z_0. Call $\dot{D}(z_0,r)=D(z_0,r)\setminus\{z_0\}$ the deleted r disk.

(ii) Let $A\subseteq\mathbb{C}$. $z_0\in A$ is an interior pt of A, if $\exists\, r>0$ s.t. $D(z_0,r)\subseteq A$.

(iii) $A\subseteq\mathbb{C}$ is open, if every pt in A is an interior pt.

(iv) $A\subseteq\mathbb{C}$ is closed, if $\mathbb{C}\setminus A$ is open.

$A=$ 

open, z_0 is an interior pt.  closed.

Prop: (i) ϕ and $\mathbb{C}$ are both open and closed.

(ii) Union of arbitrarily many open set is open.
(Intersection) (closed) (closed)

(iii) Intersection of finitely many open set is open.
(Union) (closed) (closed)

Def: Let $A\subseteq\mathbb{C}$. Define the interior of A as $\mathring{A}=\bigcup_{U\subseteq A,\, U \text{ open}} U$.

the closure of A as $\overline{A}=\bigcap_{C\supseteq A,\, C \text{ closed}} C$.

the boundary of A as $\partial A=\overline{A}\setminus\mathring{A}$.

Def: $A\subseteq\mathbb{C}$ is compact, if every open cover of A admits a finite subcover, i.e. if $A\subseteq\bigcup_{\alpha\in I} U_\alpha$, each U_α open, then $\exists$ finitely many $\alpha_j\in I$ s.t. $A\subseteq\bigcup_{j=1}^{n} U_{\alpha_j}$.

Prop: $A\subseteq\mathbb{C}$. The following are equivalent:

(i) A is compact

(ii) A is bounded and closed.

(iii) Every seq. $(z_n)\subset A$ has a convergent subseq. converging to $z\in A$.

Def: $A\subseteq\mathbb{C}$, $f: A\to\mathbb{C}$ a function. Say f is continuous at $z_0\in A$, if $\forall\varepsilon>0$, $\exists\,\delta>0$ s.t.

$$|f(z)-f(z_0)|<\varepsilon,\quad \forall\, z\in D(z_0,\delta)\cap A.$$

Basic properties: • $f,g: A\to\mathbb{C}$ continuous at $z_0\in A$. Then $f\pm g$, fg are cont. at z_0. $\frac{f}{g}$ is cont. at z_0, if $g(z_0)\neq 0$.

• If $h: B\to\mathbb{C}$ is cont. is s.t. $B\supseteq f(A)$, then $h\circ f: A\to\mathbb{C}$ is cont. at z_0.

Def: $A\subseteq\mathbb{C}$.

(i) A is called connected, if it cannot be written as a disjoint union of two non-empty, relatively open subsets A_1, A_2.

(ii) A is called path-connected, if every $z,w\in A$ has a path $\gamma:[0,1]\to A$ connecting z,w, i.e. a cont. map $\gamma:[0,1]\to A$, $\gamma(0)=z$, $\gamma(1)=w$.

(iii) A is called a domain, if it's open and connected.

Prop: • A is path-connected $\Rightarrow$ A is connected.

• If A is open, then A is connected $\Rightarrow$ A is path-connected.

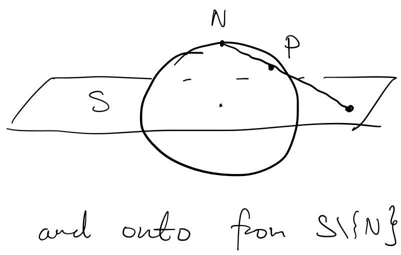
Extended complex plane $\overline{\mathbb{C}}$ = Riemann sphere.

Write $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, w/ operations:

$$\underset{\forall z \in \overline{\mathbb{C}}}{z + \infty = \infty,\ z \cdot \infty = \infty}, \quad \infty + \infty = \infty. \qquad z \in \mathbb{C} \Rightarrow \frac{z}{\infty} = 0. \quad \frac{z}{0} = \infty \ (\text{if } z \neq 0)$$

Open sets in $\overline{\mathbb{C}}$: $A \subseteq \overline{\mathbb{C}}$ s.t.: $A \cap \mathbb{C}$ is open, and if $\infty \in A$, $\exists k > 0$ s.t. $\{z \in \mathbb{C} : |z| > k\} \cap \{\infty\} \subseteq A$.

Geometric representation of $\overline{\mathbb{C}}$: Riemann sphere S



$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

$\forall P = (x,y,z) \in S$, $P \neq (0,0,1)$, assign $\overset{\varphi(P)}{\overset{\shortparallel}{w}} \in \mathbb{C}$ by

$$w = \frac{x + iy}{1 - z}.$$ This is one-to-one and onto from $S \setminus \{N\}$ to $\mathbb{C}$.

Then Define $\varphi((0,0,1)) = \infty$. Then one can check:

$\varphi : S \to \overline{\mathbb{C}}$ is a homeomorphism.

Holomorphic functions.

<u>Def</u>: A function $f : \overset{\subset \text{ open in } \mathbb{C}}{A} \to \mathbb{C}$ is differentiable at $z_0 \in A$, if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists.}$$

Say f is holom. (or analytic) at $z_0 \in A$, if $\exists\, r > 0$ s.t. f is differentiable in everyplace in $D(z_0, r)$.

<u>EX</u>: · $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^n$ is holomorphic.

Check:
$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{(z_0 + h)^n - z_0^n}{h}$$
$$= \lim_{h \to 0} \frac{1}{h} \left(n z_0^{n-1} h + \sum_{k=2}^{n} \binom{n}{k} z_0^{n-k} h^k \right) = n z_0^{n-1}.$$

$\Rightarrow f'(z) = n z^{n-1}$.

· $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \operatorname{Re} z$ is not differentiable anywhere.

$$\frac{\operatorname{Re}(z_0 + h) - \operatorname{Re}(z_0)}{h} = \frac{1}{h} \operatorname{Re}(h). \quad \text{But } \frac{1}{h} \operatorname{Re}(h) = \begin{cases} 1 & \text{if } h \in \mathbb{R} \\ 0 & \text{if } h \in i\mathbb{R}. \end{cases}$$

Basic properties: f, g holomorphic in A (open $\subset \mathbb{C}$).

- $f \pm g$, fg are holom, $\frac{f}{g}$ is hol. at $\{g \neq 0\}$.
- f is continuous on A.

<u>Def</u>: $f : \mathbb{C} \to \mathbb{C}$. If f is holomorphic on $\mathbb{C}$, say f is an entire function.

Cauchy-Riemann equations.

$f : A \to \mathbb{C}$, $z_0 \in A$. Write $f(x + iy) = u(x,y) + i v(x,y)$

Suppose f is holom. at z_0. Then $\lim_{h\to 0} \frac{1}{h}\left(f(z_0+h) - f(z_0)\right)$ exists.

Choose two diff. paths for h.

- $$\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{1}{h}\left(f(z_0+h)-f(z_0)\right) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{1}{h}\left(u(x_0+h,y_0)+iv(x_0+h,y_0)-u(x_0,y_0)-iv(x_0,y_0)\right)$$
$$= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0).$$
- $$\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{1}{ih}\left(f(z_0+ih)-f(z_0)\right) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{1}{ih}\left(u(x_0,y_0+h)+iv(x_0,y_0+h)-u(x_0,y_0)-iv(x_0,y_0)\right)$$
$$= \frac{1}{i}\left(\frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0)\right)$$
$$= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0)$$

The limits are equal $\Rightarrow$
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \qquad (*)$$

$(*)$: the Cauchy-Riemann equations.

Thm: $f: A\to\mathbb{C}$, $A\subseteq\mathbb{C}$ open. Suppose f satisfies $(*)$ in A. Then f is holomorphic.

Proof: u,v differentiable

$$\Rightarrow u(x_0+h,y_0+k) = u(x_0,y_0) + \frac{\partial u}{\partial x}\cdot h + \frac{\partial u}{\partial y}k + o\left((h^2+k^2)^{\frac12}\right).$$
$$v(x_0+h,y_0+k) = v(x_0,y_0) + \frac{\partial v}{\partial x}\cdot h + \frac{\partial v}{\partial y}\cdot k + o\left((h^2+k^2)^{\frac12}\right)$$
$$\overset{(C\text{-}R)}{=} v(x_0,y_0) - \frac{\partial u}{\partial y}\cdot h + \frac{\partial u}{\partial x}\cdot k + o\left((h^2+k^2)^{\frac12}\right).$$
$$\Rightarrow (u+iv)(x_0+h,y_0+k) - (u+iv)(x,y) = \frac{\partial u}{\partial x}(h+ik) + \frac{\partial u}{\partial y}(k-ih)$$
$$= \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h+ik)$$
$$\Rightarrow \lim_{(h,k)\to(0,0)} \frac{f(z_0+(h+ik))-f(z_0)}{h+ik} = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(x_0,y_0)$$
$\square$

Example: $f(z) = z^2 = \underbrace{x^2-y^2}_{u} + \underbrace{2ixy}_{v}$.

then: $\dfrac{\partial u}{\partial x} = 2x = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -2y = -\dfrac{\partial v}{\partial x}$.

Def: $u: A\to\mathbb{R}$, $A\subseteq\mathbb{R}^2$ open. Say u is harmonic, if $\Delta u := \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

Rmk: $f: A\to\mathbb{C}$ is holomorphic. Then u,v are harmonic.

Check: $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial y}\right)$. $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = -\dfrac{\partial}{\partial y}\left(\dfrac{\partial v}{\partial x}\right)$.