Notations

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_+$

(a, b) open intervals $= \{x \in \mathbb{R},\ a < x < b\}$

(or textbook $]a, b[$)

$[a, b]$ closed intervals

$\exists$ exist (e.g, $\exists\, x \in \mathbb{R}$)

$\forall$ for all (e.g, $\forall\, x \in (0, 1)$)

→ Complex numbers

Historically, solve $x^2 + 1 = 0$, denote by $i = \sqrt{-1}$

Generally, a complex number $z = x + iy$

$x, y \in \mathbb{R}$

For $z = x + iy$

$x =$ real # part of z

$= \operatorname{Re} z$

$y =$ imaginary part of z

$= \operatorname{Im} z$

Why do we study $\mathbb{C}$?

Thm (Fundamental thm of algebra): Any polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

i.e. $a_0, \dots, a_n \in \mathbb{C}$, the equation

$$a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 = 0$$

has a solution in $\mathbb{C}$.

Computing Real Integrals

Ex $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx \quad \text{where } \alpha \in (0,1) \quad = \frac{\pi}{\sin(\alpha\pi)}$$

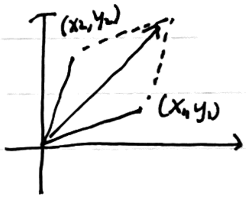
Contour integrals in $\mathbb{C}$.

Closely related.
- Analytic number Theory.
- Differential geometry
- Fluid Dynamics

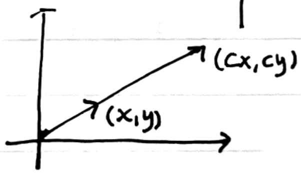
Def $\mathbb{C}$ is the set $\mathbb{R}^2$ with the following operations

For $z_j = x_j + iy_j$, $j = 1, 2$

we identify it with $(x_j, y_j) \in \mathbb{R}^2$



① Addition $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$



② Scalar Multiply $c \in \mathbb{R}$

$$cz_1 = cx_1 + i(cy_1)$$

③ Complex Multiplication

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + i(x_1y_2 + y_1x_2) - y_1y_2 \qquad (i^2 = -1) \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \end{aligned}$$

Ex) $(\pi + \sqrt{2}i)(1 - 2i) = (\pi + 2\sqrt{2}) + i(-2\pi + \sqrt{2})$

<u>Note</u> $z_1 = z_2 \iff$ (equivalent to saying)

$$\mathrm{Re}\, z_1 = \mathrm{Re}\, z_2$$
$$\mathrm{Im}\, z_1 = \mathrm{Im}\, z_2$$

<u>Field Structure of $\mathbb{C}$</u>

- $\mathbb{C}$ is closed under addition and multiplication $z_1, z_2 \in \mathbb{C}$
  $\Rightarrow z_1 + z_2 \in \mathbb{C}$
  and $z_1 z_2 \in \mathbb{C}$.

- Addition and multiplication are associative and commutative, and multiplication is distributive over addition
  $\Rightarrow z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1$
  $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad z_1 (z_2 z_3) = (z_1 z_2) z_3$
  $\Rightarrow$ [distributive] $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

- 0 is an additive identity. $z_1 + 0 = z_1$
  $(= 0 + i0)$

- $\forall z \in \mathbb{C}$, we have additive inverse.
  For $z = x + iy$, $-z = -x + i(-y)$

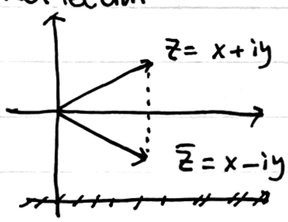
- Multiplicative identity : $1 = 1 + i0$, $\forall z \in \mathbb{C}$, $z \cdot 1 = z$.
  $\forall z \in \mathbb{C} \setminus \{0\}$ multiplicative inverse $1/z$ exists
  $z = x + iy$, $\dfrac{1}{z} = \dfrac{1}{x+iy} = \dfrac{x-iy}{(x+iy)(x-iy)} = \dfrac{x-iy}{x^2+y^2}$

$$= \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2}$$

Ex $\dfrac{1}{1-2i} = \dfrac{1+2i}{(1-2i)(1+2i)} = \dfrac{1+2i}{5} = \frac{1}{5} + i\frac{2}{5}$

More
Operation on $\mathbb{C}$

- Conjugation : $x+iy \longrightarrow x-iy$
  $(=z)$ $(=\bar{z}$: z bar)
  - $\rightarrow \bar{z} = \mathrm{Re}\, z - i\, \mathrm{Im}\, z$
  - $\rightarrow \bar{\bar{z}} = z$ (z bar bar $= z$)
  - $\rightarrow \mathrm{Re}\, z = \frac{1}{2}(z+\bar{z})$
    $\mathrm{Im}\, z = \frac{1}{2i}(z-\bar{z}) = \frac{-i}{2}(z-\bar{z})$
- Reflection



- $\bar{z}_1 + \bar{z}_2 = \overline{z_1+z_2}$
  $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

Observation Consider a polynomial eq

(1) $a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 = 0$ where $a_0, \dots, a_n \in \mathbb{R}$

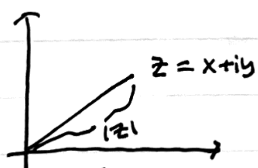
take Conjugation

$$a_n \bar{z}^n + a_{n-1} \bar{z}^{n-1} + \dots + a_0 = 0$$

$\Rightarrow$ if ~~$z \in w$~~ $z = w \in \mathbb{C}$ solves (1), then $z = \bar{w}$ also solves (1).

$\Rightarrow$ roots of a real coefficients polynomial comes in pairs of conjugation

Modulus $z = x+iy$ $|z|$ = modulus of z
$= \sqrt{x^2+y^2}$



$|z|^2 = x^2+y^2 = z\bar{z} \Longrightarrow \frac{1}{z} = \frac{\bar{z}}{|z|^2}$

(Check $z\bar{z} = (x+iy)(x-iy) = x^2+y^2$)

(Cont.) $|z_1 z_2| = |z_1| \cdot |z_2|$

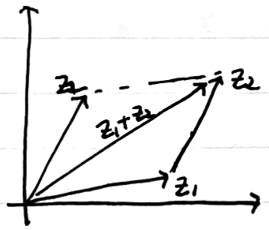
Pf1 $|z_1 z_2|^2 = z_1 z_2 \cdot \overline{z_1 z_2} = z_1 z_2 \cdot \overline{z_1} \cdot \overline{z_2}$

$= z_1 \overline{z_1} \cdot z_2 \cdot \overline{z_2}$

$= |z_1|^2 \cdot |z_2|^2$

$\Rightarrow |z_1 z_2| = |z_1| \cdot |z_2|$

- $|z_1 + z_2| \leq |z_1| + |z_2|$ triangle inequality



Pf 2 $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$

$= z_1 \cdot \overline{z_1} + z_2 \cdot \overline{z_2} + z_1 \overline{z_2} + z_2 \overline{z_1}$

$\cdot \left(\overline{z_1 \overline{z_2}} = \overline{z_1} \cdot z_2 \right)$

$= z_1 \overline{z_1} + z_2 \overline{z_2} + 2\,\mathrm{Re}(z_1 \overline{z_2})$

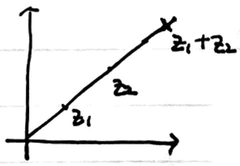
$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1 \overline{z_2})$

$\leq 2 |z_1 \overline{z_2}|$

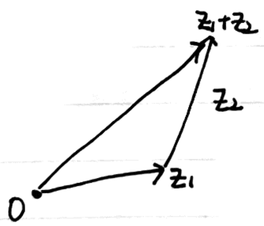
$\leq |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |z_2|$

$= (|z_1| + |z_2|)^2$

Triangle Equality $\Leftrightarrow$



$z_2 = c z_1$ where $c \in \mathbb{R}_{\geq 0}$



$|z_1 + z_2| \geq |z_1| - |z_2|$

$|z_1 + z_2| \geq |z_2| - |z_1|$

are also called Triangle Inequality.

<u>Square Root</u> Given $z = a+bi$ find $x+iy$ such that

$$(x+iy)^2 = a+ib \iff (x^2-y^2) + i(2xy) = a+bi$$

$$\iff \begin{cases} x^2-y^2 = a & ① \\ 2xy = b & ② \end{cases}$$

taking $(\text{modulus})^2$ $\quad (x^2+y^2)^2 = a^2+b^2$

$(\iff ①^2 + ②^2)$

$$x^2+y^2 = \sqrt{a^2+b^2}$$

$$① + ③ \Rightarrow \quad x^2 = \tfrac{1}{2}\left(a+\sqrt{a^2+b^2}\right)$$

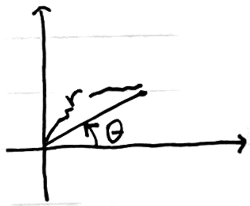
$$y^2 = \tfrac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)$$

② tells us how to choose sign

if $b \geq 0$, $(x,y) = \left(\pm\sqrt{\tfrac{1}{2}(a+\sqrt{a^2+b^2})},\ \pm\sqrt{\tfrac{1}{2}(-a+\sqrt{a^2+b^2})}\right)$

if $b < 0$, $(x,y) = \left(\pm\sqrt{\tfrac{1}{2}(a+\sqrt{a^2+b^2})},\ \mp\sqrt{\tfrac{1}{2}(-a+\sqrt{a^2+b^2})}\right)$

<u>Polar Coordinates</u>



r = length of vector $\quad r \geq 0$

θ = angle from positive x-axis to the vector

$\theta \in [0, 2\pi]$

$x = r\cos\theta$, $y = r\sin\theta$. given $z = x+iy$

write $z = r\cos\theta + ir\sin\theta$

$\quad = r(\cos\theta + i\sin\theta)$

Here, $r = |z|$, θ = argument of z = $\arg z$

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1z_2 = r_1r_2\left((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right)$$

$$= r_1r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

<u>Prop</u> $|z_1z_2| = |z_1|\cdot|z_2|$, $\arg(z_1z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}$

ex $\quad z = r(\cos\theta + i\sin\theta)$

want $\quad w^2 = z$

$$w = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

OR

$$w = \sqrt{r}\left(\cos\left(\frac{\theta}{2}+\pi\right) + i\sin\left(\frac{\theta}{2}+\pi\right)\right)$$

In Particular, $\quad z = r(\cos\theta + i\sin\theta)$

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Proposition (de Moivre) $\quad (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

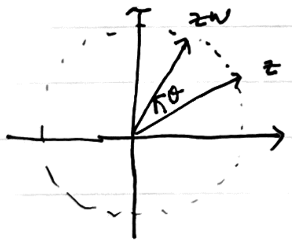
Ex) $\quad n=3 \quad \cos(3\theta) = \cos^3\theta - 3\cos\theta\cdot\sin^2\theta.$

Remark If $|w|=1$, then $w = \cos\theta + i\sin\theta$ for some θ.

The mapping $z \longmapsto zw$ is linear and

$$|zw| = |z|$$

$$\arg(zw) = \arg z + \theta \quad (\text{mod } 2\pi)$$



$\Rightarrow \quad z \mapsto zw$ is a counterclockwise rotation by $\theta = \arg w$.

nth root $\quad z = r(\cos\theta + i\sin\theta)$

n-th root $= \{ w \in \mathbb{C} \mid w^n = z \}$ $\quad (k=0,\ldots,n-1)$

$$w = r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right) = r^{\frac{1}{n}}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$$

Note $w_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$ satisfy $w_k^n = 1$

Call w_k nth roots of unity.