Complex Variables I – Problem Set 1

Due at 5 pm on Friday, September 15, 2023 via Gradescope

Problem 1

Solve the following equations

$$\text{a) } z^2 + iz + 6 = 0 \qquad \text{b) } z^6 - 64 = 0 \qquad \text{c) } z^3 + 1 = 0$$

Make sure to find all solutions!

Problem 2

Let z, w be complex numbers. Prove the parallelogram identity:

$$|z - w|^2 + |z + w|^2 = 2(|z|^2 + |w|^2).$$

Problem 3

Prove that

$$\left| \frac{z - w}{1 - \bar{z}w} \right| = 1$$

if either $|z| = 1$ or $|w| = 1$, and $\bar{z}w \neq 1$.

Problem 4

Let $c_0, c_1, \ldots, c_n$ be n complex numbers, and consider the following polynomial in z:

$$P(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0$$

Show that there exists a number $R > 0$ such that

$$\text{for all } z \in \mathbb{C} \text{ with } |z| > R, \text{ we have } \left| \frac{1}{P(z)} \right| < \frac{2}{|c_n| R^n}$$

Problem 5

Sketch the following regions of the complex plane

a) $\{z \in \mathbb{C} : |Re(z)| \geq 1\}$

b) $\{z \in \mathbb{C} : |z - i| < 1\}$

c) $\{z \in \mathbb{C} : 0 < \arg(z) < \frac{\pi}{4}\}$

d) $\{z \in \mathbb{C} : |Im(z)| \geq 1\} \cap \{z \in \mathbb{C} : |z| \leq \sqrt{2}\}$

Note: In the above, arg stands for the argument of z.

Remember to justify your answers and acknowledge collaborations and outside help!