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Lecture 7

약간안개비.

Last time: Cauchy's integral formula. ① $A \subseteq \mathbb{C}$ simply connected domain. $f: A \to \mathbb{C}$ holomorphic. $\int_C f(z)dz = 0$ $\forall$ closed $C \subseteq A$

② D disk. f holom in a neighbor of $\bar{D}$.

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z-w}\, dz = f(w)$$

Consequences. f holomorphic $\Rightarrow$ f is complex differentiable of any orders. $f^{(n)}(w) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{(z-w)^{n+1}}\, dz$

③ Liouville

④ Morera's THM: $f: A \to \mathbb{C}$ cont. $\int_C f(z)dz = 0$ for any closed curve $\Rightarrow$ f is holomorphic

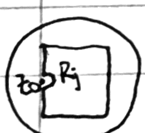
Cor $f: A \to \mathbb{C}$ is continuous. holomorphic in $A \setminus \{z_0\}$

$\Rightarrow$ f is holomorphic in A

outline. Verify $\int_{\partial R} f(z)dz = 0$. $\forall$ rectangles $R \subseteq A$

A= 

- Case 1: $z_0 \notin \bar{R}$ ($\Rightarrow$ f has a complex primitive)
- Case 2: $z_0 \in R$



HW if $\lim_{z \to z_0} (z-z_0) f(z) = 0 \Rightarrow \int_{\partial R} f(z)dz = 0$

- Case 3: $z_0 \in \partial R$ ⇝ use C^1 contours to approx ∂R

$\int_{\partial R_j} f(z)dz \to \int_{\partial R} f(z)dz$.

One more application: define log on more general domains

Recall: log is holomorphic on $\mathbb{C} \setminus L$. L = ray starting at 0

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Thm $A \subseteq \mathbb{C}$ open connected. Simply connected
$f: A \to \mathbb{C} \setminus \{0\}$ is holomorphic. Then $\exists g. A \to \mathbb{C}$
s.t $e^{g(z)} = f(z)$

Ex If $f(z) = z$ then $\log z$ exists and is holom.
If $A \subseteq \mathbb{C} \setminus \{0\}$ and is simply connected.
e.g. $A =$

Origin is not inside of contour.

Pf Want $g'(z) e^{g(z)} = f'(z)$
use $e^{g(z)} = f(z)$ $\quad g'(z) = f'(z)/f(z)$.
$\Leftrightarrow$ g is a complex primitive of f'/f
f'/f is holomorphic and A is simply connected
$\Rightarrow \exists \tilde{g}$ a complex primitive. Take $z_0 \in A$, set $g(z) = \tilde{g}(z) + C$
Such that $e^{g(z_0)} = f(z_0) \Leftrightarrow$ Choose C
Such that $e^{C} \cdot e^{\tilde{g}(z_0)} = f(z_0)$

observe: $\left(\dfrac{e^{g(z)}}{f(z)}\right)' = 0 \Rightarrow e^{g(z)} = f(z)$

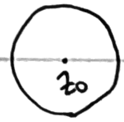
$$= e^{g(z)} \frac{g'(z) \cdot f(z) - f'(z)}{[f(z)]^2} = e^{g(z)}\left[\frac{g'(z)}{f(z)} - \frac{f'(z)}{f(z) \cdot f(z)}\right] = e^{g}\left[\frac{g'}{f} - \frac{g'}{f}\right]$$

$\boxed{\text{Maximum Modulus Principle}}$

- Mean Value Property

Prop $A \subseteq \mathbb{C}$ open. $f: A \to \mathbb{C}$ holomorphic $\overline{D(z_0, r)} \subseteq A$

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$



Pf Cauchy $\Rightarrow f(z_0) = \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(z)}{z-z_0}\,dz$

Parameterize : $z(\theta) = z_0 + re^{i\theta} \quad 0 \le \theta \le 2\pi$

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{\cancel{r}\cancel{e^{i\theta}}} \cdot \cancel{r} \cdot i \cdot \cancel{e^{i\theta}}\,d\theta$$ $\boxed{\checkmark}$

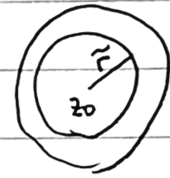
Thm (local max modulus principle) $A \subseteq \mathbb{C}$ open $f: A \to \mathbb{C}$ holom

Suppose $|f(z)|$ has a local max (a max in $D(z_0,r)$) at $z_0 \in A$.

and $\overline{D(z_0,r)} \subseteq A \Rightarrow f$ is constant in $D(z_0,r)$

Pf $\forall \tilde{r} \le r$ $|f(z_0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0+\tilde{r}e^{i\theta})\,d\theta\right|$

$$\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\tilde{r}e^{i\theta})|\,d\theta$$



$$\le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta = |f(z_0)|$$

Equality Holds $\Rightarrow |f(z_0)| = |f(z_0+\tilde{r}e^{i\theta})|$ for every $0 \le \theta \le 2\pi$ every $0 \le \tilde{r} \le r$

$\Rightarrow |f(z)| =$ Constant in $D(z_0,r)$

HW $= f =$ Constant in $D(z_0,r)$ $\square$

Rmk max $\to$ min. Standard fails. $f(z) = z$ in $D(0,1)$

$|f(0)| = 0$ is the ~~main~~ min of $|f(z)|$

But if we additionally assume that $f(z) \ne 0$ in A,

then $|f(z)|$ attains a local min $\Rightarrow f$ is constant

(apply thm to $g(z) = 1/f(z)$)

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<u>Thm</u> (Global max modulus principle) : A is bounded open connected
$f: A \to \mathbb{C}$ holomorphic. and $f: \bar{A} \to \mathbb{C}$ is continuous
if $\max_{z \in \bar{A}} |f(z)|$ is attained at $z_0 \in A$ then f is constant.

<u>Pf</u> Set $U = \{ z \in A : |f(z)| = |f(z_0)| \}$

$$\left.\begin{array}{l} - U \neq \phi \text{ (empty)} : z_0 \in U \\ - U \text{ is open} \\ - U \text{ is closed} \end{array}\right\} \Rightarrow U = A$$

U is open. $z_1 \in U \Rightarrow |f(z_1)| = |f(z_0)| = \max_{z \in \bar{A}} |f(z)|$

local version $\Rightarrow f$ is constant in $D(z_1, r) \subseteq A$

$\Rightarrow D(z, r) \subseteq U$

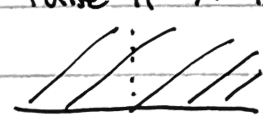
- U is closed, if $z_n \in U$, $z_n \to z_1$

$f(z) = \lim f(z_n) = f(z_0) \Rightarrow z \in U$

<u>Cor</u> $A \subseteq \mathbb{C}$ open bounded connected $f: A \to \mathbb{C}$ holom
$f: \bar{A} \to \mathbb{C}$ is continuous.

$$\max_{z \in \bar{A}} |f(z)| = \max_{z \in \partial A} |f(z)|$$

<u>Rmk</u> false if A is unbounded eg. $f(z) = e^{-z^2}$ on $A = \{ z : \operatorname{Im} z \geq 0 \}$



<u>Application</u> Schwartz lemma. $\operatorname{Aut}(D(0,1))$

<u>Lemma</u> (Schwartz lemma) $A = \{ |z| < 1 \}$. $f: A \to \mathbb{C}$ holomorphic
Suppose $f(0) = 0$ $|f(z)| \leq 1$ in A
Then : $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ Moreover, if $|f'(0)| = 1$ or
$|f(z)| = |z|$ for some $z \in A$. then $f(z) = e^{i\theta} z$

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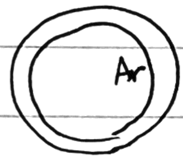
<u>Pf</u> $g(z) = \begin{cases} f(z)/z & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \quad (\because [f(z)-f(0)]/(z-0)) \end{cases}$

$g(z)$ is continuous in A and holomorphic in $A \setminus \{0\}$

Remarkable singularity $\Rightarrow$ g is holom in A.

want prove $|g(z)| \leq 1$ in A.

$A =$ 

Let $A_r = \{|z| < r\}$ $\quad g : \bar{A}_r \to \mathbb{C}$ is cont.

$g : A_r \to \mathbb{C}$ is holom

max modulus principle $\Rightarrow \max_{z \in \bar{A}_r} |g(z)| \leq \max_{z \in \partial A_r} |g(z)|$

$$= \max_{z \in \partial A_r} \frac{|f(z)|}{|z|} = \frac{1}{r} \max_{z \in \partial A_r} |f(z)| \leq \frac{1}{r}$$

Send $r \nearrow 1$ $\Rightarrow |g(z)| \leq 1$

Moreover, if either $|f'(0)| = 1$ or $|f(z)| = |z|$

$\Rightarrow |g(z_0)| = 1$ in A $\Rightarrow |g(z_0)| = \max_{z \in A} |g(z)| = 1$

$\Rightarrow g = \text{const}$ $\square$

<u>Thm</u> Suppose $g : D(0,1) \to D(0,1)$ is a holomorphic automorphism (i.e. g is a bijection g^{-1} is a holomorphic). Then g is a Mobius function $g(z) = e^{i\theta} \cdot (z - z_0)/(1 - \bar{z}_0 z)$ for some $z_0 \in D(0,1)$

<u>Rmk</u> HW (1 or 2) $z \mapsto e^{i\theta} (z - z_0)/(1 - \bar{z}_0 z)$ maps $\partial D(0,1)$ to $\partial D(0,1)$

- $z_0 \in D(0,1) \Rightarrow$ it maps $D(0,1) \to D(0,1)$
- $e^{i\theta}$ = rotating by θ $\quad z \mapsto (z - z_0)/(1 - \bar{z}_0 z)$ maps $\begin{cases} 0 \mapsto -z_0 \\ z_0 \mapsto 0 \end{cases}$

Pf set $g(0) = a \in D(0,1)$ consider $\ell(z) = (z-a)/(1-\bar{a}z)$

$f(z) = \ell(g(z))$ check:

- $f(0) = \ell(g(0)) = \ell(a) = 0$
- $f(D(0,1)) = D(0,1)$, since g, ℓ $D(0,1) \to D(0,1)$ holom bijections.

$\Rightarrow |f(z)| < 1$, $\forall z \in D(0,1)$.

Schwartz $\Rightarrow |f'(0)| \le 1$

But f^{-1} satisfies $(*)$ $\Rightarrow |(f^{-1})'(0)| \le 1$

$$z = f(f^{-1}(z)) \overset{\text{diff}}{\Longrightarrow} 1 = f'(f^{-1}(z)) \cdot (f^{-1})'(z)$$
$$= f'(0) \cdot (f^{-1})'(0)$$

$\Rightarrow$ equality case in Schwartz lemma $f(z) = e^{i\theta} z$

$\Rightarrow \ell(g(z)) = e^{i\theta} z \Rightarrow g(z) = (e^{i\theta}z + a)/(1 + \bar{a}e^{i\theta}z)$

$$= e^{i\theta}\left[\frac{z + ae^{-i\theta}}{1 + \bar{a}e^{i\theta}z}\right]$$

$\square$

Similarly, consider automorphism $\mathbb{H}_+ \to \mathbb{H}_+$, $\mathbb{H}_+ = \{z : \operatorname{Im} z > 0\}$

auto $S \to S$

$\uparrow$

Riemann sphere

max modulus principle Lec1

Midterm is easier than homework.