Harmonic function

Lecture 4

$\boxed{\text{Harmonic Functions}}$ (real-valued functions)

__Def__ $A \subseteq \mathbb{R}^2$ open. $f: A \to \mathbb{R}$ is harmonic

if $\underset{\text{Laplacian}}{\Delta f} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

__Remark__ ∇ = gradient $\nabla f = (\partial f/\partial x, \partial f/\partial y)$

div = divergence $\vec{X}$ is a vector field

$\vec{X}(x,y) = (a(x,y), b(x,y))$

$\operatorname{div} \vec{X} = \partial a/\partial x + \partial b/\partial y$

Suppose $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic.

u, v are twice countably differentiable

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \overset{C\text{-}R}{=} \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \overset{C\text{-}R}{=} \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial \cancel{x}}\right) = \frac{-\partial^2 v}{\partial y \partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Leftrightarrow \Delta u = 0$$

Call u, v conjugate __Harmonic functions__

__Basic Property__ : $A \subseteq \mathbb{C}$ open. $f, g : A \to \mathbb{C}$ harmonic

(1) $f+g$ is holomorphic. $(f+g)' = f' + g'$

(2) fg is holomorphic. $(fg)' = f'g + fg'$

(3) f/g is holomorphic on $\{g \neq 0\}$ $(f/g)' = (f'g - fg')/g^2$

(4) $h: B \to \mathbb{C}$ holomorphic. B is open $B \supseteq f(A)$ then

Basic Functions' Holomorphic Test

$h \cdot f : A \xrightarrow{f} B \xrightarrow{h} \mathbb{C}$ is holomorphic   Lecture 4

$h(f(z))' = h'(f(z)) \cdot f'(z)$

$\boxed{\text{Basic Functions}}$

<u>Prop</u> $\exp : \mathbb{C} \to \mathbb{C}$ is holomorphic $(\exp)'(z) = \exp(z)$

<u>Pf</u> $z = x + iy$.

$$e^z = e^x(\cos y + i\sin y) = \underbrace{e^x \cos y}_{u} + i\,\underbrace{e^x \sin y}_{v}$$

Verify C-R. $\dfrac{\partial u}{\partial x} = e^x \cos y = \dfrac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

$$f'(x+iy) = \partial u/\partial x + i\,\partial v/\partial x = e^x \cos y + i e^x \sin y$$
$$= e^z$$

<u>Cor</u> $\sin' z = \cos z$. $\cos' z = -\sin z$

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$(\sin z)' = \frac{1}{2i}\left((e^{iz})' - (e^{-iz})'\right) = \frac{1}{2i}\left(ie^{iz} + ie^{-iz}\right) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cos z$$

$$(\cos z)' = \frac{1}{2}\left((e^{iz})' + (e^{-iz})'\right) = \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right) = \frac{-1}{2i}\left(e^{iz} - e^{-iz}\right) = -\sin z$$

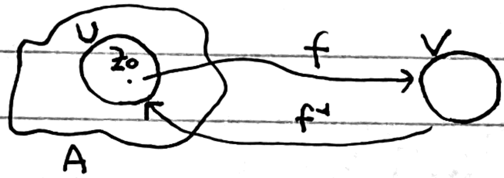
<u>Inverse Function Thm</u> $f : A \to \mathbb{C}$ is holomorphic $f'(z_0) \neq 0$.

Then $\exists$ a neighborhood U of z_0, V of $f(z_0)$

s.t. $f : U \to V$ is a bijection $f^{-1} : V \to U$ is holomorphic

$(f^{-1})'(f(z)) = 1/f'(z) \Leftrightarrow (f^{-1})'(w) = 1/f'(f^{-1}(w)),\ w \in V$

Holomorphic, log



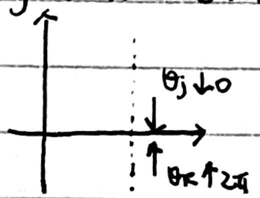
$f(f^{-1}(w)) = w$

$f'(f^{-1}(w))(f^{-1})'(w) = 1$

<u>Remark</u> U, V could be small

Take a branch of log. Say $\log : \mathbb{C} \setminus \{0\} \to A_0 = \{z : 0 \leq \operatorname{Im} z < 2\pi\}$

log is not continuous on the positive real axis



$\log(e^{x}(\cos\theta + i\sin\theta)) = \rho + i\theta \quad (0 \leq \theta < 2\pi)$

$\Rightarrow$ log is not cont. on $\{z : \operatorname{Im} z = 0, \operatorname{Re} z \geq 0\}$

Generally, $\log : \mathbb{C} \setminus \{0\} \to A_{y_0} = \{z : y_0 \leq \operatorname{Im} z < y_0 + 2\pi\}$

is not cont. on the ray $\{\arg z = y_0\}$

log with branch A_0

$$\log : \mathbb{C} \setminus \{z : \operatorname{Im} z = 0, \operatorname{Re} z \geq 0\} \to A_0$$

is holomorphic and is a bijection (the inverse of exp)

$$\log'(w) = \frac{1}{\exp'(\exp^{-1}(w))} = \frac{1}{\exp(\exp^{-1}(w))} = \frac{1}{w}$$

<u>Remark</u> On any choice of branch of log

$$\log'(z) = 1/z$$

Contour Integrals

Lecture 4

<u>Recall</u> line integrals in $\mathbb{R}^2$ $u(x,y)$ $v(x,y) : \overset{\text{open}}{A} \to \mathbb{R}$
continuous $\gamma\ [a,b] \to A$ C^1 curve
Parameterize γ by $\gamma(t) = (x(t), y(t))$

$$\int_\gamma u\,dx + v\,dy = \int_a^b \left[u(x(t),y(t))x'(t) + v(x(t),y(t))y'(t) \right] dt$$

<u>Now</u> $\int_\gamma f(z)\,dz$ $\quad dz := dx + i\,dy \quad (z = x+iy)$

Write $f(x+iy) = u(x,y) + iv(x,y)$
Then formally $\int_\gamma f(z)dz = \int_\gamma (u+iv)(dx + i\,dy)$

$$= \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy)$$

<u>Define</u> $\int_\gamma f(z)\,dz$ in 2 steps

Step 1) Integrals on Interval $f(t) = u(t) + v(t)$ $\quad t \in [a,b] \underset{\in}{\overset{\subseteq}{}} \mathbb{R}$

Define $\int_a^b f(t)dt = \int_a^b u\,dt + i\int_a^b v\,dt$

<u>Properties</u> $\int_a^b f(t)dt + \int_b^c f(t)dt = \int_a^c f(t)dt$

$\forall \lambda \in \mathbb{C}$ $\quad \int_a^b \lambda f(t)dt = \lambda \int_a^b f(t)dt$

$$\mathrm{Re}\left(\int_a^b f(t)dt\right) = \int_a^b \mathrm{Re}[f(t)]dt.$$

$$\mathrm{Im}\left(\int_a^b f(t)dt\right) = \int_a^b \mathrm{Im}[f(t)]\,dt$$

(Cont. Properties)

Triangle Inequality $\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt.$

Step 2) γ = piecewise differentiable curve

parameterize $\gamma(t)$. Let $\gamma : z = z(t)$ for $a \leq t \leq b$

$\exists\ a = a_0 < a_1 < \dots < a_n = b$ s.t. $z(t)$ is differentiable

on $[a_j, a_{j+1}]$. Then $z(t)$ is continuous on $[a, b]$

For $f(z)$ continuous on γ. Define

$$\int_\gamma f(z)\,dz = \sum_{j=0}^{n-1} \int_{a_j}^{a_{j+1}} f(z(t)) \left(\frac{dz}{dt}\right) dt$$

<u>Prop</u> $\int_\gamma f(z)\,dz$ is independent of the parametrization.

<u>Pf</u> Assume γ is differentiable. Parameterized by $z = z(t)$

Suppose $\tau : [\alpha, \beta] \to z(t(\tau))$ is another parameterization

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta f(z(t(\tau)))\, \frac{dz(t(\tau))}{d\tau}\, d\tau$$

$$= \int_\alpha^\beta f(z(t(\tau)))\, \frac{dz(t)}{dt} \cdot \underbrace{t'(\tau)\,d\tau}_{dt}$$

$u = t(\tau)$, $du = t'(\tau)\,d\tau$

$$= \int_a^b f(z(u))\, \frac{dz}{du}\, du \qquad \square$$

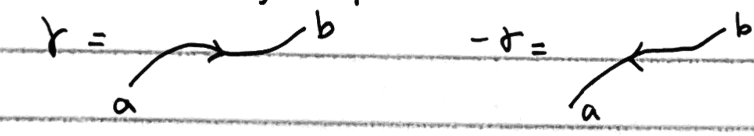

Parameterization.

Parameterization

Basic Properties

Let γ be parameterized by $z = z(t)$, $t \in [a,b]$

Define $-\gamma$ by a parameterization: $-\gamma$ $z = z(-t)$, $t \in [-b,-a]$



Then $\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz$

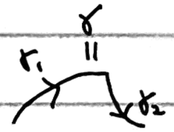
Pf $\int_{-\gamma} f(z)\,dz = \int_{-b}^{-a} f(z(-t)) \frac{dz(-t)}{dt}\,dt$ set $s = -t$

$$= \int_b^a f(z(s)) \frac{dz(s)}{d(-s)} (-ds) = -\int_a^b f(z(s)) \frac{dz}{ds} \cdot ds$$

$$= -\int_\gamma^a f(z)\,dz$$

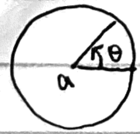
Linearity on f $\int_\gamma (af + bg)\,dz = a\int_\gamma f\,dz + b\int_\gamma g\,dz$

Linearity on γ $\int_\gamma f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz$



Ex Simple and fundamental $a \in \mathbb{C}$. γ = closed circle w/ radius R centered at a, counterclockwise $\int_\gamma \frac{1}{z-a}\,dz$ (Parametrize γ)

$z(\theta) = \underset{\text{center}}{a} + \underset{\text{Radius}}{R} e^{i\theta}$ $0 \le \theta < 2\pi$



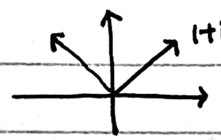
$\int_0^{2\pi} \frac{1}{z(\theta) - a} z'(\theta)\,d\theta.$ $z'(\theta) = Rie^{i\theta}$

$$= \int_0^{2\pi} \frac{1}{Re^{i\theta}} Rie^{i\theta} \cdot d\theta = 2\pi i$$

Ex $\int_\gamma z^2 dz$ $\quad \gamma$ = line segment from 0 to $1+i$

Parameterize γ by $z(t) = (1+i)t$, $0 \le t \le 1$

$$\int_\gamma z^2 dz = \int_0^1 (1+i)^2 t^2 \cdot \underbrace{(1+i)dt}_{z'(t)} = (1+i)^3 \int_0^1 t^2 dt = -2(1+i)\frac{1}{3}$$



$\boxed{\text{Fundamental Thm of Contour Integral}}$

Thm: Let $A \subseteq \mathbb{C}$ open $I = [a,b]$ $\cancel{\gamma}\ z: I \to A$ piecewise C'

$f: A \to \mathbb{C}$ is continuous, $F: B \to \mathbb{C}$ holomorphic

B open. $B \supseteq \gamma(I)$ $\boxed{F'(z) = f(z)}$

Then $\int_\gamma f(z) dz = F(z(b)) - F(z(a))$

In particular, if γ is closed $\int_\gamma f\, dz = 0$

Pf $\int_\gamma f(z) dz = \int_a^b f(z(t))\, z'(t)\, dt$

$$= \int_a^b F'(z(t))\, z'(t)\, dt = \int_a^b \frac{d}{dt}\Big(F(z(t))\Big) dt$$

$$= F(z(b)) - F(z(a))$$

Ex $z^2 = \left(\frac{1}{3}z^3\right)'$ $\quad \int_\gamma z^2 dz = \frac{1}{3}z^3 \Big|_0^{1+i}$

Ex $\int_\gamma \frac{1}{z} dz$ $\quad (\log z)' = \frac{1}{z}$

log is not defined in a ______ of γ