Complex Variables I – Problem Set 3

Due at 5 pm on Friday, September 29, 2023 via Gradescope

Problem 1

Consider two points $M_1 = (x_1, y_1, z_1)$, $M_2 = (x_2, y_2, z_2)$ on the Riemann sphere S that are not the north pole, and w_1, w_2 their stereographic projection in the complex plane. Show that $M_1 = -M_2$ (that is, they are diametrically opposite points on S) if and only if $w_1 \bar{w}_2 = -1$.

Problem 2

Which of the following functions are holomorphic in their domain of definition? Justify your answer (you may verify by definition or using the Cauchy-Riemann equation).

1. $f(z) = z^3$
2. $f(z) = |z|^2$
3. $f(z) = \frac{1}{z}$

Problem 3

Let $\Omega \subset \mathbb{C}$ be an open set, and $f : \Omega \to \mathbb{C}$ be a function. In polar coordinates $z = r(\cos\theta + i\sin\theta)$, write $f(z) = u(r,\theta) + iv(r,\theta)$ for real-valued functions u, v. Applying appropriate change of variables, deduce the Cauchy-Riemann equation for polar coordinates (r,θ): f is complex differentiable if and only if

$$\begin{cases} r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}. \end{cases} \tag{1}$$

Problem 4

Let $\Omega \subset \mathbb{C}$ be an open set, and $f : \Omega \to \mathbb{C}$ holomorphic. Consider the set

$$\tilde{\Omega} = \{z \in \mathbb{C} : \bar{z} \in \Omega\}.$$

Show that $\tilde{\Omega}$ is an open set, and that the function

$$\tilde{f} : z \in \tilde{\Omega} \mapsto \overline{f(\bar{z})} \in \mathbb{C}$$

is holomorphic.

Problem 5

1. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function, given as $f(x+iy) = u(x,y) + iv(x,y)$, where u, v are real-valued functions (you may assume that they are smooth). Show that u, v satisfy Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

2. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Show that the function

$$\phi(x,y) = \log |f(x+iy)|$$

satisfies Laplace's equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$. Here log is the real logarithm.

Remember to justify your answers and acknowledge collaborations and outside help!