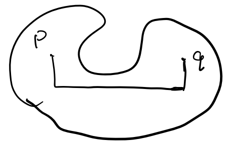
Last week: Cauchy's integral formula: $f: D \to \mathbb{C}$ holomorphic. Then for any closed curve $C \subseteq D$, $\int_C f(z)\,dz = 0$.

We proved this for $D$ = rectangles and disks.
In homework (optional problem): $D$ = region enclosed by a closed curve, and any $p, q \in D$ can be joined by a polygonal curve

e.g.: $D =$ 

(You need this for problems 5, 6.

This time: a more systematic discussion of when we can apply Cauchy.

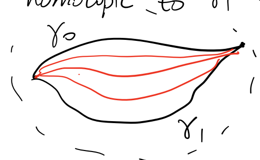
<u>EX</u>: $D$ = annulus $\{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\}$. $f: D \to \mathbb{C}$, $f(z) = \frac{1}{z}$ holomorphic.
$C$ = unit circle $\int_C f(z)\,dz = \int_C \frac{1}{z}\,dz = 2\pi i \neq 0$.

<u>Def</u>: Let $A \subseteq \mathbb{C}$ be open, $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$ continuous curves, $\gamma_0(0) = \gamma_1(0)$, $\gamma_0(1) = \gamma_1(1)$.
Say $\gamma_0, \gamma_1$ are homotopic with fixed endpoints, if $\exists\, H: [0,1] \times [0,1] \to A$ cont, s.t.

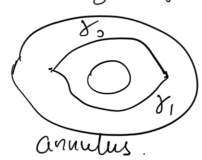
(i) $\gamma_0(t) = H(0,t)$, $\gamma_1(t) = H_1(t)$

(ii) $\forall s \in [0,1]$, $H(s,0) = \gamma_0(0) = \gamma_1(0)$, $H(s,1) = \gamma_0(1) = \gamma_1(1)$.

Say $H$ a homotopy between $\gamma_0, \gamma_1$.
Intuition: $\gamma_0$ homotopic to $\gamma_1$ $\Leftrightarrow$ can continuously deform $\gamma_0$ to $\gamma_1$.

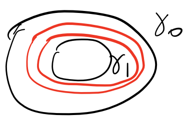
EX: 

$\gamma_0 \simeq \gamma_1$ in a disk



$\gamma_0 \not\simeq \gamma_1$ $\rightsquigarrow$ cannot deform $\gamma_0$ to $\gamma_1$ in the annulus.

<u>Rmk</u>: (1) Can also define homotopy without asking that endpts are fixed.

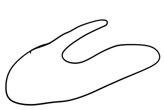
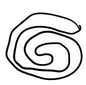
e.g. 

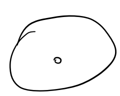
$\gamma_0 \simeq \gamma_1$ in a disk.

(2) Homotopy is an equivalent relation.
(3) If $\gamma_0, \gamma_1 : [0,1] \to A$ are $C^1$ and are homotopic, then we can also choose their homotopy $H$ to be $C^1$.

Special case: $A \subseteq \mathbb{C}$ open, say a closed curve $\gamma: [0,1] \to A$ is null-homotopic, if it is homotopic to a constant map $\gamma_0 : [0,1] \to A$, $\gamma_0(t) = z_0$ $\forall t \in [0,1]$.
i.e. each closed curve can be deformed to a pt.


<u>Def</u>: A domain $A \subseteq \mathbb{C}$ is simply connected, if every closed curve in $A$ is null-homotopic.
Intuitively, $A$ is simply connected $\Leftrightarrow$ $A$ has no holes.

<u>EX</u>:   simply connected.

 punctured disk,  annulus
not simply connected.

**Thm**: Let $A \subseteq \mathbb{C}$ be a simply connected domain. Then for any $f: A \to \mathbb{C}$ holomorphic, we have that $\int_C f(z)\,dz = 0$ along any closed curve $C \subseteq A$.

**Proof**: Fix a closed curve $C_0$, given by $\gamma_0: [0,1] \to A$. Let $H$ be a homotopy: $H: [0,1] \times [0,1] \to A$, $H(0,t) = \gamma_0(t)$; $H(1,t) = z_0$ for $t \in [0,1]$.
Set $\gamma_s = H(s,\cdot)$ a $C^1$ curve $C_s$.

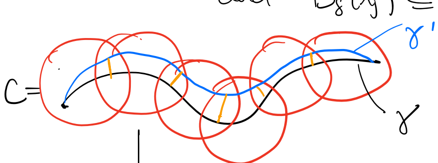
**Goal**: Prove $F(s) := \int_{C_s} f(z)\,dz$ is constant for $s \in [0,1]$.

Approach: method of continuity. Set $I = \{ s \in [0,1] : \int_{C_s} f(z)\,dz = \int_{C_0} f(z)\,dz \}$.
We prove that $I$ is both open and closed.

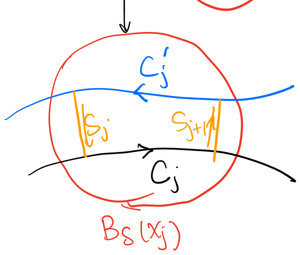
- $I$ is closed: if $S_j \in I$, $S_j \to S$, then $\int_{C_s} f(z)\,dz = \int_0^1 f(\gamma_s(t))\, \gamma_s'(t)\,dt$ is cont. in $S$. $\Rightarrow$ $F(s) = \lim_{j\to\infty} F(S_j) = F(0)$.
- $I$ is open. We prove the following: for any closed $C$, $\exists\, \varepsilon > 0$ s.t. any other curve $C'$ in an $\varepsilon$-nbhd of $C$ satisfies

$$\int_C f(z)\,dz = \int_{C'} f(z)\,dz.$$

Reason: $C$ is compact $\Rightarrow$ cover $C$ by finitely many $\{B_\delta(x_j)\}_{j=1}^n$, $x_j \in C$, and $B_\delta(x_j) \subseteq A$.




For any other curve $\gamma' \subseteq \bigcup B_\delta(x_j)$, we add straight line segments $S_j$, $j = 1, \cdots, n-1$ s.t. $S_j$ partition $C$, $C'$ into $\underbrace{\text{pieces}}_{C_j,\, C_j'}$ contained in a single $B_\delta(x_j)$.




Note: The curve $C_j - C_j' + S_j - S_{j+1}$ is closed, and is contained in a disk

Cauchy $\Rightarrow$ $\displaystyle\int_{C_j - C_j' + S_j - S_{j+1}} f(z)\,dz = 0$. Add over $j$

$\Rightarrow$ $\int_C f(z)\,dz = \int_{C'} f(z)\,dz$.

Thus, $I$ is open. $\Rightarrow$ $I = [0,1]$.

However, for at $s = 1$, $C_1 = \{pt\}$ $\Rightarrow$ $\int_{C_1} f(z)\,dz = 0$. $\Rightarrow$ $\int_{C_0} f(z)\,dz = 0$

$\square$

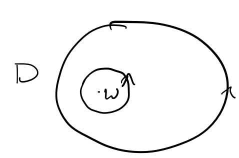
**Rmk**: The above proves: if $C_0$ is homotopic to $C_1$ then

$$\int_{C_0} f(z)\,dz = \int_{C_1} f(z)\,dz.$$

$A \subseteq \mathbb{C}$ open,

Important example: $D$ is an open disk, $w \in \mathbb{C} \setminus \partial D$. Then:

Then:
$$\int_{\partial D} \frac{1}{z-w}\,dz = \begin{cases} 2\pi i & \text{if } w \in D \\ 0 & \text{if } w \notin D \end{cases}$$



Proof: If $w \notin D$, then $\frac{1}{z-w}$ is well-defined in $D$ and is holomorphic.

If $w \in D$, then $\exists\, \varepsilon > 0$ s.t. $D(w,\varepsilon) \subseteq D$. $\Rightarrow$ $\partial D(w,\varepsilon)$ and $\partial D$ are homotopic in $D \setminus D(w,\varepsilon)$.

$$\Rightarrow \int_{\partial D} \frac{1}{z-w}\, dz = \int_{\partial D(w,\varepsilon)} \frac{1}{z-w}\, dz \longrightarrow z = w + \varepsilon e^{i\theta}$$

$$= \int_0^{2\pi} \frac{\varepsilon i e^{i\theta}}{\varepsilon e^{i\theta}}\, d\theta = 2\pi i.$$

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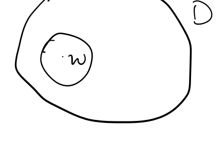
More generally

Thm (Cauchy's integral formula): $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ holomorphic. $D \subseteq A$ a disk, $w \in D$. Then

$$f(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-w}\, dz.$$




Proof: Take $\delta > 0$ s.t. $D(w,\delta) \subseteq D$. Then $\partial D$ and $\partial D(w,\delta)$ are homotopic, and $\frac{f(z)}{z-w}$ is holom. in $D \setminus D(w,\delta)$.

Thus,
$$\int_{\partial D} \frac{f(z)}{z-w}\, dz = \int_{\partial D(w,\varepsilon)} \frac{f(z)}{z-w}\, dz = \int_0^{2\pi} \frac{f(w+\delta e^{i\theta}) \cdot i e^{i\theta}}{\delta e^{i\theta}}\, d\theta$$

$$= i \int_0^{2\pi} f(w+\delta e^{i\theta})\, d\theta \quad \text{holds for all small } \delta.$$

But $f$ is continuous, hence $f(w+\delta e^{i\theta}) \to f(w)$ as $\delta \to 0$.

Thus,
$$\int_{\partial D} \frac{f(z)}{z-w}\, dz = 2\pi i\, f(w).$$

$\square$

Consequences:

Thm: $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$ holomorphic. Then $f$ has arbitrarily many complex derivatives in $A$.
Moreover, for any $w \subseteq A$, and disk $D$ with $w \in D \subseteq A$,

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-w)^{n+1}}\, dz.$$

Proof:
$$f(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-w}\, dz. \quad \text{Consider } h \in \mathbb{C},\ |h| \text{ small}.$$

$$f(w+h) - f(w) = \frac{1}{2\pi i} \int_{\partial D} f(z) \left( \frac{1}{z-(w+h)} - \frac{1}{z-w} \right) dz = \int_{\partial D} f(z) \cdot \frac{h}{(z-(w+h)) \cdot (z-w)}\, dz.$$

$$\Rightarrow \frac{1}{h}\left( f(w+h) - f(w) \right) = \frac{1}{2\pi i} \int_{\partial D} f(z) \cdot \frac{1}{(z-(w+h))(z-w)}\, dz.$$

as $h \to 0$, $\frac{1}{z-(w+h)} \to \frac{1}{z-w}$ uniformly for $z \in \partial D$.

$$\Rightarrow \lim_{h \to 0} \frac{1}{h}\left( f(w+h) - f(w) \right) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-w)^2}\, dz. \iff f'(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)\, dz}{(z-w)^2}$$

$\rightsquigarrow$ This means that one may take derivative inside contour integral.

Inductively. $f''(w) = \frac{1}{2\pi i} \int_{\partial D} 2 \cdot \frac{f(z)}{(z-w)^3} dz, \cdots$ □

**Cor** (Liouville): If $f: \mathbb{C} \to \mathbb{C}$ is entire and bounded. Then $f$ is constant.

Proof: Use Cauchy: for any $w \in \mathbb{C}$, $R>0$,

say $|f|<C$. $f'(w) = \frac{1}{2\pi i} \int_{\partial D(w,R)} \frac{f(z)}{(z-w)^2} dz$

$$\Rightarrow |f'(w)| \le \frac{1}{2\pi} \cdot \int_{\partial D(w,R)} \frac{|f(z)|}{|z-w|^2} |dz| = \frac{1}{2\pi} \cdot 2\pi R \cdot C \cdot \frac{1}{R^2} = \frac{C}{R}$$

This holds for any $R>0$. Take $R \to \infty \Rightarrow f'(w)=0. \Rightarrow f \equiv \text{Const}.$

**Thm** (Fundamental Thm of algebra). $n \ge 1$, $a_0, \cdots, a_n \in \mathbb{C}$, $a_n \ne 0$. Then

$P(z) = a_n z^n + \cdots + a_1 z + a_0$ has a root in $\mathbb{C}$.

Proof: Assume, for the sake of contradiction, that $P(z) \ne 0$ for all $z \in \mathbb{C}$.

Then $\frac{1}{P(z)}$ is entire holomorphic. It's also bounded:

(HW 1): $\exists R>0$ s.t. $\left|\frac{1}{P(z)}\right| < \frac{2}{|a_n| R^n}$, when $|z|>R$.

and $\frac{1}{|P(z)|}$ is continuous on $\{|z| \le R\} \Rightarrow$ it's bounded.

Liouville Thm $\Rightarrow \frac{1}{P(z)}$ is a const. function, contradiction. □

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**Thm** (Morera's Thm): $A \subseteq \mathbb{C}$ connected open, $f: A \to \mathbb{C}$ is continuous. Assume that for every closed curve $C \subseteq A$, $\int_C f(z) dz = 0$. Then $f$ is holomorphic.

Proof: Condition $\Rightarrow \int f(z) dz$ is path-independent. Fix $z_0 \in A$. For every $z \in A$, define $F(z) = \int_{\gamma_z} f(w) dw$, $\gamma_z$ = any piecewise $C^1$ curve connecting $z_0$ to $z$.

We have proved: $F$ is holomorphic. Cauchy $\Rightarrow F'$ is holomorphic. □