Problem 1. Regard log as a multivalued function. Find all the values of

a) $\log(-i)$

Sol) Let $z=-i$ and $w=\log z$. Then w can be written as below

$$w = \log|z| + i(\arg(z) + 2n\pi) \quad (n \in \mathbb{Z}).$$ We need to find w.

Since $|z| = |-i| = 1$ and $\arg(z) = \frac{-\pi}{2}$, w can be written as below

$$w = \log|1| + i\left(\frac{-\pi}{2} + 2n\pi\right) \quad (n \in \mathbb{Z})$$

$$= \underline{\frac{-i\pi}{2} + 2n\pi i}$$ ☑

b) $\log 2$

Sol) Let $z=2$ and $w=\log z$. w can be written as below

$$w = \log|z| + i(\arg(z) + 2n\pi) \quad n \in \mathbb{Z}.$$

To find w, we need to find $|z|$ and $\arg(z)$.

$$|z| = |2| = 2 \text{ and } \arg(z) = 0.$$

It follows that

$$w = \log|2| + i(0 + 2n\pi) = \underline{\log 2 + 2n\pi i} \quad (n \in \mathbb{Z})$$ ☑

c) $\log(1+i)$

Sol) Let $z = 1+i$ and $w = \log z$. Then, w can be written as below

$$w = \log|z| + i(\arg(z) + 2n\pi) \quad n \in \mathbb{Z}$$

To find w, we need to find $|z|$ and $\arg(z)$.

$$|z| = |1+i| = \sqrt{2} \text{ and } \arg(z) = \pi/4.$$

Thus, we end up have

$$w = \log\sqrt{2} + i(\pi/4 + 2n\pi) = \underline{\log\sqrt{2} + \frac{\pi i}{4}(1+8n)} \quad (n \in \mathbb{Z})$$ ☑

Problem 2. Solve the following equations (make sure find all solutions)

a) $\cos z = 2$

Sol) $\cos z = \dfrac{e^{iz} + e^{-iz}}{2} = 2$

$$e^{iz} + e^{-iz} = 4 \quad (\text{mult by } 2)$$
$$(e^{iz})^2 + 1 = 4e^{iz} \quad (\text{mult by } e^{iz})$$

Let $X = e^{iz}$, then the last equation can be written as

$$X^2 + 1 = 4X$$
$$X^2 - 4X + 1 = 0$$
$$X^2 - 4X + 4 = 3$$
$$(X-2)^2 = 3$$
$$X = 2 \pm \sqrt{3}$$

Since $X = e^{iz}$, the last equation can be written as

$$e^{iz} = 2 \pm \sqrt{3}$$

Taking ln on both sides yiels

$$iz = \ln(2 \pm \sqrt{3})$$
$$z = \frac{1}{i} \ln(2 \pm \sqrt{3}) + 2k\pi \quad \text{for } k \in \mathbb{Z}$$
$$= \underline{-i \ln(2 \pm \sqrt{3}) + 2k\pi} \quad (\because \tfrac{1}{i} = -i)$$

☑

Problem 2

b) $\sin z = 2$

Sol) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = 2$

$$e^{iz} - e^{-iz} = 4i \qquad \text{(mult by 2)}$$
$$(e^{iz})^2 - 1 = 4i \cdot e^{iz} \qquad \text{(mult by } e^{iz}\text{)}$$

Let $x = e^{iz}$, then we have

$$x^2 - 1 = 4ix$$
$$x^2 - 4ix - 1 = 0$$

Solve this quadratic equation

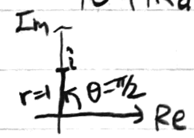
$$x = \frac{4i \pm \sqrt{(4i)^2 + 4}}{2} = \frac{4i \pm \sqrt{-12}}{2} = \frac{4i \pm 2\sqrt{3}i}{2} = i(2 \pm \sqrt{3})$$

Since $x = e^{iz}$, then it follows that

$$e^{iz} = i(2 \pm \sqrt{3})$$

$$iz = \ln[i(2\pm\sqrt{3})] = \ln(i) + \ln(2\pm\sqrt{3})$$
$$= (i\pi/2) + \ln(2\pm\sqrt{3}) \qquad (\because \ln(i) = i(\tfrac{\pi}{2}))\ (*)$$
$$z = (\tfrac{\pi}{2}) + (1/i)\ln(2\pm\sqrt{3})$$
$$= (\tfrac{\pi}{2}) - i\ln(2\pm\sqrt{3}) \qquad (\because (1/i) = -i)$$
$$= \underline{(\tfrac{\pi}{2}) - i\ln(2\pm\sqrt{3}) + 2k\pi} \quad \text{for } k \in \mathbb{Z}$$

To find $\ln(i)$, we can start with i. It can be written as



$$i = r(\cos\theta + i\sin\theta)$$
$$= 1(\cos(\tfrac{\pi}{2}) + i\sin(\tfrac{\pi}{2})) \qquad (\because r=1,\ \theta = \pi/2)$$

By Euler's formula, the last equation can be written as

$$i = 1 \cdot e^{i(\frac{\pi}{2})}$$

Taking ln on both sides yields

$$\ln(i) = \ln(1 \cdot e^{i(\frac{\pi}{2})}) = \ln 1 + \ln e^{i(\frac{\pi}{2})}$$
$$= 0 + i(\tfrac{\pi}{2}) = i(\tfrac{\pi}{2})$$

Problem 3. Prove that the function $\sin z$ maps the strip $-\pi/2 < \operatorname{Re} z < \pi/2$ onto the set $\mathbb{C} \setminus \{z : \operatorname{Im} z = 0 \text{ and } |\operatorname{Re} z| \geq 1\}$

sol) Let $z = x + iy$ $(x, y \in \mathbb{Z})$. Then we have

$$\sin z = \frac{(e^{iz} - e^{-iz})}{2i} = \frac{e^{i(x+yi)} - e^{-i(x+yi)}}{2i} = \frac{e^{-y+xi} - e^{y-xi}}{2i}$$

$$= \frac{e^{-y} \cdot e^{xi} - e^{y} \cdot e^{-xi}}{2i} = \frac{-i}{2}\left(e^{-y} \cdot e^{ix} - e^{y} \cdot e^{i(-x)}\right)$$

$$= \frac{-i}{2}\left(e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)\right)$$

$$= e^{-y}\left(\frac{-i}{2}\cos x + \frac{1}{2}\sin x\right) + e^{y}\left(\frac{i}{2}\cos x + \frac{1}{2}\sin x\right)$$

$$= \frac{1}{2}(e^{y} + e^{-y})\sin x + \frac{i}{2}(e^{y} - e^{-y})\cos x$$

Now, let $a = \left(\frac{e^{y} + e^{-y}}{2}\right) \cdot \sin x$ and $b = \left(\frac{e^{y} - e^{-y}}{2}\right) \cdot \cos x$.

To prove surjectivity, we need to show that $\exists$ x and y such that $|a| < 1$ or $b \neq 0$.

Assume that $b = 0$. Then we have

$$(1)\ \frac{e^{y} - e^{-y}}{2} = 0 \text{ and } (2)\ \cos x = 0.$$

Since $\frac{-\pi}{2} < x < \frac{\pi}{2}$, (2) is not the case.

$$\frac{e^{y} - e^{-y}}{2} = 0 \Rightarrow e^{y} = e^{-y} \Rightarrow y = 0.$$

(Cont. Problem 3)

When $y=0$, $a = \sin x$. Because it's given that $\frac{-\pi}{2} < \operatorname{Re} z = x < \frac{\pi}{2}$, we have that $|a| < 1$.

Then assume that $b \neq 0$. Because $\frac{-\pi}{2} < x < \frac{\pi}{2}$, $\cos x$ is nonzero. It follows that $y \neq 0$ because if $y=0$, then $b=0$. Because $y \neq 0$, we have $(e^{y}+e^{-y})/2 > 1$. It follows that $|\operatorname{Re} z| = |a| \geq 0$

Thus, we've just shown that there exist at least one $z \in \mathbb{C}$ that maps $\sin z$ $(-\pi/2 < \operatorname{Re} z < \pi/2)$ onto the set $\mathbb{C} \setminus \{z : \operatorname{Im} z = 0 \text{ and } |\operatorname{Re} z| \geq 1\}$

☑

Possible Combo of a, b, c, d

Problem 4. Find all Mobius transformation $f(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ such that $|f(z)| = 1$ whenever $|z| = 1$.

sol) Suppose $a \neq 0$. Dividing a on both numerator and denominator result $f(z) = \frac{z + \frac{b}{a}}{\frac{c}{a}z + \frac{d}{a}}$. Thus, we

can assume $a=1$ without loss of generality.

Let's start with $f(z) = \frac{z+b}{cz+d}$. whenever $|z|=1$, $|f(z)|=1$.

It follows that

$$|f(z)|^2 = 1 = \frac{z+b}{cz+d} \cdot \frac{\overline{z+b}}{\overline{cz+d}} = \frac{z\bar{z} + z\bar{b} + \bar{z}b + b\bar{b}}{c\bar{c}z\bar{z} + c\bar{d}z + \bar{c}d\bar{z} + d\bar{d}}$$

$$= \frac{|z|^2 + \bar{b}z + b\bar{z} + |b|^2}{|c|^2 \cdot |z|^2 + c\bar{d}z + \bar{c}d\bar{z} + |d|^2}$$

$$= \frac{1 + \bar{b}z + b\bar{z} + |b|^2}{|c|^2 + c\bar{d}z + \bar{c}d\bar{z} + |d|^2} \quad (\because |z|=1)$$

$$\Rightarrow |c|^2 + c\bar{d}z + \bar{c}d\bar{z} + |d|^2 = 1 + \bar{b}z + b\bar{z} + |b|^2$$

$$|c|^2 + (c\bar{d} - \bar{b})z + (\bar{c}d - b)\bar{z} + |d|^2 = 1 + |b|^2 \quad (1)$$

Let $c\bar{d} - \bar{b} = 0$ and $\bar{c}d - b = 0$. It follows that

$$c\bar{d} = \bar{b} \quad (2)$$

$$\bar{c}d = b \quad (3)$$

Multiplying each side, $c\bar{c}d\bar{d} = b\bar{b} \Rightarrow |c| \cdot |d| = |b|$

Applying it to equation (1),

$$|c|^2 + |d|^2 = 1 + |c|^2 \cdot |d|^2$$

$$(|c|^2 - 1)(|d|^2 - 1) = 0$$

(Cont. Problem 4)

If $|c|=1$, $|b|=|d|$. If $|d|=1$, $|b|=|c|$.

From (2) and (3), we have $\bar{b} = c\bar{d}$ and $d = \frac{b}{c}$.

Next, assume $a=0$. Then we have $f(z) = \frac{b}{cz+d}$.

It's given that $|z|=1$ follows that $|f(z)|=1$.

$$1 = \frac{b}{cz+d} \cdot \frac{\bar{b}}{\bar{c}\bar{z}+\bar{d}} = \frac{b\bar{b}}{c\bar{c}z\bar{z} + c\bar{d}z + \bar{c}d\bar{z} + d\bar{d}}$$

$$= \frac{|b|^2}{|c|^2|z|^2 + c\bar{d}z + \bar{c}d\bar{z} + |d|^2} = \frac{|b|^2}{|c|^2 + c\bar{d}z + \bar{c}d\bar{z} + |d|^2} \quad (\because |z|=1)$$

$$\Rightarrow |c|^2 + c\bar{d}z + \bar{c}d\bar{z} + |d|^2 = |b|^2 \Rightarrow |c|^2 + |d|^2 = |b|^2 \quad (1)$$

Let assume $c\bar{d}=0$ and $\bar{c}d=0$.

$$\Rightarrow c\bar{c}d\bar{d} = |c|^2 \cdot |d|^2 = 0$$

$$|c|=0 \text{ or } |d|=0$$

Applying it to the equation (1), we have

$|b| = |d|$ if $|c|=0$

$|b| = |c|$ if $|d|=0$

$|b| = 0$ if $|c|=0$ and $|d|=0$.