Complex Variables I – Problem Set 4

Due at 5 pm on Friday, Oct 6, 2023 via Gradescope

Problem 1

Let A be a connected open set in $\mathbb{C}$, $f : A \to \mathbb{C}$ is holomorphic. Suppose $|f(z)|$ is constant in A. Prove that $f(z)$ is a constant function.

Proof 1: the first proof uses the inverse function theorem. We prove that $f'(z) = 0$ for all $z \in A$. If not, suppose that $f'(z_0) \neq 0$. By the inverse function theorem, there exists a neighborhood U of z_0 and V of $f(z_0)$ such that $f : U \to V$ is a homeomorphism. In particular, $f(U)$ is not a constant, contradiction. Thus $f' = 0$ in A. Since A is open and connected, it's also path connected. For any two points $p, q \in A$, find a path γ connecting p to q. Then the fundamental theorem of contour integral gives

$$f(q) - f(p) = \int_\gamma f'(z)dz = 0.$$

Proof 2: we can also use the Cauchy-Riemann equation to prove that $f' = 0$ in A. For this, write $f(x + iy) = u(x, y) + iv(x, y)$. Then we have that

$$C = u^2 + v^2$$

for some constant C. Taking partial derivatives, we have that

$$0 = uu_x + vv_x, \quad 0 = uu_y + vv_y.$$

Take another derivative on these equations, we have:

$$0 = u_x^2 + uu_{xx} + v_x^2 + vv_{xx}, \quad 0 = u_y^2 + uu_{yy} + v_y^2 + vv_{yy}.$$

Adding them together and use that u, v are both harmonic, we have that

$$0 = u_x^2 + u_y^2 + v_x^2 + v_y^2.$$

Therefore the partial derivatives of u, v are all zero. Therefore u, v are constant in the connected open set A.

Problem 2

Let $f(z)$ be holomorphic in the disk $|z-1| < 1$, and suppose that $f'(z) = \frac{1}{z}$, $f(1) = 0$. Prove that $f(z) = \log z$ in the disk, where log is the branch that takes value in $A_{-\pi} = \{z : -\pi < \operatorname{Im} z < \pi\}$.

Proof: Note that f is holomorphic and the disk $|z-1| < 1$ is connected. Also, the function $f(z) = \log z$, where log is the branch that takes values in A_π, satisfies all the assumptions. Now suppose g is a holomorphic function satisfying all the assumptions. Let $h = f - g$. Then $h(1) = 0$ and $h'(z) = 0$. Since the disk is connected, we may connect any point p in the disk from 1 by a C^1 curve γ in the disk. By the fundamental theorem, we have

$$h(p) - h(0) = \int_\gamma h'(z)dz = 0.$$

Hence $h(p) = 0$ for all points p in the disk, and thus $g = f$ everywhere.

Problem 3

1. Verify that the function $u(x,y) = \sin x \cosh y$ is harmonic in $\mathbb{R}^2$.
2. Find the harmonic conjugate v of u such that $v(0,0) = 3$.

Solution:
1. We check: $u_{xx} = -\sin x \cosh y$, $u_{yy} = \sin x \cosh y$. Thus $\Delta u = u_{xx} + u_{yy} = 0$.
2. We would need:

$$v_x = -u_y = -\sin x \sinh y, \quad v_y = u_x = \cos x \cosh y.$$

Thus we can solve $v(x,y) = \cos x \sinh y + C$ for some constant C. Since $v(0,0) = 3$, we have that $C = 3$. Hence $v(x,y) = \cos x \sinh y + 3$.

Problem 4

Find the general form of a holomorphic function $f(z)$ whose real parts only depend on $|z|$.

Hint: the real part of a holomorphic function is harmonic.

Solution: Let $u(x,y) = g(x^2+y^2)$ be the real part of f, here $g : \mathbb{R} \to \mathbb{R}$ is a real-valued single variable function. Then u is harmonic. We have that $u_{xx} + u_{yy} = 0$. Thus,

$$g'(x^2+y^2) + (x^2+y^2)g''(x^2+y^2) = 0.$$

Setting $t = x^2+y^2$, we have that $g' + tg'' = 0$. Integrating a first time we obtain: $g'(t) = \frac{c_1}{t}$, $c_1 \in \mathbb{R}$. Integrating a second time, we obtain: $g(t) = c_1 \log t + c_2$. Therefore, we conclude that $u(x,y) = c_1 \log(x^2+y^2) + c_2$, here log stands for the real logarithm.

Next we find the imaginary part v of f. We do so using the Cauchy-Riemann equation. We have:

$$v_x = -u_y = -\frac{2c_1 y}{x^2+y^2}, \quad v_y = u_x = \frac{2c_1 x}{x^2+y^2}.$$

Solving these equations, we have that

$$v(x,y) = c_1 \arctan(\tfrac{y}{x}) + c_3.$$

For some constant $c_3 \in \mathbb{R}$. Thus, we conclude that

$$f(x+iy) = c_1(\log(x^2+y^2) + \arctan(\tfrac{y}{x})) + c_0, \quad c_1 \in \mathbb{R}, c_0 \in \mathbb{C}.$$

Note that $\arctan \frac{y}{x}$ gives the argument of $x+iy$. Thus f has to be some branch of the complex logarithm.

Problem 5

Find the following line integral.

1. $\int_C \log z dz$, here C is the unit circle oriented counterclockwise, and log is the branch which takes value in $A_0 = \{z : 0 \leq \operatorname{Im} z < 2\pi\}$.
2. $\int_C \frac{1}{z} dz$, here C is the line segment from $\frac{-\sqrt{3}-i}{2}$ to $\frac{1+\sqrt{3}i}{2}$.

Solution:

1. Note that here $\log z$ has no holomorphic primitive in an open neighborhood of the unit circle, so we cannot apply the fundamental theorem for contour integrals. Instead, we compute by definition. Parametrize C by $z(t) = \cos t + i \sin t = e^{it}$, $0 \leq t < 2\pi$. Then $\log z(t) = it$ (note that log takes values in A_0), and $z'(t) = ie^{it}$. Thus,

$$\int_C \log z dz = \int_0^{2\pi} -te^{it} dt = ite^{it} - e^{it}\Big|_0^{2\pi} = 2\pi i.$$

2. Use the fundamental theorem: note that $\log z$ with the branch in A_0 is a primitive of $1/z$. Thus,

$$\int_C \frac{1}{z} dz = \log \frac{1+\sqrt{3}i}{2} - \log \frac{-\sqrt{3}-i}{2} = \frac{\pi}{3}i - \frac{7\pi}{6}i = -\frac{5\pi}{6}i.$$

Problem 6

Consider a function f which is holomorphic in the open unit disk centered at the origin $D(0,1)$, and f satisfies

$$\forall z \in D(0,1), |f'(z)| \leq M,$$

for some $M > 0$. Show that for every $z_1, z_2 \in D(0,1)$, $|f(z_1) - f(z_2)| \leq M|z_1 - z_2|$.

Proof: use the fundamental theorem for contour integrals: for any $z_1, z_2 \in D(0,1)$, let γ be the line segment connecting them. We can parametrize γ by $t \in [0,1] \mapsto z(t) = (1-t)z_1 + tz_2$. By the fundamental theorem, $f(z_2) - f(z_1) = \int_\gamma f'(z)dz$. Thus, we have:

$$\begin{aligned} |f(z_2) - f(z_1)| &= \left| \int_0^1 f'(z(t))(z_2 - z_1)dt \right| \\ &\leq \int_0^1 |f'(z(t))||z_2 - z_1|dt \\ &\leq M|z_2 - z_1|. \end{aligned}$$

Problem 7 - bonus

Let $f : \mathbb{C} \to \mathbb{R}$ be a continuous real valued function such that $|f(z)| \leq 1$ for all $z \in \mathbb{C}$. Show that

$$\left| \int_C f(z)dz \right| \leq 4,$$

where C is the unit circle oriented counterclockwise.

Solution: parametrize C by $t \in [0, 2\pi) \mapsto z(t) = e^{it}$. Then

$$\int_C f(z)dz = \int_0^{2\pi} f(e^{it})ie^{it}dt.$$

Denote by $g(t) = f(e^{it})$ and extend g to be a 2π-periodic function on $\mathbb{R}$. Suppose that

$$\rho e^{i\alpha} = \int_C g(t)e^{it}dt,$$

for $\rho \geq 0$ and $\alpha \in \mathbb{R}$. We then would like to estimate the modulus - or equivalently - ρ. For this, we note that

$$\begin{aligned} \rho &= \int_0^{2\pi} g(t)e^{i(t-\alpha)}dt \\ &= \int_0^{2\pi} g(t)(\cos(t-\alpha) + i\sin(t-\alpha))dt \\ &= \int_0^{2\pi} g(t+\alpha)(\cos t + i\sin t)dt. \end{aligned}$$

Where in the last equality we have used that the functions $g, \cos, \sin$ are all 2π-periodic. Now observe that ρ is real, and g is also everywhere real. Thus, it is automatically true that the imaginary part of the integral vanishes, and we have:

$$\rho = \int_0^{2\pi} g(t+\alpha)\cos t dt.$$

We therefore estimate:

$$|\rho| \leq \int_0^{2\pi} |\cos t|dt = 4.$$

(Note: this question will not count in the homework grade.)

Remember to justify your answers and acknowledge collaborations and outside help!