36-402 Homework 5

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Question 1

Q1 a)

$$\frac{1}{2h}\int_{x_0-h}^{x_0+h}(m+t(x-x_0)+c(x-x_0)^2)dx = \frac{1}{2h}\int_{-h}^{h}(m+tu+cu^2)du,\ \ u = x-x_0 \tag{1}$$

$$= \frac{1}{2h}\left[mu+\frac{t}{2}u^2+\frac{c}{3}u^3\right]_{-h}^{h} \tag{2}$$

$$= \frac{1}{2h}(2mh+\frac{2}{3}ch^3) \tag{3}$$

$$= m+\frac{1}{3}ch^2 \tag{4}$$

$$\implies k = \frac{1}{3} \tag{5}$$

Q1 b)

$$\mathbb{E}\left[m(X)\right] = \int_{x_0-h}^{x_0+h} m(x)\frac{1}{2h}dx \tag{6}$$

$$= \frac{1}{2h}\int_{x_0-h}^{x_0+h} m(x)dx \tag{7}$$

$$\approx \frac{1}{2h}\int_{x_0-h}^{x_0+h} m(x_0)+m'(x_0)(x-x_0)+\frac{1}{2}m''(x_0)(x-x_0)^2dx \tag{8}$$

$$= m(x_0)+\frac{m''(x_0)}{3}h^2 \tag{9}$$

Q1 c) Let $A_{x_0} = \{i : |X_i - x_0| \leq h\}$, so $\hat{\mu}(x_0) = \frac{1}{|A_{x_0}|}\sum_{i \in A_{x_0}} Y_i$

$$\begin{aligned}
\mathbb{E}\left[\hat{\mu}(x_0)\right] &= \mathbb{E}\left[\frac{1}{|A_{x_0}|}\sum_{i \in A_{x_0}} Y_i\right] && (10)\\
&= \mathbb{E}\left[\frac{1}{|A_{x_0}|}\sum_{i \in A_{x_0}} \mu(X_i) + \epsilon_i\right] && (11)\\
&= \frac{1}{|A_{x_0}|}\sum_{i \in A_{x_0}} \mathbb{E}\left[\mu(X_i)\right] + \mathbb{E}\left[\epsilon_i\right] && (12)\\
&= \mathbb{E}\left[\mu(X) \mid |X - x_0| \leq h\right] + \mathbb{E}\left[\epsilon \mid |X - x_0| \leq h\right] && (13)\\
&\approx \mu(x_0) + \frac{\mu''(x_0)}{3}h^2 + 0 && (14)\\
&= \mu(x_0) + O(h^2) && (15)\\
\text{Bias}[\hat{\mu}(x_0)] &= \mathbb{E}\left[\hat{\mu}(x_0)\right] - \hat{\mu}(x_0) && (16)\\
&= O(h^2) && (17)
\end{aligned}$$

Q1 d)

$$\begin{aligned}
\mathbb{E}\left[m(X)\right] &= \int_{x_0-h}^{x_0+h} m(x)f(x)dx && (18)\\
&\approx \int_{x_0-h}^{x_0+h} \left(m(x_0) + m'(x_0)(x-x_0) + \frac{1}{2}m''(x_0)(x-x_0)^2\right)\left(f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2\right)dx && (19)\\
&\approx \int_{x_0-h}^{x_0+h} \left(m(x_0)f(x_0) + \left(\frac{1}{2}m(x_0)f''(x_0) + m'(x_0)f'(x_0) + \frac{1}{2}m''(x_0)f(x_0)\right)(x-x_0)^2\right)dx && (20)\\
&= m(x_0)f(x_0) + \frac{\frac{1}{2}m(x_0)f''(x_0) + m'(x_0)f'(x_0) + \frac{1}{2}m''(x_0)f(x_0)}{3}h^2 && (21)
\end{aligned}$$