36-402 Homework 5

Eu Jing Chua
eujingc

February 19, 2019

Question 1

Q1 a)

$$
\begin{aligned}
\frac{1}{2h}\int_{x_0-h}^{x_0+h}(m+t(x-x_0)+c(x-x_0)^2)dx &= \frac{1}{2h}\int_{-h}^{h}(m+tu+cu^2)du,\ u=x-x_0 && (1)\\
&= \frac{1}{2h}\left[mu+\frac{t}{2}u^2+\frac{c}{3}u^3\right]_{-h}^{h} && (2)\\
&= \frac{1}{2h}(2mh+\frac{2}{3}ch^3) && (3)\\
&= m+\frac{1}{3}ch^2 && (4)\\
\implies k &= \frac{1}{3} && (5)
\end{aligned}
$$

Q1 b)

$$
\begin{aligned}
\mathbb{E}\left[m(X)\right] &= \int_{x_0-h}^{x_0+h} m(x)\frac{1}{2h}dx && (6)\\
&= \frac{1}{2h}\int_{x_0-h}^{x_0+h} m(x)dx && (7)\\
&\approx \frac{1}{2h}\int_{x_0-h}^{x_0+h} m(x_0)+m'(x_0)(x-x_0)+\frac{1}{2}m''(x_0)(x-x_0)^2dx && (8)\\
&= m(x_0)+\frac{m''(x_0)}{3}h^2 && (9)
\end{aligned}
$$

Q1 c) Let $A_{x_0} = \{i : |X_i - x_0| \leq h\}$, so $\hat{\mu}(x_0) = \frac{1}{|A_{x_0}|}\sum_{i \in A_{x_0}} Y_i$

$$\begin{aligned}
\mathbb{E}\left[\hat{\mu}(x_0)\right] &= \mathbb{E}\left[\frac{1}{|A_{x_0}|}\sum_{i \in A_{x_0}} Y_i\right] && (10)\\
&= \mathbb{E}\left[\frac{1}{|A_{x_0}|}\sum_{i \in A_{x_0}} \mu(X_i) + \epsilon_i\right] && (11)\\
&= \frac{1}{|A_{x_0}|}\sum_{i \in A_{x_0}} \mathbb{E}\left[\mu(X_i)\right] + \mathbb{E}\left[\epsilon_i\right] && (12)\\
&= \mathbb{E}\left[\mu(X) \mid |X - x_0| \leq h\right] + \mathbb{E}\left[\epsilon \mid |X - x_0| \leq h\right] && (13)\\
&\approx \mu(x_0) + \frac{\mu''(x_0)}{3}h^2 + 0 && (14)\\
&= \mu(x_0) + O(h^2) && (15)\\
\text{Bias}[\hat{\mu}(x_0)] &= \mathbb{E}\left[\hat{\mu}(x_0)\right] - \mu(x_0) && (16)\\
&\approx O(h^2) && (17)
\end{aligned}$$

Q1 d)

$$\begin{aligned}
\mathbb{E}\left[m(X)\right] &= \int_{x_0-h}^{x_0+h} m(x)f(x)dx && (18)\\
&\approx \int_{x_0-h}^{x_0+h} \left(m(x_0) + m'(x_0)(x-x_0) + \frac{1}{2}m''(x_0)(x-x_0)^2\right)\left(f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2\right)dx && (19)\\
&\approx \int_{x_0-h}^{x_0+h} \left(m(x_0)f(x_0) + \left(\frac{1}{2}m(x_0)f''(x_0) + m'(x_0)f'(x_0) + \frac{1}{2}m''(x_0)f(x_0)\right)(x-x_0)^2\right)dx && (20)\\
&= 2hm(x_0)f(x_0) + \frac{m(x_0)f''(x_0) + 2m'(x_0)f'(x_0) + m''(x_0)f(x_0)}{3}h^3 && (21)\\
&\text{Let } B \text{ be the event that } |X - x_0| \leq h && (22)\\
&f(x \mid B) \approx \frac{f(x)}{2hf(x_0)} && (23)\\
&f'(x \mid B) \approx \frac{f'(x)}{2hf(x_0)} && (24)\\
&f''(x \mid B) \approx \frac{f''(x)}{2hf(x_0)} && (25)\\
\mathbb{E}\left[\hat{\mu}(x_0)\right] &= \mathbb{E}\left[\mu(X) \mid B\right] \text{ as shown in c)} && (26)\\
&= 2h\mu(x_0)f(x_0 \mid B) + \frac{\mu(x_0)f''(x_0 \mid B) + 2\mu'(x_0)f'(x_0 \mid B) + \mu''(x_0)f(x_0 \mid B)}{3}h^3 && (27)\\
&\approx \mu(x_0) + \frac{\mu(x_0)f''(x_0) + 2\mu'(x_0)f'(x_0) + \mu''(x_0)f(x_0)}{6f(x_0)}h^2 && (28)\\
&= \mu(x_0) + O(h^2) && (29)\\
\text{Bias}[\hat{\mu}(x_0)] &= \mathbb{E}\left[\hat{\mu}(x_0)\right] - \mu(x_0) && (30)\\
&\approx O(h^2) && (31)
\end{aligned}$$

Question 2

Q2 a)

Let $u = x_1 - x_{01}$ and $v = x_2 - x_{02}$.

$$\frac{1}{(2h)^2}\int_{B_h} u(\vec{x})d\vec{x} \tag{32}$$

$$= \frac{1}{(2h)^2}\int_{x_{01}-h}^{x_{01}+h}\int_{x_{02}-h}^{x_{02}+h} u(\vec{x})d\vec{x} \tag{33}$$

$$= \frac{1}{(2h)^2}\int_{-h}^{h}\int_{-h}^{h}(m + t_1u + t_2v + c_1u^2 + c_2v^2 + c_3uv)dudv \tag{34}$$

$$= \frac{1}{(2h)^2}\int_{-h}^{h}(2hm + 2ht_2v + \frac{2c_1}{3}h^3 + 2hc_2v^2)dv \tag{35}$$

$$= \frac{1}{2h}\int_{-h}^{h}(m + t_2v + \frac{c_1}{3}h^2 + c_2v^2)dv \tag{36}$$

$$= \frac{1}{2h}(2hm + \frac{2c_1}{3}h^3 + \frac{2c_2}{3}h^3) \tag{37}$$

$$= m + \left(\frac{c_1}{3} + \frac{c_2}{3}\right)h^2 \tag{38}$$

Hence $k_1 = \frac{1}{3}$ and $k_2 = \frac{1}{3}$ and $k_3 = 0$.

Q2 b)

$$m(\vec{x}) \approx m(\vec{x_0}) + (x_1 - x_{01})m_{x_1}(\vec{x_0}) + (x_2 - x_{02})m_{x_2}(\vec{x_0}) + \frac{1}{2}\left((x_1 - x_{01})^2 m_{x_1x_1}(\vec{x_0}) + (x_2 - x_{02})^2 m_{x_2x_2}(\vec{x_0})\right) \tag{39}$$

$$+ (x_1 - x_{01})(x_2 - x_{02})m_{x_1x_2}(\vec{x_0}) \tag{40}$$

$$\mathbb{E}\left[m(\vec{X})\right] = \int_{B_h} m(\vec{x})f(\vec{x})d\vec{x} \tag{41}$$

$$= \frac{1}{(2h)^2}\int_{B_h} m(\vec{x})d\vec{x} \tag{42}$$

$$\approx m(\vec{x_0}) + \frac{1}{3}\left(\frac{m_{x_1x_1}(\vec{x_0})}{2} + \frac{m_{x_2x_2}(\vec{x_0})}{2}\right)h^2 \tag{43}$$

$$= m(\vec{x_0}) + O(h^2) \tag{44}$$

Q2 c)

Let $A_{\vec{x}_0} = \{i : |X_{i1} - x_{01}| \leq h \wedge |X_{i2} - x_{02}| \leq h\}$, so $\hat{\mu}(x_0) = \frac{1}{|A_{\vec{x}_0}|} \sum_{i \in A_{\vec{x}_0}} Y_i$

$$\begin{aligned}
\mathbb{E}\left[\hat{\mu}(\vec{x_0})\right] &= \mathbb{E}\left[\frac{1}{|A_{\vec{x}_0}|} \sum_{i \in A_{\vec{x}_0}} Y_i\right] && (45)\\
&= \mathbb{E}\left[\frac{1}{|A_{\vec{x}_0}|} \sum_{i \in A_{\vec{x}_0}} \mu(\vec{X}_i) + \epsilon_i\right] && (46)\\
&= \frac{1}{|A_{\vec{x}_0}|} \sum_{i \in A_{\vec{x}_0}} \mathbb{E}\left[\mu(\vec{X}_i)\right] + \mathbb{E}\left[\epsilon_i\right] && (47)\\
&= \mathbb{E}\left[\mu(\vec{X}) \mid |X_1 - x_{01}| \leq h \wedge |X_2 - x_{02}| \leq h\right] + \mathbb{E}\left[\epsilon \mid |X_1 - x_{01}| \leq h \wedge |X_2 - x_{02}| \leq h\right] && (48)\\
&\approx \mu(\vec{x}_0) + O(h^2) && (49)\\
\text{Bias}[\hat{\mu}(\vec{x}_0)] &= \mathbb{E}\left[\hat{\mu}(\vec{x}_0)\right] - \mu(\vec{x}_0) && (50)\\
&\approx O(h^2) && (51)
\end{aligned}$$

Q3 a)

$$\begin{aligned}
P(\vec{X} \in B_h) &= \int_{B_h} f(\vec{x}) d\vec{x} && (52)\\
&\approx \int_{B_h} f(\vec{x}_0) d\vec{x} && (53)\\
&= f(\vec{x}_0)(2h)^p && (54)
\end{aligned}$$

This approximation is only valid for small h, and if the pdf was smooth. Then, we can approximate the surface around $\vec{x}_0$ as “flat”, having the same density as the center, which is $f(\vec{x}_0)$. The integral just becomes this average density multiplied by the p-dimensional volume of B_h.

Q3 b)

By linearity of expectations, if $P(\vec{x} \in B_h) \approx f(\vec{x}_0)(2h)^p$ for one point, then for n samples, the expected number of points N is:

$$\begin{aligned}
N &= \sum_{i=1}^{n} N_i && (55)\\
\mathbb{E}\left[N_i\right] &= 1 \times P(\vec{X}_i \in B_h) + 0 \times (1 - P(\vec{X}_i \in B_h)) = P(\vec{X}_i \in B_h) && (56)\\
\mathbb{E}\left[N\right] &= \sum_{i=1}^{n} \mathbb{E}\left[N_i\right] && (57)\\
&\approx n f(\vec{x}_0)(2h)^p && (58)
\end{aligned}$$

Q3 c)

Let $A_{\vec{x}_0} = \{i : \vec{X}_i \in B_h\}$, where $N = |A_{\vec{x}_0}|$, so $\hat{\mu}(x_0) = \frac{1}{|A_{\vec{x}_0}|} \sum_{i \in A_{\vec{x}_0}} Y_i$

$$\begin{aligned}
\mathrm{Var}\left[\hat{\mu}(\vec{x}_0)\right] &= \frac{1}{N^2} \mathrm{Var}\left[\sum_{i \in A_{\vec{x}_0}} \mu(\vec{x}_i) + \epsilon_i\right] && (59) \\
&= \frac{1}{N^2} \sum_{i \in A_{\vec{x}_0}} \mathrm{Var}\left[\epsilon_i\right] && (60) \\
&= \frac{\sigma^2}{N}, \text{ assuming that for all } i, \mathrm{Var}\left[\epsilon_i\right] = \sigma^2 && (61) \\
&\approx \frac{\sigma^2}{n f(\vec{x}_0)(2h)^p} && (62) \\
&= O(n^{-1} h^{-p}) && (63)
\end{aligned}$$