36-467 Homework 6

Eu Jing Chua

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Question 1

Q1 a)

$$\begin{aligned} x_t &= bx_{t-1} && (1)\\ &= b^2x_{t-2} && (2)\\ &= b^tx_0 && (3) \end{aligned}$$

Then we can see that $\lim_{t\to\infty} x_t = \lim_{t\to\infty} b^tx_0 = 0, |b| < 1$

Q1 b)

By a similar argument as above, note that $x_t = b^tx_0$.
Then we can see that:

$$\begin{aligned} \lim_{t\to\infty} x_t &= \lim_{t\to\infty} b^tx_0, b > 1 && (4)\\ &= \begin{cases} -\infty, & x_0 < 0 \\ \infty, & x_0 > 0 \end{cases} && (5) \end{aligned}$$

Q1 c) If $b < -1$, then x_t will not have a limit as it will alternate between positive and negative values depending on whether t is even or odd. The answer does not depend on x_0 as there is no limit regardless, unless $x_0 = 0$.

Question 2

Q2 a)

$$\begin{aligned} by_t &= bx_t - \frac{ba}{1-b} && (6)\\ &= a + bx_t - a - \frac{ba}{1-b} && (7)\\ &= x_{t+1} - \frac{a - ba + ba}{1-b} && (8)\\ &= x_{t+1} - \frac{a}{1-b} && (9)\\ &= y_{t+1}, b \neq 1 && (10) \end{aligned}$$

Q2 b) Since $x_t = y_t + \frac{a}{1-b}$,

$$\begin{aligned} \lim_{t\to\infty} x_t &= \lim_{t\to\infty} y_t + \frac{a}{1-b} && (11)\\ &= \lim_{t\to\infty} b^ty_0 + \frac{a}{1-b} && (12)\\ &= \frac{a}{1-b}, |b| < 1 && (13) \end{aligned}$$

Q2 c) When $b > 0$,

$$
\begin{aligned}
\lim_{t\to\infty} x_t &= \lim_{t\to\infty} y_t + \frac{a}{1-b} && (14)\\
&= \lim_{t\to\infty} b^t y_0 + \frac{a}{1-b} && (15)\\
&= \begin{cases} -\infty, & x_0 < \frac{a}{1-b} \\ \infty, & x_0 > \frac{a}{1-b} \end{cases} && (16)
\end{aligned}
$$

Question 3

Q3 a) Since the eigenvalues of $\mathbf{b}$ form a basis, we can write $x_0 = c_1\vec{v_1} + c_2\vec{v_2}$, where $c_1, c_2 \in \mathbb{C}$

Since $\vec{x_t} = \mathbf{b}^t\vec{x_0}$,

$$
\begin{aligned}
\vec{x_t} &= \mathbf{b}^t\left(c_1\vec{v_1} + c_2\vec{v_2}\right) && (17)\\
&= \mathbf{b}^t c_1\vec{v_1} + \mathbf{b}^t c_2\vec{v_2} && (18)\\
&= c_1\mathbf{b}^t\vec{v_1} + c_2\mathbf{b}^t\vec{v_2} && (19)\\
&= c_1\lambda_1^t\vec{v_1} + c_2\lambda_2^t\vec{v_2} && (20)
\end{aligned}
$$

Q3 b) When $|\lambda_i| < 1$,

$$
\begin{aligned}
\lim_{t\to\infty} \vec{x_t} &= \lim_{t\to\infty} c_1\lambda_1^t\vec{v_1} + c_2\lambda_2^t\vec{v_2} && (21)\\
&= \vec{0} && (22)
\end{aligned}
$$

Q3 c) When $\lambda_1 > 0$ and $|\lambda_2| < 0$, the component of $\vec{x_0}$ in the direction of $\vec{v_1}$ grows in magnitude, while the component in the direction of $\vec{v_2}$ shrinks in magnitude, as $t \to \infty$.

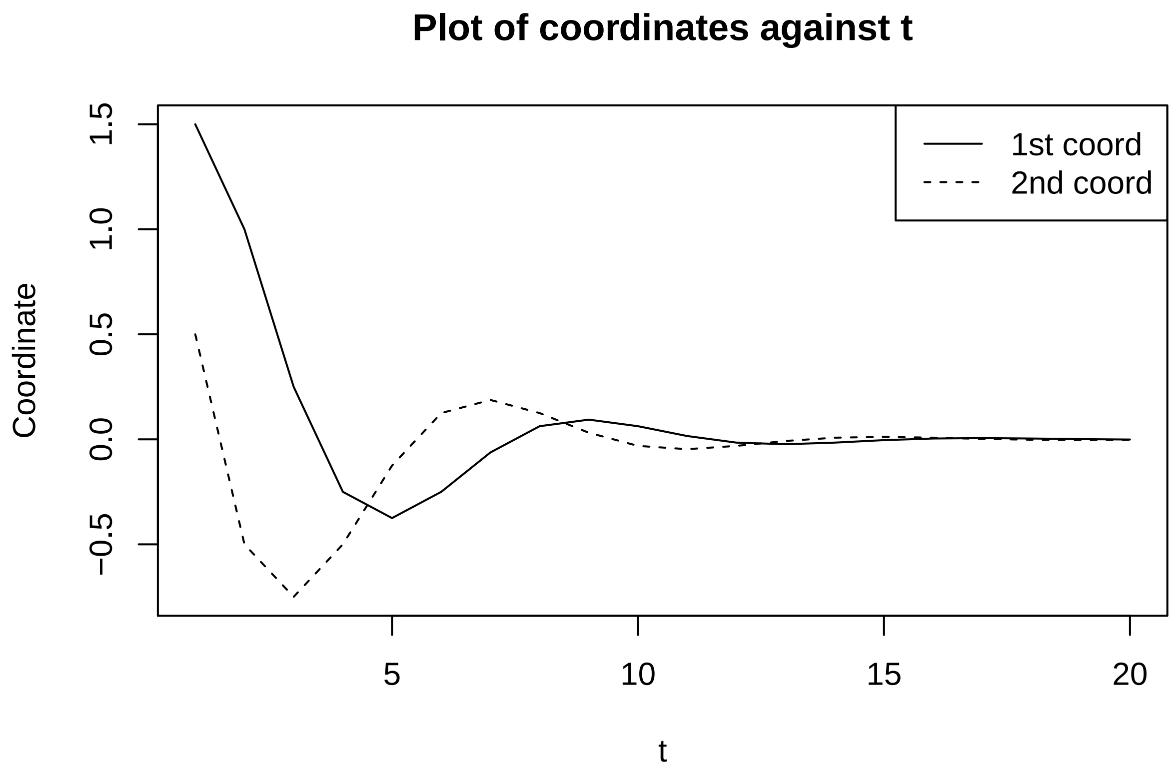
Question 4

Q4 a)

```
## Loading required package: Matrix

##
## Attaching package: 'expm'

## The following object is masked from 'package:Matrix':
##
##     expm
```



Both coordinates seem to undergo oscillation that is exponentially damped.

Q4 b)

Table 1: Matrix b (b.mat)

0.5	0.5
-0.5	0.5

Eigenvalue 1 (lambda.1)
0.5+0.5i

Eigenvalue 2 (lambda.2)
0.5-0.5i

Eigenvector v_1 (v.1)
0-0.707i
0.707+0i

Eigenvector v_2 (v.2)
0+0.707i
0.707+0i

```r
all.equal(lambda.1 * v.1, b.mat %*% v.1)
```

```
## [1] TRUE
```

```r
all.equal(lambda.2 * v.2, b.mat %*% v.2)
```

```
## [1] TRUE
```

Q4 c)

$$\vec{x_0} = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{23}$$

$$\implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix}^{-1} \vec{x_0} \tag{24}$$

$$\tag{25}$$

	Coefficients
c_1	1.41+0.71i
c_2	1.41-0.71i

Q4 d)

We know that when we have complex eigenvectors and eigenvalues from a real-valued matrix that form a basis, the eigenvalues λ_1, λ_2 have to occur as a complex-conjugate pair. Their corresponding eigenvectors $\vec{v_1}, \vec{v_2}$ thus will have conjugate entries. As we have seen above, the coefficients are also a conjugate pair. Thus, $c_1\lambda_1\vec{v_1}, c_2\lambda_2\vec{v_2}$ will also be a conjugate pair. Hence, their sum, x_1, will have $\text{Im}(x_1) = 0$.

Q4 e) Since we know that the eigenvalues of **b** are complex, then **b** represents a linear transform that both scales and rotates. The rotation factor explains why the coordinates have an oscillatory nature, and the magnitude of each eigenvalue $|\lambda_1| = |\lambda_2| = \frac{1}{\sqrt{2}} < 1$, hence the coordinates are shrunken towards 0 with increasing $t \to \infty$

Question 5

Q5 a)

$$\begin{aligned}
\mathbb{E}[X_1] &= \mathbb{E}[a + bX_0 + \epsilon_1] && (26)\\
&= a + b \cdot \mathbb{E}[X_0] + \mathbb{E}[\epsilon_1] && (27)\\
&= a + b\mu + 0 && (28)\\
&= a + b\mu && (29)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[X_0] &= \mathbb{E}[X_1] && (30)\\
\mu &= a + b\mu && (31)\\
\mu - b\mu &= a && (32)\\
\mu &= \frac{a}{1-b} && (33)
\end{aligned}$$

Q5 b)

$$\begin{aligned}
Var[X_1] &= Var[a + bX_0 + \epsilon_1] && (34)\\
&= b^2 \cdot Var[X_0] + Var[\epsilon_1] + 2 \cdot Cov[X_0, \epsilon_1] && (35)\\
&= b^2\sigma^2 + \tau^2 + 0 && (36)\\
&= b^2\sigma^2 + \tau^2 && (37)
\end{aligned}$$

$$\begin{aligned}
Var[X_0] &= Var[X_1] && (38)\\
\sigma^2 &= b^2\sigma^2 + \tau^2 && (39)\\
\sigma^2 - b^2\sigma^2 &= \tau^2 && (40)\\
\sigma^2 &= \frac{\tau^2}{1-b^2} && (41)
\end{aligned}$$

Q5 c)

$$\begin{aligned}
Cov[X_0, X_1] &= Cov[X_0, a + bX_0 + \epsilon_1] && (42)\\
&= Cov[X_0, bX_0 + \epsilon_1] && (43)\\
&= b \cdot Cov[X_0, X_0] + Cov[X_0, \epsilon_1] && (44)\\
&= b \cdot Var[X_0] + 0 && (45)\\
&= \frac{b\tau^2}{1-b^2} && (46)
\end{aligned}$$

Q5 d)

$$\begin{aligned}
\mathbb{E}[X_2] &= \mathbb{E}[a + bX_1 + \epsilon_2] && (47)\\
&= a + b \cdot \mathbb{E}[X_1] + \mathbb{E}[\epsilon_2] && (48)\\
&= a + b\frac{a}{1-b} + 0 && (49)\\
&= \frac{a - ab + ab}{1-b} && (50)\\
&= \frac{a}{1-b} && (51)
\end{aligned}$$

$$\begin{aligned}
Var[X_2] &= Var[a + bX_1 + \epsilon_2] && (52)\\
&= b^2 \cdot Var[X_1] + Var[\epsilon_2] + 2 \cdot Cov[X_1, \epsilon_2] && (53)\\
&= b^2\sigma^2 + \tau^2 + 0 && (54)\\
&= \frac{b^2\tau^2}{1-b^2} + \tau^2 && (55)\\
&= \frac{\tau^2}{1-b^2} && (56)
\end{aligned}$$

$$
\begin{aligned}
Cov[X_1, X_2] &= Cov[X_1, a + bX_1 + \epsilon_2] && (57)\\
&= b \cdot Cov[X_1, X_1] + Cov[X_1, \epsilon_2] && (58)\\
&= b \cdot Var[X_1] + Cov[a + bX_0 + \epsilon_1, \epsilon_2] && (59)\\
&= \frac{b\tau^2}{1 - b^2} && (60)
\end{aligned}
$$

$$
\begin{aligned}
Cov[X_0, X_2] &= Cov[X_0, a + bX_1 + \epsilon_2] && (61)\\
&= b \cdot Cov[X_0, X_1] + Cov[X_0, \epsilon_2] && (62)\\
&= b\frac{b\tau^2}{1 - b^2} + 0 && (63)\\
&= \frac{b^2\tau^2}{1 - b^2} && (64)
\end{aligned}
$$

Q5 e)

$$
\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}[a + b\mathbb{E}[X_{t-1}] + \epsilon_t] && (65)\\
&= a + b \cdot \mathbb{E}[X_{t-1}] && (66)\\
&= a + b(a + b \cdot \mathbb{E}[X_{t-2}]) && (67)\\
&= a + ab + b^2 \cdot \mathbb{E}[X_{t-2}]) && (68)\\
&= a + ab + b^2(a + b \cdot \mathbb{E}[X_{t-3}]) && (69)\\
&= a + ab + ab^2 + b^3\mathbb{E}[X_{t-3}] && (70)\\
&= \dots && (71)\\
&= \sum_{i=0}^{t-1} ab^i + b^t\mathbb{E}[X_0], |b| < 1 && (72)\\
&= a\frac{1 - b^t}{1 - b} + b^t\frac{a}{1 - b} && (73)\\
&= \frac{a}{1 - b} && (74)
\end{aligned}
$$

$$
\begin{aligned}
Var[X_t] &= Var[a + bX_{t-1} + \epsilon_t] && (75)\\
&= b^2Var[X_{t-1}] + Var[\epsilon_t] + 2 \cdot Cov[X_{t-1}, \epsilon_t] && (76)\\
&= b^2Var[X_{t-1}] + \tau^2 && (77)\\
&= b^2\left(b^2Var[X_{t-2}] + \tau^2\right) + \tau^2 && (78)\\
&= \dots && (79)\\
&= (b^2)^t \cdot Var[X_0] + \tau^2\sum_{i=0}^{t-1}(b^2)^i && (80)\\
&= (b^2)^t\frac{\tau^2}{1 - b^2} + \tau^2\frac{1 - (b^2)^t}{1 - b^2} && (81)\\
&= \frac{\tau^2}{1 - b^2} && (82)
\end{aligned}
$$

Q5 f)

$$
\begin{aligned}
Cov[X_t, X_{t+h}] &= Cov[X_t, a\sum_{i=0}^{h-1} b^i + \sum_{i=0}^{h-1} \epsilon_{t+h-i} b^i + b^h X_t], h > 0 && (83) \\
&= Cov[X_t, a\sum_{i=0}^{h-1} b^i] + Cov[X_t, \sum_{i=0}^{h-1} \epsilon_{t+h-i} b^i] + Cov[X_t, b^h X_t] && (84) \\
&= 0 + \sum_{i=0}^{h-1} b^i \cdot Cov[X_t, \epsilon_{t+h-i}] + b^h \cdot Cov[X_t, X_t] && (85) \\
&= 0 + 0 + b^h \cdot Var[X_t] && (86) \\
&= \frac{b^h \tau^2}{1 - b^2} && (87)
\end{aligned}
$$

Q5 g) Since the $\mathbb{E}[X_t] = \frac{a}{1-b} \quad \forall t$, and $Cov[X_t, X_{t+h}] = Cov[X_s, X_{s+h}] = \frac{b^h \tau^2}{1-b^2} \quad \forall t, s$, which is independent of time, we can assume this is a stationary process, given these conditions.