36-467 Homework 8

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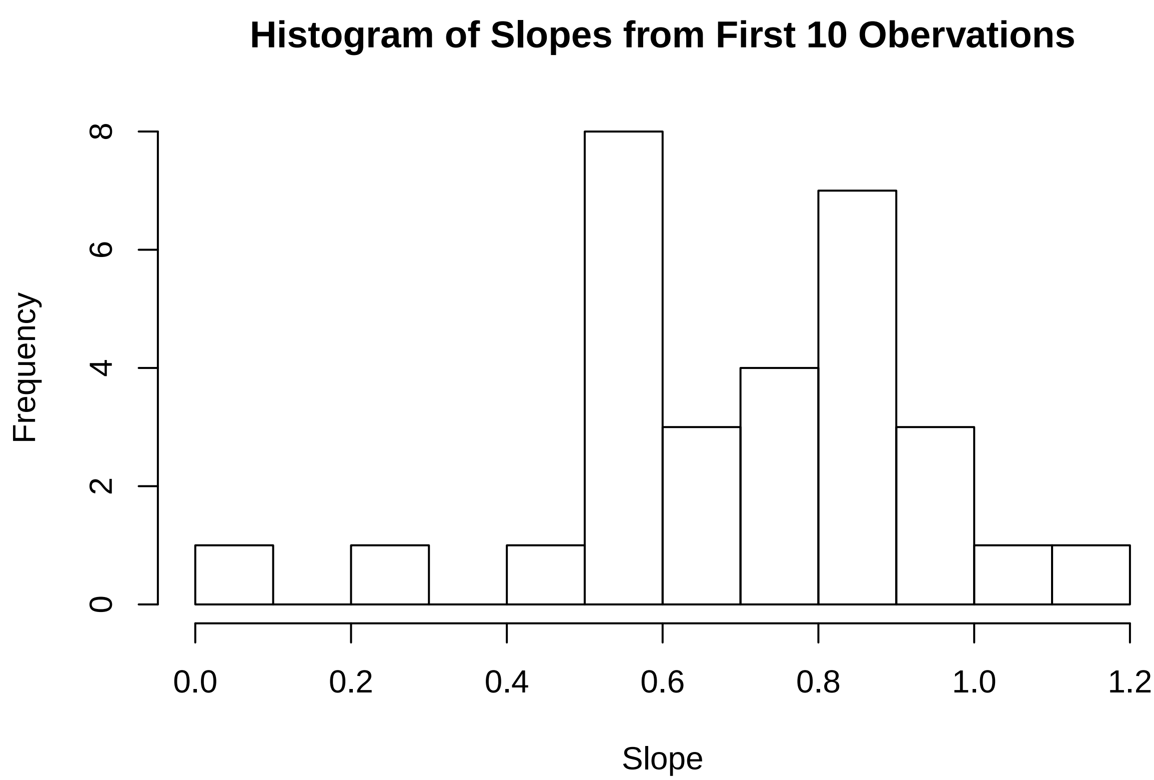
Question 1

Q1 a)

Table 1: First 10 observations

	Statistics
Mean	0.7069422
Std. Dev.	0.2355414



Q1 b)

Table 2: First 100 observations

	Statistics
Mean	0.7834464
Std. Dev.	0.0546427

Q1 c)

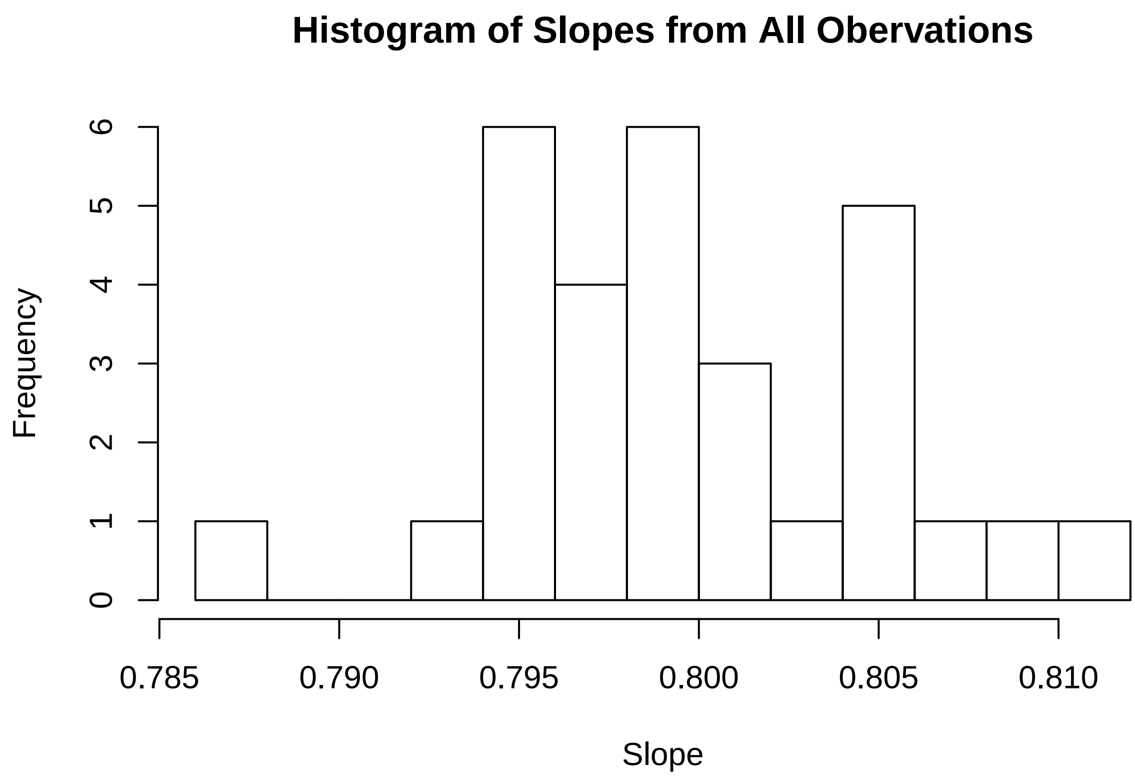
Table 3: First 1000 observations

	Statistics
Mean	0.7914340
Std. Dev.	0.0201159

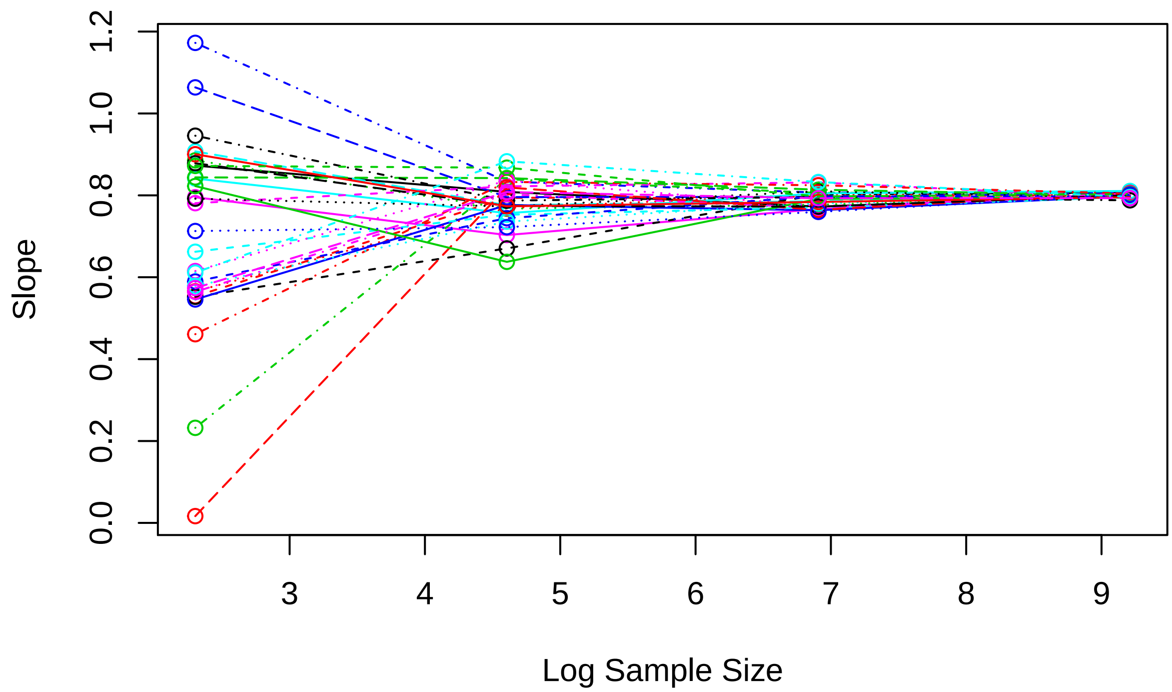
Q1 d)

Table 4: All observations

	Statistics
Mean	0.7996834
Std. Dev.	0.0051938

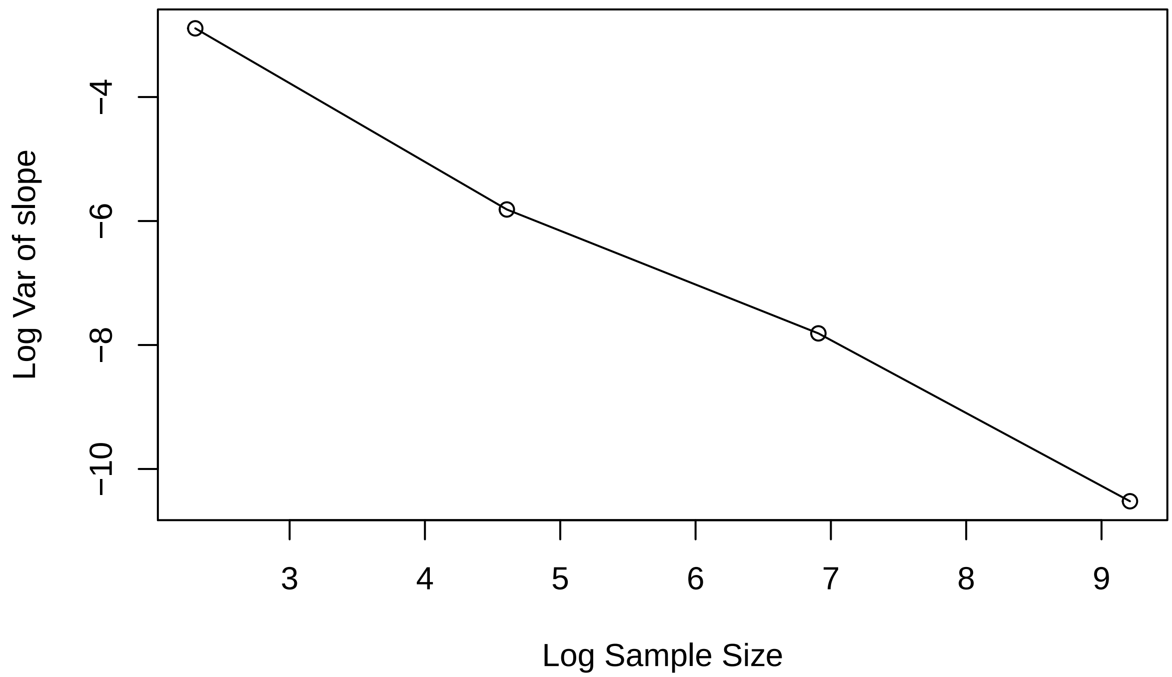


Q1 e)



The estimates are converging, as the variance at each increasing sample size is decreasing.

Q1 f)



This shows that $Var[\hat{b}_n] \propto \frac{1}{n}$, as the plot of the log quantities is roughly a negative linear relationship.

Q1 g) The best guess would be the mean of the slopes from using all the 10000 runs of the simulation, which is 0.8.

Question 2

Q2 a)

$$\begin{aligned} X(t+1) - bX(t) &= \beta X(t) + \epsilon(t+1) - bX(t) && (1) \\ &= (\beta - b)X(t) + \epsilon(t+1) && (2) \end{aligned}$$

Q2 b)

$$\begin{aligned} \mathbb{E}\left[X(t+1) - bX(t)\right] &= \mathbb{E}\left[(\beta - b)X(t) + \epsilon(t+1)\right] && (3) \\ &= (\beta - b)\mathbb{E}\left[X(t)\right] + \mathbb{E}\left[\epsilon(t+1)\right] && (4) \\ &= 0 && (5) \end{aligned}$$

Q2 c)

$$\mathbb{E}\left[(X(t+1)-bX(t))^2\right] = \left(\mathbb{E}\left[X(t+1)-bX(t)\right]\right)^2 + \mathrm{Var}\left[X(t+1)-bX(t)\right] \tag{6}$$
$$= 0 + \mathrm{Var}\left[X(t+1)\right] + b^2\mathrm{Var}\left[X(t)\right] - 2\cdot\mathrm{Cov}\left[X(t+1), bX(t)\right] \tag{7}$$
$$= \frac{\tau^2}{1-\beta^2} + b^2\frac{\tau^2}{1-\beta^2} - 2b\frac{\beta\tau^2}{1-\beta^2} \tag{8}$$
$$= \tau^2\left(\frac{b^2-2b\beta+1}{1-\beta^2}\right) \tag{9}$$
$$= \tau^2\left(\frac{1-\beta^2+b^2-2b\beta+\beta^2}{1-\beta^2}\right) \tag{10}$$
$$= \tau^2\left(1+\frac{(\beta-b)^2}{1-\beta^2}\right) \tag{11}$$

Q2 d)

$$\frac{dm}{db} = \frac{d}{db}\left[\tau^2\left(1+\frac{(\beta-b)^2}{1-\beta^2}\right)\right] \tag{12}$$
$$= \tau^2(-2)\frac{(\beta-b)}{1-\beta^2} \tag{13}$$
$$= -2\tau^2\frac{\beta-b}{1-\beta^2} \tag{14}$$
$$= 0 \iff \beta = b \tag{15}$$

Q2 e)

$$\frac{d^2m}{db^2} = \frac{d}{db}\left[-2\tau^2\frac{\beta-b}{1-\beta^2}\right] \tag{16}$$
$$= \frac{2\tau^2}{1-\beta^2} \tag{17}$$

Q2 f)

$$\frac{dM_n}{db}(b) = \frac{1}{n-1}\frac{d}{db}\left[\sum_{t=1}^{n-1}(X(t+1)-bX(t))^2\right] \tag{18}$$
$$= \frac{1}{n-1}\left[\sum_{t=1}^{n-1}\frac{d}{db}(X(t+1)-bX(t))^2\right] \tag{19}$$
$$= \frac{1}{n-1}\left[\sum_{t=1}^{n-1}(-2X(t))(X(t+1)-bX(t))\right] \tag{20}$$
$$= \frac{-2}{n-1}\left[\sum_{t=1}^{n-1}X(t)(X(t+1)-bX(t))\right] \tag{21}$$
$$\tag{22}$$

Q2 g) If $b=\beta$,

$$\frac{dM_n}{db}(\beta) = \frac{-2}{n-1}\left[\sum_{t=1}^{n-1} X(t)\left(X(t+1) - \beta X(t)\right)\right] \tag{23}$$

$$= \frac{-2}{n-1}\left[\sum_{t=1}^{n-1} X(t)\left(\beta X(t) + \epsilon(t+1) - \beta X(t)\right)\right] \tag{24}$$

$$= \frac{-2}{n-1}\sum_{t=1}^{n-1} X(t)\epsilon(t+1) \tag{25}$$

Q2 h) Since $X(t)$ and $\epsilon(t+1)$ are independent,

$$\text{Var}\left[X(t)\epsilon(t+1)\right] = \text{Var}\left[X(t)\right]\cdot \text{Var}\left[\epsilon(t+1)\right] \tag{26}$$

$$= \frac{\tau^2}{1-\beta^2}\cdot \tau^2 \tag{27}$$

$$= \frac{\tau^4}{1-\beta^2} \tag{28}$$

Q2 i)

$$\text{Cov}\left[X(t)\epsilon(t+1), X(t+h)\epsilon(t+h+1)\right] \tag{29}$$

$$= \mathbb{E}\left[X(t)\epsilon(t+1)X(t+h)\epsilon(t+h+1)\right] + \mathbb{E}\left[X(t)\epsilon(t+1)\right]\mathbb{E}\left[X(t+h)\epsilon(t+h+1)\right] \tag{30}$$

$$= \mathbb{E}\left[X(t)\epsilon(t+1)X(t+h)\epsilon(t+h+1)\right] + \mathbb{E}\left[X(t)\right]\mathbb{E}\left[\epsilon(t+1)\right]\mathbb{E}\left[X(t+h)\right]\mathbb{E}\left[\epsilon(t+h+1)\right] \tag{31}$$

$$= \mathbb{E}\left[X(t)\epsilon(t+1)X(t+h)\epsilon(t+h+1)\right] + 0, \qquad \mathbb{E}\left[\epsilon(t+1)\right] = 0 \tag{32}$$

$$= \mathbb{E}\left[\mathbb{E}\left[X(t)\epsilon(t+1)X(t+h)\epsilon(t+h+1)|X(t+h)\right]\right] \tag{33}$$

$$= \mathbb{E}\left[X(t)\epsilon(t+1)X(t+h)\cdot\mathbb{E}\left[\epsilon(t+h+1)|X(t+h)\right]\right] \tag{34}$$

$$= \mathbb{E}\left[X(t)\epsilon(t+1)X(t+h)\cdot 0\right] \tag{35}$$

$$= 0 \tag{36}$$

Q2 j)
Note that $\frac{dM_n}{db}(\beta) = -2\mathbb{E}\left[X(t)\epsilon(t+1)\right]$
Also from above, we know that $\sum_{h=1}^{\infty}\text{Cov}\left[X(t)\epsilon(t+1), X(t+h)\epsilon(t+h+1)\right] = 0 < \infty$
Thus as $n\to\infty$, $\mathbb{E}\left[X(t)\epsilon(t+1)\right]\to 0 \implies \frac{dM_n}{db}\to 0$

Q2 k) Let $\hat{b}_n = \text{argmin}_b M_n(b)$
We know that $\lim_{n\to\infty} M_n(b) = \mathbb{E}\left[(X(t+1) - bX(t))^2\right] = m(b)$.
From 2d), we know that $b=\beta$ minimizes $m(b)$ and it is a minimum as in 2e), we know $\frac{d^2m}{db^2} > 0$.
Thus, we can conclude that as $n\to\infty$, $\hat{b}_n\to\beta$

Q2 l)

$$\text{Var}\left[\frac{dM_n}{db}(\beta)\right] = \text{Var}\left[\frac{-2}{n-1}\sum_{t=1}^{n-1} X(t)\epsilon(t+1)\right] \tag{37}$$

$$= \frac{4}{(n-1)^2}\sum_{t=1}^{n-1}\text{Var}\left[X(t)\epsilon(t+1)\right], \text{as the covariances are } 0 \tag{38}$$

$$= \frac{4}{(n-1)^2}(n-1)\frac{\tau^4}{1-\beta^2} \tag{39}$$

$$= \frac{4}{(n-1)}\frac{\tau^4}{1-\beta^2} \tag{40}$$

Q2 m)

$$\operatorname{Var}\left[\hat{b}_n\right] \approx \left(\frac{d^2 m}{db^2}\right)^{-2} \operatorname{Var}\left[\frac{dM_n}{db}(\beta)\right] \tag{41}$$

$$= \frac{(1-\beta^2)^2}{4\tau^4} \frac{4}{(n-1)} \frac{\tau^4}{1-\beta^2} \tag{42}$$

$$= \frac{1-\beta^2}{n-1} \tag{43}$$

Q2 n)

	Est. Std. Err.	Std. Dev.
10	0.23576	0.23554
100	0.06246	0.05464
1000	0.01934	0.02012
10000	0.00600	0.00519

The estimates match the actual standard deviations quite well, and they should match as the sum of covariances is finite and n becomes relatively large, and since we also know the underlying model was AR(1) which we model it with.

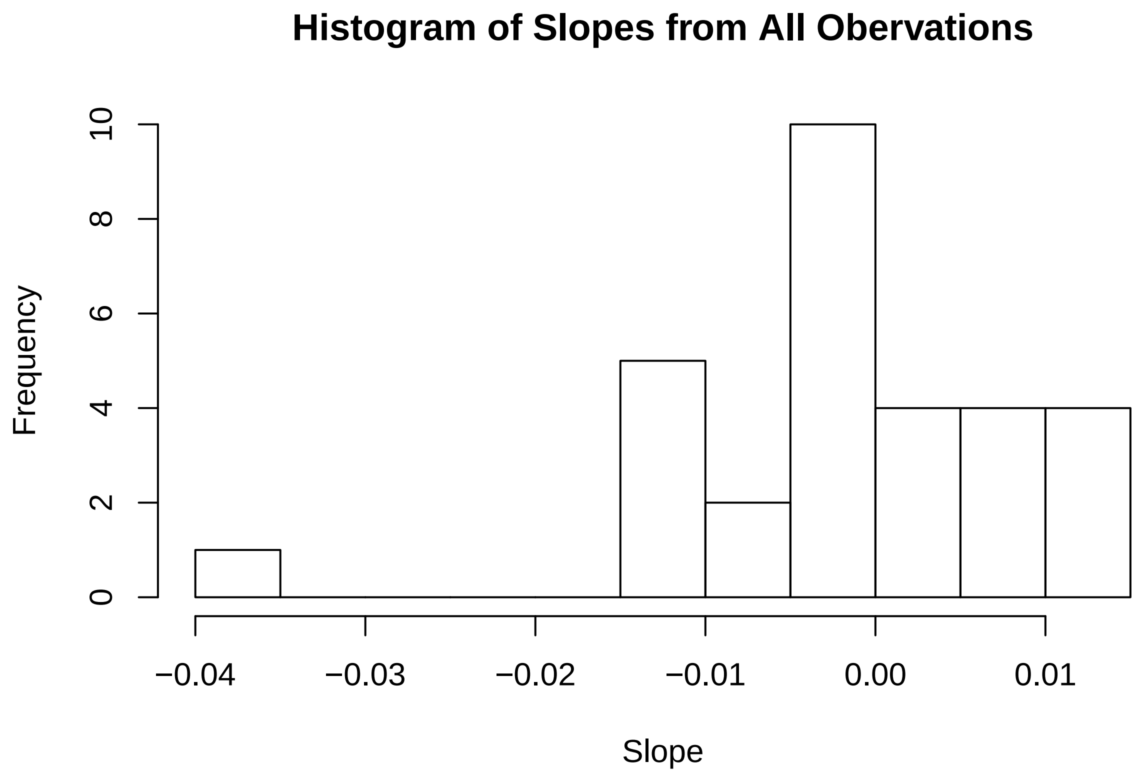
Q2 o) We know that $\operatorname{Var}[\epsilon(t)] = \tau^2$, and $\frac{d^2m}{db^2} \propto \tau^2$. When there is more noise, this increase in variance causes each point to potentially be perturbed more at each step. Thus, as we can see from the equation for $\frac{d^2m}{db^2}$, increased variance in noise causes the curvature to increase too.

Question 3

Q3 a)

Table 6: All observations

	Statistics
Mean	-0.001389
Std. Dev.	0.010438



Q3 b)

$$\frac{d^2M_n}{db^2}(b) = \frac{d}{db}\left[\frac{-2}{n-1}\sum_{t=1}^{n-1} X(t)(X(t+1) - bX(t))\right] \tag{44}$$

$$= \frac{-2}{n-1}\sum_{t=1}^{n-1} \frac{d}{db}\left[X(t)(X(t+1) - bX(t))\right] \tag{45}$$

$$= \frac{-2}{n-1}\sum_{t=1}^{n-1} -X^2(t) \tag{46}$$

$$= \frac{2}{n-1}\sum_{t=1}^{n-1} X^2(t) \tag{47}$$

b does not appear in this formula as the curvature of $M_n(b)$ is independent of b, as it is a parabolic function with respect to b.

Q3 c)

$\frac{d^2M_n}{db^2} = 0.24815$

Q3 d) Regardless of the real distribution of $X(t)$, we can still find the best fitting AR(1) model with no intercept by minimizing the mean-squared error, or $M_n(b)$, hence 2f) is still relevant in this process.

Q3 e)

Estimated Variance
6.1e-06

This is a reasonable approximation of the variance as since we know that the runs are stationary with expectation 0, the first moment $\mathbb{E}\left[\frac{dM_n}{db}(\beta)\right] \approx 0$ as $X(t+1) \approx \hat{b}_n X(t)$. Thus, $\mathrm{Var}\left[\frac{dM_n}{db}(\beta)\right] \approx \mathbb{E}\left[(\frac{dM_n}{db}(\beta))^2\right]$.

Q3 f)

	Slope variance
Estimated	0.0000995
Across Runs	0.0001089

The estimated variance closely matches that found across simulation runs. They should match as n is large, so $M_n'' \to m''$, as found in 3c). Combining with the result in 3e), we can get a good approximation of $\mathrm{Var}\left[\hat{b}_n\right]$ with large n.