# Notes on Kähler-Ricci Flow

Yuang Shi

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**Abstract**

Lecture notes on Kähler Ricci flow. We firstly give a fundamental introduction to Kähler Ricci flow. Then we present some of their applications to Kähler-Einstein metrics, Minimal Model Program (MMP) and Yau's uniformization conjecture.

## Contents

# 1 Preliminaries on Fully Nonlinear Parabolic Equations

## 1.1 Main Objects and Notation

### 1.1.1 Geometric Objects

Firstly, we consider a connected open bounded set $\Omega \subset \mathbb{R}^d$. We refer to such a set as a domain.

**Definition 1.1.1.** *A domain $\Omega$ is $C^{2,\alpha}$ if locally the boundary of the domain can be represented as the graph of a function with two derivatives that are $\alpha$-Hölder continuous.*

Parabolic equations are considered in cylindrical domain of the form $(0,T) \times \Omega$.

**Definition 1.1.2.** *The parabolic boundary of $(0,T) \times \Omega$ is denoted by $\partial_p(0,T) \times \Omega$; it's defined as follows*

$$\partial_p(0,T) \times \Omega = \{0\} \times \Omega \cup (0,T) \times \partial\Omega. \tag{1.1}$$

The following elementary cylindrical domains play a central role in the theory: for all $\rho > 0$ and $x \in \mathbb{R}^d$, we define

$$Q_\rho(t,x) = (t-\rho,t) \times B_\rho(x). \tag{1.2}$$

When we write $Q_\rho$, we mean $Q_\rho(0,0)$. Therefore $Q_\rho(t,x) = (t,x) + Q_\rho$ and $Q_\rho = \rho Q_1$.

### 1.1.2 A Linear Operator

The general parabolic equation involves the following linear operator

$$Lu = \sum_{i,j} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(t,x) \frac{\partial u}{\partial x} + c(t,x)u. \tag{1.3}$$

where unknown function $u : (0,T) \times \Omega \to \mathbb{R}$ depends on two variables: $t \in \mathbb{R}$. and $x \in \mathbb{R}^d$.

Also, the linear operator introduced above can be written as follows

$$Lu = \mathrm{trace}(AD^2u) + b \cdot Du + cu. \tag{1.4}$$

where $A = (a_{ij})_{ij}$.

## 1.2 Spaces of Hölder Functions

Because now we study parabolic equations, Hölder continuity of solutions refers to uniform continuity with respect to

$$\rho(X,Y) = \sqrt{|t-s|} + |x-y|. \tag{1.5}$$

where $X = (t,x)$ and $Y = (s,y)$. In other words, solutions are always twice more regular with respect to the space variable than with respect to the time variable.

**Definition 1.2.1.** *Let $Q \subset (0,T) \times \Omega$ and $\alpha \in (0,1]$.*

*(a) We say that $u \in C^{0,\alpha}(Q)$ if $u$ is $\frac{\alpha}{2}$-Hölder continuous in $t$ and $\alpha$-Hölder continuous in $x$, and*

$$
\begin{aligned}
[u]_{\alpha,Q} &= \sup_{X,Y\in Q, X\neq Y} \frac{u(X)-u(Y)|}{\rho(X,Y)^{\alpha}};\\
|u|_{0,Q} &= \sup_{X\in Q} |u|(x);\\
|u|_{\alpha,Q} &= |u|_{0,Q} + [u]_{\alpha,Q};
\end{aligned} \tag{1.6}
$$

*(b) We say that $u \in C^{1,\alpha}(Q)$ if $u$ is $\frac{\alpha+1}{2}$-Hölder continuous in $t$ and $Du$ is $\alpha$-Hölder continuous in $x$.*

*(c) We say that $u \in C^{2,\alpha}(Q)$ if $u_t$ is $\frac{\alpha}{2}$-Hölder continuous in $t$ and $D^2u$ is $\alpha$-Hölder continuous in $x$, and*

$$
\begin{aligned}
[u]_{2+\alpha,Q} &= [u_t]_{\alpha,Q} + [D^2u]_{\alpha,Q};\\
|u|_{2+\alpha,Q} &= |u|_{0,Q} + |Du|_{0,Q} + |D^2u|_{0,Q} + |u_t|_{0,Q} + [u]_{2+\alpha,Q};
\end{aligned} \tag{1.7}
$$

We will use repeatedly the following elementary proposition.

**Proposition 1.2.1.**

$$[uv]_{\alpha,Q} \le |u|_{0,Q}[v]_{\alpha,Q} + |v|_{0,Q}[u]_{\alpha,Q}. \tag{1.8}$$

*and for $k = 0, 2$,*

$$[u+v]_{k+\alpha,Q} \le [u]_{k+\alpha,Q} + [v]_{k+\alpha,Q}. \tag{1.9}$$

The following proposition implies that in particular that in order to control the norm $|u|_{2+\alpha,Q}$, it's enough to control $|u|_{0,Q}$ and $[u]_{2+\alpha,Q}$.

**Proposition 1.2.2** (Interpolation Inequalities)**.** *For all $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that for all $u \in C^{2,\alpha}(Q)$,*

$$|u_t|_{0,Q} \le \varepsilon\,[u_t]_{\alpha,Q} + C(\varepsilon)|u|_{0,Q}; \tag{1.10a}$$

$$|u_{xx}|_{0,Q} \le \varepsilon[u]_{2+\alpha,Q} + C(\varepsilon)|u|_{0,Q}; \tag{1.10b}$$

$$|u_x|_{0,Q} \le \varepsilon[u]_{2+\alpha,Q} + C(\varepsilon)|u|_{0,Q}; \tag{1.10c}$$

$$[u_x]_{\alpha,Q} \le \varepsilon[u]_{2+\alpha,Q} + C(\varepsilon)|u|_{0,Q}; \tag{1.10d}$$

$$[u]_{\alpha,Q} \le \varepsilon[u]_{2+\alpha,Q} + C(\varepsilon)|u|_{0,Q}; \tag{1.10e}$$

The following proposition is a precise parabolic statement of the following elliptic fact: in order to control the Hölder modulus of continuity of the gradient of $u$, it's enough to make sure that around each point, $u$ can be perturbed linearly so that the oscillation of $u$ in a ball of radius $r > 0$ is of order $r^{1+\alpha}$.

**Proposition 1.2.3** (An Equivalent Seminorm)**.** *There exists $C \ge 1$ such that for all $u \in C^{2,\alpha}(Q)$,*

$$C^{-1}[u]'_{2+\alpha,Q} \le [u]_{2+\alpha,Q} \le C[u]'_{2+\alpha,Q}. \tag{1.11}$$

*where*

$$[u]'_{2+\alpha,Q} = \sup_{X\in Q}\sup_{\rho>0} \rho^{-2-\alpha} \inf_{p\in\mathscr{P}_2} |u-P|_{0,Q_\rho(X)\cap Q}. \tag{1.12}$$

*where*

$$\mathscr{P}_2 = \{\alpha t + p\cdot x + \frac{1}{2}Xx\cdot x + c \mid \alpha, c \in \mathbb{R}, p \in \mathbb{R}^d, X \in \mathbb{S}_d\}. \tag{1.13}$$

The proofs of the tow previous propositions are showed in [13].

## 1.3 Schauder Estimates for Linear Parabolic Equations

In this first section, we state a fundamental existence and uniqueness result for linear parabolic equations with Hölder continuous coefficients. We only focus on two particular aspects: uniqueness and interior estimates.

### 1.3.1 Linear Parabolic Equations

Then general form of a linear parabolic equation with variable coefficients is the following

$$\frac{\partial u}{\partial t} - \sum_{i,j} a_{ij}(X) \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_i b_i(X) \frac{\partial u}{\partial x_i} - c(X)u = 0. \tag{1.14}$$

where $c \leq 0$ and $A = (a_{ij}(X))_{i,j}$ is a symmetric matrix satisfying one of the following assumptions

- (Degenerate ellipticity) for all $X$, $A(X) \geq 0$;
- (Strict ellipticity) there exists $\lambda > 0$ s.t. for all $X$, $A(X) \geq \lambda I$;
- (Uniform ellipticity) there exists $\Lambda \geq \lambda > 0$ s.t. for all $X$, $\lambda I \leq A(X) \leq \Lambda I$;

and we define the linear differential operator $L$ as follows

$$Lu = \sum_{i,j} a_{ij}(X) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(X) \frac{\partial u}{\partial t} + c(X)u. \tag{1.15}$$

### 1.3.2 Global Schauder A Priori Estimates

In this subsection, we state a fundamental existence and uniqueness result for linear parabolic equation with Hölder continuous coefficients. Such a result together with its proof can be found in [13].

In the following, $\mathbb{R}^{d+1}_+$ denotes $[0, +\infty) \times \mathbb{R}^d$.

**Theorem 1.3.1** (Global Schauder A Priori Estimates). *If $\Omega$ is a $C^{2,\alpha}$ domain and the coefficients $A, b, c \in C^\alpha((0,T) \times \Omega)$ and $f \in C^\alpha(\mathbb{R}^{d+1}_+)$, $g \in C^{2,\alpha}((0,T) \times \Omega)$, $h \in C^{2,\alpha}(\mathbb{R}^d)$, and $g$ and $h$ are compatible, then there exists a unique solution $u \in C^{2,\alpha}(Q)$ of*

$$\begin{cases} \frac{\partial u}{\partial t} - Lu = f & \text{in } (0,T) \times \Omega \\ u = g & \text{on } (0, +\infty) \times \partial\Omega \\ u = h & \text{on } \{0\} \times \bar{\Omega} \end{cases} \tag{1.16}$$

*In addition,*

$$|u|_{2+\alpha,(0,T)\times\Omega} \leq C(|f|_{\alpha,\mathbb{R}^{d+1}_+} + |g|_{2+\alpha,(0,T)\times\Omega} + |h|_{2+\alpha,\mathbb{R}^d}). \tag{1.17}$$

*where $C = C(d, \lambda, K, \alpha, \rho_0, \mathrm{diam}(\Omega))$ and $K = |A|_{\alpha,(0,T)\times\Omega} + |b|_{\alpha,(0,T)\times\Omega} + |c|_{\alpha,(0,T)\times\Omega}$ and $\rho_0$ is related to the $C^{2,\alpha}$ regularity of the boundary of $\Omega$.*

### 1.3.3 Maximum and Comparison Principles

**Theorem 1.3.2** (Maximum Principle)**.** *Consider a bounded continuous function* $u : (0,T) \times \Omega \to \mathbb{R}$ *such that* $\frac{\partial u}{\partial t}$ *exists at each point of* $(0,T) \times \Omega$ *and* $Du, D^2u$ *exists and are continuous in* $(0,T) \times \Omega$.

*If*

$$\frac{\partial u}{\partial t} - Lu \leq 0 \text{ in } (0,T) \times \Omega. \tag{1.18}$$

*and*

$$u \leq 0 \text{ on } \partial_p(0,T) \times \Omega. \tag{1.19}$$

*then* $u \leq 0$ *in* $(0,T) \times \Omega$.

*Proof.* Fix $\gamma > 0$ and consider the function $v(t,x) = u(t,x) - \frac{\gamma}{T-t}$. Assume that $v$ is not non-positive. Then the maximal can be only taken at some $t \in (0,T)$ and $x \in \Omega$. In particular,

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t}(t,x) = \frac{\partial u}{\partial t}(t,x) - \frac{\gamma}{(T-t)^2}; \\ 0 &= Dv(t,x) = Du(t,x); \\ 0 &\geq D^2v(t,x) = D^2u(t,x). \end{aligned} \tag{1.20}$$

Since $A$ is elliptic, $Lu(t,x) \leq 0$. Since $u(t,x) \geq v(t,x) > 0, c \leq 0$, $A \geq 0$ and $D^2u \leq 0$. Therefore

$$\frac{\gamma}{(T-t)^2} = \frac{\partial u}{\partial t}(t,x) \leq Lu(t,x) \leq 0. \tag{1.21}$$

Contradiction! Since $\gamma$ is arbitrary, the proof is complete. □

We state two direct corollaries.

**Corollary 1.3.1** (Comparison Principle I)**.** *Consider two bounded continuous functions* $u$ *and* $v$ *which are differentiable with respect to time and such that first and second derivatives with respect to space are continuous. If*

$$\begin{aligned} &\frac{\partial u}{\partial t} - Lu \leq f \text{ in } (0,T) \times \Omega; \\ &\frac{\partial v}{\partial t} - Lv \geq f \text{ in } (0,T) \times \Omega; \end{aligned} \tag{1.22}$$

*and* $u \leq v$ *in* $\partial_p Q$, *then* $u \leq v$ *in* $(0,T) \times \Omega$.

The next result is a first estimate for solutions of linear parabolic equations.

**Corollary 1.3.2** (A First Estimate)**.** *Consider a bounded continuous solution* $u$ *of* $\frac{\partial u}{\partial t} - Lu = f$ *in* $(0,T) \times \Omega$. *Assume moreover that it is differentiable with respect to time and continuously twice differentiable with respect to space. Then*

$$|u|_{0,(0,T)\times\Omega} \leq T|f|_{0,(0,T)\times\Omega} + |g|_{0,\partial_p(0,T)\times\Omega}. \tag{1.23}$$

*Proof.* Consider $v^{\pm} = u \pm (|g|_{0,\partial_p(0,T)\times\Omega} + t|f|_{0,(0,T)\times\Omega})$, then $v^+$ is a supersolution and $v^-$ is a subsolution. Then Corollary 1.3.1 yields the result. □

### 1.3.4 Schauder Estimate For the Heat Equation

The "interior" Schauder estimate for the heat equation takes the following form.

**Theorem 1.3.3.** *Let $\alpha \in (0,1)$ and consider a $C^\infty$ function $u : \mathbb{R}^{d+1} \to \mathbb{R}$ with compact support and define $f = \frac{\partial u}{\partial t} - \Delta u$. There there exists a constant $C > 0$ only depending on dimension and $\alpha$ such that*

$$[u]_{2+\alpha,\mathbb{R}^{d+1}} \leq C[f]_{\alpha,\mathbb{R}^{d+1}}. \tag{1.24}$$

Before proving Theorem 1.3.3, we recall two facts about heat equation. Firstly, a solution $u \in C^\infty$ of

$$\frac{\partial u}{\partial t} - \Delta u = f. \tag{1.25}$$

with compact support included in $(0, +\infty) \times \mathbb{R}^d$, can be represented as

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(s,y) f(t-s, x-y) ds dy. \tag{1.26}$$

where

$$G(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}. \tag{1.27}$$

we write in short hand

$$u = G * f. \tag{1.28}$$

**Fact 1.** For any $0 \leq \rho \leq R$,

$$|G * \chi_{Q_R(Z_0)}|_{0,Q_p(Z_0)} \leq CR^2. \tag{1.29}$$

**Fact 2.** There exists a constant $C > 0$ such that any solution of $\frac{\partial h}{\partial t} = \Delta h$ in $Q_R(0)$ satisfies

$$\left| \frac{\partial^n}{\partial t^n} D^\alpha h(0) \right| \leq C \frac{|h|_{0,Q_R(0)}}{R^{2n+|\alpha|}}. \tag{1.30}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \sum\limits_i \alpha_i$ and $D^\alpha h = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} h$.

The proof of the second fact is referred to Theorem 8.4.4 in [13], using Bernstein's techniques.

Now we start to prove Theorem 1.3.3:

*Proof.* WLOG, assume that the compact support of $u$ is included in $(0, +\infty) \times \mathbb{R}^d$.

Take $X_0 \in \mathbb{R}^{d+1}$, $\rho > 0$ and $K \geq 1$ to be specified later. Let $Q$ denote $Q_{(K+1)\rho}(X_0)$ and take $\xi \in C^\infty(\mathbb{R}^{d+1})$ with compact support and such that $\xi \equiv 1$ in $Q$.

We consider the "second order" Taylor polynomial associated with a function $w$ at a point $X = (t,x)$

$$T_X w(s,y) = w(X) + w_t(X)(s-t) + Dw(X) \cdot (y-x) + \frac{1}{2} D^2 w(X)(y-x) \cdot (y-x). \tag{1.31}$$

We now consider

$$g = (\xi T_{X_0} u)_t - \Delta(\xi T_{X_0} u). \tag{1.32}$$

Thus $g \equiv f(X_0)$ in $Q$. Then for $X \in Q$,

$$u - T_{X_0}u = u - \xi T_{X_0}u = G * (f - g) = h + r. \tag{1.33}$$

with

$$h = G * ((f - g)\chi_{Q^c} \text{ and } r = G * ((f - f(X_0))\chi_Q). \tag{1.34}$$

Remark in particular that

$$h_t - \Delta h = 0 \text{ in } Q. \tag{1.35}$$

Now we estimate

$$\left|u - T_{X_0}u - T_{X_0}h\right|_{0,Q_\rho(X_0)} \le \left|h - T_{X_0}h\right|_{0,Q_\rho(X_0)} + |r|_{0,Q_\rho(X_0)}. \tag{1.36}$$

and we study the two terms of the right hand side. We use Fact 1 to get first

$$\begin{aligned} |r|_{0,Q_\rho(X_0)} &\le [f]_{\alpha,Q}(K+1)^\alpha \rho^\alpha \left|G * \chi_Q\right|_{0,Q_\rho(X_0)} \\ &\le C(K+1)^{2+\alpha}\rho^{2+\alpha}[f]_{\alpha,Q}. \end{aligned} \tag{1.37}$$

We now write for $X \in Q_\rho(X_0)$,

$$h(X) = h(X_0) + h_t(\theta, x)(t - t_0) + Dh(X_0) \cdot (x - x_0) + \frac{1}{2}D^2h(\Theta)(x - x_0) \cdot (x - x_0). \tag{1.38}$$

for some $\theta \in (t_0, t)$ and $\Theta = (t_0, y_0) \in Q_\rho(X_0)$. Hence, we have

$$\begin{aligned} h(X) - T_{X_0}h(X) = &\left(h_t(\theta, x) - h_t(X_0)\right)(t - t_0) \\ &+ \frac{1}{2}\left(D^2h(\Theta) - D^2h(X_0)\right)(x - x_0) \cdot (x - x_0). \end{aligned} \tag{1.39}$$

from which we deduce

$$\left|h(X) - T_{X_0}h(X)\right| \le \rho^2 \left|h_t(\theta, x) - h_t(X_0)\right| + \rho^2 \left|D^2h(\Theta) - D^2h(X_0)\right|. \tag{1.40}$$

We now use Fact 2 in order to get

$$\begin{aligned} \left|h - T_{X_0}h\right|_{0,Q_\rho(X_0)} \le &\rho^2 \left(\rho^2 \left|\frac{\partial^2}{\partial t^2}h\right|_{0,Q_\rho(X_0)} + \rho \left|\frac{\partial}{\partial t}Dh\right|_{0,Q_\rho(X_0)}\right) \\ &+ C\rho^3 \left|D^3h\right|_{0,Q_\rho(X_0)} \\ \le &C\left(\rho^4(K\rho)^{-4} + \rho^3(K\rho)^{-3} + \rho^3(K\rho)^{-3}\right)|h|_{0,Q} \\ \le &C\left(K^{-4} + 2K^{-3}\right)|h|_{0,Q} \\ \le &CK^{-3}|h|_{0,Q}. \end{aligned} \tag{1.41}$$

by choosing $K \ge 1$. We next estimate $|h|_{0,Q}$ as follows

$$\begin{aligned} |h|_{0,Q} &\le \left|u - T_{X_0}u - r\right|_{0,Q} \le \left|u - T_{X_0}u\right|_{0,Q} + |r|_{0,Q} \\ &\le C(K+1)^{2+\alpha}\rho^{2+\alpha}\left([u]_{2+\alpha,Q} + [f]_{\alpha,Q}\right). \end{aligned} \tag{1.42}$$

where we used 1.40 for $u$ instead of $h$ and we used 1.37. Then, we have

$$\left|h - T_{X_0} h\right|_{0,Q_\rho(X_0)} \le C\frac{(K+1)^{2+\alpha}}{K^3}\rho^{2+\alpha}\left([u]_{2+\alpha,Q} + [f]_{\alpha,Q}\right). \tag{1.43}$$

Combining 1.36, 1.37 and 1.42, we finally get

$$\begin{aligned}\rho^{-(2+\alpha)}\left|u - T_{X_0}u - T_{X_0}h\right|_{0,Q_\rho(X_0)} &\le C(K+1)^{2+\alpha}[f]_{\alpha,Q} \\ &\quad + C\frac{(K+1)^{2+\alpha}}{K^3}\left([u]_{2+\alpha,Q} + [f]_{\alpha,Q}\right).\end{aligned} \tag{1.44}$$

In view of Proposition 1.2.3, it is enough to choose $K \ge 1$ large enough so that

$$C\frac{(K+1)^{2+\alpha}}{K^3} \le \frac{1}{2}. \tag{1.45}$$

to conclude the proof of the theorem. □

**Corollary 1.3.3.** *Let $\alpha \in (0,1)$ and assume that $A \equiv A_0$ in $\mathbb{R}^{d+1}$ and $b \equiv 0$, $c \equiv 0$. Then there exists a constant $C > 0$ only depending on dimension and $\alpha$ such that for any $C^\infty$ function $u$ with compact support*

$$[u]_{2+\alpha,\mathbb{R}^{d+1}} \le C[f]_{\alpha,\mathbb{R}^{d+1}} \tag{1.46}$$

*where $f = \frac{\partial u}{\partial t} - Lu$.*

*Proof.* The proof consists in performing an appropriate change of coordinates. Precisely, we choose $P \in \mathbb{S}_d$ such that $A_0 = P^2$ and consider $v(t,x) = u(t,Px)$. Then check that $\Delta v = \mathrm{trace}\left(A_0 D^2 u\right) = Lu$ and use Theorem 1.3.3. □

### 1.3.5 Schauder Estimate in the Case of Variable Coefficients

**Theorem 1.3.4.** *Consider a function $u \in C^{2,\alpha}\left((0,T)\times\mathbb{R}^d\right)$ for some $\alpha \in (0,1)$. Then there exists $C = C(d,\alpha)$ such that*

$$[u]_{2+\alpha,(0,T)\times\mathbb{R}^d} \le C\left([f]_{\alpha,(0,T)\times\mathbb{R}^d} + |u|_{0,(0,T)\times\mathbb{R}^d}\right) \tag{1.47}$$

*where $f = \frac{\partial u}{\partial t} - Lu$.*

*Proof.* We first assume that $b \equiv 0$ and $c \equiv 0$. Let $f$ denote $\frac{\partial u}{\partial t} - Lu$.

Let $\varepsilon \in (0, T/2)$ and $\gamma \le \varepsilon/2$ be a positive real number to be fixed later and consider $X_1$ and $X_2$ such that

$$[u_t]_{\alpha,(\varepsilon,T-\varepsilon)\times\mathbb{R}^d} \le 2\rho\left(X_1,X_2\right)^{-\alpha}\left|u_t\left(X_1\right) - u_t\left(X_2\right)\right| \tag{1.48}$$

where we recall that $\rho\left(X_1,X_2\right) = \sqrt{|t_1 - t_2|} + |x_1 - x_2|$ if $X_i = (t_i, x_i)\,, i = 1,2$. If $\rho\left(X_1,X_2\right) \ge \gamma$, then we use interpolation inequalities in order to get

$$\begin{aligned}[u_t]_{\alpha,(\varepsilon,T-\varepsilon)\times\mathbb{R}^d} &\le 2\gamma^{-\alpha}\left|u_t\right|_0 \\ &\le \frac{1}{4}[u]_{2+\alpha} + C(\gamma)|u|_0.\end{aligned} \tag{1.49}$$

If $\rho(X_1, X_2) < \gamma$, we consider $\zeta \in C^\infty\left(\mathbb{R}^{d+1}\right)$ with compact support such that $\zeta(X) = 1$ if $\rho(X, 0) \leq 1$ and $\zeta(X) = 0$ if $\rho(X, 0) \geq 2$. We next define $\xi(t, x) = \zeta\left(\gamma^{-2}(t - t_1), \gamma^{-1}(x - x_1)\right)$. In particular, $\xi(X) = 1$ if $\rho(X, X_1) \leq \gamma$ and $\xi(X) = 0$ if $\rho(X, X_1) \geq 2\gamma$.

Now we use Corollary 1.3.3 in order to get

$$\begin{aligned} [u_t]_{\alpha,(\varepsilon,T-\varepsilon)\times\mathbb{R}^d} &\leq 2\rho(X_1, X_2)^{-\alpha} |u_t(X_1) - u_t(X_2)| . \\ &\leq 2[(u\xi)]_{2+\alpha} \\ &\leq 2C\left[(u\xi)_t - L(X_1)(u\xi)\right]_\alpha \\ &\leq 2C\left[(u\xi)_t - L(u\xi)\right]_\alpha + 2C\left[(L(X_1) - L)(u\xi)\right]_\alpha . \end{aligned} \tag{1.50}$$

We estimate successively the two terms of the right hand side of the last line. First, we write

$$(u\xi)_t - L(u\xi) = \xi f + u(\xi_t - L\xi) - 2ADu \cdot D\xi. \tag{1.51}$$

since $L(u\xi) = uL\xi + \xi Lu + 2ADu \cdot D\xi$. Using interpolation inequalities, this implies

$$\begin{aligned} \left[(u\xi)_t - L(u\xi)\right]_\alpha &\leq C(\gamma)\left([f]_\alpha + [u]_\alpha + [Du]_\alpha\right) \\ &\leq \gamma^\alpha [u]_{2+\alpha} + C(\gamma)\left([f]_\alpha + |u|_0\right). \end{aligned} \tag{1.52}$$

We next write

$$(L(X_1) - L)(u\xi) = \operatorname{trace}\left[(A(X_1) - A(X))\, D^2(u\xi)\right]. \tag{1.53}$$

and for $X$ such that $\rho(X_1, X) \leq 2\gamma$, we thus get thanks to interpolation inequalities

$$\begin{aligned} \left[(L(X_1) - L)(u\xi)\right]_\alpha &\leq C\gamma^\alpha \left[D^2(u\xi)\right]_\alpha + C\left|D^2(u\xi)\right|_0 \\ &\leq C\gamma^\alpha [u]_{2+\alpha} + C(\gamma)|u|_0. \end{aligned} \tag{1.54}$$

Combining 1.52-1.54, we finally get in the case where $\rho(X_1, X_2) \leq \gamma$,

$$[u_t]_{\alpha,(\varepsilon,T-\varepsilon)\times\mathbb{R}^d} \leq C\gamma^\alpha [u]_{2+\alpha} + C(\gamma)\left([f]_\alpha + |u|_0\right). \tag{1.55}$$

We conclude that we have in both cases

$$[u_t]_{\alpha,(\varepsilon,T-\varepsilon)\times\mathbb{R}^d} \leq \left(C\gamma^\alpha + 1/4\right)[u]_{2+\alpha} + C(\gamma)\left([f]_\alpha + |u|_0\right). \tag{1.56}$$

We can argue in a similar way to get

$$\left[D^2 u\right]_{\alpha,(\varepsilon,T-\varepsilon)\times\mathbb{R}^d} \leq \left(C\gamma^\alpha + 1/4\right)[u]_{2+\alpha} + C(\gamma)\left([f]_\alpha + |u|_0\right). \tag{1.57}$$

$\square$

# 2 Preliminaries on Kähler Geometry

## 2.1 Kähler Manifolds

Let $M$ be a complex manifold of complex dimension $n$. We will often work in a holomorphic coordinate chart $U$ with coordinates $(z^1, \cdots, z^n)$ and write a tensor in terms of its components in such a coordinate system.

A Hermitian metric on $M$ is a smooth tensor $g = g_{i\bar{j}}$ which is a Hermitian symmetric $(1,1)$ tensor on the holomorphic tangent bundle $T^{1,0}M$. Also $\left(g_{i\bar{j}}\right)$ is a positive definite Hermitian matrix at each point in the coordinate chart. Associated to $g$ there is a $(1,1)$ real form $\omega$ given by

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

If $d\omega = 0$ then we say that $g$ is a Kähler metric and that $\omega$ is the Kähler form associated to $g$. Abusing terminology slightly, we will often refer to a Kähler form $\omega$ as a Kähler metric.

In fact, there are many equivalent conditions of the Kähler condition:

**Lemma 2.1.1.** *Under above settings of such manifold $(M, \omega)$, and denote $\nabla$ be the Levi-Civita connection with respect to $g$, then the following are equivalent:*

*(a) $d\omega = 0$, $\nabla\omega = 0$ or $\nabla J = 0$, where $J$ is the complex structure;*

*(b) $\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}$ for all $i, j, k$ in a coordinate chart $(z^1, \cdots, z^n)$;*

*(c) For any $p \in M$, there exists a complex coordinate such that $\omega = \frac{\sqrt{-1}}{2} \sum\limits_{i,j=1}^{n} (\delta_{i\bar{j}} + [2]) dz^i \wedge d\bar{z}^j$ near $p$ where $[2]$ represents terms of order $\geq 2$;*

*(d) The holonomy group is contained in $U(n)$ for complex $n$-dim $M$, or equivalently, parallel transports preserve type of tangent vectors, or equivalently, $JX$ is still parallel for any parallel $X$;*

*(e) The Levi-Civita connection on $TM$ coincides with the Chern connection on $T^{1,0}M$;*

## 2.2 Curvatures on Kähler manifolds

Now we consider the Chern connection and curvatures on Kähler manifolds in detail.

First of all, if we write $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k + \Gamma_{ij}^{\bar{k}} \bar{\partial}_k$. Since $\nabla(J) = 0$, we have $\nabla_{\partial_i}(J\partial_j) = J(\nabla_{\partial_i}\partial_j) =$, which means $\Gamma_{ij}^{\bar{k}} = 0$ for any $i, j, k = 1, \cdots, n$. Similarly, any mixed terms of $\Gamma$ are vanishing.

To compute $\Gamma_{ij}^k$, we have

$$\partial_i g_{j\bar{l}} = \langle \nabla_{\partial_i}\partial_j, \partial_{\bar{l}} \rangle = \Gamma_{ij}^k g_{k\bar{l}}$$

Therefore $\Gamma_{ij}^k = g^{\bar{l}k} \partial_i g_{j\bar{l}}$, and $\Gamma_{\bar{i}\bar{j}}^{\bar{k}} = \overline{\Gamma_{ij}^k}$.

The curvature tensor is defined as followings with commuting formulae with respect to $J$,

$$R(X,Y)(Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,\ R(X,Y)(JZ) = JR(X,Y)Z;$$
$$R(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle,\ R(X,Y,JZ,JW) = R(X,Y,Z,W);$$

In coordinate, we have

$$R_{i\bar{j}k}{}^{l}\partial_l = R(\partial_i, \bar{\partial}_j)\partial_k = -\nabla_{\bar{j}}\nabla_i\partial_k = -\partial_{\bar{j}}\Gamma_{ik}^l \partial_l$$

Thus $R_{i\bar{j}k}{}^{l} = -\partial_{\bar{j}}\Gamma_{ik}^l, R_{i\bar{j}k\bar{l}} = -\left(\bar{\partial}_j \Gamma_{ik}^t\right) g_{t\bar{l}} = -\partial_i\partial_{\bar{j}} g_{k\bar{l}} + \left(\partial_{\bar{j}} g_{r\bar{l}}\right)\left(\partial_i g_{k\bar{q}}\right) g^{\bar{q}r}$.

The holomorphic (bi)sectional curvature is defined as followings, and the sectional curvatures are the same as Riemannian cases,

$$\xi = \frac{u - \sqrt{-1}Ju}{\sqrt{2}} \in T^{1,0}M,\ \|\xi\| = 1,\ R(\xi, \bar{\xi}, \xi, \bar{\xi}) = R(u, Ju, Ju, u);$$

$$\xi = \frac{u - \sqrt{-1}Ju}{\sqrt{2}}, \eta = \frac{v - \sqrt{-1}Jv}{\sqrt{2}} \in T^{1,0}M,\ \|\xi\| = 1, \|\eta\| = 1,\ R(\xi, \bar{\xi}, \eta, \bar{\eta}) = R(u, Jv, Jv, u) + R(u, v, v, u);$$

The Ricci curvature form and scalar curvature are defined as followings,

$\operatorname{Ric}(X,Y) = \operatorname{Tr}(Z \longrightarrow R(Z,X)Y), \operatorname{Ric}(JX,JY) = \operatorname{Tr}(JZ \longrightarrow R(JZ,JX)JY) = \operatorname{Tr}(JZ \longrightarrow JR(Z,X)Y) = \operatorname{Ric}(X,Y);$
$\operatorname{Ric}_{i\bar{j}} = R_p{}^p{}_{i\bar{j}} = -\partial_{\bar{j}}\Gamma^p_{pi} = -\partial_{\bar{j}}(g^{\bar{l}k}\partial_i g_{k\bar{l}}) = -\partial_i\partial_{\bar{j}}\log\det(g_{k\bar{l}}), \operatorname{Ric}_{i\bar{j}} = g^{\bar{l}k}R_{i\bar{j}k\bar{l}};$
$\operatorname{Ric}(\omega) = \sqrt{-1}\operatorname{Ric}_{i\bar{j}}\, dz^i \wedge d\bar{z}^j = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{k\bar{l}});$
$S = g^{\bar{j}i}\operatorname{Ric}_{i\bar{j}};$

Since from Bianchi identity we get

$$\begin{aligned}
\operatorname{Ric}(u_i,\bar{u}_i) &= \sum_j R(u_i,\bar{u}_i,u_j,\bar{u}_j)\\
&= \frac{1}{2}\sum_j R(e_i - \sqrt{-1}Je_i, e_i + \sqrt{-1}Je_i, u_j, \bar{u}_j)\\
&= \sum_j \sqrt{-1}R(e_i, Je_i, u_j, \bar{u}_j)\\
&= -\sum_j R(e_i, Je_i, e_j, Je_j)\\
&= \sum_j R(Je_i, e_j, e_i, Je_j) + \sum_j R(e_j, e_i, Je_i, Je_j)\\
&= \sum_j R(e_i, Je_j, Je_j, e_i) + \sum_j R(e_j, e_i, e_i, e_j)\\
&= \operatorname{Ric}(e_i, e_i).
\end{aligned}$$

This shows that the Ricci curvature defined above is the same as the one in Riemannian geometry.

## 2.3 Maximum Principle

There are various notions of "maximum principle". In the setting Ricci flow, Hamilton introduced his maximum principle for tensors. For our purposes however, we need only a simple version of the maximum principle.

Suppose $(M,\omega)$ is a compact Kähler manifold with Kähler form $\omega$.

**Proposition 2.3.1** (parabolic maximum principle)**.** *Fix $T > 0$. Let $f = f(x,t)$ be a smooth function on $M \times [0,T]$. If $f$ achieves its maximum (minimum) at $(x_0,t_0) \in M \times [0,T]$, then either $t_0 = 0$ or at $(x_0,t_0)$,*

$$\begin{aligned}
\frac{\partial f}{\partial t} &\geq 0 (\leq 0);\\
df &= 0;\\
\sqrt{-1}\partial\bar{\partial} f &\leq 0 (\geq 0);
\end{aligned}$$

We end this subsection with a useful application of the maximum principle in the case where $f$ satisfies a heat-type differential inequality:

**Theorem 2.3.1.** *Fix $T$ with $0 < T \leq \infty$. Suppose that $f = f(x,t)$ is smooth on $M \times [0,T)$ satisfying the differential inequality*

$$(\frac{\partial}{\partial t} - \Delta) f \leq 0$$

*Then* $\sup_{(x,t)\times M\times[0,T)} f(x,t) \leq \sup_{x\in M} f(x,0)$.

*Proof.* Fix $T_0 \in (0,T)$. For $\varepsilon > 0$, define $f_\varepsilon = f - \varepsilon t$. Suppose $f_\varepsilon$ achieve its maximum at $(x_0, t_0)$. If $t_0 > 0$, then by Proposition 1.3.1,

$$0 \leq (\frac{\partial}{\partial t} - \Delta) f_\varepsilon(x_0, t_0) \leq -\varepsilon.$$

Contradiction! Therefore $f_\varepsilon$ can only achieve its maximum at $t_0 = 0$ and

$$\sup_{(x,t)\times M\times[0,T_0]} f(x,t) \leq \sup_{(x,t)\in M\times[0,T_0]} f_\varepsilon(x,t) + \varepsilon T_0 \leq \sup_{x\in M} f(x,0) + \varepsilon T_0.$$

Let $\varepsilon \to 0$. Since $T_0$ is arbitrary, the proof is complete. □

## 2.4 Poincaré Inequality and Sobolev Inequality

In this subsection, we list a number of analytic inequality, which we will need later. For some good references, see [2],[12].

Let $(M, \omega)$ be a compact Kähler manifold of complex dimension $n$. For the rest of this section, we assume all functions and tensors on $M$ are smooth.

**Theorem 2.4.1** (Poincaré Inequality)**.** *There exists a constant $C_P$ such that for any real-valued function $f$ on $M$ with $\int_M f\omega^n = 0$, we have*

$$\int_M f^2 \omega^n \leq C_P \int_M |\partial f|^2 \omega^n.$$

**Remark.** The constant $C_P$ is (up to scaling by some universal factor) equal to $\lambda^{-1}$ where $\lambda$ is the first nonzero eigenvalue of the operator $-\Delta$ associated to $g$.

Next, we have the Sobolev inequality.

**Theorem 2.4.2.** *Assume $n > 1$. There exists a uniform constant $C_S$ such that for any real-valued function $f$ on $M$, we have*

$$\left(\int_M |f|^{2\beta} \omega^n\right)^{\frac{1}{\beta}} \leq C_S \left(\int_M |\partial f|^2 \omega^n + \int_M |f|^2 \omega^n\right),$$

*for $\beta = \frac{n}{n-1} > 1$.*

# 3 An Introduction to the Kähler-Ricci Flow

## 3.1 Kähler-Ricci Flow

Now we give the equation of the Kähler-Ricci flow:

**Definition 3.1.1.** *Suppose $(M, \omega)$ be a compact Kähler manifold of complex dimension $n$, a solution of the Kähler-Ricci flow on $M$ starting at $\omega_0$ is a family of Kähler metrics $\omega = \omega(t)$ solving*

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega), \qquad \omega|_{t=0} = \omega_0; \tag{3.1}$$

For later use, it will be convenient to consider a more general equation than 3.1, namely

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega) - \nu\omega, \qquad \omega|_{t=0} = \omega_0; \tag{3.2}$$

where $\nu$ is a fixed real number which we take to be either $\nu = 0$ or $\nu - 1$. As we will discuss later, the case $\nu = 1$ corresponds to a rescaling of 3.1. and we call it the normalized Kähler-Ricci flow.

We have the following existence and uniqueness result.

**Theorem 3.1.1.** *There exists a unique solution $\omega = \omega(t)$ to 3.2 on some maximal time interval $[0, T)$ for some $T$ with $0 < T \leq \infty$.*

## 3.2 Some Metric Variation Formulas

### 3.2.1 Evolution of Scalar Curvature

Suppose $\omega = \omega(t)$ is a solution to the Kähler-Ricci flow 3.2 on $[0, T)$ for $T$ with $0 < T \leq \infty$. We compute the well-known evolution of the scalar curvature.

**Theorem 3.2.1.** *The scalar curvature $R$ of $\omega = \omega(t)$ evolves by*

$$\frac{\partial}{\partial t}R = \Delta R + \|\operatorname{Ric}(\omega)\|^2 + \nu R, \tag{3.3}$$

*where $\|\operatorname{Ric}\|^2 = g^{\bar{l}i}g^{\bar{j}k}\operatorname{Ric}_{i\bar{j}}\operatorname{Ric}_{k\bar{l}}$. Hence the scalar curvature has a lower bound*

$$R(t) \geq -\nu n - C_0 e^{-\nu t}, \tag{3.4}$$

*for $C_0 = -\inf_M R(0) - \nu n$.*

*Proof.* Taking the trace of the evolution equation 3.2 gives

$$g^{\bar{l}k}\frac{\partial}{\partial t}g_{k\bar{l}} = -R - \nu n, \tag{3.5}$$

Since $R = -g^{\bar{j}i}\partial_i\partial_{\bar{j}}\log\det g$ we have

$$\frac{\partial}{\partial t}R = -g^{\bar{j}i}\partial_i\partial_{\bar{j}}\left(g^{\bar{l}k}\frac{\partial}{\partial t}g_{k\bar{l}}\right) - \left(\frac{\partial}{\partial t}g^{\bar{j}i}\right)\partial_i\partial_{\bar{j}}\log\det g \tag{3.6a}$$

$$= \Delta R + g^{\bar{l}i}g^{\bar{j}k}\operatorname{Ric}_{k\bar{l}}\operatorname{Ric}_{i\bar{j}} + \nu R, \tag{3.6b}$$

as required. Since $n\|\operatorname{Ric}(\omega)\|^2 \geq R^2$ by Cauchy-Schwarz inequality, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)R \geq \nu R + \frac{1}{n}R^2. \tag{3.7}$$

Hence
$$\left(\frac{\partial}{\partial t} - \Delta\right)(e^{\nu t}(R + \nu n)) = \frac{1}{n}(n\nu + R)^2 \geq 0. \tag{3.8}$$
By the parabolic maximum principle (Proposition 2.3.1) we get $e^{\nu t}(R+\nu n) \geq \inf_M R(0) + \nu n$. Therefore the proof is complete. $\square$

Theorem 3.2.1 implies a bound on the volume form of the metric.

**Corollary 3.2.1.** *Let $\omega = \omega(t)$ be a solution of 3.2 on $[0, T)$ and $C_0$ as in Theorem 3.2.1.*
*(i) If $\nu = 0$ then*
$$\omega^n(t) \leq e^{C_0 t}\omega^n(0). \tag{3.9}$$
*In particular, if $T$ is finite then the volume form $\omega^n(t)$ is uniformly bounded from above for $t \in [0, T)$.*
*(ii) If $\nu = 1$ there exists a uniform constant $C$ such that*
$$\omega^n(t) \leq e^{C_0(1-e^{-t})}\omega^n(0). \tag{3.10}$$
*In particular, the volume form $\omega^n(t)$ is uniformly bounded from above for $t \in [0, T)$.*

*Proof.* Since
$$\frac{\partial}{\partial t}\log\frac{\omega^n(t)}{\omega^n(0)} = g^{\bar{j}i}\frac{\partial}{\partial t}g_{i\bar{j}} = -R - \nu n \leq C_0 e^{-\nu t}. \tag{3.11}$$
The consequences are obtained by integrating in time. $\square$

### 3.2.2 Evolution of the Trace of the Metric

We now prove an estimate for the trace of the metric along the Kähler-Ricci flow, which is originally due to Cao [5] and is the parabolic version of an estimate for the complex Monge–Ampère equation due to Yau and Aubin [1, 26]. We begin by computing the evolution of $\operatorname{tr}_{\hat{\omega}}\omega$, the trace of $\omega$ with respect to a fixed metric $\hat{\omega}$.

**Proposition 3.2.1.** *Suppose $\hat{\omega}$ is a fixed Kähler metric on $M$, and $\omega = \omega(t)$ is a solution of 3.2. Then*
$$\left(\frac{\partial}{\partial t} - \Delta\right)\operatorname{tr}_{\hat{\omega}}\omega = -\nu\operatorname{tr}_{\hat{\omega}}\omega - g^{\bar{l}k}\hat{R}_{k\bar{l}}{}^{\bar{j}i}g_{i\bar{j}} - \hat{g}^{\bar{j}i}g^{\bar{q}p}g^{\bar{l}k}\hat{\nabla}_i g_{p\bar{l}}\hat{\nabla}_{\bar{j}}g_{k\bar{q}}, \tag{3.12}$$
*where $\hat{R}_{k\bar{l}}{}^{\bar{j}i}, \hat{\nabla}$ denote the curvature and covariant derivative with respect to $\hat{g}$.*

*Proof.* Using normal coordinates for $\hat{g}$ we have
$$\begin{aligned}\Delta\operatorname{tr}_{\hat{\omega}}\omega &= g^{\bar{l}k}\partial_k\partial_{\bar{l}}(\hat{g}^{\bar{j}i}g_{i\bar{j}})\\ &= g^{\bar{l}k}\left(\partial_k\partial_{\bar{l}}(\hat{g}^{\bar{j}i})\right)g_{i\bar{j}} + g^{\bar{l}k}\hat{g}^{\bar{j}i}\partial_k\partial_{\bar{l}}(g_{i\bar{j}})\\ &= g^{\bar{l}k}\hat{R}_{k\bar{l}}{}^{\bar{j}i}g_{i\bar{j}} - \hat{g}^{\bar{j}i}R_{i\bar{j}} + \hat{g}^{\bar{j}i}g^{\bar{q}p}g^{\bar{l}k}\partial_i g_{p\bar{l}}\partial_{\bar{j}}g_{k\bar{q}}.\end{aligned} \tag{3.13}$$
from the coordinate formulas of curvature. And
$$\frac{\partial}{\partial t}\operatorname{tr}_{\hat{\omega}}\omega = -\hat{g}^{\bar{j}i}R_{i\bar{j}} - \nu\operatorname{tr}_{\hat{\omega}}\omega. \tag{3.14}$$
and combining these gives 3.12. $\square$

Proposition 3.2.1 gives the following estimate.

**Proposition 3.2.2.** *Suppose $\hat{\omega}$ is a fixed Kähler metric on $M$, and let $\omega = \omega(t)$ be a solution to 3.2. Then there exists a constant $\hat{C}$ depending only on the lower bound of the holomorphic bisectional curvature for $\hat{g}$ such that*

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \operatorname{tr}_{\hat{\omega}} \omega \leq \hat{C} \operatorname{tr}_{\omega} \hat{\omega} - \nu. \tag{3.15}$$

*Proof.* Observe that for positive function $f$ we have

$$\Delta \log f = \frac{\Delta f}{f} - \frac{\|\partial f\|^2}{f^2}. \tag{3.16}$$

From Proposition 3.2.1 we get that

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \operatorname{tr}_{\hat{\omega}} \omega = \frac{1}{\operatorname{tr}_{\hat{\omega}} \omega} \left(-\nu \operatorname{tr}_{\hat{\omega}} \omega - g^{\bar{l}k} \hat{R}_{k\bar{l}}{}^{\bar{j}i} g_{i\bar{j}} - \hat{g}^{\bar{j}i} g^{\bar{q}p} g^{\bar{l}k} \hat{\nabla}_i g_{p\bar{l}} \hat{\nabla}_{\bar{j}} g_{k\bar{q}} + \frac{\|\partial \operatorname{tr}_{\hat{\omega}} \omega\|_g^2}{\operatorname{tr}_{\hat{\omega}} \omega}\right). \tag{3.17}$$

We claim that

$$-\hat{g}^{\bar{j}i} g^{\bar{q}p} g^{\bar{l}k} \hat{\nabla}_i g_{p\bar{l}} \hat{\nabla}_{\bar{j}} g_{k\bar{q}} + \frac{\|\partial \operatorname{tr}_{\hat{\omega}} \omega\|_g^2}{\operatorname{tr}_{\hat{\omega}} \omega} \leq 0. \tag{3.18}$$

To prove this, we choose normal coordinates for $\hat{g}$ for which $g$ is diagonal. Compute using Cauchy-Schwarz inequality

$$\begin{aligned}
\| \operatorname{tr}_{\hat{\omega}} \omega\|_g^2 &= \sum_i g^{\bar{i}i} \partial_i \left(\sum_j g_{j\bar{j}}\right) \partial_{\bar{i}} \left(\sum_k g_{k\bar{k}}\right) \\
&= \sum_{j,k} \sum_i g^{\bar{i}i} (\partial_i g_{j\bar{j}})(\partial_{\bar{i}} g_{k\bar{k}}) \\
&\leq \sum_{j,k} \left(\sum_i g^{\bar{i}i} \left|\partial_i g_{j\bar{j}}\right|^2\right)^{\frac{1}{2}} \left(\sum_i g^{\bar{i}i} \left|\partial_i g_{k\bar{k}}\right|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_j \left(\sum_i g^{\bar{i}i} \left|\partial_i g_{j\bar{j}}\right|^2\right)^{\frac{1}{2}}\right)^2 \\
&= \left(\sum_j \sqrt{g_{j\bar{j}}} \left(\sum_i g^{\bar{i}i} g^{\bar{j}j} \left|\partial_i g_{j\bar{j}}\right|^2\right)^{\frac{1}{2}}\right)^2 \\
&= \left(\sum_j g_{j\bar{j}}\right) \left(\sum_{i,j} g^{\bar{i}i} g^{\bar{j}j} \left|\partial_j g_{i\bar{j}}\right|^2\right) \\
&\leq \left(\sum_j g_{j\bar{j}}\right) \left(\sum_{i,j,k} g^{\bar{i}i} g^{\bar{j}j} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}}\right).
\end{aligned} \tag{3.19}$$

where in the second-to-last line we use the Kähler condition which implies that $\partial_i g_{j\bar{j}} = \partial_j g_{i\bar{j}}$. Thus the claim is proved.

Set a constant $\hat{C} = -\inf_{x\in M}\{\hat{R}_{i\bar{i}j\bar{j}}(x)|\{\partial_{z^1}, \cdots, \partial_{z^n}\}$ is orthonormal w.r.t. $\hat{g}$ at $x,\ i,j = 1,\cdots,n\}$. $\hat{C}$ is finite since $M$ is compact.

Then computing at a point using normal coordinates for $\hat{g}$ for which the metric $g$ is diagonal we have

$$g^{\bar{l}k}\hat{R}_{k\bar{l}}{}^{\bar{j}i}g_{i\bar{j}} = \sum_{k,i} g^{\bar{k}k}\hat{R}_{k\bar{k}i\bar{i}}g_{i\bar{i}} \geq -\hat{C}\sum_{k} g^{\bar{k}k}\sum_{i} g_{i\bar{i}} = -\hat{C}(\operatorname{tr}_\omega \hat{\omega})(\operatorname{tr}_{\hat{\omega}} \omega). \tag{3.20}$$

Combining 3.17, 3.18, 3.20 yields 3.15. □

### 3.2.3 The Parabolic Schwarz Lemma

In this section we prove the parabolic Schwarz lemma in [22], which is a parabolic version of Yau's Schwarz lemma [25] as follows:

**Theorem 3.2.2** (Yau's Schwarz lemma)**.** *Let $M$ be a complex Kähler manifold with scalar curvature bounded from below by $K_1$. Let $N$ be another Hermitian manifold with Ricci curvature bounded from above by a negative constant $K_2$. Suppose the Ricci curvature of $M$ is bounded from below and $\dim M = \dim N$. Then the existence of a non-degenerate holomorphic map $f$ from $M$ into $N$ implies that $K_1 \leqslant 0$ and*

$$f^* dV_N \leqslant \frac{K_1}{K_2} dV_M, \tag{3.21}$$

*where $dV_M, dV_N$ are volume elements of $M$ and $N$ respectively.*

We state it here in the form of an evolution inequality.

**Theorem 3.2.3.** *Let $f : M \to N$ be a holomorphic map between compact complex manifolds $M$ and $N$ of complex dimension $n$ and $k$ respectively. Let $\omega_0$ and $\omega_N$ be Kähler metrics on $M$ and $N$ respectively and let $\omega = \omega(t)$ be a solution of 3.2 on $M \times [0,T)$, namely,*

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega) - \nu\omega, \qquad \omega|_{t=0} = \omega_0. \tag{3.22}$$

*for $t \in [0,T)$, with either $\nu = 0$ or $\nu = 1$. Then for all points of $M \times [0,T)$ with $\operatorname{tr}_\omega(f^*\omega_N)$ positive we have*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\log \operatorname{tr}_\omega(f^*\omega_N) \leq C_N \operatorname{tr}_\omega(f^*\omega_N) + \nu. \tag{3.23}$$

*where $C_N$ is an upper bound for the holomorphic bisectional curvature of $\omega_N$.*

*Proof.* Fix $x \in M$ with $f(x) = y \in N$, and choose normal coordinate system $(z^i)_{i=1,\cdots,n}$ for $g$ centered at $x$ and $(\omega^\alpha)_{\alpha_1,\cdots,k}$ for $g_N$ centered at $y$. The map $f$ is given locally by $(f^1,\cdots,f^k)$. We get $\mathrm{tr}_\omega(f^*\omega_N) = g^{\bar{j}i}(g_N)_{\alpha\bar{\beta}} \left(\frac{\partial f^\alpha}{\partial z^i}\right)\overline{\left(\frac{\partial f^\beta}{\partial z^j}\right)}$. Write $u = \mathrm{tr}_\omega(f^*\omega_N) > 0$, we compute at $x$,

$$\begin{aligned}\Delta u &= g^{\bar{l}k}\partial_k\partial_{\bar{l}}\left(g^{\bar{j}i}(g_N)_{\alpha\bar{\beta}}\left(\frac{\partial f^\alpha}{\partial z^i}\right)\overline{\left(\frac{\partial f^\beta}{\partial z^j}\right)}\right)\\ &= \mathrm{Ric}^{\bar{j}i}(g_N)_{\alpha\bar{\beta}}\left(\frac{\partial f^\alpha}{\partial z^i}\right)\overline{\left(\frac{\partial f^\beta}{\partial z^j}\right)} + g^{\bar{l}k}g^{\bar{j}i}\left(\frac{\partial^2 f^\alpha}{\partial z^i\partial z^k}\right)\overline{\left(\frac{\partial^2 f^\beta}{\partial z^j\partial z^l}\right)}(g_N)_{\alpha\bar{\beta}}\\ &\quad - g^{\bar{l}k}g^{\bar{j}i}S_{\alpha\bar{\beta}\gamma\bar{\delta}}\left(\frac{\partial f^\alpha}{\partial z^i}\right)\overline{\left(\frac{\partial f^\beta}{\partial z^j}\right)}\left(\frac{\partial f^\gamma}{\partial z^k}\right)\overline{\left(\frac{\partial f^\delta}{\partial z^l}\right)}.\end{aligned} \tag{3.24}$$

for $S_{\alpha\bar{\beta}\gamma\bar{\delta}}$ the curvature tensor of $g_N$. Next,

$$\frac{\partial}{\partial t}u = \mathrm{Ric}^{\bar{j}i}\left(\frac{\partial f^\alpha}{\partial z^i}\right)\overline{\left(\frac{\partial f^\beta}{\partial z^j}\right)}(g_N)_{\alpha\bar{\beta}} + \nu u. \tag{3.25}$$

Combining 3.16, 3.24, 3.25, we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\log u &= \frac{1}{u}g^{\bar{l}k}g^{\bar{j}i}S_{\alpha\bar{\beta}\gamma\bar{\delta}}\left(\frac{\partial f^\alpha}{\partial z^i}\right)\overline{\left(\frac{\partial f^\beta}{\partial z^j}\right)}\left(\frac{\partial f^\gamma}{\partial z^k}\right)\overline{\left(\frac{\partial f^\delta}{\partial z^l}\right)}\\ &\quad + \frac{1}{u}\left(\frac{\|\partial u\|^2}{u} - g^{\bar{l}k}g^{\bar{j}i}\left(\frac{\partial^2 f^\alpha}{\partial z^i\partial z^k}\right)\overline{\left(\frac{\partial^2 f^\beta}{\partial z^j\partial z^l}\right)}(g_N)_{\alpha\bar{\beta}}\right) + \nu.\end{aligned} \tag{3.26}$$

If $C_N$ is an upper bound of the holomorphic bisectional curvature of $g_N$ we see that

$$g^{\bar{j}i}S_{\alpha\bar{\beta}\gamma\bar{\delta}}\left(\frac{\partial f^\alpha}{\partial z^i}\right)\overline{\left(\frac{\partial f^\beta}{\partial z^j}\right)}\left(\frac{\partial f^\gamma}{\partial z^k}\right)\overline{\left(\frac{\partial f^\delta}{\partial z^l}\right)} \le C_N u^2. \tag{3.27}$$

Therefore 3.23 follows from the claim

$$\frac{\|\partial u\|^2}{u} - g^{\bar{l}k}g^{\bar{j}i}\left(\frac{\partial^2 f^\alpha}{\partial z^i\partial z^k}\right)\overline{\left(\frac{\partial^2 f^\beta}{\partial z^j\partial z^l}\right)}(g_N)_{\alpha\bar{\beta}} \le 0. \tag{3.28}$$

Indeed, at the point $x$,

$$
\begin{aligned}
\|\partial u\|^2 &= \sum_{i,j,k,\alpha,\beta} \overline{\left(\frac{\partial f^\alpha}{\partial z^i}\right)} \left(\frac{\partial f^\beta}{\partial z^j}\right) \left(\frac{\partial^2 f^\alpha}{\partial z^i \partial z^k}\right) \overline{\left(\frac{\partial^2 f^\beta}{\partial z^j \partial z^k}\right)} \\
&\leq \sum_{i,j,\alpha,\beta} \left|\frac{\partial f^\alpha}{\partial z^i}\right| \left|\frac{\partial f^\beta}{\partial z^j}\right| \left(\sum_k \left|\frac{\partial^2 f^\alpha}{\partial z^i \partial z^k}\right|^2\right)^{\frac{1}{2}} \left(\sum_l \left|\frac{\partial^2 f^\beta}{\partial z^j \partial z^l}\right|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i,\alpha} \left|\frac{\partial f^\alpha}{\partial z^i}\right| \left(\sum_k \left|\frac{\partial^2 f^\alpha}{\partial z^i \partial z^k}\right|^2\right)^{\frac{1}{2}}\right)^2 \\
&\leq \left(\sum_{j,\beta} \left|\frac{\partial f^\beta}{\partial z^j}\right|^2\right) \left(\sum_{i,k,\alpha} \left|\frac{\partial^2 f^\alpha}{\partial z^i \partial z^k}\right|^2\right) \\
&\leq u g^{\bar{l}k} g^{\bar{j}i} \left(\frac{\partial^2 f^\alpha}{\partial z^i \partial z^k}\right) \overline{\left(\frac{\partial^2 f^\beta}{\partial z^j \partial z^l}\right)} (g_N)_{\alpha\bar{\beta}}.
\end{aligned}
\tag{3.29}
$$

□

Then a simple maximum principle argument immediately gives the following consequence.

**Corollary 3.2.2.** *If the holomorphic bisectional curvature of $\omega_N$ has a negative upper bound $C_N < 0$ on $N$ then there exists a constant $C > 0$ depending only on $C_N, \omega_0, \omega_N$ and $\nu$ such that $\mathrm{tr}_\omega(f^*\omega_N) \leq C$ on $M \times [0,T)$ and hence*

$$
\omega \geq \frac{1}{C} f^*\omega_N, \qquad \text{on } M \times [0,T). \tag{3.30}
$$

*Proof.* From Theorem 3.2.3, it's easy to deduce that

$$
\left(\frac{\partial}{\partial t} - \Delta\right) u \leq u - K u^2, \tag{3.31}
$$

for some $K > 0$ Let $u_{\max}(t) = \max_X u(t,\cdot) = u\,(t, z_t)$ for some $z_t \in X$. By the maximum principle, $\Delta u\,(t, z_t) \leq 0$ so that we have

$$
\frac{d}{dt} u_{\max} \leq u_{\max} - K u_{\max}^2. \tag{3.32}
$$

Thus $u_{\max}(t) \leq \frac{1}{K}$ if $u_{\max}(0) \leq \frac{1}{K}$ and

$$
u_{\max}(t) \leq \frac{1}{K - Ce^{-t}}. \tag{3.33}
$$

for some $C < K$ if $u_{\max}(0) > \frac{1}{K}$. This complete the proof. □

### 3.2.4 The Third Order Estimate

Here we will prove the so-called "third order" estimate for the Kähler-Ricci flow assuming that the metric is uniformly bounded. By third order estimate we mean an estimate on the first derivative of the Kähler metric, which is of order 3 in terms of the potential function. Since the work of Yau [26] on the elliptic Monge–Ampère equation, such estimates have often been referred to as Calabi estimates in reference to a well-known calculation of Calabi [3]. There are now many generalizations of the Calabi estimate. A parabolic Calabi estimate was applied to the Kähler-Ricci flow in [5]. Phong-Šešum-Sturm [18] later gave a succinct and explicit formula.

Suppose $\omega = \omega(t)$ is a solution of 3.2 on $[0, T)$ for $0 < T \leq \infty$ and $\hat{\omega}$ is a fixed Kähler metric on $M$. Let $S = \|\hat{\nabla} g\|_g^2$. where $g$ is the evolving metric. Namely,

$$S = g^{\bar{j}i} g^{\bar{l}k} g^{\bar{q}p} \hat{\nabla}_i g_{k\bar{q}} \overline{\hat{\nabla}_j g_{l\bar{p}}}. \tag{3.34}$$

Define a tensor $\Psi_{ij}^k$ by

$$\Psi_{ij}^k := \Gamma_{ij}^k - \hat{\Gamma}_{ij}^k = g^{\bar{l}k} \hat{\nabla}_i g_{j\bar{l}}. \tag{3.35}$$

Thus we can rewrite $S$ as

$$S = \|\Psi\|^2 = g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \Psi_{ip}^k \overline{\Psi_{jq}^l}. \tag{3.36}$$

**Proposition 3.2.3.** *With the notion above, $S$ evolves by*

$$\left(\frac{\partial}{\partial t} - \Delta\right) S = -\|\overline{\nabla}\Psi\|^2 - \|\nabla\Psi\|^2 + \nu\|\Psi\|^2 - 2\,\mathrm{Re}\left(g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \nabla^{\bar{b}} \hat{R}_{i\bar{b}p}{}^k \overline{\Psi_{jq}^l}\right). \tag{3.37}$$

*where $\nabla^{\bar{b}} = g^{\bar{b}a}\nabla_a$ and $\hat{R}_{i\bar{b}p}{}^k := \hat{g}^{\bar{m}k}\hat{R}_{i\bar{b}p\bar{m}}$.*

*Proof.* Compute

$$\Delta S = g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \left((\Delta\Psi)_{ip}^k \overline{\Psi_{jq}^l} + \Psi_{ip}^k \overline{(\overline{\Delta}\Psi)_{jq}^l}\right) + \|\overline{\nabla}\Psi\|^2 + \|\nabla\Psi\|^2. \tag{3.38}$$

where $\Delta = g^{\bar{b}a}\nabla_a\nabla_{\bar{b}}$ is the "rough" laplacian and $\overline{\Delta} = g^{\bar{b}a}\nabla_{\bar{b}}\nabla_a$.

By commutation formulae,

$$\bar{\Delta}\Psi_{jq}^l = \Delta\Psi_{jq}^l + \mathrm{Ric}_j{}^b \Psi_{bq}^l + \mathrm{Ric}_q{}^b \Psi_{jb}^l - \mathrm{Ric}_b{}^l \Psi_{jq}^b. \tag{3.39}$$

Combining 3.38 and 3.39 we have

$$\begin{aligned} \Delta S = {} & 2\,\mathrm{Re}\left(g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \left(\Delta\Psi_{ip}^k\right) \overline{\Psi_{jq}^l}\right) + |\bar{\nabla}\Psi|^2 + |\nabla\Psi|^2 \\ & + \mathrm{Ric}^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \Psi_{ip}^k \overline{\Psi_{jq}^l} + g^{\bar{j}i}\,\mathrm{Ric}^{\bar{q}p} g_{k\bar{l}} \Psi_{ip}^k \overline{\Psi_{jq}^l} - g^{\bar{j}i} g^{\bar{q}p}\,\mathrm{Ric}_{k\bar{l}} \Psi_{ip}^k \overline{\Psi_{jq}^l}. \end{aligned} \tag{3.40}$$

We claim that

$$\frac{\partial}{\partial t}\Psi_{ip}^k = \Delta\Psi_{ip}^k - \nabla^{\bar{b}}\hat{R}_{i\bar{b}p}{}^k. \tag{3.41}$$

Since

$$\frac{\partial}{\partial t}\Psi_{ip}^k = \frac{\partial}{\partial t}\Gamma_{ip}^k = -\nabla_i \mathrm{Ric}_p{}^k. \tag{3.42}$$

On the other hand,

$$\nabla_{\bar{b}} \Psi_{ip}^{k} = \partial_{\bar{b}} (\Gamma_{ip}^{k} - \hat{\Gamma}_{ip}^{k}) = \hat{R}_{i\bar{b}p}{}^{k} - R_{i\bar{b}p}{}^{k}, \tag{3.43}$$

hence

$$\Delta \Psi_{ip}^{k} = g^{\bar{b}a} \nabla_a \nabla_{\bar{b}} \Psi_{ip}^{k} = \nabla^{\bar{b}} \hat{R}_{i\bar{b}p}{}^{k} - \nabla_i \operatorname{Ric}_p{}^{k}. \tag{3.44}$$

The claim is obtained.

Given this, together with

$$\frac{\partial}{\partial t} g^{\bar{j}i} = \operatorname{Ric}^{\bar{j}i} + \nu g^{\bar{j}i}, \quad \frac{\partial}{\partial t} g_{k\bar{l}} = -\operatorname{Ric}_{k\bar{l}} - \nu g_{k\bar{l}}. \tag{3.45}$$

we obtain

$$\begin{aligned} \frac{\partial}{\partial t} S = {} & 2 \operatorname{Re} \left( g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \left( \Delta \Psi_{ip}^{k} - \nabla^{\bar{b}} \hat{R}_{i\bar{i}p}{}^{k} \right) \overline{\Psi_{jq}^{l}} \right) + \operatorname{Ric}^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \Psi_{ip}^{k} \overline{\Psi_{jq}^{l}} \\ & + g^{\bar{j}i} \operatorname{Ric}^{\bar{q}p} g_{k\bar{l}} \Psi_{ip}^{k} \overline{\Psi_{jq}^{l}} - g^{\bar{j}i} g^{\bar{q}p} \operatorname{Ric}_{k\bar{l}} \Psi_{ip}^{k} \overline{\Psi_{jq}^{l}} + \nu |\Psi|^2. \end{aligned} \tag{3.46}$$

Combining 3.40 and 3.46 yields 3.37. □

Using this evolution equation together with Proposition 3.2.1, we obtain a third order estimate assuming a metric bound.

**Theorem 3.2.4.** *Let $\omega = \omega(t)$ solve 3.2 and assume that there exists a constant $C_0 > 0$ such that*

$$\frac{1}{C_0} \omega_0 \leq \omega \leq C_0 \omega_0. \tag{3.47}$$

*Then there exists a constant $C$ depending only on $C_0$ and $\omega_0$ such that*

$$S := \left\| \nabla_{g_0} g \right\|^2 \leq C. \tag{3.48}$$

*In addition, there exists a constant $C'$ depending only on $C_0$ and $\omega_0$ such that*

$$\left( \frac{\partial}{\partial t} - \Delta \right) S \leq -\frac{1}{2} \| \operatorname{Rm} \|^2 + C', \tag{3.49}$$

*where $\| \operatorname{Rm} \|^2$ denotes the norm squared of the curvature tensor $R_{i\bar{j}k\bar{\ell}}$.*

*Proof.* We apply 3.37. First note that

$$\nabla^{\bar{b}} \hat{R}_{i\bar{b}p}{}^{k} = g^{\bar{b}r} \hat{\nabla}_r \hat{R}_{i\bar{b}p}{}^{k} - g^{\bar{b}r} \Psi_{ir}^{a} \hat{R}_{a\bar{b}p}{}^{k} - g^{\bar{b}r} \Psi_{pr}^{a} \hat{R}_{i\bar{b}a}{}^{k} + g^{\bar{b}r} \Psi_{ar}^{k} \hat{R}_{i\bar{b}p}{}^{a}. \tag{3.50}$$

Then with $\hat{g} = g_0$, we have, by 3.47,

$$\left| 2 \operatorname{Re} \left( g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \nabla^{\bar{b}} \hat{R}_{i\bar{b}p}{}^{k} \overline{\Psi_{jq}^{l}} \right) \right| \leq C_1 (S + \sqrt{S}) \leq 2 C_1 (S + 1), \tag{3.51}$$

for some uniform constant $C_1$. Hence for a uniform $C_2$,

$$\left( \frac{\partial}{\partial t} - \Delta \right) S \leq - \| \overline{\nabla} \Psi \|^2 - \| \nabla \Psi \|^2 + C_2 S + C_2. \tag{3.52}$$

On the other hand, from Proposition 3.2.1 and the assumption 3.47 again,

$$\left(\frac{\partial}{\partial t} - \Delta\right) \operatorname{tr}_{\hat{\omega}} \omega \leq C_3 - \frac{1}{C_3} S. \tag{3.53}$$

for a uniform $C_3 > 0$. Define $Q = S + C_3(1 + C_2) \operatorname{tr}_{\hat{\omega}} \omega$ and compute

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q \leq -S + C_4. \tag{3.54}$$

for a uniform constant $C_4$. It follows that $S$ is bounded from above at a point at which $Q$ achieves a maximum, and 3.48 follows. For 3.49, observe from 3.43 that

$$\|\overline{\nabla}\Psi\|^2 = \left\|\hat{R}_{i\bar{b}p}{}^{k} - R_{i\bar{b}p}{}^{k}\right\|^2 \geq \frac{1}{2}\| \operatorname{Rm} \|^2 - C_5. \tag{3.55}$$

Then 3.49 follows from 3.48, 3.52 and 3.54. □

### 3.2.5 Curvature and Higher Derivative Bounds

Now we assume that we have a solution $\omega = \omega(t)$ on $[0, T)$ with $0 < T \leq \infty$ which satisfies the estimates

$$\frac{1}{C_0}\omega_0 \leq \omega \leq C_0 \omega_0. \tag{3.56}$$

for some uniform constant $C_0$. We show that the curvature and all derivatives of the curvature of $\omega$ are uniformly bound, and that we have uniform $C^\infty$ estimates of $g$ with respect to the fixed metric $\omega_0$.

Firstly, we compute the evolution of the curvature tensor.

**Lemma 3.2.1.** *Along the flow 3.2, the curvature tensor evolves by*

$$\begin{aligned}\frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} = & \frac{1}{2}\Delta_{\mathbb{R}} R_{i\bar{j}k\bar{l}} - \nu R_{i\bar{j}k\bar{l}} + R_{i\bar{j}a\bar{b}} R^{\bar{b}a}{}_{k\bar{l}} + R_{i\bar{b}a\bar{l}} R^{\bar{b}}{}_{\bar{j}k}{}^{a} - R_{i\bar{a}k\bar{b}} R^{\bar{a}}{}_{\bar{j}}{}^{\bar{b}}{}_{\bar{l}} \\ & - \frac{1}{2}\left(\operatorname{Ric}_i{}^{a} R_{a\bar{j}k\bar{l}} + \operatorname{Ric}^{\bar{a}}{}_{\bar{j}} R_{i\bar{a}k\bar{l}} + \operatorname{Ric}_k{}^{a} R_{i\bar{j}a\bar{l}} + \operatorname{Ric}^{\bar{a}}{}_{\bar{l}} R_{i\bar{j}k\bar{a}}\right).\end{aligned} \tag{3.57}$$

*where we write $\Delta_{\mathbb{R}} = \Delta + \overline{\Delta}$ and $\Delta = g^{\bar{q}p}\nabla_p\nabla_{\bar{q}}$.*

*Proof.* Using the formula $\frac{\partial}{\partial t}\Gamma^p_{ik} = -\nabla_i \operatorname{Ric}_k{}^{p}$ and the Bianchi identity we get

$$\frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} = -\left(\frac{\partial}{\partial t} g_{p\bar{j}}\right)\partial_{\bar{l}}\Gamma^p_{ik} - g_{p\bar{j}}\partial_{\bar{l}}\left(\frac{\partial}{\partial t}\Gamma^p_{ik}\right) = -\operatorname{Ric}^{\bar{a}}{}_{\bar{j}} R_{i\bar{a}k\bar{l}} - \nu R_{i\bar{j}k\bar{l}} + \nabla_{\bar{l}}\nabla_k \operatorname{Ric}_{i\bar{j}}. \tag{3.58}$$

Also by Bianchi identity and the commutation formulae, we obtain

$$\begin{aligned}\Delta R_{i\bar{j}k\bar{l}} &= g^{\bar{b}a}\nabla_a\nabla_{\bar{l}} R_{i\bar{j}k\bar{b}} \\ &= g^{\bar{b}a}\nabla_{\bar{l}}\nabla_a R_{i\bar{j}k\bar{b}} + g^{\bar{b}a}\left[\nabla_a, \nabla_{\bar{l}}\right] R_{i\bar{j}k\bar{b}} \\ &= \nabla_{\bar{l}}\nabla_k \operatorname{Ric}_{i\bar{j}} - R^{\bar{b}}{}_{\bar{l}k}{}^{a} R_{a\bar{b}i\bar{j}} + \operatorname{Ric}^{\bar{b}}{}_{\bar{l}} R_{k\bar{b}i\bar{j}} - R^{\bar{b}}{}_{\bar{l}i}{}^{a} R_{k\bar{b}a\bar{j}} + R^{\bar{b}}{}_{\bar{l}}{}^{\bar{a}}{}_{\bar{j}} R_{k\bar{b}i\bar{a}}.\end{aligned} \tag{3.59}$$

And

$$
\begin{aligned}
\overline{\Delta} R_{i\bar{j}k\bar{\ell}} &= g^{\bar{b}a}\nabla_{\bar{b}}\nabla_k R_{i\bar{j}a\bar{l}} \\
&= g^{\bar{b}a}\nabla_k\nabla_{\bar{b}} R_{i\bar{j}a\bar{l}} + g^{\bar{b}a}\left[\nabla_{\bar{b}},\nabla_k\right] R_{i\bar{j}a\bar{l}} \\
&= \nabla_{\bar{l}}\nabla_k \operatorname{Ric}_{i\bar{j}} + \left[\nabla_k,\nabla_{\bar{l}}\right]\operatorname{Ric}_{i\bar{j}} + g^{\bar{b}a}\left[\nabla_{\bar{b}},\nabla_k\right] R_{i\bar{j}a\bar{l}} \\
&= \nabla_{\bar{l}}\nabla_k \operatorname{Ric}_{i\bar{j}} - R_{k\bar{l}i}{}^{a}\operatorname{Ric}_{a\bar{j}} + R_{k\bar{l}}{}^{\bar{b}}{}_{\bar{j}}\operatorname{Ric}_{i\bar{b}} \\
&\quad + R_k{}^{a}{}_{i}{}^{b} R_{b\bar{j}a\bar{l}} - R_k{}^{a\bar{b}}{}_{\bar{j}} R_{i\bar{b}a\bar{\ell}} + R_k{}^{b} R_{i\bar{j}b\bar{l}} - R_k{}^{a\bar{b}}{}_{\bar{l}} R_{i\bar{j}a\bar{b}}.
\end{aligned}
\tag{3.60}
$$

Combining 3.58, 3.59 and 3.60 yields 3.57. □

In fact we merely need the general form

$$
\frac{\partial}{\partial t}\operatorname{Rm} = \frac{1}{2}\Delta_{\mathbb{R}}\operatorname{Rm} - \nu\operatorname{Rm} + \operatorname{Rm} * \operatorname{Rm} + \operatorname{Rc} * \operatorname{Rm}. \tag{3.61}
$$

where $\operatorname{Rc}$ denotes the Ricci tensor. To clarify notation: if $A$ and $B$ are tensors, we write $A * B$ for any combination of products of the tensors $A$ and $B$ formed by contractions on $A_{i_1\cdots i_k}$ and $B_{j_i\cdots j_l}$ using the metric $g$.

**Lemma 3.2.2.** *There exists a universal constant $C$ such that*

$$
\left(\frac{\partial}{\partial t} - \Delta\right)\|\operatorname{Rm}\|^2 \le -\|\nabla\operatorname{Rm}\|^2 - \|\overline{\nabla}\operatorname{Rm}\|^2 + C\|\operatorname{Rm}\|^3 - \nu\|\operatorname{Rm}\|^2. \tag{3.62}
$$

*and for all points of $M \times [0,T)$ where $\|\operatorname{Rm}\|$ is not zero,*

$$
\left(\frac{\partial}{\partial t} - \Delta\right)\|\operatorname{Rm}\| \le \frac{C}{2}\|\operatorname{Rm}\|^2 - \frac{\nu}{2}\|\operatorname{Rm}\|. \tag{3.63}
$$

*Proof.* 3.62 comes from 3.61, and when $\|\operatorname{Rm}\| \neq 0$,

$$
\left(\frac{\partial}{\partial t} - \Delta\right)\|\operatorname{Rm}\| = \frac{1}{2\|\operatorname{Rm}\|}\left(\frac{\partial}{\partial t} - \Delta\right)\|\operatorname{Rm}\|^2 + \frac{1}{4\|\operatorname{Rm}\|^3} g^{\bar{j}i}\nabla_i\|\operatorname{Rm}\|^2\nabla_{\bar{j}}\|\operatorname{Rm}\|^2. \tag{3.64}
$$

and

$$
g^{\bar{j}i}\nabla_i\|\operatorname{Rm}\|^2\nabla_{\bar{j}}\|\operatorname{Rm}\|^2 \le 2\|\operatorname{Rm}\|^2(\|\nabla\operatorname{Rm}\|^2 + \|\overline{\nabla}\operatorname{Rm}\|^2). \tag{3.65}
$$

3.63 comes from 3.62 and 3.65. □

Combining this result with the third order estimate in Theorem 3.2.4 we get

**Theorem 3.2.5.** *Let $\omega = \omega(t)$ solve 3.2 and assume that there exists a constant $C_0 > 0$ such that*

$$
\frac{1}{C_0}\omega_0 \le \omega \le C_0\omega_0. \tag{3.66}
$$

*Then there exists a constant $C$ depending only on $C_0$ and $\omega_0$ such that*

$$
\|\operatorname{Rm}\|^2 \le C. \tag{3.67}
$$

*In addition, there exists a constant $C'$ depending only on $C_0$ and $\omega_0$ such that*

$$
\left(\frac{\partial}{\partial t} - \Delta\right)\|\operatorname{Rm}\|^2 \le -\|\nabla\operatorname{Rm}\|^2 - \|\overline{\nabla}\operatorname{Rm}\|^2 + C'. \tag{3.68}
$$

*Proof.* From Theorem 3.2.4, $S = \|\nabla_{g_0} g\|^2$ is uniformly bounded, and from 3.49 and 3.63, if we set $Q = \|\operatorname{Rm}\| + AS$ for a constant $A$. If $A$ is chosen to be sufficiently large, we obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q \leq -\|\operatorname{Rm}\|^2 + C'. \tag{3.69}$$

for a uniform constant $C'$. Then the upper bound of $\|\operatorname{Rm}\|^2$ follows from the maximum principle. Finally, 3.68 comes from 3.62. □

Once we have bounded curvature, it's a result of Hamilton [10] that bounds on all derivatives of curvature follow. For convenience we change to a real coordinate system.

**Theorem 3.2.6.** *Let $\omega = \omega(t)$ solve 3.2 on $[0, T)$ with $0 < T \leq \infty$ and assume that there exists a constant $C > 0$ such that*

$$\|\operatorname{Rm}\|^2 \leq C. \tag{3.70}$$

*Then there exist uniform constants $C_m$ for $m = 1, 2, \ldots$ depending only on $\omega_0$ such that*

$$\|\nabla_{\mathbb{R}}^m \operatorname{Rm}\|^2 \leq C_m. \tag{3.71}$$

*Proof.* From Lemma 3.2.1 and an induction argument (see Theorem 13.2 in [10])

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta_{\mathbb{R}}\right) \nabla_{\mathbb{R}}^m \operatorname{Rm} = \sum_{p+q=m} \nabla_{\mathbb{R}}^p \operatorname{Rm} * \nabla_{\mathbb{R}}^q \operatorname{Rm}. \tag{3.72}$$

It follows that

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta_{\mathbb{R}}\right) \|\nabla_{\mathbb{R}}^m \operatorname{Rm}\|^2 = -\left\|\nabla_{\mathbb{R}}^{m+1} \operatorname{Rm}\right\|^2 + \sum_{p+q=m} \nabla_{\mathbb{R}}^p \operatorname{Rm} * \nabla_{\mathbb{R}}^q \operatorname{Rm} * \nabla_{\mathbb{R}}^m \operatorname{Rm}. \tag{3.73}$$

Since $\|\operatorname{Rm}\|^2$ is bounded, from 3.62 we get

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta_{\mathbb{R}}\right) \|\operatorname{Rm}\|^2 \leq -\|\nabla_{\mathbb{R}} \operatorname{Rm}\|^2 + C'. \tag{3.74}$$

for some uniform constant $C'$. For the case $m = 1$, set $Q = \|\nabla_{\mathbb{R}} \operatorname{Rm}\|^2 + A\|\operatorname{Rm}\|^2$ for $A > 0$ sufficiently large, from 3.73 we have

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta_{\mathbb{R}}\right) Q \leq -\|\nabla_{\mathbb{R}} \operatorname{Rm}\|^2 + C''. \tag{3.75}$$

and it follows from the maximum principle that $\|\nabla_{\mathbb{R}} \operatorname{Rm}\|^2$ is uniformly bounded from above. In addition,

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta_{\mathbb{R}}\right) \|\nabla_{\mathbb{R}} \operatorname{Rm}\|^2 \leq -\left\|\nabla_{\mathbb{R}}^2 \operatorname{Rm}\right\|^2 + C''', \tag{3.76}$$

and an induction completes the proof. □

Next, we show that once we have a uniform bound on a metric evolving by the Kähler-Ricci flow, together with bounds on derivatives of curvature, then we have $C^\infty$ bounds for the metric. Moreover, the result is local:

**Theorem 3.2.7.** *Let $\omega = \omega(t)$ solve 3.2 on $U \times [0,T)$ with $0 \le T \le \infty$, where $U$ is an open subset of $M$. Assume that there there exist constants $C_m$ for $m = 0, 1, 2 \dots$ such that*

$$\frac{1}{C_0}\omega_0 \le \omega \le C_0\omega_0, \quad S \le C_0 \quad \textit{and} \quad |\nabla_{\mathbb{R}}^m \mathrm{Rm}|^2 \le C_m. \tag{3.77}$$

*Then for any compact subset $K \subset U$ and for $m = 1, 2, \dots$, there exist constants $C'_m$ depending only on $\omega_0, K, U$ and $C_m$ such that*

$$\|\omega(t)\|_{C^m(K,g_0)} \le C'_m. \tag{3.78}$$

*Proof.* It suffices to prove the result on the ball $B$ say, in a fixed holomorphic coordinate chart. We will obtain the $C^\infty$ estimates for $\omega(t)$ on a slightly smaller ball. Fix a time $t \in (0,T]$. Consider the equations

$$\Delta_{\mathrm{E}} g_{i\bar{j}} = -\sum_k R_{k\bar{k}i\bar{j}} + \sum_{k,p,q} g^{q\bar{p}} \partial_k g_{i\bar{q}} \partial_{\bar{k}} g_{p\bar{j}} =: Q_{i\bar{j}}. \tag{3.79}$$

where $\Delta_{\mathrm{E}} = \sum_k \partial_k \partial_{\bar{k}}$. For each fixed $i, j$, we can regard 3.79 as Poisson's equation $\Delta_{\mathrm{E}} g_{i\bar{j}} = Q_{i\bar{j}}$.

Fix $p > 2n$. From our assumptions, each $\|Q_{i\bar{j}}\|_{L^p(B)}$ is uniformly bounded. Applying the standard elliptic estimates (see Theorem 9.11 of [9] for example) to 3.79 we see that the Sobolev norm $\|g_{i\bar{j}}\|_{W^{2,p}}$ is uniformly bounded on a slightly smaller ball. From now on, the estimates that we state will always be modulo shrinking the ball slightly. Morrey's embedding theorem gives that $\|g_{i\bar{j}}\|_{C^{1+\beta}}$ is uniformly bounded for some $0 < \beta < 1$.

The key observation we now need is as follows: the $m$ th derivative of $Q_{i\bar{j}}$ can be written in the form $A * B$ where each $A$ or $B$ represents either a covariant derivative of $\mathrm{Rm}$ or a quantity involving derivatives of $g$ up to order at most $m+1$. Hence if $g$ is uniformly bounded in $C^{m+1+\beta}$ then each $Q_{i\bar{j}}$ is uniformly bounded in $C^{m+\beta}$.

Applying this observation with $m = 0$ we see that each $\|Q_{i\bar{j}}\|_{C^\beta}$ is uniformly bounded. The standard Schauder estimates give that $\|g_{i\bar{j}}\|_{C^{2+\beta}}$ is uniformly bounded.

We can now apply a bootstrapping argument. Applying the observation with $m = 1$ we see that $Q_{i\bar{j}}$ is uniformly bounded in $C^{1+\beta}$, and so on. This completes the proof. □

Combining Theorems 3.2.5-3.2.7, we obtain:

**Corollary 3.2.3.** *Let $\omega = \omega(t)$ solve 3.2 on $M \times [0,T)$ with $0 \le T \le \infty$. Assume that there exists a constant $C_0$ such that*

$$\frac{1}{C_0}\omega_0 \le \omega \le C_0\omega_0. \tag{3.80}$$

*Then for $m = 1, 2, \dots$, there exist uniform constants $C_m$ such that*

$$\|\omega(t)\|_{C^m(g_0)} \le C_m. \tag{3.81}$$

In fact, there is a local version of Corollary 2.2.3:

**Corollary 3.2.4.** *Let $\omega = \omega(t)$ solve 3.2 on $U \times [0,T)$ with $0 \le T \le \infty$, where $U$ is an open subset of $M$. Assume that there there exists a constant $C_0$ for such that*

$$\frac{1}{C_0}\omega_0 \le \omega \le C_0\omega_0. \tag{3.82}$$

*Then for any compact subset $K \subset U$ and for $m = 1, 2, \dots$, there exist constants $C'_m$ depending only on $\omega_0, K$ and $U$ such that*

$$\|\omega(t)\|_{C^m(K,g_0)} \le C'_m. \tag{3.83}$$

## 3.3 Maximal Existence Time for the Kähler-Ricci Flow

In this section, we prove Theorem 3.1.1 by rewriting the Kähler-Ricci flow as a parabolic complex Monge–Ampère equation.

### 3.3.1 The Parabolic Monge–Ampère equation

Let $\omega = \omega(t)$ be a solution of the Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega), \qquad \omega|_{t=0} = \omega_0. \tag{3.84}$$

Consider the evolution of the cohomology class $[\omega(t)]$:

$$\frac{\partial}{\partial t}[\omega(t)] = -2\pi c_1(M), \qquad [\omega(0)] = [\omega_0]. \tag{3.85}$$

Therefore ODE argument tells us that $[\omega(t)] = [\omega_0] - 2\pi t c_1(M)$. Immediately, a necessary condition for the Kähler-Ricci flow to exist for $t \in [0, t')$ is that $[\omega_0] - 2\pi t c_1(M) > 0$ for $t \in [0, t')$. Amazingly, this necessary condition is in fact sufficient. If we define

$$T = \sup\{t > 0 | [\omega_0] - 2\pi t c_1(M) > 0\}. \tag{3.86}$$

then we have

**Theorem 3.3.1.** *There exists a unique maximal solution $g(t)$ of the Kähler-Ricci flow 3.84 for $t \in [0, T)$.*

The theorem was prove by Cao[5] in the special case when $c_1(M)$ is zero or definite. In this generality, the result is due to Tian-Zhang[24].

*Proof.* Fix $T' < T$, we will show that there exists a solution to 3.84 on $[0, T')$. First we observe that 3.84 can be rewritten as a parabolic complex Monge–Ampère equation.

To do this, we choose a reference metrics $\hat{\omega}_t$ in the cohomology class $[\omega_0] - 2\pi t c_1(M)$. Since $[\omega_0] - 2\pi T' c_1(M)$ is a Kähler class, there exists a Kähler form $\eta$ in $[\omega_0] - 2\pi T' c_1(M)$. Set $\chi = \frac{1}{T'}(\eta - \omega_0) \in -2\pi c_1(M)$, and

$$\hat{\omega}_t = \omega_0 + t\chi = \frac{1}{T'}\big((T' - t)\omega_0 + t\eta\big) \in [\omega_0] - 2\pi t c_1(M). \tag{3.87}$$

Fix a volume form $\Omega$ on $M$ with

$$\sqrt{-1}\partial\bar{\partial}\log\Omega = \chi = \frac{\partial}{\partial t}\hat{\omega}_t \in -2\pi c_1(M). \tag{3.88}$$

which exists by $2\pi c_1(M) = [\operatorname{Ric}(M)]$ and $\partial\bar{\partial}$-lemma.

We now consider the parabolic complex Monge–Ampère equation, for $\phi = \phi(t)$ a real-valued function on $M$,

$$\frac{\partial}{\partial t}\phi = \log\frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\Omega}, \qquad \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\phi > 0, \qquad \phi|_{t=0} = 0. \tag{3.89}$$

This equation is equivalent to the Kähler-Ricci flow 3.84:

Given a smooth solution $\phi$ of 3.89 on $[0, T')$, we can obtain a solution $\omega = \omega(t)$ of 3.84 as follows. Define $\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\phi$ and $\omega(0) = \hat{\omega}_0 = \omega_0$ and

$$\frac{\partial}{\partial t}\omega = \frac{\partial}{\partial t}\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\left(\frac{\partial}{\partial t}\phi\right) = -\operatorname{Ric}(\omega). \tag{3.90}$$

as required.

Conversely, suppose $\omega = \omega(t)$ solves 3.84 on $[0, T')$. Then since $\hat{\omega}_t \in [\omega(t)]$, we can apply $\partial\bar{\partial}$-lemma to find a family of potential functions $\tilde{\phi}(t)$ such that $\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\tilde{\phi}(t)$ and $\int_M \tilde{\phi}(t)\omega_0^n = 0$. By standard elliptic regularity theory the family $\tilde{\phi}(t)$ is smooth on $M \times [0, T')$. Then

$$\sqrt{-1}\partial\bar{\partial}\log\omega^n = \frac{\partial}{\partial t}\omega = \sqrt{-1}\partial\bar{\partial}\log\Omega + \sqrt{-1}\partial\bar{\partial}\left(\frac{\partial}{\partial t}\tilde{\phi}\right). \tag{3.91}$$

Since the only pluriharmonic functions on $M$ are the constants, we have

$$\frac{\partial}{\partial t}\tilde{\phi} = \log\frac{\omega^n}{\Omega} + c(t). \tag{3.92}$$

where $c : [0, T') \to \mathbb{R}$ is smooth. Now set $\phi(t) = \tilde{\phi}(t) - \int_0^t c(s)ds - \tilde{\phi}(0)$. It follows that $\phi = \phi(t)$ solves the parabolic complex Monge–Ampère equation 3.89.

Therefore it suffices to study 3.89. Since the linearization of the right hand side of 3.89 is the Laplace operator $\Delta_{g(t)}$, which is elliptic, it follows that 3.89 is a strictly parabolic PDE for $\phi$. By standard parabolic theory, there exists a unique maximal solution of 3.89 for some time interval $[0, T_{\max})$ with $0 < T_{\max} \leq \infty$. WLOG, $T_{\max} < T'$. Then we will obtain a contradiction by showing that a solution of 3.89 exists beyond $T_{\max}$. This will be done in the next two subsections. □

### 3.3.2 Estimates for the Potential Function and the Volume Form

We assume now that we have a solution $\varphi = \varphi(t)$ to the parabolic complex Monge-Ampère equation 3.89 on $[0, T_{\max})$, for $0 < T_{\max} < T' < T$. Our goal is to establish uniform estimates for $\varphi$ on $[0, T_{\max})$. In this subsection we will prove a $C^0$ estimate for $\varphi$ and a lower bound for the volume form.

**Lemma 3.3.1.** *There exists a uniform $C$ such that for all $t \in [0, T_{\max})$,*

$$\|\phi(t)\|_{C^0(M)} \leq C. \tag{3.93}$$

*Proof.* For the upper bound of $\phi$, we will apply the maximum principle to $\theta := \phi - At$ for $A > 0$ a uniform constant to be determined later. From 3.89 we have

$$\frac{\partial\theta}{\partial t} = \log\frac{\left(\hat{\omega}_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\theta\right)^n}{\Omega} - A. \tag{3.94}$$

Fix $t' \in (0, T_{\max})$. Since $M \times [0, t']$ is compact, $\theta$ attains a maximum at some point $(x_0, t_0) \in M \times [0, t']$. We claim that if $A$ is sufficiently large we have $t_0 = 0$. Otherwise $t_0 > 0$. Then by Proposition 2.3.1, at $(x_0, t_0)$,

$$0 \leq \frac{\partial\theta}{\partial t} = \log\frac{\left(\hat{\omega}_{t_0} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\theta\right)^n}{\Omega} - A \leq \log\frac{\hat{\omega}_{t_0}^n}{\Omega} - A \leq -1. \tag{3.95}$$

a contradiction, where we have chosen $A \geq 1 + \sup_{M\times[0,T_{\max}]} \log\left(\hat{\omega}_t^n/\Omega\right)$. Hence we have proved the claim that $t_0 = 0$, giving $\sup_{M\times[0,t']} \theta \leq \sup_M \theta\Big|_{t=0} = 0$ and thus

$$\varphi(x,t) \leq At \leq AT_{\max}, \quad \text{for } (x,t) \in M \times [0,t']. \tag{3.96}$$

Since $t' \in (0, T_{\max})$ was arbitrary, this gives a uniform upper bound for $\phi$ on $[0, T_{\max})$. We apply a similar argument to $\psi = \phi + Bt$ for $B$ a positive constant with $B \geq 1 - \inf_{M\times[0,T_{\max}]} \log\left(\hat{\omega}_t^n/\Omega\right)$ and obtain

$$\phi(x,t) \geq -BT_{\max}, \quad \text{for } (x,t) \in M \times [0,t']. \tag{3.97}$$

giving the lower bound. □

Next we prove a lower bound for the volume form along the flow, or equivalently a lower bound for $\dot{\varphi} = \partial\varphi/\partial t$. This argument is due to Tian-Zhang [24].

**Lemma 3.3.2.** *There exists a uniform $C > 0$ such that on $M \times [0, T_{\max})$,*

$$\frac{1}{C}\Omega \leq \omega^n(t) \leq C\Omega. \tag{3.98}$$

*or equivalently, $\|\dot{\phi}\|_{C^0(M)}$ is uniformly bounded.*

*Proof.* The upper bound of $\omega^n$ follows from part (i) of Corollary 3.3.1. Note that since this is equivalent to an upper bound of $\dot{\phi}$, we have given an alternative proof of the upper bound part of Lemma 3.3.1. For the lower bound of $\omega^n$, differentiate 3.89:

$$\frac{\partial\dot{\phi}}{\partial t} = \Delta\dot{\phi} + \operatorname{tr}_\omega \chi. \tag{3.99}$$

Define a quantity $Q = \left(T' - t\right)\dot{\phi} + \phi + nt$ and compute using 3.99,

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q = \left(T' - t\right)\operatorname{tr}_\omega \chi + n - \Delta\phi = \operatorname{tr}_\omega\left(\hat{\omega}_t + \left(T'-t\right)\chi\right) = \operatorname{tr}_\omega \hat{\omega}_{T'} > 0. \tag{3.100}$$

where we have used the fact that

$$\Delta\phi = \operatorname{tr}_\omega\left(\omega - \hat{\omega}_t\right) = n - \operatorname{tr}_\omega \hat{\omega}_t \tag{3.101}$$

Then by the maximum principle (Proposition 2.3.1), $Q$ is uniformly bounded from below on $M \times [0, T_{\max})$ by its infimum at the initial time. Thus

$$\left(T'-t\right)\dot{\phi} + \phi + nt \geq T' \inf_M \log\frac{\omega_0^n}{\Omega}, \quad \text{on } M \times [0, T_{\max}), \tag{3.102}$$

and since $\phi$ is uniformly bounded from Lemma 3.3.1 and $T' - t \geq T' - T_{\max} > 0$, this gives the desired lower bound of $\dot{\phi}$. □

### 3.3.3 A Uniform Bound for the Evolving Metric

Again we assume that we have a solution $\phi = \phi(t)$ to 3.89 on $[0, T_{\max})$, for $0 < T_{\max} < T' < T$. From Lemma 3.3.1, we have a uniform bound for $\|\phi\|_{C^0(M)}$ and we will use this together with Proposition 3.2.2 to obtain an upper bound for the quantity $\operatorname{tr}_{\omega_0} \omega$ on $[0, T_{\max})$. We will then complete the proof of Theorem 3.3.1.

**Lemma 3.3.3.** *There exists a uniform $C$ such that on $M \times [0, T_{\max})$,*

$$\operatorname{tr}_{\omega_0} \omega \leq C. \tag{3.103}$$

*Proof.* Consider the quantity

$$Q = \log \operatorname{tr}_{\omega_0} \omega - A\phi, \tag{3.104}$$

for $A > 0$ a uniform constant to be determined later. For a fixed $t' \in (0, T_{\max})$, assume that $Q$ on $M \times [0, t']$ attains a maximum at a point $(x_0, t_0)$. WiLOG suppose that $t_0 > 0$. Then at $(x_0, t_0)$, applying Proposition 3.2.2 with $\hat{\omega} = \omega_0$,

$$\begin{aligned} 0 \leq \left(\frac{\partial}{\partial t} - \Delta\right) Q &\leq C_0 \operatorname{tr}_\omega \omega_0 - A\dot{\phi} + A\Delta\phi \\ &= \operatorname{tr}_\omega \left(C_0 \omega_0 - A\hat{\omega}_{t_0}\right) - A \log \frac{\omega^n}{\Omega} + An. \end{aligned} \tag{3.105}$$

for $C_0$ depending only on the lower bound of the bisectional curvature of $g_0$. Choose $A$ sufficiently large so that $A\hat{\omega}_{t_0} - (C_0 + 1)\,\omega_0$ is Kähler on $M$. Then

$$\operatorname{tr}_\omega \left(C_0 \omega_0 - A\hat{\omega}_{t_0}\right) \leq -\operatorname{tr}_\omega \omega_0. \tag{3.106}$$

and so at $(x_0, t_0)$

$$\operatorname{tr}_\omega \omega_0 + A \log \frac{\omega^n}{\Omega} \leq An. \tag{3.107}$$

and hence

$$\operatorname{tr}_\omega \omega_0 + A \log \frac{\omega^n}{\omega_0^n} \leq C. \tag{3.108}$$

for some uniform constant $C$. At $(x_0, t_0)$, choose coordinates so that

$$(g_0)_{i\bar{j}} = \delta_{ij} \quad \text{and} \quad g_{i\bar{j}} = \lambda_i \delta_{ij}, \quad \text{for } i, j = 1, \ldots, n, \tag{3.109}$$

for positive $\lambda_1, \ldots, \lambda_n$. Then 3.108 is precisely

$$\sum_{i=1}^{n} \left(\frac{1}{\lambda_i} + A \log \lambda_i\right) \leq C. \tag{3.110}$$

Since the function $x \mapsto \frac{1}{x} + A \log x$ for $x > 0$ is uniformly bounded from below, we have,

$$\left(\frac{1}{\lambda_i} + A \log \lambda_i\right) \leq C, \quad \text{for } i = 1, \ldots, n. \tag{3.111}$$

Then $A \log \lambda_i \leq C$, giving a uniform upper bound for $\lambda_i$ and hence $\left(\operatorname{tr}_{\omega_0} \omega\right)(x_0, t_0)$. Since $\phi$ is uniformly bounded on $M \times [0, T_{\max})$ we see that $Q(x_0, t_0)$ is uniformly bounded from above. Hence $Q$ is bounded from above on $M \times [0, t']$ for any $t' < T_{\max}$. Using again that $\phi$ is uniformly bounded we obtain the required estimate 3.103. □

As a consequence of Lemma 3.3.3, we have:

**Corollary 3.3.1.** *There exists a uniform $C > 0$ such that on $M \times [0, T_{\max})$,*

$$\frac{1}{C}\omega_0 \leq \omega \leq C\omega_0. \tag{3.112}$$

*Proof.* The upper bound follows from Lemma 3.3.3. For the lower bound,

$$\operatorname{tr}_\omega \omega_0 \leq \frac{1}{(n-1)!}\left(\operatorname{tr}_{\omega_0} \omega\right)^{n-1} \frac{\omega_0^n}{\omega^n} \leq C \tag{3.113}$$

using Lemma 3.3.2. To verify the first inequality of 3.113, choose coordinates as in 3.109 and observe that

$$\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} \leq \frac{1}{(n-1)!}\frac{(\lambda_1 + \cdots + \lambda_n)^{n-1}}{\lambda_1 \cdots \lambda_n}, \tag{3.114}$$

for positive $\lambda_i$. □

Now the proof of Theorem 3.3.1 is clear:

**Proof of Theorem 3.3.1**. Combining Corollary 3.3.1 with Corollary 3.2.4, we obtain uniform $C^\infty$ estimates for $g(t)$ on $[0, T_{\max})$. Hence as $t \to T_{\max}$, the metrics $g(t)$ converge in $C^\infty$ to a smooth Kähler metric $g(T_{\max})$ and thus we obtain a smooth solution to the Kähler-Ricci flow on $[0, T_{\max}]$. But we have already seen that we can always find a smooth solution of the Kähler-Ricci flow on some, possibly short, time interval with any initial Kähler metric. Applying this to $g\,(T_{\max})$, we obtain a solution of the Kähler-Ricci flow $g(t)$ on $[0, T_{\max} + \varepsilon)$ for $\varepsilon > 0$. But this contradicts the definition of $T_{\max}$, and completes the proof of Theorem 3.3.1.

## 3.4 Convergence of the Flow

In this section we show that the Kähler-Ricci flow converges, after some appropriate normalization, to a Kähler-Einstein metric in the case of $c_1(M) \leq 0$, which was proved by Cao [5].

### 3.4.1 The Normalized Kähler-Ricci Flow When $\mathbf{c_1(M) < 0}$

Firstly we consider a manifold $M$ of general type, i.e. $c_1(M) < 0$. We restrict to the case when $[\omega_0] = -2\pi c_1(M)$. By theorem 3.3.1, there exists a solution to the Kähler-Ricci flow 3.1 for $t \in [0, \infty)$. Then the Kähler class evolves as $[\omega(t)] = (1 + 2\pi t)[\omega_0]$ which diverges as $t \to \infty$. To avoid this we consider instead the normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega) - \omega, \quad \omega|_{t=0} = \omega_0. \tag{3.115}$$

This is just a rescaling of 3.1. Indeed

If $\tilde{\omega}(s)$ solves 3.1 for $s \in [0, \infty)$ then $\omega(t) = \frac{\tilde{\omega}(s)}{s+1}$ with $t = \log(s+1)$ solves 3.115. Conversely, given a solution to 3.115 we can rescale to obtain a solution to 3.1.

In this case we have $[\omega(t)] = [\omega_0]$ for all $t$. The following result is due to Cao [5].

**Theorem 3.4.1.** *The solution $\omega = \omega(t)$ to 3.115 converges in $C^\infty$ to the unique Kähler-Einstein metric $\omega_{\mathrm{KE}} \in -2\pi c_1(M)$.*

*Proof.* Proof of the uniqueness: In this case, a metric is if and only if $\mathrm{Ric}(\omega_{\mathrm{KE}}) = -\omega_{\mathrm{KE}}$. The uniqueness is due to Calabi [4], which follows from the maximum principle. Suppose $\omega'_{\mathrm{KE}}, \omega_{\mathrm{KE}} \in 2\pi c_1(M)$ are both Kähler-Einstein. Writing $\omega'_{\mathrm{KE}} = \omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar{\partial}\phi$, then $\mathrm{Ric}(\omega'_{\mathrm{KE}}) = -\omega'_{\mathrm{KE}} = \mathrm{Ric}(\omega_{\mathrm{KE}}) - \sqrt{-1}\partial\bar{\partial}\phi$ and hence

$$\log \frac{(\omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\omega_{\mathrm{KE}}^n} = \phi + C. \tag{3.116}$$

for some constant $C$. By considering the maximum and minimum of $\phi + C$ on $M$ we find that $\phi + C = 0$ thus $\omega'_{\mathrm{KE}} = \omega_{\mathrm{KE}}$. □

To prove theorem 3.4.1, we reduce 3.115 to a parabolic Monge-Ampère equation. Let $\Omega$ be a volume form on $M$ satisfying

$$\sqrt{-1}\partial\bar{\partial}\log\Omega = \omega_0 \in -c_1(M), \quad \int_M \Omega = \int_M \omega_0^n. \tag{3.117}$$

Then we consider the normalized parabolic complex Monge-Ampère equation,

$$\frac{\partial}{\partial t}\phi = \log\frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\Omega} - \phi, \quad \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0, \quad \phi|_{t=0} = 0. \tag{3.118}$$

Similarly as the previous section, given a solution $\phi = \phi(t)$ of 3.118, the metrics $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$ solve 3.115. Conversely, given a solution $\omega = \omega(t)$ of 3.115 we can obtain via the $\partial\bar{\partial}$-lemma a solution $\phi = \phi(t)$ of 3.118.

We get some estimates for $\phi$ solving 3.118:

**Lemma 3.4.1.** *We have*

*(i) There exists a uniform constant $C$ such that for $t$ in $[0, \infty)$,*

$$\|\dot{\phi}(t)\|_{C^0(M)} \le Ce^{-t}. \tag{3.119}$$

*(ii) There exists a continuous real-valued function $\phi_\infty$ on $M$ such that for $t \in [0, \infty)$,*

$$\|\phi(t) - \phi_\infty\|_{C^0(M)} \le Ce^{-t}. \tag{3.120}$$

*(iii) $\|\phi(t)\|_{C^0(M)}$ is uniformly bounded for $t \in [0, \infty)$.*

*(iv) There exists a uniform constant $C'$ such that on $M \times [0, \infty)$, the volume form of $\omega = \omega(t)$ satisfies*

$$\frac{1}{C'}\omega_0^n \le \omega^n \le C'\omega_0^n. \tag{3.121}$$

*Proof.* Since

$$\frac{\partial}{\partial t}\dot{\phi} = \Delta\dot{\phi} - \dot{\phi}. \tag{3.122}$$

hence

$$\frac{\partial}{\partial t}(e^t\dot{\phi}) = \Delta(e^t\dot{\phi}). \tag{3.123}$$

By maximum principle (i) holds.

For (ii), let $s, t \geq 0$ and $x \in M$, then

$$\begin{aligned} |\phi(x,s) - \phi(x,t)| = \left| \int_s^t \dot{\phi}(x,u) du \right| &\leq \int_t^s |\dot{\phi}(x,u)| du \\ &\leq \int_t^s C e^{-u} du = C(e^{-t} - e^{-s}). \end{aligned} \tag{3.124}$$

which shows that $\phi(t)$ converges uniformly to a continuous function $\phi_\infty$ on $M$. Taking $s \to \infty$ in 3.124 gives (ii).

(iii) follows immediately from (ii) and (iv) follows from (i), (iii) and 3.118. □

Then we use the $C^0$ bound on $\phi$ to obtain an upper bound on the evolving metric.

**Lemma 3.4.2.** *There exists a uniform constant $C$ such that on $M \times [0, \infty)$, $\omega = \omega(t)$ satisfies*

$$\frac{1}{C}\omega_0 \leq \omega \leq C\omega_0. \tag{3.125}$$

*Proof.* By lemma 3.4.1 (iv) and arguments in corollary 3.3.1, it suffices to obtain a uniform upper bound for $\operatorname{tr}_{\omega_0} \omega$.

Applying proposition 3.2.2 we have

$$\left( \frac{\partial}{\partial t} - \Delta \right) \log \operatorname{tr}_{\omega_0} \omega \leq C_0 \operatorname{tr}_\omega \omega_0 - 1, \tag{3.126}$$

for some $C_0$ depending only on $\omega_0$, apply the maximum principle to $Q = \log \operatorname{tr}_\omega \omega_0 - A\phi$, we have

$$\left( \frac{\partial}{\partial t} - \Delta \right) Q \leq C_0 \operatorname{tr}_\omega \omega_0 - 1 - A\dot{\phi} + An - A \operatorname{tr}_\omega \omega_0. \tag{3.127}$$

Assume that $Q$ achieves a maximum at a point $(x_0, t_0)$ with $t_0 > 0$. Choosing $A = C_0 + 1$ and using $\dot{\phi}$ is uniformly bounded, we see that $\operatorname{tr}_\omega \omega_0$ is uniformly bounded at $(x_0, t_0)$. Arguing as in 3.114 we have

$$(\operatorname{tr}_{\omega_0} \omega)(x_0, t_0) \leq \frac{1}{(n-1)!} (\operatorname{tr}_\omega \omega_0)^{n-1}(x_0, t_0) \frac{\omega^n}{\omega_0^n}(x_0, t_0) \leq C. \tag{3.128}$$

using (iv) in lemma 3.4.1. Since $\phi$ is uniformly bounded, this shows that $Q$ is bounded from above at $(x_0, t_0)$. Hence $\operatorname{tr}_{\omega_0} \omega$ is uniformly bounded from above. □

Now we can complete the proof of theorem 3.4.1:

**Proof of theorem 3.4.1**. By Corollary 3.2.3 we have uniform $C^\infty$ estimates on $\omega(t)$. Since $\phi(t)$ is bounded in $C^0$ it follows that we have uniform $C^\infty$ estimates on $\phi(t)$. Since by lemma 3.4.1 (ii) $\phi(t)$ converges to a continuous function $\phi_\infty$ on $M$ as $t \to \infty$. By Arzela-Ascoli lemma and the uniqueness of the limmit, there exists $t_k \to \infty$ such that $\phi(t_k)$ converges in $C^\infty$ to $\phi_\infty$, which is smooth. In fact, we have the convergence without passing to a subsequence by the $C^0$ convergence.

It remains to show that the limit metric $\omega_\infty = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_\infty$ is Kähler-Einstein. From lemma 3.4.1 (i) we have $\dot{\phi}(t) \to 0$ as $t \to \infty$, thus from 3.118 we get $\log \frac{\omega_\infty^n}{\Omega} = \phi_\infty$. Apply $\sqrt{-1}\partial\bar{\partial}$ on both sides gives $\operatorname{Ric}(\omega_\infty) = -\omega_\infty$ as required. The proof is now complete.

### 3.4.2 The Case of $\mathbf{c_1(M) = 0}$: Yau's zeroth Order Estimate

In this section we discuss the Kähler-Ricci flow on a Kähler manifold $(M, \omega_0)$ with vanishing first Chern class. By theorem 3.3.1 there exists a solution $\omega(t)$ of the Kähler-Ricci flow 3.84 for $t \in [0, \infty)$ and we have $[\omega(t)] = [\omega_0]$. The following result is due to Cao [5] and makes use of Yau's celebrated zeroth order estimate.

**Theorem 3.4.2.** *The solution $\omega(t)$ to 3.84 converges in $C^\infty$ to the unique Kähler-Einstein metric $\omega_{\mathrm{KE}} \in [\omega_0]$.*

*Proof.* Proof of the uniqueness: In this case, a metric is Kähler-Einstein if and only if it's Ricci flat. The uniqueness is due to Calabi [4]. Suppose $\omega'_{\mathrm{KE}} = \omega_{\mathrm{KE}} + \sqrt{-1}\partial\bar{\partial}\phi$ is another Kähler-Einstein metric. Then the equation $\mathrm{Ric}(\omega'_{\mathrm{KE}}) = \mathrm{Ric}(\omega_{\mathrm{KE}})$ gives

$$\log \frac{\omega'^n_{\mathrm{KE}}}{\omega^n_{\mathrm{KE}}} = C. \tag{3.129}$$

for some constant $C$. Exponentiating and then integrating gives $C = 1$ and hence $\omega'^n_{\mathrm{KE}} = \omega^n_{\mathrm{KE}}$. Then compute

$$\begin{aligned} 0 = \int_M \phi(\omega^n_{\mathrm{KE}} - \omega'^n_{\mathrm{KE}}) &= -\int_M \phi\sqrt{-1}\partial\bar{\partial}\phi \wedge (\sum_{i=0}^{n-1} \omega^i_{\mathrm{KE}} \wedge \omega'^{n-1-i}_{\mathrm{KE}}) \\ &= \int_M \sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \wedge (\sum_{i=0}^{n-1} \omega^i_{\mathrm{KE}} \wedge \omega'^{n-1-i}_{\mathrm{KE}}) \\ &\geq \frac{1}{n}\int_M |\partial\phi|^2_{\omega_{\mathrm{KE}}} \omega^n_{\mathrm{KE}}. \end{aligned} \tag{3.130}$$

which implies that $\phi$ is a constant and hence $\omega'_{\mathrm{KE}} = \omega_{\mathrm{KE}}$. □

As usual, we reduce 3.84 to a parabolic complex Monge-Ampère equation. Since $c_1(M) = 0$ there exists a unique volume form $\Omega$ satisfying

$$\sqrt{-1}\partial\bar{\partial}\log\Omega = 0, \quad \int_M \Omega = \int_M \omega_0^n. \tag{3.131}$$

Then solving 3.84 is equivalent to solving complex Monge-Ampère equation

$$\frac{\partial}{\partial t}\phi = \log\frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\Omega}, \quad \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0, \quad \phi|_{t=0} = 0. \tag{3.132}$$

Firstly we observe

**Lemma 3.4.3.** *We have*

*(i) There exists a uniform constant $C$ such that for $t \in [0, \infty)$*

$$\|\dot{\phi}(t)\|_{C^0(M)} \leq C. \tag{3.133}$$

*(ii) There exists a uniform constat $C'$ such that on $M \times [0, \infty)$ the volume form of $\omega = \omega(t)$ satisfies*

$$\frac{1}{C'}\omega_0^n \leq \omega^n \leq C'\omega_0^n. \tag{3.134}$$

*Proof.* Differentiating 3.132 with respect to $t$ we otain

$$\frac{\partial}{\partial t}\dot{\phi} = \Delta\dot{\phi}. \tag{3.135}$$

and (i) follows immediately from the maximum principle. Part (ii) follows from (i). □

Then we obtain a bound on the oscillation of $\phi(t)$ using Yau's zeroth order estimate for the elliptic complex Monge-Ampère equation:

**Theorem 3.4.3.** *Let $(M, \omega_0)$ be a compact Kähler manifold and let $\phi$ be a smooth function on $M$ satisfying*

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^F\omega_0^n, \quad \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0. \tag{3.136}$$

*for some smooth function $F$. Then there exists a uniform $C$ depending only on $\sup\limits_M F$ and $\omega_0$ such that*

$$\mathrm{osc}_M\, \phi = \sup_M \phi - \inf_M \phi \leq C. \tag{3.137}$$

*Proof.* WLOG assume that $\int_M \phi\omega_0^n = 0$.

Write $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$. Then

$$\begin{aligned} C\int_M |\phi|\omega_0^n &\geq \int_M \phi(\omega_0^n - \omega^n) \\ &= \int_M \sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \wedge (\sum_{i=0}^{n-1}\omega_0^n \wedge \omega^{n-1-i}) \\ &\geq \frac{1}{n}\int_M |\partial\phi|^2_{\omega_0}\omega_0^n. \end{aligned} \tag{3.138}$$

By Poincaré Inequality (Theorem 2.4.1) and Cauchy Schwarz inequality we have

$$\int_M |\phi|^2\omega_0^n \leq C\int_M |\partial\phi|^2_{\omega_0}\omega_0^n \leq C'\int_M |\phi|\omega_0^n \leq C''\left(\int_M |\phi|^2\omega_0^n\right)^{\frac{1}{2}}. \tag{3.139}$$

Therefore $\|\phi\|_{L^2(\omega_0)} \leq C$. Now repeat this argument with $\phi$ replaced by $\phi|\phi|^\alpha$ for $\alpha \geq 0$:

$$\begin{aligned} C\int_M |\varphi|^{\alpha+1}\omega_0^n &\geq \int_M \varphi|\varphi|^\alpha\,(\omega_0^n - \omega^n) \\ &= -\int_M \varphi|\varphi|^\alpha\sqrt{-1}\partial\bar{\partial}\varphi \wedge \sum_{i=0}^{n-1}\omega_0^i \wedge \omega^{n-1-i} \\ &= (\alpha+1)\int_M |\varphi|^\alpha\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \sum_{i=0}^{n-1}\omega_0^i \wedge \omega^{n-1-i} \\ &= \frac{(\alpha+1)}{\left(\frac{\alpha}{2}+1\right)^2}\int_M \sqrt{-1}\partial\left(\varphi|\varphi|^{\alpha/2}\right) \wedge \bar{\partial}\left(\varphi|\varphi|^{\alpha/2}\right) \wedge \sum_{i=0}^{n-1}\omega_0^i \wedge \omega^{n-1-i}. \end{aligned} \tag{3.140}$$

Therefore for some uniform $C > 0$,

$$\int_M \left|\partial\left(\varphi|\varphi|^{\alpha/2}\right)\right|^2_{\omega_0} \omega_0^n \le C(\alpha+1)\int_M |\varphi|^{\alpha+1}\omega_0^n. \tag{3.141}$$

Now apply Sobolev inequality (Theorem 2.4.2) to $f = \phi|\phi|^{\alpha/2}$. Then for $\beta = \frac{n}{n-1}$ we have

$$\left(\int_M |\varphi|^{(\alpha+2)\beta}\omega_0^n\right)^{1/\beta} \le C\left((\alpha+1)\int_M |\varphi|^{\alpha+1}\omega_0^n + \int_M |\varphi|^{\alpha+2}\omega_0^n\right). \tag{3.142}$$

By Hölder's inequality and Yang's inequality we have

$$\int_M |\varphi|^{\alpha+1}\omega_0^n \le 1 + C\int_M |\varphi|^{\alpha+2}\omega_0^n. \tag{3.143}$$

Now substituting $p = \alpha+2$ we have from 3.143,

$$\|\varphi\|^p_{L^{p\beta}(\omega_0)} \le Cp\max\left(1, \|\varphi\|^p_{L^p(\omega_0)}\right). \tag{3.144}$$

Raising to the power $\frac{1}{p}$ we have for all $p \ge 2$,

$$\max\left(1, \|\varphi\|_{L^{p\beta}(\omega_0)}\right) \le C^{1/p}p^{1/p}\max\left(1, \|\varphi\|_{L^p(\omega_0)}\right). \tag{3.145}$$

Fix an integer $k > 0$. Replace $p$ in by $p\beta^k$ and then $p\beta^{k-1}$ and so on, to obtain

$$\begin{aligned}\max\left(1, \|\varphi\|_{L^{p\beta^{k+1}}(\omega_0)}\right) &\le C^{\frac{1}{p\beta^k}}\left(p\beta^k\right)^{\frac{1}{p\beta^k}}\max\left(1, \|\varphi\|_{L^{p\beta^k}(\omega_0)}\right) \le \cdots \\ &\le C^{\frac{1}{p\beta^k}+\frac{1}{p\beta^{k-1}}+\cdots+\frac{1}{p}}\left(p\beta^k\right)^{\frac{1}{p\beta^k}}\left(p\beta^{k-1}\right)^{\frac{1}{p\beta^{k-1}}}\cdots p^{\frac{1}{p}}\max\left(1, \|\varphi\|_{L^p(\omega_0)}\right) \\ &= C_k\max\left(1, \|\varphi\|_{L^p(\omega_0)}\right)\end{aligned} \tag{3.146}$$

for

$$C_k = C^{\frac{1}{p}\left(\frac{1}{\beta^k}+\frac{1}{\beta^{k-1}}+\cdots+1\right)}p^{\frac{1}{p}\left(\frac{1}{\beta^k}+\frac{1}{\beta^{k-1}}+\cdots+1\right)}\beta^{\frac{1}{p}\left(\frac{k}{\beta^k}+\frac{k-1}{\beta^{k-1}+\cdots+\frac{1}{\beta}}\right)}. \tag{3.147}$$

Since the infinite sums $\sum\frac{1}{\beta^i}$ and $\sum\frac{i}{\beta^i}$ converge for $\beta = n/(n-1) > 1$ we see that for any fixed $p$, the constants $C_k$ are uniformly bounded from above, independent of $k$. Setting $p = 2$ and letting $k \to \infty$ in 3.147 we finally obtain

$$\max\left(1, \|\varphi\|_{C^0}\right) \le C\max\left(1, \|\varphi\|_{L^2(\omega_0)}\right) \le C'. \tag{3.148}$$

and hence 3.138. □

Combining lemma 3.4.3 and theorem 3.4.3 we have

**Lemma 3.4.4.** *There exists a uniform constant $C$ such that for $t \in [0,\infty)$,*

$$\operatorname{osc}_M \phi \le C. \tag{3.149}$$

*Proof.* From lemma 3.4.3 (i) we have uniform bound for $\dot{\phi}$. Rewrite the parabolic Monge-Ampère equation 3.132 as

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi(t))^n = e^{F(t)}\omega_0^n \quad \text{with} \quad F(t) = \log\frac{\Omega}{\omega_0^n} + \dot{\phi}(t). \tag{3.150}$$

and apply theorem 3.4.3. The proof is now complete. □

### 3.4.3 Higher Order Estimates and Convergence When $\mathbf{c_1(M) = 0}$

Now we complete the proof of theorem 3.4,2. As above, let $\phi(t)$ solve the parabolic complex Monge-Ampère equation 3.132 on $M$ with $c_1(M) = 0$ and write $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$.

**Lemma 3.4.5.** *There exists a uniform constant $C$ such that on $M \times [0, \infty)$, $\omega = \omega(t)$ satisfies*

$$\frac{1}{C}\omega_0 \le \omega \le C\omega_0. \tag{3.151}$$

*Proof.* Similar as before, we just need to show $\operatorname{tr}_{\omega_0}\omega$ has a uniform upper bound. Define $Q = \log\operatorname{tr}_{\omega_0}\omega - A\phi$, by Proposition 3.2.2 we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)Q \le C_0\operatorname{tr}_\omega\omega_0 - A\dot{\varphi} + An - A\operatorname{tr}_\omega\omega_0. \tag{3.152}$$

for some $C_0$ depending only on $\omega_0$. Choosing $A = C_0 + 1$, since $\dot{\phi}$ is uniformly bounded

$$\left(\frac{\partial}{\partial t} - \Delta\right)Q \le -\operatorname{tr}_\omega\omega_0 + C. \tag{3.153}$$

We claim that for any $(x, t) \in M \times [0, \infty)$,

$$\left(\operatorname{tr}_{\omega_0}\omega\right)(x, t) \le Ce^{A\left(\varphi(x,t) - \inf_{M\times[0,t]}\varphi\right)}. \tag{3.154}$$

this comes from maximum principle.

Define

$$\tilde{\phi} := \phi - \frac{1}{V}\int_M \phi\Omega, \quad \text{where} \quad V := \int_M \Omega = \int_M \omega^n. \tag{3.155}$$

From Lemma 3.4.4, $\|\tilde{\phi}\|_{C^0(M)} \le C$. The estimate 3.154 cam be rewritten as

$$\begin{aligned}\left(\operatorname{tr}_{\omega_0}\omega\right)(x, t) &\le Ce^{A\left(\tilde{\varphi}(x,t) + \frac{1}{V}\int_M\varphi(t)\Omega - \inf_{M\times[0,t]}\tilde{\varphi} - \inf_{[0,t]}\frac{1}{V}\int_M\varphi\Omega\right)} \\ &\le Ce^{C' + \frac{A}{V}\left(\int_M\varphi(t)\Omega - \inf_{[0,t]}\int_M\varphi\Omega\right)}.\end{aligned} \tag{3.156}$$

Using Jensen's inequality,

$$\frac{d}{dt}\left(\frac{1}{V}\int_M\varphi\Omega\right) = \frac{1}{V}\int_M\dot{\varphi}\Omega = \frac{1}{V}\int_M\log\left(\frac{\omega^n}{\Omega}\right)\Omega \le \log\left(\frac{1}{V}\int_M\omega^n\right) = 0, \tag{3.157}$$

and hence $\inf_{[0,t]}\int_M\varphi\Omega = \int_M\varphi(t)\Omega$. The required upper bound of $\operatorname{tr}_{\omega_0}\omega$ follows then from 3.156. □

Now from Corollary 3.2.3 we have uniform $C^\infty$ estimates on $g(t)$ and the normalized potential function $\tilde{\phi}(t) = \phi(t) - V^{-1}\int_M \phi(t)\Omega$. To show the $C^\infty$ convergence, we follow the method of Phong-Strum [17] and use a functional known as the Mabuchi energy. It's noted that in [6],[8] that the monotonicity of the Mabuchi functional along the Kähler-Ricci flow was established in unpublished work of H.-D. Cao in 1991.

**Definition 3.4.1.** *Fix a metric $\omega_0$. The Mabuchi functional $\mathrm{Mab}_{\omega_0}$ on the space*

$$\mathrm{PSH}(M,\omega_0) = \left\{\phi \in C^\infty(M) \mid \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\right\}. \tag{3.158}$$

*with the property that if $\phi_t$ is any smooth path in $\mathrm{PSH}(M,\omega_0)$ then*

$$\frac{d}{dt}\mathrm{Mab}_{\omega_0}(\phi_t) = -\int_M \dot{\phi}_t R_{\phi_t}\omega_{\phi_t}^n. \tag{3.159}$$

*where $\omega_{\phi_t} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_t$, and $R_{\phi_t}$ is the scalar curvature of $\omega_{\phi_t}$.*

If $\phi_\infty$ is a critical point of $\mathrm{Mab}_{\omega_0}$ then $\omega_\infty$ must have zero scalar curvature and hence Ricci flat. (Since $c_1(M) = 0$, then $\mathrm{Ric}(\omega_\infty) = \sqrt{-1}\partial\bar{\partial}h_\infty$ for some smooth function $h_\infty$. Taking the trace we have $\Delta h_\infty = 0$ which implies $h$ is constant, hence $\mathrm{Ric}(\omega_\infty) = 0$.)

There is an explicit formula as derived in [23]:

$$\mathrm{Mab}_{\omega_0}(\phi) = \int_M \log(\frac{\omega_\phi^n}{\omega_0^n})\omega_\phi^n - \int_M h_0(\omega_\phi^n - \omega_0^n). \tag{3.160}$$

where $\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$ and $h_0$ is the Ricci potential for $\omega_0$ given by

$$\mathrm{Ric}(\omega_0) = \sqrt{-1}\partial\bar{\partial}h_0, \quad \int_M e^{h_0}\omega_0 = \int_M \omega_0^n. \tag{3.161}$$

Note that $\mathrm{Mab}_{\omega_0}$ can be regarded as a functional on the Kähler class of $\omega_0$. Let $\phi_t$ be a smooth path in $\mathrm{PSH}(M,\omega_0)$. Using integrating by parts, we compute

$$\begin{aligned}
\frac{d}{dt}\mathrm{Mab}_{\omega_0}(\phi_t) &= \int_M \Delta\dot{\phi}_t\omega_{\phi_t}^n + \int_M \log\frac{\omega_{\phi_t}^n}{\omega_0^n}\Delta\dot{\phi}_t\omega_{\phi_t}^n - \int_M h_0\Delta\dot{\phi}_t\omega_{\phi_t}^n \\
&= \int_M \dot{\phi}_t\left(-R_{\phi_t} + \mathrm{tr}_\omega \mathrm{Ric}(\omega_0)\right)\omega_{\phi_t}^n - \int_M \dot{\phi}_t\Delta h_0\omega_{\phi_t}^n \\
&= -\int_M \dot{\phi}_t R_{\varphi_t}\omega_{\phi_t}^n.
\end{aligned} \tag{3.162}$$

The key fact is the following lemma:

**Lemma 3.4.6.** *Let $\phi = \phi(t)$ solve the Kähler-Ricci flow 3.131 3.132. Then*

$$\frac{d}{dt}\mathrm{Mab}_{\omega_0}(\phi) = -\int_M |\partial\dot{\phi}|_\omega^2\omega^n. \tag{3.163}$$

*In particular, the Mabuchi energy is decreasing along the Kähler-Ricci flow. Moreover, there exists a uniform constant $C$ such that*

$$\frac{d}{dt}\int_M |\partial\dot{\phi}|_\omega^2\omega^n \leq C\int_M |\partial\dot{\phi}|_\omega^2\omega^n. \tag{3.164}$$

*Proof.* We have $\sqrt{-1}\partial\bar{\partial}\dot{\phi} = -\operatorname{Ric}(\omega)$ and taking the trace of this gives $\Delta\dot{\phi} = -R$. Then

$$\frac{d}{dt}\operatorname{Mab}_{\omega_0}(\phi) = -\int_M \dot{\phi} R\omega^n = \int_M \dot{\phi}\Delta\dot{\phi}\omega^n = -\int_M |\partial\dot{\phi}|^2_\omega \omega^n, \tag{3.165}$$

giving 3.163. For 3.164, compute

$$\begin{aligned}\frac{d}{dt}\int_M |\partial\dot{\phi}|^2_\omega\omega^n &= \int_M \left(\frac{\partial}{\partial t} g^{\bar{j}i}\right)\partial_i\dot{\phi}\partial_{\bar{j}}\dot{\phi}\omega^n + 2\operatorname{Re}\left(\int_M g^{\bar{j}i}\partial_i(\Delta\dot{\phi})\partial_{\bar{j}}\dot{\phi}\omega^n\right) + \int_M |\partial\dot{\phi}|^2\Delta\dot{\phi}\omega^n \\ &= \int_M R^{\bar{j}i}\partial_i\dot{\phi}\partial_{\bar{j}}\dot{\phi}\omega^n - 2\int_M (\Delta\dot{\phi})^2\omega^n - \int_M |\partial\dot{\phi}|^2 R\omega^n \\ &\le C\int_M |\partial\dot{\phi}|^2_\omega\omega^n,\end{aligned} \tag{3.166}$$

using 3.135 and the fact that we have uniform bounds of the Ricci curvature and scalar curvature of $\omega$. $\square$

Now we complete the proof of Theorem 3.4.1:

**Proof of Theorem 3.4.1.** Since we have uniform estimates for $\omega(t)$ along the flow, from 3.160 the Mabuchi energy is uniformly bounded. From 3.163 there exists a sequence of times $t_i \in [i, i+1]$ for which

$$\left(\int_M \left|\partial\log\frac{\omega^n}{\Omega}\right|^2_\omega \omega^n\right)(t_i) = \left(\int_M |\partial\dot{\phi}|^2_\omega\omega^n\right)(t_i) \to 0, \quad \text{as } i\to\infty. \tag{3.167}$$

By 3.164

$$\left(\int_M \left|\partial\log\frac{\omega^n}{\Omega}\right|^2_\omega \omega^n\right)(t) \to 0, \quad \text{as } t\to\infty. \tag{3.168}$$

But since we have $C^\infty$ estimates for $\phi(t)$ we can apply the Arzela-Ascoli Theorem to obtain a sequence of times $t_j$ such that $\phi\left(t_j\right)$ converges in $C^\infty$ to $\phi_\infty$, say. Writing $\omega_\infty = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_\infty > 0$, we have from 3.168,

$$\left(\int_M \left|\partial\log\frac{\omega^n_\infty}{\Omega}\right|^2_{\omega_\infty} \omega^n_\infty\right) = 0, \tag{3.169}$$

and hence

$$\log\frac{\omega^n_\infty}{\Omega} = C, \tag{3.170}$$

for some constant $C$. Taking $\sqrt{-1}\partial\bar{\partial}$ of 3.170 gives $\operatorname{Ric}(\omega_\infty) = 0$. Hence for a sequence of times $t_j \to \infty$ the Kähler-Ricci flow converges to $\omega_\infty$, the unique Kähler-Einstein metric in the cohomology class $[\omega_0]$.

The argument to show the convergence of the metrics $\omega(t)$ is in $C^\infty$ without passing to a subsequence is the same as Theorem 3.3.1.

**Remark.** One can also use the functional $\int_M h\omega^n$ where $h$ is the Ricci potential of the evolving metric.

# 4 Kähler-Ricci Flow on Noncompact Manifolds and Yau's Uniformization Conjecture

In this chapter, we consider the famous Yau's uniformization conjecture, which was proposed by Yau in [27] about 1970s.

**Conjecture 4.0.1.** *Suppose $M^n$ is a complete noncompact Kähler manifold with positive bisectional curvature, then $M$ is biholomorphic to $\mathbb{C}^n$.*

Over 50 years, several results concerning this conjecture were obtained, see [20, 16, 7, 14, 15]. Many different methods have been used, such as Kähler(Chern)-Ricci flow, Cheeger-Colding theory on Ricci limit spaces, Hörmander's $L^2$ estimate for $\bar{\partial}$-operator and Ni-Tam's heat flow method and so on.

We will give a comprehensive introduction to the applications of Kähler-Ricci flow in Yau's uniformization conjecture.

## 4.1 Shi's Uniformization Theorem

In [21], Shi Wan-Xiong firstly considered the conjecture through Ricci flow arguments. In fact, the main theorem in [21] was given as follows:

**Theorem 4.1.1.** *Suppose $(M^n, \omega)$ is a complete noncompact Kähler manifold with positive bounded bisectional curvature, $p \in M$ is a fixed point. If $M$ satisfies:*

*(a) $M$ has maximal volume growth, i.e $\lim\limits_{r\to\infty} \frac{\mathrm{vol}\left(B_p(r)\right)}{r^{2n}} > 0$;*

*(b) $\int_{B_p(r)} R\omega^n \le Cr^{2n-2}$ holds for any $r > 0$, where $R$ is the scalar curvature of $M$, $C$ is a positive constant;*

*Then $M$ is biholomorphic to $\mathbb{C}^n$.*

### 4.1.1 Short Time Existence

The main idea is to run the evolution equation (3.1) and show that under some modification, the flow converges to the flat metric and preserves the Euclidean volume growth. However, since $M$ is noncompact, the short time existence for the solution of (3.1) is not true in general. If we assume that the curvature tensor is bounded by some constant, then the short time existence theorem of Ricci flow was proved by Shi in [19].

**Theorem 4.1.2.** *Suppose $(M, g)$ is an n-dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor* $\mathrm{Rm}$ *satisfying*

$$\|\mathrm{Rm}\| \le K_0 \quad \text{on } M \tag{4.1}$$

*where $K_0 > 0$ is a constant, then there exists a constant $T = T(n, K_0) > 0$ such that the evolution equation*

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2\,\mathrm{Ric}_{ij}(x,t) \\ g_{ij}(x,0) = g_{ij}(x) \end{cases} \tag{4.2}$$

*has a smooth solution $g_{ij}(x,t) > 0$ for $t \in [0,T]$. Moreover, for any $m \geq 0$, there exists constant $C(n,m,K_0) > 0$ depending only on $n, m$ and $K_0$ such that*

$$\sup_{x \in M} \|\nabla^m R_{ijkl}(x,t)\|^2 \leq \frac{C(n,m,K_0)}{t^m}, \quad 0 \leq t \leq T. \tag{4.3}$$

*Proof.* This is Theorem 1.1 in [20]. □

Using a rescaling argument, it's easy to see that:

**Corollary 4.1.1.** *Suppose $(M,g)$ is an $n$-dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor $\mathrm{Rm}$ satisfying*

$$\|\mathrm{Rm}\| \leq K_0 \quad \textit{on } M \tag{4.4}$$

*where $K_0 > 0$ is a constant, then there exists a constant $\theta = \theta(n) > 0$ such that the evolution equation*

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2\,\mathrm{Ric}_{ij}(x,t) \\ g_{ij}(x,0) = g_{ij}(x) \end{cases} \tag{4.5}$$

*has a smooth solution $g_{ij}(x,t) > 0$ for $t \in [0, \frac{\theta}{\sqrt{K_0}}]$. Moreover, for any $m \geq 0$, there exists constant $C(n,m) > 0$ depending only on $n$ and $m$ such that*

$$\sup_{x \in M} \|\nabla^m R_{ijkl}(x,t)\|^2 \leq \frac{C(n,m)K_0}{t^m}, \quad 0 \leq t \leq \frac{\theta}{\sqrt{K_0}}. \tag{4.6}$$

**Lemma 4.1.1.** *Suppose $M$ is an $n$-dimensional complete noncompact Riemannian manifold and $g_{ij}(x,t)$ are smooth Riemannian metric defined on $M \times [0,T]$, where $0 < T < +\infty$ is an arbitrary constant. Suppose the following assumptions hold:*

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2\,\mathrm{Ric}_{ij}(x,t) \quad \textit{on } M \times [0,T]; \tag{4.7a}$$

$$\sup_{M \times [0,T]} \|\mathrm{Rm}\|^2 \leq K_0; \tag{4.7b}$$

*where $0 < K_0 < +\infty$. Then for any integer $m \geq 0$ there exists constant $0 < C(n,m) < +\infty$ depending only on $n$ and $m$ such that*

$$e^{-2n\sqrt{K_0}T} g_{ij}(x,0) \leq g_{ij}(x,t) \leq e^{2n\sqrt{K_0}T} g_{ij}(x,0) \quad 0 \leq t \leq T; \tag{4.8a}$$

$$\sup_{x \in M} \|\nabla^m R_{ijkl}(x,t)\|^2 \leq C(n,m) \left[ K_0 \cdot \left(\frac{1}{t}\right)^m + K_0^{\frac{m}{2}+1} \right] \quad 0 \leq t \leq T. \tag{4.8b}$$

*Proof.* WLOG, suppose $K_0 = 1$. Since $\sup_{M \times [0,T]} \|\mathrm{Rm}\|^2 \leq 1$, we get

$$\|\mathrm{Ric}\|^2 \leq n^2 \quad \text{on } M \times [0,T]. \tag{4.9}$$

Therefore

$$\|\frac{\partial}{\partial t}g_{ij}\| \leq 2n; \tag{4.10a}$$

$$-2ng_{ij}(x,t) \leq \frac{\partial}{\partial t}g_{ij}(x,t) \leq 2ng_{ij}(x,t); \tag{4.10b}$$

$$e^{-2nt}g_{ij}(x,0) \leq g_{ij}(x,t) \leq e^{2nt}g_{ij}(x,0); \tag{4.10c}$$

Since $0 \leq t \leq T$, from (4.10) we get

$$e^{-2nT}g_{ij}(x,0) \leq g_{ij}(x,t) \leq e^{2nT}g_{ij}(x,0) \quad \text{on } M \times [0,T]. \tag{4.11}$$

By Corollary 4.1.1 we have there exists $\theta = \theta(n)$ such that

$$\sup_{x\in M} \|\nabla^m R_{ijkl}(x,t)\|^2 \leq \frac{C(n,m)}{t^m}, \quad 0 \leq t \leq \theta. \tag{4.12}$$

If $T \leq \theta$, then we are done. For $T \geq \theta$, for any $t_0 \in [\theta, T]$, define

$$\tilde{g}_{ij}(x,t) = g_{ij}(x, t + t_0 - \theta) \quad \text{for } x \in M, \theta - t_0 \leq t \leq T + \theta - t_0; \tag{4.13}$$

then

$$\frac{\partial}{\partial t}\tilde{g}_{ij}(x,t) = -2\widetilde{\mathrm{Ric}}_{ij}(x,t) \quad \text{on } M \times [0, T + \theta - t_0]; \tag{4.14a}$$

$$\|\widetilde{\mathrm{Rm}}\|^2 \leq 1 \quad \text{on } M \times [0, T + \theta - t_0]; \tag{4.14b}$$

where $\widetilde{\mathrm{Rm}}$ is the Riemannian curvature tensor of $\tilde{g}$ on $M$. By Corollary 4.1.1 we have

$$\sup_{x\in M} \|\widetilde{\nabla}^m \tilde{R}_{ijkl}(x,t)\|^2 \leq \frac{C(n,m)}{t^m}, \quad 0 \leq t \leq \theta, m \geq 0. \tag{4.15}$$

thus

$$\sup_{x\in M} \|\nabla^m R_{ijkl}(x,t)\|^2 \leq \frac{C(n,m)}{(t - t_0 + \theta)^m}, \quad t_0 - \theta \leq t \leq t_0, m \geq 0. \tag{4.16}$$

Take $t = t_0$ we have

$$\sup_{x\in M} \|\nabla^m R_{ijkl}(x,t_0)\|^2 \leq \frac{C(n,m)}{\theta^m}. \tag{4.17}$$

Since $t_0 \in [\theta, T]$ is arbitrary, the proof is complete. □

Applying the results to Kähler manifolds, we have

**Theorem 4.1.3.** *Suppose $(M^n, \omega)$ is a complete noncompact Kähler manifold with bounded and nonnegative holomorphic bisectional curvature:*

$$0 \leq R_{i\bar{i}j\bar{j}}(x) \leq K_0, \quad \forall x \in M. \tag{4.18}$$

*Then there exists a constant $\theta = \theta(n) > 0$ such that the Kähler-Ricci flow equation has a smooth solution $g_{ij}(x,t) > 0$ for a short time*

$$0 \leq t \leq \frac{\theta}{K_0}. \tag{4.19}$$

*and satisfying that for any $m \geq 0$, there exists constants $C(n,m) > 0$ such that*

$$\sup_{x\in M} \|\nabla^m R_{ijkl}(x,t)\|^2 \leq \frac{C(n,m)K_0^2}{t^m} \quad 0 \leq t \leq \frac{\theta}{K_0}. \tag{4.20}$$

*Proof.* Since the boundness of bisectional curvature implies the boundness of Riemannian curvature tensor, then apply Corollary 4.1.1. □

### 4.1.2 The Consruction of the Exhaustion Function

To control the solution and prove the long time existence for the solution of Ricci flow, the first thing we have to do is to establish the maximum principle for the parabolic system on complete noncompact manifolds.

Suppose $(M,g)$ is an n-dimensional complete noncompact Riemannian manifold, to establish the maximal principle for the solution of Ricci flow on $M$, the key point is to construct a smooth function $\varphi(x) \in C^\infty(M)$ such that

$$\begin{cases} \frac{1}{C}(1+r(x,x_0)) \leq \varphi(x) \leq C(1+r(x,x_0)); \\ \|\nabla\varphi(x)\| \leq C \quad \forall x \in M; \\ \|\nabla_i\nabla_j\varphi(x)\| \leq Cg_{ij}(x) \quad \forall x \in M; \end{cases} \tag{4.21}$$

where $C$ is a positive constant.

Shi used the arguments derived by R.Shoen and S,T.Yau. In the case that $M$ is a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq -K_0$, they constructed a smooth function $\varphi \in C^\infty(M)$ such that

$$\begin{cases} \frac{1}{C}(1+r(x,x_0)) \leq \varphi(x) \leq C(1+r(x,x_0)); \\ \|\nabla\varphi(x)\| \leq C \quad \forall x \in M; \\ \|\Delta\varphi(x)\| \leq C \quad \forall x \in M; \end{cases} \tag{4.22}$$

where $C$ is a positive constant.

**Theorem 4.1.4.** *Suppose $(M,g)$ is an n-dimensional complete noncompact Riemannian manifold with bounded Riemannian tensor $\mathrm{Rm}$ satisfying*

$$\|\mathrm{Rm}\|^2 \leq K_0. \tag{4.23}$$

*Suppose $x_0 \in M$ is fixed, then there exists a smooth function $\varphi$ on $M$ such that:*

$$\begin{cases} \frac{1}{C}(1+r(x,x_0)) \leq \varphi(x) \leq C(1+r(x,x_0)); \\ \|\nabla\varphi(x)\| \leq C \quad \forall x \in M; \\ \|\nabla_i\nabla_j\varphi(x)\| \leq Cg_{ij}(x) \quad \forall x \in M; \end{cases} \tag{4.24}$$

*where $C$ is a positive constant.*

*Proof.* Suppose $\lambda > 0$ is a constant to be determined later and $\gamma > 1$, we solve the following Dirichlet problem:

$$\begin{cases} \Delta u_\gamma(x) = \lambda u_\gamma(x) & \forall x \in B(x_0,\gamma)\backslash B(x_0,1); \\ u_\gamma(x) \equiv 0 & \forall x \in \partial B(x_0,\gamma); \\ u_\gamma(x) \equiv 1 & \forall x \in \partial B(x_0,1); \end{cases} \tag{4.25}$$

By classical theory of the second order elliptic equations we know that (4.25) has a smooth solution $u_\gamma(x)$ on $B(x_0,\gamma)\backslash B(x_0,1)$. By the maximum principle we have

$$0 < u_\gamma(x) < 1 \quad \forall x \in B(x_0,\gamma)\backslash\overline{B(x_0,1)}; \tag{4.26}$$

For $\gamma_2 > \gamma_2$, by maximum principle it's easy to find that

$$u_{\gamma_2} > u_{\gamma_1} \quad \forall x \in B(x_0,\gamma_1)\backslash\overline{B(x_0,1)}; \tag{4.27}$$

By (4.26) and (4.27) we have as $\gamma \to +\infty$ the limit

$$u(x) = \lim_{\gamma\to+\infty} u_\gamma(x). \tag{4.28}$$

exists for all $x \in M\backslash B(x_0,1)$ and

$$0 < u(x) \le 1. \tag{4.29}$$

for any $x \in M\backslash B(x_0,1)$.

For any $x\_1inM$ and $\gamma > 1$ if

$$B(x_1,1) \subset B(x_0,\gamma)\backslash\overline{B(x_0,1)}. \tag{4.30}$$

then by (4.25) and (4.26) we have

$$\begin{cases} \Delta u_\gamma(x) = \lambda u_\gamma(x) & \forall x \in B(x_1,1); \\ 0 < u_\gamma(x) < 1 & \forall x \in B(x_1,1); \end{cases} \tag{4.31}$$

By Cheng-Yau's gradient estimate we have

$$\|\nabla u_\gamma(x)\| \le C(n,K_0,\lambda)u_\gamma(x) \quad \forall x \in B(x_1,\frac{3}{4}). \tag{4.32}$$

by (4.31) we have $0 < u_\gamma(x) < 1$, thus

$$\sup_{x\in B(x_1,\frac{3}{4})} \|\nabla u_\gamma(x)\| \le C(n,K_0,\lambda). \tag{4.33}$$

By (4.31) and (4.33) and the classical Schauder estimates we know that for any integer $m \ge 2$, there exists constants $0 < C(n,m,g_{ij}\mid_{B(x_1,1)}) < +\infty$ such that

$$\sup_{x\in B(x_1,\frac{1}{2}+(\frac{1}{2})^{m+1})} \|\nabla^m u_\gamma(x)\| \le C(n,m,g_{ij}\mid_{B(x_1,1)}). \tag{4.34}$$

Therefore by Arzela-Ascoli lemma we know that there exists a subsequence $\gamma_i \to +\infty$ as $i \to \infty$ such that

$$u_{\gamma_i}(x) \xrightarrow{C^\infty} u(x) \quad \text{on } M\backslash B(x_0,2), \text{ as } i\to\infty. \tag{4.35}$$

Thus

$$\begin{cases} u(x) \in C^\infty(M\backslash B(x_0,2)); \\ \Delta u(x) = \lambda u(x) \quad \forall x \in M\backslash B(x_0,2); \end{cases} \tag{4.36}$$

Also we still have

$$\sup_{x\in M\backslash B(x_0,2)} \|\nabla \log u(x)\| \le C(n,K_0,\lambda). \tag{4.37}$$

Now we are going to show that $u(x)$ actually tends to zero exponentially as $x \in \infty$ if $\lambda$ is big enough.

**Lemma 4.1.2.** *Suppose $M$ is an $n$-dimensional complete noncompact Riemannian manifold with*

$$\mathrm{Ric} \geq -K \quad \forall x \in M. \tag{4.38}$$

*for some $K > 0$. Then there exists a constant $C = C(n, K) > 0$ such that*

$$\mathrm{Vol}(B(x,1)) \geq e^{-Cr(x,x_0)} \mathrm{Vol}(B(x_0,1)). \tag{4.39}$$

*for all $x, x_0 \in M$.*

*Proof.* Fix $x \in M$, for any $y \in M$ define

$$\rho(y) = r(x,y). \tag{4.40}$$

By laplacian comparison we know that

$$\begin{cases} \Delta\rho \leq \frac{n-1}{\rho} + \sqrt{(n-1)K}; \\ \|\nabla\rho\| \leq 1; \end{cases} \tag{4.41}$$

Thus

$$\Delta\rho^2 \leq 2n + 2\sqrt{(n-1)K}\rho. \tag{4.42}$$

For any $t > 0$ we have

$$\begin{aligned} \int_{B(x,t)} \Delta\rho^2 &\leq \int_{B(x,t)} 2n + 2\sqrt{(n-1)K} \int_{B(x,t)} \rho \\ &\leq 2n \,\mathrm{Vol}(B(x,t)) + 2t\sqrt{(n-1)K}\,\mathrm{Vol}(B(x,t)). \end{aligned} \tag{4.43}$$

On the other hand, by Stokes theorem we have

$$\begin{aligned} \int_{B(x,t)} \Delta\rho^2 &= \int_{\partial B(x,t)} \frac{\partial\rho^2}{\partial t} \\ &= 2t\,\mathrm{Vol}(\partial B(x,t)) \\ &= 2t\frac{\partial}{\partial t}\mathrm{Vol}(B(x,t)). \end{aligned} \tag{4.44}$$

Combining (4.43) and (4.44) we get

$$\frac{\partial}{\partial t}\left[t^{-n}e^{-\sqrt{(n-1)K}t}\,\mathrm{Vol}(B(x,t))\right] \leq 0, \quad 0 \leq t < \infty. \tag{4.45}$$

Thus if $t \geq 1$ we get

$$t^{-n}e^{-\sqrt{(n-1)K}t}\,\mathrm{Vol}(B(x,t)) \leq e^{-\sqrt{(n-1)K}}\,\mathrm{Vol}(B(x,1)). \tag{4.46}$$

Now choose $t = r(x,x_0) + 1$ we have

$$(r(x,x_0)+1)^{-n}e^{-\sqrt{(n-1)K}(r(x,x_0)+1)}\,\mathrm{Vol}(B(x,r(x,x_0)+1)) \leq e^{-\sqrt{(n-1)K}}\,\mathrm{Vol}(B(x,1)). \tag{4.47}$$

Since $B(x_0,1) \subset B(x, 1 + r(x,x_0))$, from (4.47) we have

$$\mathrm{Vol}(B(x,1)) \geq (r(x,x_0)+1)^{-n}e^{-\sqrt{(n-1)K}r(x,x_0)}\,\mathrm{Vol}(B(x_0,1)). \tag{4.48}$$

Thus the proof is complete. □

Coming back to the proof of theorem 4.1.4, from (4.39) we have

$$\operatorname{Vol}(B(x,1)) \geq e^{-Cr(x,x_0)} \operatorname{Vol}(B(x_0,1)) \quad \forall x, x_0 \in M. \tag{4.49}$$

Suppose $a > 0$ is a constant to be determined later and $\gamma > 3$, by Stokes theorem and (4.25) we have

$$\begin{aligned}
\int_{B(x_0,\gamma)\setminus B(x_0,2)} e^{ar(x,x_0)} u_\gamma(x) \Delta u_\gamma(x) dx &= \int_{\partial B(x_0,2)} e^{ar(x,x_0)} u_\gamma(x) \frac{\partial u_\gamma}{\partial \nu}(x) \\
&- \int_{B(x_0,\gamma)\setminus B(x_0,2)} \langle \nabla (e^{ar(x,x_0)} u_\gamma(x)), \nabla u_\gamma(x) \rangle dx \\
&= e^{2a} \int_{\partial B(x_0,2)} u_\gamma(x) \frac{\partial u_\gamma}{\partial \nu}(x) - \int_{B(x_0,\gamma)\setminus B(x_0,2)} e^{ar(x,x_0)} \|\nabla u_\gamma(x)\|^2 dx \\
&- a \int_{B(x_0,\gamma)\setminus B(x_0,2)} e^{ar(x,x_0)} u_\gamma(x) \langle \nabla r(x,x_0), \nabla u_\gamma(x) \rangle dx.
\end{aligned} \tag{4.50}$$

Since we have

$$\left| \frac{\partial u_\gamma(x)}{\partial \nu} \right| \leq \|\nabla u_\gamma(x)\| \leq C(n, K_0, \lambda), \quad \forall x \in \partial B(x_0, 2). \tag{4.51}$$

Since $0 \leq u_\gamma \leq 1$, we have

$$e^{2a} \int_{\partial B(x_0,2)} u_\gamma(x) \frac{\partial u_\gamma}{\partial \nu}(x) \leq C(n, K_0, \lambda) e^{2a} \operatorname{Vol}(\partial B(x_0, 2)). \tag{4.52}$$

Combining (4.50) and (4.52) we have

$$\begin{aligned}
\lambda \int_{B(x_0,\gamma)\setminus B(x_0,2)} e^{ar(x,x_0)} u_\gamma(x)^2 dx &= \int_{B(x_0,\gamma)\setminus B(x_0,2)} e^{ar(x,x_0)} u_\gamma(x) \Delta u_\gamma(x) dx \\
&\leq C(n, K_0, \lambda) e^{2a} \operatorname{Vol}(\partial B(x_0, 2)) - \int_{B(x_0,\gamma)\setminus B(x_0,2)} e^{ar(x,x_0)} \|\nabla u_\gamma(x)\|^2 dx \\
&- a \int_{B(x_0,\gamma)\setminus B(x_0,2)} e^{ar(x,x_0)} u_\gamma(x) \langle \nabla r(x,x_0), \nabla u_\gamma(x) \rangle dx \\
&\leq C(n, K_0, \lambda) e^{2a} \operatorname{Vol}(\partial B(x_0, 2)) - \int_{B(x_0,\gamma)\setminus B(x_0,2)} e^{ar(x,x_0)} \|\nabla u_\gamma(x)\|^2 dx \\
&+ a \int_{B(x_0,\gamma)\setminus B(x_0,2)} e^{ar(x,x_0)} u_\gamma(x) \|\nabla u_\gamma(x)\| dx \\
&\leq C(n, K_0, \lambda) e^{2a} \operatorname{Vol}(\partial B(x_0, 2)) + \frac{a^2}{4} \int_{B(x_0,\gamma)\setminus B(x_0,2)} e^{ar(x,x_0)} u_\gamma(x)^2 dx.
\end{aligned} \tag{4.53}$$

Thus

$$(\lambda - \frac{a^2}{4}) \int_{B(x_0,\gamma)\setminus B(x_0,2)} e^{ar(x,x_0)} u_\gamma(x)^2 dx \leq C(n, K_0, \lambda) e^{2a} \operatorname{Vol}(\partial B(x_0, 2)). \tag{4.54}$$

Set $\lambda = 1 + \frac{a^2}{4}$ we get

$$\int_{B(x_0,\gamma)\setminus B(x_0,2)} e^{ar(x,x_0)} u_\gamma(x)^2 dx \leq C(n, K_0, \lambda) e^{2a} \operatorname{Vol}(\partial B(x_0, 2)). \tag{4.55}$$

Take $\gamma \to \infty$ we have

$$\int_{M\backslash B(x_0,2)} e^{ar(x,x_0)}u(x)^2 dx \leq C(n,K_0,\lambda)e^{2a}\operatorname{Vol}(\partial B(x_0,2)). \tag{4.56}$$

For any $y \in M\backslash B(x_0,3)$, we have

$$\begin{aligned} &B(y,1) \subset M\backslash B(x_0,2);\\ &r(x,x_0) \geq r(y,x_0) - 1; \end{aligned} \tag{4.57}$$

Thus

$$\int_{B(y,1)} e^{ar(x,x_0)}u(x)^2 dx \leq C(n,K_0,\lambda)e^{2a}\operatorname{Vol}(\partial B(x_0,2)). \tag{4.58}$$

From (4.57) we know that

$$\int_{B(y,1)} u(x)^2 dx \leq C(n,K_0,\lambda)e^{3a-ar(y,x_0)}\operatorname{Vol}(\partial B(x_0,2)). \tag{4.59}$$

From (4.37) we have

$$u(x) \geq e^{-C(n,K_0,\lambda)}u(y), \quad \forall x \in B(y,1). \tag{4.60}$$

Combining (4.59) and (4.60) we get

$$e^{-2C(n,K_0,\lambda}u(y)^2\operatorname{Vol}(B(y,1)) \leq C(n,K_0,\lambda)e^{3a-ar(y,x_0)}\operatorname{Vol}(\partial B(x_0,2)). \tag{4.61}$$

Thus

$$u(y) \leq C(n,K_0,\lambda)e^{\frac{3a}{2}+C(n,K_0,\lambda)}e^{-\frac{a}{2}r(y,x_0)}\left(\frac{\operatorname{Vol}(\partial B(x_0,2))}{\operatorname{Vol}(B(y,1))}\right)^{\frac{1}{2}}, \quad \forall y \in M\backslash B(x_0,3). \tag{4.62}$$

From (4.39), 4.43) and (4.44) we get $\frac{\operatorname{Vol}(\partial B(x_0,2))}{\operatorname{Vol}(B(y,1))} \leq C(n,K_0)e^{-Cr(y,x_0)}$.

Thus

$$u(y) \leq C(n,K_0,\lambda,a)e^{(\frac{1}{2}C-\frac{a}{2})r(y,x_0)}, \quad \forall y \in M\backslash B(x_0,3). \tag{4.63}$$

Choose $a = C+2$ then $\lambda = 1 + \frac{1}{4}(2+C)^2$, and

$$u(y) \leq C(n,K_0)e^{-r(y,x_0)}, \quad \forall y \in M\backslash B(x_0,3). \tag{4.64}$$

In a word, we construct a smooth function $u \in C^\infty(M\backslash B(x_0,2))$ which satisfying

**Lemma 4.1.3.** *Under the curvature assumption of Theorem 4.1.4, for any point $x_0 \in M$, there exists a smooth function $u(x) \in C^\infty(M\backslash B(x_0,2))$ such that*

$$\begin{cases} 0 < u \leq 1;\\ \Delta u = \lambda u;\\ \|\nabla \log u(x)\| \leq C(n,K_0);\\ u(x) \leq C(n,K_0)e^{-r(x,x_0)}; \end{cases} \tag{4.65}$$

*holds for any $x \in M\backslash B(x_0,3)$.*

Now we are going to control the second order covariant derivatives $\|\nabla^2 u\|^2$ only in terms of $n$ and $K_0$.

**Lemma 4.1.4.** *Suppose $u$ is the smooth function we obtained in Lemma 4.1.3, then there exists a constant $C = C(n, K_0) > 0$ such that*

$$\frac{\|\nabla^2 u(x)\|}{u(x)} \leq C(n, K_0). \tag{4.66}$$

*for any $x \in M \backslash B(x_0, 3 + \pi \left(\frac{1}{K_0}\right)^{\frac{1}{4}})$.*

*Proof.* We use the iteration arguments of Moser for the solutions of elliptic equations.

For any $y \in M \backslash B(x_0, 3 + \pi \left(\frac{1}{K_0}\right)^{\frac{1}{4}})$, we have

$$B(y, \frac{1}{8}\left(\frac{1}{K_0}\right)^{\frac{1}{4}}) \subset M \backslash B(x_0, 3). \tag{4.67}$$

Thus from (4.65) we get

$$\|\nabla \log u(x)\| \leq C(n, K_0), \quad \forall x \in B(y, \frac{1}{8}\left(\frac{1}{K_0}\right)^{\frac{1}{4}}). \tag{4.68}$$

Thus

$$\frac{1}{C} \leq \frac{u(x)}{u(y)} \leq C, \quad \forall x \in B(y, \frac{1}{8}\left(\frac{1}{K_0}\right)^{\frac{1}{4}}). \tag{4.69}$$

Combining (**??**) and (4.69) we have

$$\frac{\|\nabla u(x)\|}{u(y)} \leq C, \quad \forall x \in B(y, \frac{1}{8}\left(\frac{1}{K_0}\right)^{\frac{1}{4}}). \tag{4.70}$$

Since

$$\Delta u(x) = \lambda u(x), \quad \forall x \in B(y, \frac{1}{8}\left(\frac{1}{K_0}\right)^{\frac{1}{4}}). \tag{4.71}$$

Differentiating both sides of (4.71) we get

$$\nabla_i(\Delta u) = \lambda \nabla_i u, \quad \text{on } B(y, \frac{1}{8}\left(\frac{1}{K_0}\right)^{\frac{1}{4}}). \tag{4.72}$$

By Ricci identity we get

$$\Delta(\nabla_i u) - \mathrm{Ric}_{ip} \nabla_p u - \lambda \nabla_i u = 0, \quad \text{on } B(y, \frac{1}{8}\left(\frac{1}{K_0}\right)^{\frac{1}{4}}). \tag{4.73}$$

Thus

$$\Delta(\|\nabla_i u\|^2) - 2\,\mathrm{Ric}_{ip} \nabla_p u \nabla_i u - 2\|\nabla_i \nabla_j u\|^2 - 2\lambda \|\nabla_i u\|^2 = 0 \quad \text{on } B(y, \frac{1}{8}\left(\frac{1}{K_0}\right)^{\frac{1}{4}}). \tag{4.74}$$

Suppose $\xi(x)$ is a cut-off function such that

$$\begin{cases} \xi \in C_0^{\infty}(M); \\ \xi(x) \equiv 1 \quad \forall x \in B(y, \frac{3}{4}r_0); \\ \xi(x) \equiv 0 \quad \forall x \in M \backslash B(y, \frac{7}{8}r_0); \\ \|\nabla \xi\| \leq \frac{16}{r_0}; \\ 0 \leq \xi \leq 1; \end{cases} \tag{4.75}$$

where $r_0 = \frac{1}{8}\left(\frac{1}{K_0}\right)^{\frac{1}{4}}$.

From (4.74) we have

$$\begin{aligned} \int_{B(y,r_0)} \|\nabla^2 u(x)\|^2 \xi(x)^2 dx &= \frac{1}{2}\int_{B(y,r_0)} \left[\Delta(\|\nabla_i u\|^2) - 2\operatorname{Ric}_{ip} \nabla_p u \nabla_i u - 2\lambda \|\nabla_i u\|^2\right] \xi(x)^2 dx \\ &\leq \frac{1}{2}\int_{B(y,r_0)} \Delta(\|\nabla_i u\|^2)\xi(x)^2 dx - \int_{B(y,r_0)} \operatorname{Ric}_{ip} \nabla_p u \nabla_i u \xi(x)^2 dx. \end{aligned} \tag{4.76}$$

Since

$$\begin{aligned} -\int_{B(y,r_0)} \operatorname{Ric}_{ip} \nabla_i u \nabla_p u \xi(x)^2 dx &\leq n\sqrt{K_0}\int_{B(y,r_0)} \|\nabla u\|^2 \xi(x)^2 dx \\ &\leq Cn\sqrt{K_0}\int_{B(y,r_0)} u(y)^2 \xi(x)^2 dx \\ &\leq Cn\sqrt{K_0}\int_{B(y,r_0)} u(y)^2 dx \\ &= Cn\sqrt{K_0} u(y)^2 \operatorname{Vol}(B(y,r_0)). \end{aligned} \tag{4.77}$$

Also, by integrating by part, we have

$$\begin{aligned} \frac{1}{2}\int_{B(y,r_0)} \Delta(\|\nabla_i u\|^2)\xi(x)^2 dx &= -\frac{1}{2}\int_{B(y,r_0)} \nabla_k(\|\nabla_i u\|^2)\nabla_k \xi(x)^2 dx \\ &= -2\int_{B(y,r_0)} \nabla_k \nabla_i u \nabla_i u \xi(x) \nabla_k \xi(x) dx \\ &\leq \frac{1}{2}\int_{B(y,r_0)} \|\nabla_k \nabla_i u\|^2 \xi(x)^2 dx + C\int_{B(y,r_0)} \|\nabla_i u\|^2 \|\nabla_k \xi(x)\|^2 dx. \end{aligned} \tag{4.78}$$

Together with (4.75) implies

$$\begin{aligned} \frac{1}{2}\int_{B(y,r_0)} \Delta(\|\nabla_i u\|^2)\xi(x)^2 dx &\leq \frac{1}{2}\int_{B(y,r_0)} \|\nabla_k \nabla_i u\|^2 \xi(x)^2 dx + \frac{C}{r_0^2}\int_{B(y,r_0)} \|\nabla u\|^2 dx \\ &\leq \frac{1}{2}\int_{B(y,r_0)} \|\nabla_k \nabla_i u\|^2 \xi(x)^2 dx + \frac{C}{r_0^2} u(y)^2 \operatorname{Vol}(B(y,r_0)). \end{aligned} \tag{4.79}$$

Combining (4.79), (4.77) and (4.76) we get

$$\int_{B(y,r_0)} \|\nabla^2 u(x)\|^2 \xi(x)^2 dx \leq \frac{1}{2}\int_{B(y,r_0)} \|\nabla_k \nabla_i u\|^2 \xi(x)^2 dx + C(n,K_0) u(y)^2 \operatorname{Vol}(B(y,r_0)). \tag{4.80}$$

Thus

$$\int_{B(y,r_0)} \|\nabla^2 u(x)\|^2 \xi(x)^2 dx \leq C(n, K_0) u(y)^2 \operatorname{Vol}(B(y, r_0)). \tag{4.81}$$

Which implies that

$$\int_{B(y,\frac{3}{4}r_0)} \|\nabla^2 u(x)\|^2 dx \leq C(n, K_0) u(y)^2 \operatorname{Vol}(B(y, r_0)). \tag{4.82}$$

Now we apply the iteration arguments of Moser to estimate the $L^p$ norm of $\nabla^2 u$ on $B(y_0, \frac{1}{2}r_0)$ for any $p \geq 2$. WLOG, $n \geq 3$, if $n \leq 2$, consider a new space $\widetilde{B} = B(y, r_0) \times [0,1]^2$ and a new function $\tilde{u}(x, a, b) = u(x)$.

For any point $x \in M$, assume WLOG

$$\operatorname{inj}(x) \geq \pi \left(\frac{1}{K_0}\right)^{\frac{1}{4}}. \tag{4.83}$$

Since if (4.83) is not true for $y \in M$, just consider $\hat{B}(0, \pi \left(\frac{1}{K_0}\right)^{\frac{1}{4}}) \subset T_y M$.

For any subdomain $\Omega \subset B(y, r_0)$, there exists a positive $C = C(n)$ such that

$$C \left(\int_\Omega \varphi(x)^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{2n}} \leq \left(\int_\Omega \|\nabla \varphi\|^2 dx\right)^{\frac{1}{2}}. \tag{4.84}$$

holds for any $\varphi \in C_0^\infty(\Omega)$.

Differentiating both sides of (4.73), we have

$$\nabla_j \Delta(\nabla_i u) - \operatorname{Ric}_{ip} \nabla_j \nabla_p u - \nabla_j \operatorname{Ric}_{ip} \nabla_p u - \lambda \nabla_j \nabla_i u = 0. \tag{4.85}$$

Using Ricci identity we have

$$\begin{aligned} &\Delta(\nabla_j \nabla_i u) + 2R_{jkip} \nabla_k \nabla_p u - \operatorname{Ric}_{ip} \nabla_j \nabla_p u - \operatorname{Ric}_{jp} \nabla_p \nabla_i u \\ &+ \left[\nabla_p \operatorname{Ric}_{ij} - \nabla_i \operatorname{Ric}_{jp} - \nabla_j \operatorname{Ric}_{ip}\right] \nabla_p u - \lambda \nabla_j \nabla_i u = 0, \quad \text{on } B(y, r_0). \end{aligned} \tag{4.86}$$

For any two tensor $A$ and $B$ on $M$, we use $A \star B$ to denote the linear combination of the tensor product of $A$ and $B$. Thus (4.86) can be written as

$$\Delta(\nabla_j \nabla_i u) + \operatorname{Rm} \star \nabla^2 u + \nabla \operatorname{Rm} \star \nabla u - \lambda \nabla_j \nabla_i u = 0. \tag{4.87}$$

which implies

$$\Delta(\|\nabla^2 u\|^2) - 2\|\nabla^2 u\|^2 + \operatorname{Rm} \star \nabla^2 u \star \nabla^2 u + \nabla \operatorname{Rm} \star \nabla u \star \nabla^2 u - 2\lambda \|\nabla^2 u\|^2 = 0. \tag{4.88}$$

Define a new function $\psi(x)$ as follows:

$$\psi(x) = \left(1 + \frac{\|\nabla^2 u(x)\|^2}{u(y)^2}\right)^{\frac{1}{2}}, \quad \forall x \in B(y, r_0). \tag{4.89}$$

From (4.82) we get

$$\int_{B(y,\frac{3}{4}r_0)} (\psi(x)^2 - 1)dx \le C(n, K_0) \operatorname{Vol}(B(y, r_0)). \tag{4.90}$$

Thus

$$\int_{B(y,\frac{3}{4}r_0)} \psi(x)^2 dx \le (1 + C(n, K_0)) \operatorname{Vol}(B(y, r_0)). \tag{4.91}$$

A simple computation implies that

$$\|\nabla_k \psi(x)\|^2 \le n \frac{\|\nabla^3 u(x)\|^2}{u(y)^2}, \quad \forall x \in B(y, r_0). \tag{4.92}$$

From (4.88) we get

$$\Delta \left( \frac{\|\nabla^2 u\|^2}{u(y)^2} \right) - 2\frac{\|\nabla^3 u\|^2}{u(y)^2} + \mathrm{Rm} \star \frac{\nabla^2 u}{u(y)} \star \frac{\nabla^2 u}{u(y)} + \nabla \mathrm{Rm} \star \frac{\nabla u}{u(y)} \star \frac{\nabla^2 u}{u(y)} - 2\lambda \frac{\|\nabla^2 u\|^2}{u(y)^2} = 0, \quad \text{on } B(y, r_0). \tag{4.93}$$

Combining (4.89), (4.92) and (4.93) we have

$$\Delta \psi^2 - \frac{1}{n}\|\nabla_k \psi\|^2 + \mathrm{Rm} \star \frac{\nabla^2 u}{u(y)} \star \frac{\nabla^2 u}{u(y)} - \frac{\|\nabla^3 u\|^2}{u(y)^2} + \nabla \mathrm{Rm} \star \frac{\nabla u}{u(y)} \star \frac{\nabla^2 u}{u(y)} \ge 0, \quad \text{on } B(y, r_0). \tag{4.94}$$

By assumption $\|\mathrm{Rm}\|^2 \le K_0$, thus

$$\mathrm{Rm} \star \frac{\nabla^2 u}{u(y)} \star \frac{\nabla^2 u}{u(y)} \le C(n, K_0) \frac{\|\nabla^2 u\|^2}{u(y)^2} \le C(n, K_0)\psi^2(x), \quad \text{on } B(y, r_0). \tag{4.95}$$

substituting into (4.94) we have

$$\Delta \psi^2 - \frac{1}{n}\|\nabla \psi\|^2 - \frac{\|\nabla^3 u\|^2}{u(y)^2} + C(n, K_0)\psi^2 + \nabla \mathrm{Rm} \star \frac{\nabla u}{u(y)} \star \frac{\nabla^2 u}{u(y)} \ge 0, \quad \text{on } B(y, r_0). \tag{4.96}$$

Define $\theta = \frac{n}{n-2}$, since $n \ge 3$, we get $1 < \theta \le 3$.

Suppose $k \ge 1$ is an integer, let

$$\Omega = B(y, \frac{r_0}{2} + \left(\frac{1}{4}\right)^k r_0). \tag{4.97}$$

Find a cut-off function $\zeta \in C_0^\infty(M)$ such that

$$\begin{cases} \zeta(x) \equiv 1 \quad \forall x \in B(y, \frac{r_0}{2} + \left(\frac{1}{4}\right)^{k+1} r_0); \\ \zeta(x) \equiv 0 \quad \forall x \in M \backslash B(y, \frac{r_0}{2} + \left(\frac{1}{4}\right)^{k} r_0); \\ \|\nabla \zeta\| \le \frac{4^{k+2}}{r_0}; \\ 0 \le \zeta \le 1; \end{cases} \tag{4.98}$$

Since $\psi^{\theta^{k-1}}\zeta$ vanishes on $\partial\Omega$, from (4.84) we get

$$
\begin{aligned}
C\left(\int_\Omega \psi(x)^{2\theta^k}\zeta(x)^{2\theta}dx\right)^{\frac{1}{\theta}} &\le \int_\Omega \|\nabla(\psi^{\theta^{k-1}}\zeta)\|^2 dx \\
&\le 2\int_\Omega \|\nabla\psi^{\theta^{k-1}}\|^2\zeta^2 dx + 2\int_\Omega \psi^{2\theta^{k-1}}\|\nabla\zeta\|^2 dx.
\end{aligned}
\tag{4.99}
$$

From (4.98) we get

$$
C\left(\int_\Omega \psi(x)^{2\theta^k}\zeta(x)^{2\theta}dx\right)^{\frac{1}{\theta}} \le 2\int_\Omega \|\nabla\psi^{\theta^{k-1}}\|^2\zeta^2 dx + \frac{4^{2k+5}}{r_0^2}\int_\Omega \psi^{2\theta^{k-1}}dx. \tag{4.100}
$$

A simple computation and (4.96) implies that

$$
\begin{aligned}
\Delta\psi^{2\theta^{k-1}} \ge\ & \frac{1}{n}\theta^{k-1}\psi^{2\theta^{k-1}-2}\|\nabla\psi\|^2 + 4\theta^{k-1}(\theta^{k-1}-1)\psi^{2\theta^{k-1}-2}\|\nabla\psi\|^2 \\
&+ \theta^{k-1}\psi^{2\theta^{k-1}-2}\left[-C\psi^2 - \nabla\operatorname{Rm}\star\frac{\nabla u}{u(y)}\star\frac{\nabla^2 u}{u(y)} + \frac{\|\nabla^3 u\|^2}{u(y)^2}\right], \quad \text{on } B(y,r_0).
\end{aligned}
\tag{4.101}
$$

Thus

$$
\begin{aligned}
\|\nabla\psi^{\theta^{k-1}}\|^2 &\le \frac{\theta^{k-1}}{\frac{1}{n}+4(\theta^{k-1}-1)}\left[\Delta\psi^{2\theta^{k-1}} + C\theta^{k-1}\psi^{2\theta^{k-1}} + \theta^{k-1}\psi^{2\theta^{k-1}-2}\nabla\operatorname{Rm}\star\frac{\nabla u}{u(y)}\star\frac{\nabla^2 u}{u(y)} - \theta^{k-1}\psi^{2\theta^{k-1}-2}\frac{\|\nabla^3 u\|^2}{u(y)^2}\right]. \\
&\le 2n\left[\Delta\psi^{2\theta^{k-1}} + C\theta^{k-1}\psi^{2\theta^{k-1}} - \theta^{k-1}\psi^{2\theta^{k-1}-2}\frac{\|\nabla^3 u\|^2}{u(y)^2} + \theta^{k-1}\psi^{2\theta^{k-1}-2}\nabla\operatorname{Rm}\star\frac{\nabla u}{u(y)}\star\frac{\nabla^2 u}{u(y)}\right], \quad \text{on } B(y,r_0).
\end{aligned}
\tag{4.102}
$$

Thus

$$
\begin{aligned}
\int_\Omega |\nabla\psi^{\theta^{k-1}}\|^2\zeta(x)^2 dx \le\ & 2n\int_\Omega \Delta\psi^{2\theta^{k-1}}\zeta^2 dx + 2nC\theta^{k-1}\int_\Omega \psi^{2\theta^{k-1}}\zeta^2 dx \\
&- 2n\theta^{k-1}\int_\Omega \psi^{2\theta^{k-1}-2}\frac{\|\nabla^3 u\|^2}{u(y)^2}\zeta^2 dx + 2n\theta^{k-1}\int_\Omega \psi^{2\theta^{k-1}-2}\nabla\operatorname{Rm}\star\frac{\nabla u}{u(y)}\star\frac{\nabla^2 u}{u(y)}\zeta^2 dx.
\end{aligned}
\tag{4.103}
$$

Since $0 \le \zeta \le 1$, we have

$$
2nC\theta^{k-1}\int_\Omega \psi^{2\theta^{k-1}}\zeta^2 dx \le 2nC\theta^{k-1}\int_\Omega \psi^{2\theta^{k-1}}dx. \tag{4.104}
$$

Integrating by part, we have

$$
\begin{aligned}
2n\int_\Omega \Delta(\psi^{2\theta^{k-1}})\zeta^2 dx &= -2n\int_\Omega \nabla_p(\psi^{2\theta^{k-1}})\nabla(\zeta^2)dx \\
&\le \frac{1}{4}\int_\Omega \|\nabla(\psi^{\theta^{k-1}})\|^2\zeta^2 dx + 64n^2\int_\Omega \psi^{2\theta^{k-1}}\|\nabla_p\zeta\|^2 dx.
\end{aligned}
\tag{4.105}
$$

Together with (4.98) implies that

$$
2n\int_\Omega \Delta(\psi^{2\theta^{k-1}})\zeta^2 dx \le \frac{1}{4}\int_\Omega \|\nabla(\psi^{\theta^{k-1}})\|^2\zeta^2 dx + \frac{4^{2k+7}n^2}{r_0^2}\int_\Omega \psi^{2\theta^{k-1}}dx. \tag{4.106}
$$

Integrating by part we have

$$
\begin{aligned}
2n\theta^{k-1}\int_\Omega \psi^{2\theta^{k-1}-2}\nabla\operatorname{Rm}\star\frac{\nabla u}{u(y)}\star\frac{\nabla^2 u}{u(y)}\zeta^2 dx &= \frac{2n\theta^{k-1}}{u(y)^2}\int_\Omega \psi^{2\theta^{k-1}-2}\nabla\operatorname{Rm}\star\nabla u\star\nabla^2 u\star\zeta^2 dx\\
&= -\frac{2n\theta^{k-1}}{u(y)^2}\int_\Omega \operatorname{Rm}\cdot\nabla\left(\psi^{2\theta^{k-1}-2}\star\nabla u\star\nabla^2 u\star\zeta^2\right)dx\\
&= -\frac{2n\theta^{k-1}}{u(y)^2}\int_\Omega \operatorname{Rm}\star(\psi^{2\theta^{k-1}-2}\star\nabla^2 u\star\nabla^2 u\star\zeta^2\\
&+\psi^{2\theta^{k-1}-2}\star\nabla u\star\nabla^3 u\star\zeta^2+\psi^{2\theta^{k-1}-2}\star\nabla u\star\nabla^2 u\star\zeta\star\nabla\zeta\\
&+2(\theta^{k-1}-1)\psi^{2\theta^{k-1}-3}\nabla\psi\star\nabla u\star\nabla^2 u\star\zeta^2)dx\\
&\le 2n\theta^{k-1}\int_\Omega \psi^{2\theta^{k-1}-2}\frac{\|\nabla^3 u\|^2}{u(y)^2}\zeta^2 dx+\frac{1}{4}\int_\Omega \|\nabla\psi^{\theta^{k-1}}\|^2\zeta^2 dx\\
&+C(n,K_0)(4+\theta)^{2k}\int_\Omega \psi(x)^{2\theta^{k-1}}dx.
\end{aligned}
\tag{4.107}
$$

Now combining (4.107), (4.104), (4.106) and (4.103) we get

$$
\int_\Omega |\nabla\psi^{\theta^{k-1}}\|^2\zeta(x)^2 dx\le\frac{1}{2}\int_\Omega \|\nabla\psi^{\theta^{k-1}}\|^2\zeta^2 dx+2nC\theta^{k-1}\int_\Omega \psi^{2\theta^{k-1}}dx+\left(\frac{4^{2k+7}n^2}{r_0^2}+C(4+\theta)^{2k}\right)\int_\Omega \psi^{2\theta^{k-1}}dx.
\tag{4.108}
$$

Thus

$$
\int_\Omega |\nabla\psi^{\theta^{k-1}}\|^2\zeta(x)^2 dx\le C(n,K_0)(4+\theta)^{2k}\int_\Omega \psi^{2\theta^{k-1}}dx.
\tag{4.109}
$$

Together with (4.100) yield

$$
C\left(\int_\Omega \psi(x)^{2\theta^k}\zeta(x)^{2\theta}dx\right)^{\frac{1}{\theta}}\le C(n,K_0)(4+\theta)^{2k}\int_\Omega \psi^{2\theta^{k-1}}dx.
\tag{4.110}
$$

which together with (4.97) and (4.98) yield

$$
\left(\int_{B(y,\frac{r_0}{2}+\left(\frac{1}{4}\right)^k r_0)}\psi(x)^{2\theta^k}\zeta(x)^{2\theta}dx\right)\le C(n,K_0)(4+\theta)^{2\theta k}\left(\int_{B(y,\frac{r_0}{2}+\left(\frac{1}{4}\right)^k r_0)}\psi^{2\theta^{k-1}}dx\right)^\theta.
\tag{4.111}
$$

Define $H_k=\|\psi\|^{2\theta^k}_{L^{2\theta^k}(B(y,\frac{r_0}{2}+\left(\frac{1}{4}\right)^k r_0)}$, we have from (4.111),

$$
H_k\le C(n,K_0)(4+\theta)^{2\theta k}H_{k-1}^\theta.
\tag{4.112}
$$

From (**??**, we have

$$
H_0\le(1+C)\operatorname{Vol}(B(y,r_0)).
\tag{4.113}
$$

Thus

$$
\lim_{k\to\infty}H_k^{\frac{1}{\theta^k}}\le C(n,K_0)\operatorname{Vol}(B(y,r_0)).
\tag{4.114}
$$

Since $\lim_{k\to\infty} H_k^{\frac{1}{\theta^k}} = \sup_{B(y,\frac{1}{2}r_0)} \psi^2(x)$, and

$$\lim_{t\to 0} t^{-n} e^{-\sqrt{n(n-1)\sqrt{K_0}t}} \operatorname{Vol}(B(y,t)) = C(n). \tag{4.115}$$

From (4.115) we have

$$\frac{\partial}{\partial t}\left( t^{-n} e^{-\sqrt{n(n-1)\sqrt{K_0}t}} \operatorname{Vol}(B(y,t)) \right) \le 0. \tag{4.116}$$

Combining (4.115) and (4.116) we get

$$\operatorname{Vol}(B(y,r_0)) \le C(n,K_0). \tag{4.117}$$

Combining (4.117) and (4.114) we get

$$\sup_{B(y,\frac{1}{2}r_0)} \psi^2(x) \le C(n,K_0). \tag{4.118}$$

By definition of $\psi$ and then let $x=y$ we get

$$\frac{\nabla^2 u(y)}{u(y)^2} leq C(n,K_0). \tag{4.119}$$

Since $y \in M\backslash B(x_0, 3+\pi\left(\frac{1}{K_0}\right)^{\frac{1}{4}})$ is arbitrary, we complete the proof of the lemma. □

**Lemma 4.1.5.** *Under the curvature assumption of Theorem 4.1.4, for any $x_0 \in M$, there exists a constant $C = C(n,K_0) > 0$ and a smooth function $\omega(x) \in C^\infty(M\backslash B(x_0, 3+8\pi r_0))$ such that*

$$\begin{cases} 1 + r(x,x_0) \le \omega(x) \le C(1+r(x,x_0)); \\ \nabla_i \omega\| \le C; \\ \|\nabla^2\omega\| \le C; \end{cases} \tag{4.120}$$

*Proof.* Suppose $u(x) \in C^\infty(M\backslash B(x_0,2))$ is the smooth function we obtained above, from (4.65) we get

$$r(x,x_0) - \log C(n,K_0) \le -\log u(x) < +\infty, \quad \text{on } M\backslash B(x_0,3). \tag{4.121}$$

Now we just define

$$\tilde{\omega}(x) = -\log u(x) + \log C(n,K_0) + 1, \quad \text{on } M\backslash B(x_0,3). \tag{4.122}$$

Thus $\tilde{\omega} \in C^\infty(M\backslash B(x_0,3))$ and

$$\tilde{\omega}(x) \ge 1 + r(x,x_0), \quad \text{on } M\backslash B(x_0,3). \tag{4.123}$$

By definition

$$\begin{aligned} \nabla_i\tilde{\omega}(x) &= -\nabla_i \log u(x); \\ \nabla_i\nabla_j\tilde{\omega}(x) &= -\frac{\nabla_i\nabla_j u(x)}{u(x)} + \frac{\nabla_i u(x)\nabla_j u(x)}{u(x)^2}; \end{aligned} \tag{4.124}$$

which together with Lemma 4.1.3 and Lemma 4,1,4 yield

$$\begin{aligned} \|\nabla_i \tilde{\omega}(x)\| \leq C(n, K_0); \\ \|\nabla_i \nabla_j \tilde{\omega}(x)\| \leq C(n, K_0); \end{aligned} \tag{4.125}$$

Since $0 < u \leq 1$ on $M \backslash B(x_0, 3)$, we have

$$0 < \sup_{M \backslash B(x_0, 3+\pi r_0)} u(x) \leq 1; \tag{4.126}$$

But $M \backslash B(x_0, 3+\pi r_0)$ is closed and $u(x) \to 0$ as $r(x, x_0) \to \infty$, thus there exists a point $x_1 \in M \backslash B(x_0, 3+\pi r_0)$ such that

$$u(x_1) = \sup_{M \backslash B(x_0, 3+\pi r_0)} u(x). \tag{4.127}$$

However, since $\Delta u = \lambda u$ on $M \backslash B(x_0, 3)$, by the maximum principle we get

$$x_1 \in \partial M \backslash B(x_0, 3+\pi r_0). \tag{4.128}$$

Therefore

$$\begin{cases} 0 < \frac{u(x)}{u(x_1)} \leq 1, & \forall x \in M \backslash B(x_0, 3+\pi r_0); \\ \Delta\left(\frac{u(x)}{u(x_1)}\right) = \lambda \frac{u(x)}{u(x_1)}, & \forall x \in M \backslash B(x_0, 3+\pi r_0); \\ \frac{u(x)}{u(x_1)} \leq \frac{C(n, K_0)}{u(x_1)} e^{-r(x, x_0)}, & \forall x \in M \backslash B(x_0, 3+\pi r_0); \\ \|\nabla \log \frac{u(x)}{u(x_1)}\| \leq C(n, K_0), & \forall x \in M \backslash B(x_0, 3+\pi r_0); \end{cases} \tag{4.129}$$

For any $\gamma > 3 + \pi r_0$ , by Stokes theorem we get

$$\begin{aligned} \lambda \int_{B(x_0,\gamma) \backslash B(x_0, 3+\pi r_0)} e^{ar(x,x_0)} \left(\frac{u(x)}{u(x_1)}\right)^2 dx &= \int_{B(x_0,\gamma) \backslash B(x_0, 3+\pi r_0)} e^{ar(x,x_0)} \left(\frac{u(x)}{u(x_1)}\right) \Delta \left(\frac{u(x)}{u(x_1)}\right) dx \\ &= \int_{\partial B(x_0, 3+\pi r_0)} e^{ar(x,x_0)} \frac{u(x)}{u(x_1)} \frac{\partial}{\partial \nu} \left(\frac{u(x)}{u(x_1)}\right) \\ &- \int_{\partial B(x_0,\gamma)} e^{ar(x,x_0)} \frac{u(x)}{u(x_1)} \frac{\partial}{\partial \nu} \left(\frac{u(x)}{u(x_1)}\right) - \\ &\int_{B(x_0,\gamma) \backslash B(x_0, 3+\pi r_0)} \nabla_i \left(e^{ar(x,x_0)} \frac{u(x)}{u(x_1)}\right) \nabla_i \left(\frac{u(x)}{u(x_1)}\right) dx. \end{aligned} \tag{4.130}$$

By similar arguments in (4.56) we get

$$\begin{aligned} \int_{B(x_0,\gamma) \backslash B(x_0, 3+\pi r_0)} e^{ar(x,x_0)} \left(\frac{u(x)}{u(x_1)}\right)^2 dx &\leq C(n, K_0, \lambda) e^{2a} \operatorname{Vol}(\partial B(x_0, 3+\pi r_0)) \\ &- \int_{\partial B(x_0,\gamma)} e^{ar(x,x_0)} \frac{u(x)}{u(x_1)} \frac{\partial}{\partial \nu} \left(\frac{u(x)}{u(x_1)}\right). \end{aligned} \tag{4.131}$$

From (4.56) we have

$$\int_{M \backslash B(x_0,2)} e^{ar(x,x_0)} \left(\frac{u(x)}{u(x_1)}\right)^2 dx \leq \frac{C(n, K_0, \lambda)}{u(x_1)^2} e^{2a} \operatorname{Vol}(\partial B(x_0, 2)). \tag{4.132}$$

Thus there exists a sequence $\gamma_i \to \infty$ such that

$$\lim_{i\to\infty} \int_{\partial B(x_0,\gamma_i)} e^{ar(x,x_0)} \left(\frac{u(x)}{u(x_1)}\right)^2 dx = 0. \tag{4.133}$$

From (4.129) we have

$$\begin{aligned} | \int_{\partial B(x_0,\gamma)} e^{ar(x,x_0)} \frac{u(x)}{u(x_1)} \frac{\partial}{\partial t}\left(\frac{u(x)}{u(x_1)}\right) | &\le \int_{\partial B(x_0,\gamma)} e^{ar(x,x_0)} \frac{u(x)}{u(x_1)} \|\nabla\left(\frac{u(x)}{u(x_1)}\right)\| \\ &\le C(n,K_0) \int_{\partial B(x_0,\gamma)} e^{ar(x,x_0)} \left(\frac{u(x)}{u(x_1)}\right)^2. \end{aligned} \tag{4.134}$$

which together with (4.133) imply

$$\lim_{i\to\infty} \int_{\partial B(x_0,\gamma)} e^{ar(x,x_0)} \frac{u(x)}{u(x_1)} \frac{\partial}{\partial t}\left(\frac{u(x)}{u(x_1)}\right) = 0. \tag{4.135}$$

Then let $\gamma \to \infty$ in (4.131) we get

$$\int_{M\setminus B(x_0,3+\pi r_0)} e^{ar(x,x_0)} \left(\frac{u(x)}{u(x_1)}\right)^2 dx \le C(n,K_0,\lambda) e^{2a} \operatorname{Vol}(\partial B(x_0, 3+\pi r_0)). \tag{4.136}$$

Then use the same arguments as in (4.63) we have

$$\frac{u(x)}{u(x_1)} \le C(n,K_0) e^{-r(x,x_0)}, \quad \forall x \in M\setminus B(x_0,3+\pi r_0). \tag{4.137}$$

Fro any $y \in M\setminus B(x_0, 3+\pi r_0)$, there exists a curve $\theta(s)$ connected $y$ and $x_1$ such that

$$\begin{cases} \theta \subset M\setminus B(x_0, 3+\pi r_0); \\ \|\theta\| \le r(y,x_0) + 100 + 8\pi r_0; \end{cases} \tag{4.138}$$

Then by integrating we get

$$| \log \frac{u(y)}{u(x_1)} | \le C(n,K_0)(r(y,x_0) + 100 + 8\pi r_0). \tag{4.139}$$

which implies that

$$\frac{u(x)}{u(x_1)} \ge e^{-C(n,K_0)(1+r(x,x_0)}, \quad \forall x \in M\setminus B(x_0,3+\pi r_0). \tag{4.140}$$

Therefore

$$r(x,x_0) - C(n,K_0) - \log u(x_1) \le \tilde\omega(x) \le C(n,K_0)(1+r(x,x_0)) - \log u(x_1), \quad \forall x \in M\setminus B(x_0, 3+\pi r_0). \tag{4.141}$$

Now we just define

$$\omega(x) = \tilde\omega(x) + \log u(x_1) + C(n,K_0) + 1, \quad \forall x \in M\setminus B(x_0, 3+\pi r_0). \tag{4.142}$$

The proof of the lemma is complete. □

To complete the proof of Theorem 4.1.1. Fix $x_0 \in M$, from Lemma 4.1.5, there exists a smooth function $\omega$ on $M\backslash B(x_0, 3 + \pi r_0)$ satisfying (4.120).

Suppose $y \in M$ such that $r(y, x_0) = (1 + C)(5 + 8\pi r_0)$, by Lemma 4.1.5 we construct a smooth function $v(x)$ on $M\backslash B(y, 3 + \pi r_0)$ satisfying (4.120).

Take a smooth function $\theta \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} \theta(t) = 0, & -\infty < t \leq C(5 + 8\pi r_0); \\ 0 \leq \theta(x) \leq 1, & C(5 + 8\pi r_0) \leq t \leq 2 + C(5 + 8\pi r_0); \\ \theta(t) = 1, & 2 + C(5 + 8\pi r_0) \leq t < +\infty; \end{cases} \tag{4.143}$$

and

$$\begin{cases} \mid \theta' \mid \leq 1; \\ \mid \theta'' \mid \leq 4; \end{cases} \tag{4.144}$$

Now we just define

$$\begin{cases} \varphi(x) = v(x), & x \in B(x_0, \frac{7}{2} + 8\pi r_0); \\ \varphi(x) = \theta(\omega(x))\omega(x) + (1 - \theta(\omega(x)))v(x), & x \in B(x_0, \frac{3}{2} + C(5 + 8\pi r_0))\backslash B(x_0, \frac{7}{2} + 8\pi r_0); \\ \varphi(x) = \omega(x), & x \in M\backslash B(x_0, \frac{3}{2} + C(5 + 8\pi r_0)) \end{cases} \tag{4.145}$$

Then $\varphi$ is the function that we want. $\square$

### 4.1.3 Maximum Principles on Noncompact Manifolds

In this section, we are going to use the exhaustion function before to establish the maximal principles on complete noncompact manifolds for the solutions of parabolic equations. In this section we always make the following assumption:

**Assumption A.** Suppose $(M, \tilde{g})$ is an n-dimensional complete noncompact Riemannian manifold. Suppose $0 < T, K_0 < +\infty$ are some constants and $g$ is the smooth solution of the evolution equation

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2\,\mathrm{Ric}_{ij}(x,t), & \text{on } M \times [0, T]; \\ g_{ij}(x, 0) = \tilde{g}_{ij}(x), & \forall x \in M; \end{cases} \tag{4.146}$$

and satisfies the following estimate:

$$\sup_{M\times[0,T]} \| \mathrm{Rm} \|^2 \leq K_0. \tag{4.147}$$

We use

$$\begin{aligned} d\tilde{s}^2 &= \tilde{g}_{ij} dx^i dx^j; \\ ds_t^2 &= g_{ij}(x,t) dx^i dx^j; \end{aligned} \tag{4.148}$$

to denote the metrics on $M$, and use $\widetilde{\nabla}$ to denote the covariant derivatives w.r.t. $d\tilde{s}^2$, use $\nabla$ to denote the covariant derivatives w.r.t. $ds_t^2$, use $\Delta$ to denote the laplacian w.r.t. $ds_t^2$, use $r_t$ to denote the distance w.r.t. $ds_t^2$.

**Lemma 4.1.6.** *Under Assumption A, we have*

$$\begin{aligned} e^{-2\sqrt{nK_0}t}d\tilde{s}^2 \le ds_t^2 \le e^{2\sqrt{nK_0}t}d\tilde{s}^2; \\ e^{-\sqrt{nK_0}t}r_0(x,y) \le r_t(x,y) \le e^{\sqrt{nK_0}t}r_0(x,y); \end{aligned} \tag{4.149}$$

*Proof.* This is just a simple computation. □

From the lemma we know that for any $t \in [0,T]$, $ds_t^2$ is equivalent to $d\tilde{s}^2$, thus also a complete Riemannian metric.

By the previous section, we know

**Lemma 4.1.7.** *Under Assumption A, for any fixed point $x_0 \in M$, there exists a function $\varphi \in C^\infty(M)$ such that*

$$\begin{cases} \frac{1}{C}(1 + r_0(x,x_0)) \le \varphi(x) \le C(1 + r_0(x,x_0)); \\ \|\widetilde{\nabla}\varphi(x)\| \le C \quad \forall x \in M; \\ \|\widetilde{\nabla}_i\widetilde{\nabla}_j\varphi(x)\| \le Cg_{ij}(x) \quad \forall x \in M; \end{cases} \tag{4.150}$$

*where $C = C(n, K_0)$ is positive.*

**Lemma 4.1.8.** *The function $\varphi$ in Lemma 4.1.7 satisfies*

$$\begin{cases} \frac{1}{C}(1 + r_t(x,x_0)) \le \varphi(x) \le C(1 + r_t(x,x_0)); \\ \|\nabla\varphi(x)\| \le C \quad \forall x \in M; \\ \|\nabla_i\nabla_j\varphi(x)\| \le Cg_{ij}(x) \quad \forall x \in M; \end{cases} \tag{4.151}$$

*where $C = C(n, K_0, T)$ is positive.*

*Proof.*

$$\frac{1}{C}(1 + r_t(x,x_0)) \le \varphi(x) \le C(1 + r_t(x,x_0)); \tag{4.152}$$

and

$$\|\nabla\varphi(x)\| \le C \quad \forall x \in M; \tag{4.153}$$

comes easily from (4.149) and (4.150).

For the third inequality, we have

$$\widetilde{\nabla}_i\widetilde{\nabla}_j\varphi(x) = \frac{\partial^2\varphi}{\partial x^i\partial x^j} - \Gamma_{ij}^k(x,0)\frac{\partial\varphi(x)}{\partial x^k}; \tag{4.154a}$$

$$\nabla_i\nabla_j\varphi(x) = \frac{\partial^2\varphi}{\partial x^i\partial x^j} - \Gamma_{ij}^k(x,t)\frac{\partial\varphi(x)}{\partial x^k}; \tag{4.154b}$$

Thus

$$\nabla_i\nabla_j\varphi(x) = \widetilde{\nabla}_i\widetilde{\nabla}_j\varphi(x) - \left[\Gamma_{ij}^k(x,t) - \Gamma_{ij}^k(x,0)\right]\widetilde{\nabla}_k\varphi(x); \tag{4.155}$$

Then the third inequality comes from

$$\left[\Gamma_{ij}^k(x,t) - \Gamma_{ij}^k(x,0)\right]^2 \le C(n, K_0, T), \quad \text{on } M \times [0,T]. \tag{4.156}$$

which was developed in the proof of Lemma 4.3 in [19]. □

**Lemma 4.1.9.** *Under Assumption A, for any constant $0 < C < +\infty$, we can find a function $\theta(x,t) \in C^\infty(M \times [0,T])$ and a constant $0 < C' < +\infty$ depending only on $n, K_0, T$ and $C$ such that*

$$\begin{cases} 0 < \theta(x,t) \le 1; \\ \frac{C'^{-1}}{1+r_0(x,x_0)} \le \theta(x,t) \le \frac{C'}{1+r_0(x,x_0)}; \\ \frac{\partial\theta}{\partial t} \le \Delta\theta - \frac{2\|\nabla\theta\|^2}{\theta} - C\theta; \end{cases} \tag{4.157}$$

*Proof.* From Lemma 4.1.7 we know that

$$\Delta\varphi \le C_1. \tag{4.158}$$

Let

$$\xi(x,t) = e^{Ct}(\varphi(x) + C_1 t). \tag{4.159}$$

Then

$$\frac{\partial\xi}{\partial t} = C\xi + C_1 e^{Ct}. \tag{4.160}$$

Since

$$\Delta\xi = e^{Ct}\Delta(\varphi(x) + C_1 t) = e^{Ct}\Delta\varphi \le C_1 e^{Ct}. \tag{4.161}$$

From (4.160) we get

$$\frac{\partial\xi}{\partial t} \ge \Delta\xi + C\xi. \tag{4.162}$$

By (4.159),

$$\varphi(x) \le \xi(x,t) \le e^{CT}\varphi(x) + e^{CT}C_1 T. \tag{4.163}$$

and therefore,

$$C_2(1 + r_0(x,x_0)) \le \xi(x,t) \le (C_3 e^{CT} + C_1 T e^{CT})(1 + r_0(x,x_0)). \tag{4.164}$$

Let

$$\tilde{\theta}(x,t) = \frac{1}{\xi(x,t)}, \quad \text{on } M \times [0,T]. \tag{4.165}$$

Then

$$\frac{\partial\tilde{\theta}}{\partial t} = -\frac{1}{\xi^2}\frac{\partial\xi}{\partial t} \le -\frac{1}{\xi^2}\left[\Delta\xi + C\xi\right] = \Delta\tilde{\theta} - \frac{2}{\tilde{\theta}}\|\nabla\tilde{\theta}\|^2 - C\tilde{\theta}. \tag{4.166}$$

Also from (4.164),

$$\tilde{\theta} \le \frac{1}{C_4}. \tag{4.167}$$

Thus take

$$\theta = C_4\tilde{\theta}. \tag{4.168}$$

Thus

$$\frac{C_4}{(C_3 + C_1 T)e^{CT}(1 + r_0(x,x_0))} \le \theta(x,t) \le \frac{1}{1 + r(x,x_0)}. \tag{4.169}$$

Therefore take $C' \ge 1 + \frac{C_3 + C_1 T}{C_4} e^{CT}$, the proof is complete. □

Now we can prove the following maximum principle on noncompact manifold $M$.

**Lemma 4.1.10.** *Under Assumption A, suppose* $\varphi(x,t) \in C^\infty(M \times [0,T])$ *such that*

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \Delta\varphi + Q(x,t); \\ \mid \varphi(x,t) \mid \leq C; \\ \varphi(x,0) \leq 0; \\ Q(\varphi, x, t) \leq 0; \end{cases} \tag{4.170}$$

*Thus we have*

$$\varphi(x,t) \leq 0. \tag{4.171}$$

*Proof.* Suppose not, then there exists $(x_0, t_0) \in M \times [0,T]$ such that

$$\varphi(x_0, t_0) > 0. \tag{4.172}$$

Suppose $\theta \in C^\infty(M \times [0,T])$ is the function obtained in Lemma 4.1.9, and define

$$\tilde{\varphi}(x,t) = \theta(x,t)\varphi(x,t). \tag{4.173}$$

Since $0 < \theta \leq 1$ and $|\varphi| \leq C$, we have

$$\begin{aligned} |\tilde{\varphi}| &\leq C; \\ \tilde{\varphi}(x_0, t_0) &> 0; \end{aligned} \tag{4.174}$$

Let

$$\alpha = \sup_{M \times [0,T]} \tilde{\varphi}(x,t). \tag{4.175}$$

Then

$$0 < \alpha \leq C. \tag{4.176}$$

so that

$$|\tilde{\varphi}| \leq C\theta(x,t) \leq \frac{CC'}{1 + r(x, x_0)}. \tag{4.177}$$

Let

$$D = \{x \in M \mid r(x, x_0) \leq \frac{CC'}{\alpha}\}. \tag{4.178}$$

Then $D \subset M$ is compact.

If $(x,t) \notin D \times [0,T]$, then $r(x,x_0) > \alpha^{-1}CC'$. Thus

$$|\tilde{\varphi}(x,t)| < \alpha, \quad \text{for } x \notin D \times [0,T]. \tag{4.179}$$

Since $D \times [0,T]$ is compact, there exists a point $(x_1, t_1) \in D \times [0,T]$ such that $\tilde{\varphi}(x_1, t_1) = \alpha$. And

$$\begin{aligned} \frac{\partial \tilde{\varphi}}{\partial t}(x_1, t_1) &\geq 0; \\ \Delta\tilde{\varphi}(x_1, t_1) &\leq 0; \\ \nabla\tilde{\varphi}(x_1, t_1) &= 0; \end{aligned} \tag{4.180}$$

where the first inequality comes from that $\tilde{\varphi}(x,0) = \theta(x,0)\varphi(x,0) \leq 0$. Therefore we always have $t_1 > 0$.

A simple computation shows that

$$\frac{\partial \tilde{\varphi}}{\partial t} = \Delta \tilde{\varphi} - \frac{2}{\theta}\langle \nabla \theta, \nabla \tilde{\varphi}\rangle + \theta Q(\varphi, x, t) + \left(\frac{\partial \theta}{\partial t} - \Delta\theta + \frac{2}{\theta}\|\nabla \theta\|^2\right)\varphi. \tag{4.181}$$

Since $\tilde{\varphi}(x_1, t_1) = \theta(x_1, t_1)\varphi(x_1, t_1) > 0$ and $\theta(x_1, t_1) > 0$, we get

$$\varphi(x_1, t_1) > 0. \tag{4.182}$$

Let $C = 1$ in Lemma 4.1.9. Then

$$\frac{\partial \theta}{\partial t} \leq \Delta\theta - \frac{2}{\theta}\|\nabla\theta\|^2 - \theta. \tag{4.183}$$

Thus

$$\left(\frac{\partial \theta}{\partial t} - \Delta\theta + \frac{2}{\theta}\|\nabla \theta\|^2\right)\varphi(x_1, t_1) \leq -\theta\varphi(x_1, t_1) = -\tilde{\varphi}(x_1, t_1). \tag{4.184}$$

and

$$\theta Q(\varphi, x, t) \leq 0. \tag{4.185}$$

Substituting (4.180), (4.184) and (4.185) into (4.181) we get

$$\frac{\partial \tilde{\varphi}}{\partial t}(x_1, t_1) \leq -\tilde{\varphi}(x_1, t_1) < 0. \tag{4.186}$$

which contradicts (4.180). The proof is complete. □

**Theorem 4.1.5.** *Under Assumption A, suppose $\varphi(x,t)$ is a $C^\infty$ function on $M \times [0,T]$ such that*

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \Delta\varphi + C\|\nabla\varphi\|^2 + Q(\varphi, x, t), & \text{on } M \times [0,T]; \\ \varphi(x,t) \leq C, & \text{on } M \times [0,T]; \\ \varphi(x,0) \leq 0, & \text{on } M; \\ Q(\varphi, x, t) \leq C\varphi, & \text{if } \varphi \geq 0; \end{cases} \tag{4.187}$$

*where $C > 0$ is a positive constant. Then we have*

$$\varphi(x,t) \leq 0 \quad \text{on } M \times [0,T]. \tag{4.188}$$

*Proof.* Let

$$\omega(x,t) = e^{-\beta}[e^{C\varphi(x,t)} - 1] \quad \text{on } M \times [0,T]. \tag{4.189}$$

where $\beta > 0$ is a constant to be determined later. Then

$$\frac{\partial \omega}{\partial t} = \Delta\omega - \beta\omega + Ce^{-\beta t}e^{C\varphi}Q(\varphi, x, t). \tag{4.190}$$

If $\omega(x,t) \geq 0$, it follows from (4.189) that $\varphi(x,t) \geq 0$. Therefore

$$Q(\varphi, x, t) \leq C\varphi(x,t). \tag{4.191}$$

Since $\varphi(x,t) \le C$, there exists a constant $\delta > 0$ such that

$$\varphi(x,t) \le \delta[e^{C\varphi(x,t)} - 1] \quad \text{for } 0 \le \varphi(x,t) \le C. \tag{4.192}$$

which together with (4.191) implies

$$Q(\varphi, x, t) \le C\delta[e^{C\varphi(x,t)} - 1], \tag{4.193}$$

thus

$$Ce^{-\beta t} e^{C\varphi} Q(\varphi, x, t) \le C^2 \delta e^{C^2} \omega. \tag{4.194}$$

Let

$$\tilde{Q}(\omega, x, t) = -\beta\omega + Ce^{-\beta t} e^{C\varphi} Q(\varphi, x, t). \tag{4.195}$$

Then

$$\frac{\partial \omega}{\partial t} = \Delta\omega + \tilde{Q}(\omega, x, t). \tag{4.196}$$

If $\omega(x,t) \ge 0$, then from (4.194) we get

$$\tilde{Q}(\omega, x, t) \le [C^2 \delta e^{C^2} - \beta]\omega. \tag{4.197}$$

Choose

$$\beta = C^2 \delta e^{C^2}. \tag{4.198}$$

then

$$\tilde{Q}(\omega, x, t) \le 0 \quad \text{for } \omega \ge 0. \tag{4.199}$$

Since $\varphi(x,t) \le 0$, from (4.188),

$$\begin{aligned} &\omega(x,0) \le 0 \quad \text{on } M \times [0,T], \\ &-1 \le \omega(x,t) \le e^{C^2} - 1 \quad \text{on } M \times [0,T]. \end{aligned} \tag{4.200}$$

so that

$$|\omega(x,t)| \le e^{C^2}. \tag{4.201}$$

Then from (4.196), (4.199) and (4.201) and Lemma 4.1.10, we get

$$\omega(x,t) \le 0 \quad \text{on } M \times [0,T]. \tag{4.202}$$

so that

$$\varphi(x,t) \le 0 \quad \text{on } M \times [0,T]. \tag{4.203}$$

□

Now Shi established another kind of maximum principle:

**Lemma 4.1.11.** *Under Assumption A, for any fixed point $x_0 \in M$ and constants $\varepsilon > 0, h \ge 4$, there exists a function $\theta \in C^\infty(M)$ and a constant $C_5 = C_5(n, K_0, \varepsilon) > 0$ such that*

$$\begin{cases} 0 \le \theta \le 1 & \text{on } M; \\ \theta(x) \equiv 1 & \forall x \in \mathring{B}(x_0, h); \\ \theta(x) \equiv 0 & \forall x \in M \backslash \mathring{B}(x_0, 2C^2 h); \end{cases} \tag{4.204}$$

$$\begin{cases} \|\widetilde{\nabla}\left(\frac{1}{\theta}\right)\| \leq \frac{C_5}{h}\left(\frac{1}{\theta}\right)^{1+\varepsilon}, & \forall x \in \Omega; \\ |\widetilde{\nabla}_i\widetilde{\nabla}_j\left(\frac{1}{\theta}\right)| \leq \frac{C_5}{h}\left(\frac{1}{\theta}\right)^{1+\varepsilon}, & \forall x \in \Omega; \end{cases} \tag{4.205}$$

*where $C$ is the constant in (4.151) and*

$$\Omega = \{x \in M \mid \theta(x) > 0\}. \tag{4.206}$$

*Proof.* There exists two functions $\xi(t)$ and $\eta(t)$ such that

$$\begin{cases} \xi(t) \in C^\infty[0, \frac{7}{4}h) & ; \\ \xi(t) \equiv 1 & 0 \leq t \leq \frac{5}{4}h; \\ \xi(t) \geq 1 & 0 \leq t < \frac{7}{4}h; \end{cases} \tag{4.207}$$

$$\begin{cases} 0 \leq \xi'(t) \leq \frac{C}{h}\xi(t)^{1+\varepsilon}, & 0 \leq t < \frac{7}{4}h; \\ |\xi''(t)| \leq \frac{C}{h^2}\xi(t)^{1+\varepsilon}, & 0 \leq t < \frac{7}{4}h; \end{cases} \tag{4.208}$$

$$\begin{cases} \eta(t) = \frac{1}{\xi(t)}, & 0 \leq t < \frac{7}{4}h; \\ \eta(t) \equiv 0, & \frac{7}{4}h \leq t < +\infty; \\ \eta(t) \in C^\infty[0, +\infty); \end{cases} \tag{4.209}$$

where $C = C(\varepsilon) > 0$ is a positive constant. Suppose $\varphi(x) \in C^\infty(M)$ is the function we obtained in Lemma 4.1.8, we define

$$\theta(x) = \eta(\frac{\varphi(x)}{C'}) \tag{4.210}$$

since $h \geq 4$, from (4.150) we get

$$\begin{cases} \frac{\varphi(x)}{C'} \leq \frac{5}{4}h, & \forall x \in \mathring{B}(x_0, h); \\ \frac{\varphi(x)}{C'} \geq 2h, & \forall x \in M \backslash \mathring{B}(x_0, 2C'^2h); \end{cases} \tag{4.211}$$

Then it's easy to see $\theta$ is what we need. $\square$

The following results are showed by the similar arguments as above.

**Lemma 4.1.12.** *For the function $\theta$ which we obtained in Lemma 4.1.11, then there exists a constant $C = C(n, K_0, \varepsilon, T) > 0$ such that*

$$\begin{cases} \|\nabla^t\left(\frac{1}{\theta(x)}\right)\| \leq \frac{C}{h}\xi(t)^{1+\varepsilon}, & \forall (x,t) \in \Omega \times [0,T]; \\ |\nabla_i^t\nabla_j^t\left(\frac{1}{\theta(x)}\right| \leq \frac{C}{h}\xi(t)^{1+\varepsilon}, & \forall (x,t) \in \Omega \times [0,T]; \end{cases} \tag{4.212}$$

**Lemma 4.1.13.** *Under Assumption A, suppose there exists constant $\varepsilon, C > 0$ and $\varphi(x,t) \in C^\infty(M \times [0,T])$ such that*

$$\begin{cases} \frac{\partial\varphi}{\partial t} = \Delta\varphi + Q(\varphi, x, t), & \text{on } M \times [0,T]; \\ \varphi(x,0) \leq C, & \text{on } M; \\ Q(\varphi, x, t) \leq -C\varphi^{1+\varepsilon}, & \text{if } \varphi \geq C; \end{cases} \tag{4.213}$$

*Then we have*

$$\varphi(x,t) \leq C, \quad \text{on } M \times [0,T]. \tag{4.214}$$

**Lemma 4.1.14.** *Under Assumption A, suppose $\varepsilon, C > 0$ are constants and $\varphi(x,t) \in C^\infty(M \times [0,T])$ such that*

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \Delta\varphi + Q(\varphi, x, t), & \textit{on } M \times [0,T]; \\ \varphi(x,0) \leq C, & \textit{on } M; \\ Q(\varphi, x, t) \leq \frac{C}{\varphi}\|\nabla\varphi\|^2 - C\varphi^{1+\varepsilon}, & \textit{if } \varphi \geq C; \end{cases} \tag{4.215}$$

*Then we have*

$$\varphi(x,t) \leq C, \quad \textit{on } M \times [0,T]. \tag{4.216}$$

**Lemma 4.1.15.** *Under Assumption A, suppose $\varepsilon, C > 0$ are constants and $\varphi(x,t) \in C^\infty(M \times [0,T])$ such that*

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \Delta\varphi + Q(\varphi, x, t), & \textit{on } M \times [0,T]; \\ \varphi(x,0) \leq C, & \textit{on } M; \\ Q(\varphi, x, t) \leq \frac{C}{\varphi}\|\nabla\varphi\|^2 + \langle \psi, \nabla\varphi\rangle - C\|\psi\|^2\varphi - C\varphi^{1+\varepsilon}, & \textit{if } \varphi \geq C; \end{cases} \tag{4.217}$$

*where $\psi$ is a tensor. Then we have*

$$\varphi(x,t) \leq C, \quad \textit{on } M \times [0,T]. \tag{4.218}$$

*Proof.* The proof follows from Lemma 4.1.14 and the inequality

$$\langle \psi, \nabla\varphi\rangle - C\|\psi\|^2\varphi \leq \frac{\|\nabla\varphi\|^2}{4C\varphi}. \tag{4.219}$$

□

**Theorem 4.1.6.** *Under Assumption A, suppose $\varepsilon, C > 0$ are constants and $\varphi(x,t) \in C^\infty(M \times [0,T])$ such that*

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \Delta\varphi + Q(\varphi, x, t), & \textit{on } M \times [0,T]; \\ \varphi(x,0) \leq 0; & \textit{on } M; \\ Q(\varphi, x, t) \leq C\|\nabla\varphi\|^2 + C\varphi, & \textit{if } 0 \leq \varphi \leq C; \\ Q(\varphi, x, t) \leq \frac{C}{\varphi}\|\nabla\varphi\|^2 + \langle \psi, \nabla\varphi\rangle - C\|\psi\|^2\varphi - C\varphi^{1+\varepsilon}, & \textit{if } \varphi \geq C; \end{cases} \tag{4.220}$$

*where $\psi$ is a tensor. Then we have*

$$\varphi(x,t) \leq C, \quad \textit{on } M \times [0,T]. \tag{4.221}$$

*Proof.* From Lemma 4.1.15, we know that

$$\varphi(x,t) \leq C, \quad \text{on } M \times [0,T]. \tag{4.222}$$

Using Theorem 4.15 we thus complete the proof. □

### 4.1.4 Preserving the Kählerity of the Metrics

Suppose $g_{ij}(x,t) > 0$ is the smooth solution of the evolution

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2\,\mathrm{Ric}_{ij}(x,t), \quad \text{on } M \times [0,T]. \tag{4.223}$$

In this section we will show that if $g_{ij}(x,0)$ is a Kähler metric on $M$, then $g_{ij}(x,t)$ are also Kähler for any $t \in [0,T]$. This tells us that Kähler-Ricci flow is in fact just Ricci flow on Kähler manifolds.

To prove the statement, we need to use the maximum principles established in the previous section.

**Theorem 4.1.7.** *Under Assumption A of Section 4.1.3, if $M$ is a complex manifold and $\tilde{g}$ is a Kähler metric on $M$, then $g(t)$ are also Kähler for any $t \in [0,T]$.*

*Proof.* Since $M$ is complex, we suppose $M$ has complex dimension $n$. Suppose $\{z^1, \cdots, z^n\}$ is a local holomorphic coordinate on $M$, and

$$\begin{cases} z^k = x^k + \sqrt{-1}y^k; \\ x^k, y^k \in \mathbb{R}; \end{cases} \tag{4.224}$$

From now on, we use $i, j, k, j, \cdots$ to denote the indices corresponding to real vectors or real covectors, use $\alpha, \beta, \gamma, \delta, \cdots$ to denote the indices corresponding to holomorphic vectors or holomorphic covectors, and use $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \cdots$ to denote the indices corresponding to anti-holomorphic vectors or anti-holomorphic covectors. Finally we use $A, B, C, D, \cdots$ to denote both $\alpha, \beta, \gamma, \delta, \cdots$ and $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \cdots$.

By the definition of Kähler metric, we know that $ds_t^2$ is Kähler if and only if

$$\begin{cases} g_{\alpha\beta}(z,t) \equiv 0, g_{\bar{\alpha}\bar{\beta}}(z,t) \equiv 0; \\ \frac{\partial g_{\alpha\bar{\beta}}(z,t)}{\partial z^\gamma} \equiv \frac{\partial g_{\gamma\bar{\beta}}(z,t)}{\partial z^\alpha}; \end{cases} \tag{4.225}$$

From (4.223) we get

$$\frac{\partial g_{AB}}{\partial t}(z,t) = -2\,\mathrm{Ric}_{AB}(z,t). \tag{4.226}$$

**Lemma 4.1.16.** *Suppose $g_{AB}(z,t)$ satisfy (4.226) on $M \times [0,T]$, then we have*

$$\begin{aligned} \frac{\partial}{\partial t} R_{ABCD} &= \Delta R_{ABCD} + 2(B_{ABCD} - B_{ABDC} - B_{ADBC} + B_{ACBD}) \\ &\quad - g^{EF}(R_{EBCD}\,\mathrm{Ric}_{FA} + R_{AECD}\,\mathrm{Ric}_{FA} + R_{ABED}\,\mathrm{Ric}_{FC} + R_{ABCE}\,\mathrm{Ric}_{FD}). \\ \frac{\partial}{\partial t}\mathrm{Ric}_{AB} &= \Delta\,\mathrm{Ric}_{AB} + 2g^{CD}g^{EF}R_{CAEB}\,\mathrm{Ric}_{DF} - 2g^{CD}\,\mathrm{Ric}_{AC}\,\mathrm{Ric}_{BD}. \\ \frac{\partial}{\partial t} R &= \Delta R + 2g^{AB}g^{CD}\,\mathrm{Ric}_{AC}\,\mathrm{Ric}_{BD}. \end{aligned} \tag{4.227}$$

*where $B_{ABCD} = g^{EF}g^{GH}R_{EAGB}R_{FCHD}$.*

*Proof.* This follows from Theorem 3.2.1and Lemma 3.2.1. □

To avoid the complicated computation on the change of the metrics $g_{AB}(z,t)$ among the proof of Theorem 4.1.7, we use the abstract tangent vector bundle method which was originally derived by R.S.Hamilton in [11]. We pick an abstract vector bundle $V$ which is isomorphic to the complex tangent bundle $T_{\mathbb{C}}M$, but with a fixed metric $\tilde{g}_{AB}$ on the fibres of $V$. We choose an ismoetry $u = \{u_B^A\}$ between $V$ and $T_{\mathbb{C}}M$ at time $t = 0$, and we let the isometry $u$ evolve by the equation

$$\frac{\partial}{\partial t} u_B^A = g^{AC} \operatorname{Ric}_{CD} u_B^D, \quad 0 \leq t \leq T. \tag{4.228}$$

Then the pull-back metric

$$\tilde{g}_{AB}(z,t) = g_{CD}(z,t) u_A^C(z,t) u_B^D(z,t). \tag{4.229}$$

remain constant in time, it's easy to see that

$$\frac{\partial}{\partial t} \tilde{g}_{AB}(z,t) = 0, \quad 0 \leq t \leq T. \tag{4.230}$$

and $u$ remains an isometry. We use $u$ to pull-back the curvature tensor on $T_{\mathbb{C}}M$ back to $V$:

$$\tilde{R}_{ABCD}(z,t) = R_{EFGH} u_A^E u_B^F u_C^G u_D^H, \quad 0 \leq t \leq T. \tag{4.231}$$

We can also pull back the Levi-Civita connection $\Gamma = \{\Gamma_{AB}^C\}$ on $T_{\mathbb{C}}M$ to get a connection $\tilde{\Gamma} = \{\tilde{\Gamma}_{AB}^C\}$ on $V$.

In particular, we have

$$\nabla_A u_C^B \equiv 0, \nabla_A \tilde{g}_{BC} \equiv 0, \quad 0 \leq t \leq T. \tag{4.232}$$

We can also define the Laplacian operator be the trace of the second order covariant derivatives. Similar to (4.227), it's easy to show that

$$\frac{\partial}{\partial t}\tilde{R}_{ABCD} = \Delta\tilde{R}_{ABCD} - 2\tilde{g}^{EF}\tilde{g}^{GH}\tilde{R}_{EABG}\tilde{R}_{FHCD} - 2\tilde{g}^{EF}\tilde{g}^{GH}\tilde{R}_{EAGD}\tilde{R}_{FBHC} + 2\tilde{g}^{EF}\tilde{g}^{GH}\tilde{R}_{EAGC}\tilde{R}_{FBHD}. \tag{4.233}$$

For the details of this technique, one can see Hamilton [11].

Since

$$\|\tilde{R}_{ABCD}\| = \|R_{ABCD}\|, \quad \text{on } M \times [0,T]. \tag{4.234}$$

Now we define a new function $\varphi$ on $M \times [0,T]$:

$$\varphi(z,t) = \tilde{g}^{\alpha\bar{\xi}}\tilde{g}^{\beta\bar{\zeta}}\tilde{g}^{\gamma\bar{\sigma}}\tilde{g}^{\delta\bar{\eta}}\tilde{R}_{\bar{\xi}\bar{\zeta}\bar{\sigma}\bar{\eta}} + \tilde{g}^{\bar{\alpha}\xi}\tilde{g}^{\bar{\beta}\zeta}\tilde{g}^{\gamma\bar{\sigma}}\tilde{g}^{\delta\bar{\eta}}\tilde{R}_{\bar{\alpha}\bar{\beta}\gamma\delta}\tilde{R}_{\xi\zeta\bar{\sigma}\bar{\eta}} + \tilde{g}^{\bar{\alpha}\xi}\tilde{g}^{\beta\bar{\zeta}}\tilde{g}^{\gamma\bar{\sigma}}\tilde{g}^{\delta\bar{\eta}}\tilde{R}_{\bar{\alpha}\beta\gamma\delta}\tilde{R}_{\xi\bar{\zeta}\bar{\sigma}\bar{\eta}} + \tilde{g}^{\alpha\bar{\xi}}\tilde{g}^{\bar{\beta}\zeta}\tilde{g}^{\gamma\bar{\sigma}}\tilde{g}^{\delta\bar{\eta}}\tilde{R}_{\alpha\bar{\beta}\gamma\delta}\tilde{R}_{\bar{\xi}\zeta\bar{\sigma}\bar{\eta}}. \tag{4.235}$$

It's easy to see that $\varphi$ is a well-defined smooth function on $M \times [0,T]$ and is independent of the coordinate $\{z^\alpha\}$ on $M$.

By the hypothesis of Theorem 4.1.7, we know that

$$\tilde{g}_{\alpha\beta}(z,0) = 0, \tilde{g}_{\bar{\alpha}\bar{\beta}}(z,0) = 0, \quad \forall z \in M. \tag{4.236}$$

From (4.230) we know that

$$\tilde{g}_{\alpha\beta}(z,t) \equiv 0, \tilde{g}_{\bar{\alpha}\bar{\beta}}(z,t) \equiv 0, \quad \forall z \in M. \tag{4.237}$$

For any $z \in M$, there exists a local holomorphic coordinate $\{z^\alpha\}$ such that

$$\tilde{g}_{\alpha\bar{\beta}}(z,0) = \delta_{\alpha\beta}. \tag{4.238}$$

From (4.230) we know that

$$\tilde{g}_{\alpha\bar{\beta}}(z,t) = \delta_{\alpha\beta}. \tag{4.239}$$

From the Assumption A,

$$\sum_{A,B,C,D} \tilde{R}_{ABCD}\overline{\tilde{R}_{ABCD}} \leq K_0. \tag{4.240}$$

and

$$\varphi(z,t) = \sum_{A,B,\gamma,\delta} |\tilde{R}_{AB\gamma\delta}|^2 \geq 0. \tag{4.241}$$

And $\varphi(z,t) = 0$ if and only if $\tilde{R}_{AB\gamma\delta}(z,t) = 0$ for all $A, B, \gamma, \delta$. From (4.233),

$$\frac{\partial}{\partial t}\tilde{R}_{AB\gamma\delta} = \Delta\tilde{R}_{AB\gamma\delta} + \tilde{R}_{CDEF} \star \tilde{R}_{GH\alpha\beta}. \tag{4.242}$$

And

$$\frac{\partial}{\partial t}\varphi = \Delta\varphi - 2\|\nabla\tilde{R}\|^2 + 2\tilde{R}_{CDEF} \star \tilde{R}_{GH\alpha\beta} \star \tilde{R}_{AB\gamma\delta}. \tag{4.243}$$

It's easy to see that

$$2\tilde{R}_{CDEF} \star \tilde{R}_{GH\alpha\beta} \star \tilde{R}_{AB\gamma\delta} \leq C(n)|\tilde{R}_{CDEF}||\tilde{R}_{AB\gamma\delta}|^2 \leq C(n)\sqrt{K_0}\varphi. \tag{4.244}$$

Which together with (4.243) yield

$$\frac{\partial}{\partial t}\varphi \leq \Delta\varphi + C(n)\sqrt{K_0}\varphi. \tag{4.245}$$

Since $\varphi(z,0) \equiv 0$ and $0 \leq \varphi(z,t) \leq K_0$. By the maximum principle Theorem 4.1.5, we get

$$\varphi(z,t) \equiv 0. \tag{4.246}$$

Thus

$$\tilde{R}_{AB\gamma\delta} \equiv 0. \tag{4.247}$$

Similarly we have

$$\tilde{R}_{\alpha\beta AB} \equiv 0, \tilde{R}_{AB\bar{\alpha}\bar{\beta}} \equiv 0, \tilde{R}_{\bar{\alpha}\bar{\beta}AB} \equiv 0. \tag{4.248}$$

Define $\widetilde{\mathrm{Ric}}_{AB} = \tilde{g}^{CD}\tilde{R}_{CABD}$. Then

$$\frac{\partial}{\partial t}u^A_B = \tilde{g}^{EF}\widetilde{\mathrm{Ric}}_{FB}u^A_E. \tag{4.249}$$

Suppose the coordinate satisfies (4.239) at one point, we thus have

$$\frac{\partial}{\partial t}u^A_B = \widetilde{\mathrm{Ric}}_{\bar{E}B}u^A_E. \tag{4.250}$$

By the definition of $u = \{u^A_B\}$, one can choose a base of the vector bundle $V$ such that

$$u^A_B(z,0) \equiv \begin{cases} 1 & \text{if } A = B; \\ 0 & \text{if } A \neq B; \end{cases} \tag{4.251}$$

Thus

$$\frac{\partial}{\partial t} u^\alpha_{\bar\beta} = \widetilde{\mathrm{Ric}}_{\bar E\bar\beta} u^\alpha_E = \widetilde{\mathrm{Ric}}_{\gamma\bar\beta} u^\alpha_{\bar\gamma} + \widetilde{\mathrm{Ric}}_{\bar\gamma\bar\beta} u^\alpha_\gamma = \widetilde{\mathrm{Ric}}_{\gamma\bar\beta} u^\alpha_{\bar\gamma}. \tag{4.252}$$

Since

$$u^\alpha_{\bar\beta}(z,0) \equiv 0, \quad \forall \alpha,\beta. \tag{4.253}$$

Then from (4.252) and (4.253) we get

$$u^\alpha_{\bar\beta}(z,t) \equiv 0, \quad \forall \alpha,\beta. \tag{4.254}$$

Since

$$R_{ABCD}(z,t) = \tilde R_{EFGH}(z,t) v^E_A v^F_B v^G_C v^H_D. \tag{4.255}$$

where $(v^A_B) = (u^B_A)^{-1}$, we get

$$R_{AB\gamma\delta}(z,t) \equiv 0, \quad \text{on } M\times[0,T]. \tag{4.256}$$

Similarly,

$$R_{AB\bar\gamma\bar\delta} \equiv 0, R_{\gamma\delta AB} \equiv 0, R_{\bar\gamma\bar\delta AB} \equiv 0. \tag{4.257}$$

Using the same argument,

$$\mathrm{Ric}_{\alpha\beta} = \mathrm{Ric}_{\bar\alpha\bar\beta} \equiv 0. \tag{4.258}$$

From (4.226) and (4.258) we have

$$g_{\alpha\beta} = g_{\bar\alpha\bar\beta} \equiv 0. \tag{4.259}$$

By Bianchi identity,

$$\nabla_\alpha \mathrm{Ric}_{\beta\bar\gamma} = \nabla_\beta \mathrm{Ric}_{\alpha\bar\gamma}, \quad \text{on } M\times[0,T]. \tag{4.260}$$

Since

$$\begin{aligned}
\frac{\partial}{\partial t}\left[\frac{\partial g_{\alpha\bar\beta}(z,t)}{\partial z^\gamma} - \frac{\partial g_{\gamma\bar\beta}(z,t)}{\partial z^\alpha}\right] &= \frac{\partial}{\partial z^\gamma}\left(\frac{\partial}{\partial t} g_{\alpha\bar\beta}\right) - \frac{\partial}{\partial z^\alpha}\left(\frac{\partial}{\partial t} g_{\gamma\bar\beta}\right) \\
&= -2\frac{\partial \mathrm{Ric}_{\alpha\bar\beta}}{\partial z^\gamma} + 2\frac{\partial \mathrm{Ric}_{\gamma\bar\beta}}{\partial z^\alpha} \\
&= 2\nabla_\alpha \mathrm{Ric}_{\gamma\bar\beta} - 2\nabla_\gamma \mathrm{Ric}_{\alpha\bar\beta} + 2\Gamma^A_{\alpha\gamma}\mathrm{Ric}_{A\bar\beta} - 2\Gamma^A_{\gamma\alpha}\mathrm{Ric}_{A\bar\beta} + 2\Gamma^A_{\alpha\bar\beta}\mathrm{Ric}_{\gamma A} - 2\Gamma^A_{\gamma\bar\beta}\mathrm{Ric}_{\alpha A} \\
&= 2\Gamma^{\bar\delta}_{\alpha\bar\beta}\mathrm{Ric}_{\gamma\bar\delta} - 2\Gamma^{\bar\delta}_{\gamma\bar\beta}\mathrm{Ric}_{\alpha\bar\delta} \\
&= g^{\bar\delta\eta}\mathrm{Ric}_{\gamma\bar\delta}\left[\frac{\partial g_{\eta\bar\beta}}{\partial z^\alpha} - \frac{\partial g_{\alpha\bar\beta}}{\partial z^\eta}\right] - g^{\bar\delta\eta}\mathrm{Ric}_{\alpha\bar\delta}\left[\frac{\partial g_{\eta\bar\beta}}{\partial z^\gamma} - \frac{\partial g_{\gamma\bar\beta}}{\partial z^\eta}\right].
\end{aligned} \tag{4.261}$$

Since

$$\frac{\partial g_{\alpha\bar\beta}}{\partial z^\gamma}(z,0) - \frac{\partial g_{\gamma\bar\beta}}{\partial z^\alpha}(z,0) \equiv 0, \quad \text{on } M. \tag{4.262}$$

which together with (4.261) implies that

$$\frac{\partial g_{\alpha\bar\beta}}{\partial z^\gamma}(z,t) - \frac{\partial g_{\gamma\bar\beta}}{\partial z^\alpha}(z,t) \equiv 0, \quad \text{on } M\times[0,T]. \tag{4.263}$$

Thus $g_{AB}(z,t)$ is Kähler on $M$ for any $t\in[0,T]$. Therefore the proof is complete. □

## References


[1] Aubin, T. Équations du type Monge-Ampère sur les variétés kählériennes compactes. Bull. Sci. Math. (2) 102 (1978), no. 1, 63–95.

[2] Aubin.T. Nonlinear Analysis on Manifolds. Monge-Ampère Equations, Grundl. Math. Wissen. 252, Springer-Verlag, New York, 1982.

[3] Calabi, E. Improper affine hypersurfaces of convex type and a generalization of a theorem by K. Jörgens. Mich. Math. J. 5, 105–126, 1958.

[4] E. Calabi. On Kähler manifolds with vanishing canonical class, in: Algebraic Geometry and Topology. A Symposium in Honor of S. Lefschetz, Princeton University Press, Princeton, N. J., 1957, pp. 78–89.

[5] H.-D. Cao. Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds. Invent. Math. 81, 359–372 (1985).

[6] H.-D. Cao. On Harnack's inequalities for the Kähler-Ricci flow, Invent. Math. 109 (1992), no. 2, 247–263.

[7] A. Chau and L.-F. Tam, On the complex structure of Kähler manifolds with nonnegative curvature, J. Differential Geom. 73 (2006), no. 3, 491–530.

[8] W.Y. Ding, G. Tian. Kähler-Einstein metrics and the generalized Futaki invariant. Invent. Math. 110(2), 315–335 (1992)

[9] D. Gilbarg, N.S. Trudinger. Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1983.

[10] Hamilton, R.S. Three-manifolds with positive Ricci curvature. J. Differ. Geom. 17(2), 255–306 (1982).

[11] R.S. Hamilton, Four-manifolds with positive curvature operator, J. Differ. Geom. 24 (1986) 153–179.

[12] Hebey.E. Nonlinear analysis on manifolds: Sobolev Spaces and inequalities, Courant Lect. Notes Math., New York, 1990.

[13] Krylov, N. V. Lectures on elliptic and parabolic equations in Hölder spaces. Graduate Studies in Mathematics, 12. American Mathematical Society, Providence, R.I., 1996.

[14] G. Liu, On Yau's uniformization conjecture, Camb. J. Math. 7 (2019), no. 1–2, 33–70.

[15] M.-C. Lee and L.-F. Tam, Chern–Ricci flows on noncompact complex manifolds, J. Differ. Geom. 115(3) (2020) 529–564.

[16] L. Ni, L.-F. Tam, Plurisubharmonic functions and the structure of complete Khler man- ifolds with nonnegative curvature, J. Differential Geom. 64 (2003) 457–524,

[17] Phong, D.H., Sturm, J. On stability and the convergence of the Kähler–Ricci flow. J. Differ. Geometry 72(1), 149–168 (2006)

[18] Phong, D.H., Sesum, S., Sturm, J. Multiplier ideal sheaves and the Kähler–Ricci flow. Commun. Anal. Geom. 15(3), 613–632 (2007).

[19] W.-X. Shi, Ricci deformation of the metric on complete noncompact Riemannian manifolds, J. Differ. Geometry 30 (1989) 223–301.

[20] Wan-Xiong Shi, Deforming the metric on complete Riemannian manifolds, J. Differential Geom. 30 (1989), no. 1, 223–301.

[21] W. X. Shi, Ricci deformation of the metric on complete noncompact Kähler manifolds, PhD thesis, Harvard University, 1990.

[22] Song, J., Tian, G. The Kähler-Ricci flow on surfaces of positive Kodaira dimension. Invent. Math. 170(3), 609–653 (2007).

[23] G. Tian. Canonical metrics in Kähler geometry, Lectures in Mathematics ETH Zürich, Birkhäuser, 2000.

[24] G. Tian, Z. Zhang. On the Kähler-Ricci flow on projective manifolds of general type, Chin. Ann. Math., Ser. B 27 (2) (2006) 179–192.

[25] Yau, S.T. A general Schwarz lemma for Kähler manifolds. Am. J. Math. 100(1), 197–203 (1978).

[26] Yau, S.T. On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I, Comm. Pure Appl. Math. 31 (3) (1978) 339–411.

[27] Yau, S.T. Open problems in geometry, Lectures on Differential Geometry, by Schoen and Yau 1 (1994), 365–404.