Sandwich Variance of DR ATT Estimator

- 11월 4일 Version (수정 Version)

Estimating equation)

$$\psi_T(\theta) = \begin{bmatrix} v_1 - \frac{N_1}{N} \times (Y_T \cdot E_T) \\ v_0 - \frac{N_1}{N} \times \frac{\{Y_T \cdot (1-E_T) \cdot e_T + E[Y_T \mid E=0, X_{c,T}] \cdot (E_T - e_T)\}}{1-e_T} \\ X_{0,T}(y_T - X_{0,T}^T \alpha) \\ \left(E_T - \frac{e^{X_{p,T}^T\beta}}{1+e^{X_{p,T}^T\beta}}\right) \cdot X_{p,T}^T \end{bmatrix} \begin{array}{l} \text{---} \psi_{v_1} \\ \text{---} \psi_{v_0} \\ \text{---} \psi_{\alpha} \\ \text{---} \psi_{\beta} \end{array}$$

$\theta = (v_1, v_0, \underset{\sim}{\alpha}^T, \underset{\sim}{\beta}^T)^T$ / X_c = Covariates set, X_0 = Outcome model의 Design matrix

X_p = PS model의 Design matrix 라고 표기 $\Rightarrow$ $(1, 0, X_c^T) = X\_0$ 라 표기

$$\frac{\partial}{\partial\theta'}\psi_T(\hat\theta) = \begin{bmatrix} \frac{\partial}{\partial v_1}\psi_{v_1}, & \frac{\partial}{\partial v_0}\psi_{v_1}, & \frac{\partial}{\partial \alpha}\psi_{v_1}, & \frac{\partial}{\partial \beta}\psi_{v_1} \\ \frac{\partial}{\partial v_1}\psi_{v_0}, & \frac{\partial}{\partial v_0}\psi_{v_0}, & \frac{\partial}{\partial \alpha}\psi_{v_0}, & \frac{\partial}{\partial \beta}\psi_{v_0} \\ \frac{\partial}{\partial v_1}\psi_{\alpha}, & \frac{\partial}{\partial v_0}\psi_{\alpha}, & \frac{\partial}{\partial \alpha}\psi_{\alpha}, & \frac{\partial}{\partial \beta}\psi_{\alpha} \\ \frac{\partial}{\partial v_1}\psi_{\beta}, & \frac{\partial}{\partial v_0}\psi_{\beta}, & \frac{\partial}{\partial \alpha}\psi_{\beta}, & \frac{\partial}{\partial \beta}\psi_{\beta} \end{bmatrix} \begin{array}{l} \text{--(1)} \\ \text{--(2)} \\ \text{--(3)} \\ \text{--(4)} \end{array}$$

1) $\psi_{v_1} = v_1 - \frac{N_1}{N} \times Y_T \times E_T$

Outcome model의 Parameter 개수

; $\frac{\partial}{\partial v_1}\psi_{v_1} = 1$ / $\frac{\partial}{\partial v_0}\psi_{v_1} = 0$ / $\frac{\partial}{\partial \underset{\sim}{\alpha}^T}\psi_{v_1} = \underbrace{\underset{\sim}{0}^T}_{q}$ / $\frac{\partial}{\partial \underset{\sim}{\beta}^T}\psi_{v_1} = \underbrace{\underset{\sim}{0}^T}_{e}$ Propensity score model의 parameter 개수

(1) = $(0, 0, \underset{\sim}{0}^T, \underset{\sim}{0}^T)$

$$2)\ \psi_{v_0} = v_0 - \frac{N_1}{N} \times \frac{\left\{ Y_T \cdot (1-E_T) \cdot \left(\frac{e^{X_{p,T}^T \beta}}{1+e^{X_{p,T}^T\beta}} \right) + X\_0^T \cdot \alpha \cdot \left(E_T - \frac{e^{X_{p,T}^T\beta}}{1+e^{X_{p,T}^T\beta}} \right) \right\}}{\frac{1}{1+e^{X_{p,T}^T\beta}}}$$

$$= v_0 - \frac{N_1}{N} \times (1+e^{X_{p,T}^T\beta}) \left\{ Y_T(1-E_T) \cdot \frac{e^{\beta^T X_{p,T}}}{1+e^{\beta^T X_{p,T}}} + X\_0^T \cdot \alpha \left(E_T - \frac{e^{\beta^T X_{p,T}}}{1+e^{\beta^T X_{p,T}}} \right) \right\}$$

$$= v_0 - \frac{N_1}{N} \left\{ Y_T (1-E_T) \cdot e^{\beta^T X_{p,T}} + (1+e^{\beta^T X_{p,T}}) X\_0^T \alpha \cdot E_T - (X\_0^T \cdot \alpha) \cdot e^{\beta^T X_{p,T}} \right\}$$

$$= v_0 - \frac{N_1}{N} \times Y_T (1-E_T) \cdot e^{\beta^T X_{p,T}} - \frac{N_1}{N} \times (1+e^{\beta^T X_{p,T}}) X\_0^T \cdot \alpha \cdot E_T + \frac{N_1}{N} (X\_0^T \cdot \alpha) \cdot e^{\beta^T X_{p,T}}$$

$$\therefore \frac{\partial}{\partial v_1} \psi_{v_0} = 0,\quad \frac{\partial}{\partial v_0} \psi_{v_0} = 1,\quad \frac{\partial}{\partial \alpha^T} \psi_{v_0} = -\frac{N_1}{N} \times (1+e^{\beta^T X_{p,T}}) \cdot E_T \cdot (X\_0)^T$$

$$+ \frac{N_1}{N} \times e^{\beta^T X_{p,T}} \times (X\_0)^T = -\frac{N_1}{N} \left\{ (1+e^{\beta^T X_{p,T}}) \cdot E_T - e^{\beta^T X_{p,T}} \right\} (X\_0)^T$$

$$\frac{\partial}{\partial \beta^T} \psi_{v_0} = -Y_T \cdot (1-E_T) \cdot e^{X_{p,T}^T \beta} \cdot (X_{p,T})^T - e^{X_{p,T}^T\beta} \cdot (X\_0^T \cdot \alpha) \cdot E_T \cdot X_{p,T}^T + e^{X_{p,T}^T\beta} \cdot (X\_0^T \cdot \alpha) \cdot X_{p,T}^T$$

$$= e^{X_{p,T}^T \cdot \beta} \left\{ -Y_T (1-E_T) - (X\_0^T \cdot \alpha) \cdot E_T + (X\_0^T \cdot \alpha) \right\} \cdot X_{p,T}^T$$

$$= e^{X_{p,T}^T\beta} \left\{ (1-E_T)(-Y_T + (X\_0^T \cdot \alpha)) \right\} X_{p,T}^T$$

$$(2) = (0,\ 0,\ ▨\,(1, 0, X_c^T),\ △\,(1, X_c^T))$$

$$3)\ \psi_\alpha = X_{0,T} (y_T - X_{0,T}^T \cdot \alpha)$$

$$\therefore \frac{\partial}{\partial v_1} \psi_\alpha = \underbrace{\underset{\sim}{0}}_{\alpha\text{의 개수}},\quad \frac{\partial}{\partial v_0} \psi_\alpha = \overbrace{\underset{\sim}{0}}^{\alpha\text{의 개수}},\quad \frac{\partial}{\partial \underset{\sim}{\alpha}^T} \psi_\alpha = -X\_0 \cdot X\_0^T = -\begin{bmatrix} 1 \\ 0 \\ X_c \end{bmatrix} \begin{bmatrix} 1 & 0 & X_c^T \end{bmatrix}$$

$$\frac{\partial}{\partial \underset{\sim}{\beta}^T} \psi_\alpha = \underbrace{\square}_{(\alpha\text{ 개수} \times \beta\text{ 개수})} \text{ Zero matrix}$$

$$(3) = \left[\underset{\sim}{0},\ \underset{\sim}{0},\ -\begin{bmatrix} 1 \\ 0 \\ X_c \end{bmatrix} \begin{bmatrix} 1 & 0 & X_c^T \end{bmatrix},\ \boxed{0} \right]$$

4) $\psi_\beta = \left(E_T - \dfrac{e^{X_{p,T}^T\beta}}{1+e^{X_{p,T}^T\beta}} \right) \cdot X_{p,T}$

$\cdot\ \dfrac{\partial}{\partial v_1}\psi_\beta = \underset{\sim}{0}, \quad \dfrac{\partial}{\partial v_0}\psi_\beta = \underset{\sim}{0}, \quad \dfrac{\partial}{\partial \underset{\sim}{\alpha}^T}\psi_\beta = \boxed{0}$ ($\underset{\sim}{\beta}$ 개수 × $\underset{\sim}{\alpha}$ 개수)

PS model parameter 의 개수

$$\frac{\partial}{\partial \underset{\sim}{\beta}^T}\psi_\beta = -\frac{e^{X_{p,T}^T\beta}\cdot(1+e^{X_{p,T}^T\beta})\cdot X_{p,T} - e^{X_{p,T}^T\beta}\cdot e^{X_{p,T}^T\beta}\cdot X_{p,T}}{(1+e^{X_{p,T}^T\beta})^2}\cdot X_{p,T}^T = \frac{-e^{X_{p,T}^T\beta}}{(1+e^{X_{p,T}^T\beta})^2}\cdot X_{p,T}\cdot X_{p,T}^T$$

$$= -\frac{e^{X_{p,T}^T\beta}}{(1+e^{X_{p,T}^T\cdot\beta})^2}\cdot \begin{bmatrix} 1 \\ X_c \end{bmatrix} \begin{bmatrix} 1 & X_c^T \end{bmatrix}$$

$$(4) = \left[\underset{\sim}{0},\ \underset{\sim}{0},\ \boxed{0},\ -\frac{e^{X_{p,T}^T\beta}}{(1+e^{X_{p,T}^T\cdot\beta})^2}\cdot \begin{bmatrix} 1 \\ X_c \end{bmatrix} \begin{bmatrix} 1 & X_c^T \end{bmatrix} \right]$$