Sandwich Variance of DR ATT Estimator

- 11월 16일 Version (수정 Version)

Estimating equation)

$$\psi_i(\theta) = \begin{bmatrix} v_1 - \frac{N}{N_1} \times (Y_i \cdot E_i) \\ v_0 - \frac{N}{N_1} \times \frac{\{Y_i \cdot (1-E_i) \cdot e_i + E[Y_i \mid E=0, X_{c,i}] \cdot (E_i - e_i)\}}{1-e_i} \\ X_{0,i}(y_i - X_{0,i}^T \alpha) \\ \left(E_i - \frac{e^{X_{p,i}^T\beta}}{1+e^{X_{p,i}^T\beta}}\right) \cdot X_{p,i}^T \end{bmatrix} \begin{matrix} \text{--- } \psi_{v_1} \\ \text{--- } \psi_{v_0} \\ \text{--- } \psi_{\alpha} \\ \text{--- } \psi_{\beta} \end{matrix}$$

$\theta = (v_1, v_0, \underset{\sim}{\alpha}^T, \underset{\sim}{\beta}^T)^T$ / X_c = Covariates set, X_0 = Outcome model 의 Design matrix

X_p = PS model 의 Design matrix 라고 표기 $\Rightarrow (1, 0, X_c^T) = X\_0$ 라 표기

$$\frac{\partial}{\partial \theta'} \psi_i(\hat{\theta}) = \begin{bmatrix} \frac{\partial}{\partial v_1}\psi_{v_1}, & \frac{\partial}{\partial v_0}\psi_{v_1}, & \frac{\partial}{\partial \alpha}\psi_{v_1}, & \frac{\partial}{\partial \beta}\psi_{v_1} \\ \frac{\partial}{\partial v_1}\psi_{v_0}, & \frac{\partial}{\partial v_0}\psi_{v_0}, & \frac{\partial}{\partial \alpha}\psi_{v_0}, & \frac{\partial}{\partial \beta}\psi_{v_0} \\ \frac{\partial}{\partial v_1}\psi_{\alpha}, & \frac{\partial}{\partial v_0}\psi_{\alpha}, & \frac{\partial}{\partial \alpha}\psi_{\alpha}, & \frac{\partial}{\partial \beta}\psi_{\alpha} \\ \frac{\partial}{\partial v_1}\psi_{\beta}, & \frac{\partial}{\partial v_0}\psi_{\beta}, & \frac{\partial}{\partial \alpha}\psi_{\beta}, & \frac{\partial}{\partial \beta}\psi_{\beta} \end{bmatrix} \begin{matrix} \text{--(1)} \\ \text{--(2)} \\ \text{--(3)} \\ \text{--(4)} \end{matrix}$$

1) $\psi_{v_1} = v_1 - \frac{N}{N_1} \times Y_i \times E_i$

Outcome model 의 Parameter 개수

$\therefore \frac{\partial}{\partial v_1}\psi_{v_1} = 1$ / $\frac{\partial}{\partial v_0}\psi_{v_1} = 0$ / $\frac{\partial}{\partial \underset{\sim}{\alpha}^T}\psi_{v_1} = \underbrace{\underset{\sim}{0}^T}_{q}$ / $\frac{\partial}{\partial \underset{\sim}{\beta}^T}\psi_{v_1} = \underbrace{\underset{\sim}{0}^T}_{e}$ Propensity score model 의 parameter 개수

(1) $= (0, 0, \underset{\sim}{0}^T, \underset{\sim}{0}^T)$

2) $\psi_{v_0} = v_0 - \frac{N}{N_1} \times \dfrac{\left\{ Y_T \cdot (1-E_T) \cdot \left(\frac{e^{X_{p,T}^T \beta}}{1+e^{X_{p,T}^T \beta}} \right) + X\_0^T \cdot \alpha \cdot \left(E_T - \frac{e^{X_{p,T}^T \beta}}{1+e^{X_{p,T}^T \beta}} \right) \right\}}{\frac{1}{1+e^{X_{p,T}^T \beta}}}$

$$= v_0 - \frac{N}{N_1} \times \left(1+e^{X_{p,T}^T \beta}\right) \left\{ Y_T (1-E_T) \cdot \frac{e^{\beta^T X_{p,T}}}{1+e^{\beta^T X_{p,T}}} + X\_0^T \cdot \alpha \left(E_T - \frac{e^{\beta^T X_{p,T}}}{1+e^{\beta^T X_{p,T}}} \right) \right\}$$

$$= v_0 - \frac{N}{N_1} \left\{ Y_T (1-E_T) \cdot e^{\beta^T X_{p,T}} + \left(1+e^{\beta^T X_{p,T}}\right) X\_0^T \alpha \cdot E_T - (X\_0^T \cdot \alpha) \cdot e^{\beta^T X_{p,T}} \right\}$$

$$= v_0 - \frac{N}{N_1} \times Y_T (1-E_T) \cdot e^{\beta^T X_{p,T}} - \frac{N}{N_1} \times \left(1+e^{\beta^T X_{p,T}}\right) X\_0^T \cdot \alpha \cdot E_T + \frac{N}{N_1} (X\_0^T \cdot \alpha) \cdot e^{\beta^T X_{p,T}}$$

- $\frac{\partial}{\partial v_1} \psi_{v_0} = 0$, $\frac{\partial}{\partial v_0} \psi_{v_0} = 1$, $\frac{\partial}{\partial \underset{\sim}{\alpha}^T} \psi_{v_0} = -\frac{N}{N_1} \times \left(1+e^{\beta^T X_{p,T}}\right) \cdot E_T \cdot (X\_0)^T$

$$+ \frac{N}{N_1} \times e^{\beta^T X_{p,T}} \times (X\_0)^T = -\frac{N}{N_1} \left\{ \left(1+e^{\beta^T X_{p,T}}\right) \cdot E_T - e^{\beta^T X_{p,T}} \right\} (X\_0)^T$$

$$\frac{\partial}{\partial \underset{\sim}{\beta}^T} \psi_{v_0} = -\frac{N_1}{N} \times Y_T (1-E_T) \cdot e^{X_{p,T}^T \beta} \cdot (X_{p,T})^T - \frac{N}{N_1} \cdot (X\_0)^T \alpha \cdot e^{X_{p,T}^T \beta} \cdot (X_{p,T})^T \cdot E_T$$

$$+ \frac{N_1}{N} \cdot (X\_0)^T \cdot \alpha \cdot e^{X_{p,T}^T \beta} \cdot (X_{p,T})^T$$

$$= -\frac{N_1}{N} \cdot e^{X_{p,T}^T \beta} \left\{ Y_T (1-E_T) + (X\_0^T \cdot \alpha) \cdot E_T - (X\_0^T \cdot \alpha) \right\} (X_{p,T})^T$$

$$= -\frac{N_1}{N} \cdot e^{X_{p,T}^T \beta} \left\{ (1-E_T)\left(Y_T - (X\_0^T \cdot \alpha)\right) \right\} (X_{p,T})^T$$

(2) = (0, 0, ▨ $(1, 0, X_c^T)$, ▲ $(1, X_c^T)$)

3) $\psi_\alpha = X_{0,T} (Y_T - X_{0,T}^T \cdot \alpha)$

- $\frac{\partial}{\partial v_1} \psi_\alpha = \underset{\sim}{0}$ (α의 개수), $\frac{\partial}{\partial v_0} \psi_\alpha = \underset{\sim}{0}$ (α의 개수), $\frac{\partial}{\partial \underset{\sim}{\alpha}^T} \psi_\alpha = -X\_0 \cdot X\_0^T = -\begin{bmatrix} 1 \\ 0 \\ X_c \end{bmatrix} \begin{bmatrix} 1 & 0 & X_c^T \end{bmatrix}$

$\frac{\partial}{\partial \underset{\sim}{\beta}^T} \psi_\alpha = \boxed{}$ → Zero matrix (α 개수 × β 개수)

/ (3) = $\left[\underset{\sim}{0}, \underset{\sim}{0}, -\begin{bmatrix} 1 \\ 0 \\ X_c \end{bmatrix} \begin{bmatrix} 1 & 0 & X_c^T \end{bmatrix}, \boxed{0} \right]$

4) $\psi_\beta = \left(E_T - \dfrac{e^{X_{p,T}^T\beta}}{1+e^{X_{p,T}^T\beta}}\right)\cdot X_{p,T}$

- $\dfrac{\partial}{\partial v_1}\psi_\beta = \underbrace{\underset{\sim}{0}}_{\text{PS model parameter 의 개수}}$, $\dfrac{\partial}{\partial v_0}\psi_\beta = \underset{\sim}{0}$, $\dfrac{\partial}{\partial \underset{\sim}{\alpha}^T}\psi_\beta = \boxed{0}$ ($\underset{\sim}{\beta}$ 개수 × $\underset{\sim}{\alpha}$ 개수)

$$\frac{\partial}{\partial \underset{\sim}{\beta}^T}\psi_\beta = -\frac{e^{X_{p,T}^T\beta}\cdot(1+e^{X_{p,T}^T\beta})\cdot X_{p,T} - e^{X_{p,T}^T\beta}\cdot e^{X_{p,T}^T\beta}\cdot X_{p,T}}{(1+e^{X_{p,T}^T\beta})^2}\cdot X_{p,T}^T = \frac{-e^{X_{p,T}^T\beta}}{(1+e^{X_{p,T}^T\beta})^2}\cdot X_{p,T}\cdot X_{p,T}^T$$

$$= -\frac{e^{X_{p,T}^T\beta}}{(1+e^{X_{p,T}^T\cdot\beta})^2}\cdot \begin{bmatrix} 1 \\ X_c \end{bmatrix}\begin{bmatrix} 1 & X_c^T \end{bmatrix}$$

$$(4) = \left[\underset{\sim}{0},\ \underset{\sim}{0},\ \boxed{0},\ -\frac{e^{X_{p,T}^T\beta}}{(1+e^{X_{p,T}^T\cdot\beta})^2}\cdot \begin{bmatrix} 1 \\ X_c \end{bmatrix}\begin{bmatrix} 1 & X_c^T \end{bmatrix}\right]$$