Sandwich Variance Estimator of DR ATE Estimator

: Re-re update Version (9월 21일 Version)
: 미분 잘못 작성한 부분 수정

$$
\psi_T(\theta) = \begin{bmatrix}
\nu_1 - \dfrac{\{E_T Y_T - (E_T - e_T)\, E[Y_T \mid E=1, \overset{\text{Covariance term}}{X_T}]\}}{e_T} \\[2ex]
\nu_0 - \dfrac{\{(1-E_T) Y_T + (E_T - e_T)\, E[Y_T \mid E=0, X_T]\}}{1-e_T} \\[2ex]
X_T (Y_T - \alpha^T \underbrace{X_T}_{\text{Outcome regression model의 model matrix}}) \\[2ex]
E_T \cdot \underbrace{X_T}_{\text{PS model의 model matrix}} - \dfrac{e^{X_T^T \beta}}{1 + e^{X_T^T \beta}} \cdot X_T
\end{bmatrix}
\begin{array}{l}
\text{--- } (\psi_{\nu_1}) \\[2ex]
\text{--- } (\psi_{\nu_0}) \\[2ex]
\text{--- } (\psi_{\alpha}) \\[2ex]
\text{--- } (\psi_{\beta})
\end{array}
, \quad \underset{\sim}{\theta} = (\nu_1, \nu_0, \underset{\sim}{\alpha}, \underset{\sim}{\beta})
$$

$$
\frac{\partial}{\partial \theta'} \psi_T(\hat{\theta}) = \begin{bmatrix}
\frac{\partial}{\partial \nu_1} \psi_{\nu_1}, & \frac{\partial}{\partial \nu_0} \psi_{\nu_1}, & \frac{\partial}{\partial \alpha} \psi_{\nu_1}, & \frac{\partial}{\partial \beta} \psi_{\nu_1} \\[1ex]
\frac{\partial}{\partial \nu_1} \psi_{\nu_0}, & \frac{\partial}{\partial \nu_0} \psi_{\nu_0}, & \frac{\partial}{\partial \alpha} \psi_{\nu_0}, & \frac{\partial}{\partial \beta} \psi_{\nu_0} \\[1ex]
\frac{\partial}{\partial \nu_1} \psi_{\alpha}, & \frac{\partial}{\partial \nu_0} \psi_{\alpha}, & \frac{\partial}{\partial \alpha} \psi_{\alpha}, & \frac{\partial}{\partial \beta} \psi_{\alpha} \\[1ex]
\frac{\partial}{\partial \nu_1} \psi_{\beta}, & \frac{\partial}{\partial \nu_0} \psi_{\beta}, & \frac{\partial}{\partial \alpha} \psi_{\beta}, & \frac{\partial}{\partial \beta} \psi_{\beta}
\end{bmatrix}
\begin{array}{l}
\text{--- } (1) \\[1ex]
\text{--- } (2) \\[1ex]
\text{--- } (3) \\[1ex]
\text{--- } (4)
\end{array}
$$

1) $$\psi_{\nu_1} = \nu_1 - \frac{(1+e^{\beta^T X_T})\{E_T Y_T - (E_T - \frac{e^{\beta^T X_T}}{1+e^{\beta^T X_T}}) \cdot (\overset{\text{절편}}{1}, \overset{\text{trt}}{1}, X_T^T)\alpha\}}{e^{\beta^T X_T}}$$

$$= \nu_1 - \frac{(1+e^{\beta^T X_T})(E_T Y_T) - E_T(1+e^{\beta^T X_T}) \cdot (1,1,X_T^T)\alpha + e^{\beta^T X_T} \cdot (1,1,X_T^T)\alpha\}}{e^{\beta^T X_T}}$$

$$= \nu_1 - \frac{(1+e^{\beta^T X_T}) E_T Y_T}{e^{\beta^T X_T}} + \frac{E_T(1+e^{\beta^T X_T})}{e^{\beta^T X_T}} \cdot (1,1,X_T^T)\alpha - (1,1,X_T^T)\alpha$$

$\therefore \frac{\partial}{\partial \nu_1} \psi_{\nu_1} = 1$ / $\frac{\partial}{\partial \nu_0} \psi_{\nu_1} = 0$

; $\frac{\partial}{\partial \alpha}\psi_{v_1} = \frac{E_T(1+e^{\beta_T^T X_T})}{e^{\beta_T^T X_T}} \cdot (1,1,X_T^T) - (1,1,X_T^T) = \Big(\underbrace{\frac{E_T(1+e^{\beta_T^T X_T})}{e^{\beta_T^T X_T}} - 1}_{\text{Scalar}}\Big)(1,1,X_T^T)$

Our situation ; (B_T, C_T)

; ψ_{v_1} 에서 β와 관련된 부분 $\Rightarrow$ $\frac{-(1+e^{X_T^T\beta})\cdot E_T\cdot Y_T + E_T(1+e^{X_T^T\beta})\cdot(1,1,X_T^T)\alpha}{e^{X_T^T\beta}} = -E_T\cdot\frac{(1+e^{X_T^T\beta})}{e^{X_T^T\beta}}\cdot(Y_T-(1,1,X_T^T)\alpha)$

$\frac{\partial}{\partial\beta}\psi_{v_1} = -E_T(Y_T-(1,1,X_T^T)\alpha)\cdot\frac{e^{X_T^T\beta}\cdot X_T\cdot e^{X_T^T\beta}-(1+e^{X_T^T\beta})\cdot e^{X_T^T\beta}\cdot X_T}{(e^{X_T^T\beta})^2} = E_T(Y_T-(1,1,X_T^T)\alpha)\cdot\frac{1}{e^{X_T^T\beta}}\cdot X_T$

$= v_0 - (1-E_T)Y_T - e^{X_T^T\beta}(1-E_T)Y_T - E_T\cdot(1,0,X_T^T)\alpha - e^{X_T^T\beta}\cdot E_T\cdot(1,0,X_T^T)\alpha + e^{X_T^T\beta}\cdot(1,0,X_T^T)\alpha$

2) $\psi_{v_0} = v_0 - (1+e^{\beta_T^T X_T})(1-E_T)Y_T - (1+e^{\beta_T^T X_T})\cdot E_T\cdot(1,0,X_T^T)\alpha + e^{\beta_T^T X_T}\cdot(1,0,X_T^T)\alpha$

; $\frac{\partial}{\partial v_1}\psi_{v_0} = 0$, $\frac{\partial}{\partial v_0}\psi_{v_0} = 1$, $\frac{\partial}{\partial\alpha}\psi_{v_0} = -(1+e^{\beta_T^T X_T})\cdot E_T\cdot(1,0,X_T^T) + e^{\beta_T^T X_T}(1,0,X_T^T)$

$= \{-(1+e^{\beta_T^T X_T})\cdot E_T + e^{\beta_T^T X_T}\}(1,0,X_T^T)$

; ψ_{v_0} 중 β와 관련된 부분 $\Rightarrow -e^{\beta_T^T X_T}(1-E_T)Y_T - e^{\beta_T^T X_T}\cdot E_T(1,0,X_T^T)\alpha + e^{\beta_T^T X_T}(1,0,X_T^T)\alpha$

$\frac{\partial}{\partial\beta}\psi_{v_0} = -e^{\beta_T^T X_T}(1-E_T)Y_T\cdot X_T - e^{\beta_T^T X_T}\cdot E_T\cdot(1,0,X_T^T)\alpha\cdot X_T + e^{\beta_T^T X_T}(1,0,X_T^T)\alpha\cdot X_T$

$= e^{\beta_T^T X_T}\cdot X_T(-(1-E_T)Y_T - E_T(1,0,X_T^T)\alpha + (1,0,X_T^T)\alpha)$

PS model 의 model matrix

$= e^{\beta_T^T X_T}(-(1-E_T)Y_T + (1,0,X_T^T)\alpha(1-E_T))X_T = -e^{X_T^T\beta}((1-E_T)(Y_T-(1,0,X_T^T)\alpha)\cdot X_T$

3) $\psi_\alpha = X_T(Y_T - X_T^T\alpha)$ (X_T는 Outcome model 의 model matrix $\Rightarrow (1,E,\underset{\sim}{X})$)

; $\frac{\partial}{\partial v_1}\psi_\alpha = 0$, $\frac{\partial}{\partial v_0}\psi_\alpha = 0$, $\frac{\partial}{\partial\alpha}\psi_\alpha = -X_T X_T^T$, $\frac{\partial}{\partial\beta}\psi_\alpha = \underset{\sim}{0}$ ($\leftarrow$ 원소 개수는 $\underset{\sim}{\beta}$ 원소의 개수)

4) $\psi_\beta = E_T\cdot X_T - \frac{e^{X_T^T\beta}}{1+e^{X_T^T\beta}}\cdot X_T$ (X_T는 PS model 의 model matrix $\Rightarrow (1,\underset{\sim}{X})$)

; $\frac{\partial}{\partial v_1}\psi_\beta = \frac{\partial}{\partial v_0}\psi_\beta = 0$, $\frac{\partial}{\partial\alpha}\psi_\beta = \underset{\sim}{0}$ ($\leftarrow$ 원소 개수는 $\underset{\sim}{\alpha}$ 원소의 개수)

$\frac{\partial}{\partial\beta}\psi_\beta = \frac{-e^{X_T^T\beta}(1+e^{X_T^T\beta})\cdot X_T + e^{X_T^T\beta}\cdot e^{X_T^T\beta}\cdot X_T}{(1+e^{X_T^T\beta})^2} = \frac{-e^{X_T^T\beta}}{(1+e^{X_T^T\beta})^2}\cdot X_T$