Sandwich Variance Estimator of DR ATE Estimator

: Update Version (9월 14일 Version)

Covariance term

$$\psi_T(\theta) = \begin{bmatrix} \nu_1 - \dfrac{\{E_T Y_T - (E_T - e_T)\, E[Y_T \mid E=1, X_T]\}}{e_T} \\ \nu_0 - \dfrac{\{(1-E_T) Y_T - (E_T - e_T)\, E[Y_T \mid E=0, X_T]\}}{1-e_T} \\ X_T(Y_T - \alpha^T X_T) \\ E_T \cdot X_T - \dfrac{e^{X_T^T \beta}}{1+e^{X_T^T\beta}} \cdot X_T \end{bmatrix} \begin{matrix} \text{--- } (\psi_{\nu_1}) \\ \text{--- } (\psi_{\nu_0}) \\ \text{--- } (\psi_\alpha) \\ \text{--- } (\psi_\beta) \end{matrix}$$

Outcome regression model의 model matrix

PS model의 model matrix

$$, \ \underset{\sim}{\theta} = (\nu_1, \nu_0, \underset{\sim}{\alpha}, \underset{\sim}{\beta})$$

$$\frac{\partial}{\partial \theta'} \psi_T(\hat\theta) = \begin{bmatrix} \frac{\partial}{\partial \nu_1}\psi_{\nu_1}, & \frac{\partial}{\partial \nu_0}\psi_{\nu_1}, & \frac{\partial}{\partial \alpha}\psi_{\nu_1}, & \frac{\partial}{\partial \beta}\psi_{\nu_1} \\ \frac{\partial}{\partial \nu_1}\psi_{\nu_0}, & \frac{\partial}{\partial \nu_0}\psi_{\nu_0}, & \frac{\partial}{\partial \alpha}\psi_{\nu_0}, & \frac{\partial}{\partial \beta}\psi_{\nu_0} \\ \frac{\partial}{\partial \nu_1}\psi_{\alpha}, & \frac{\partial}{\partial \nu_0}\psi_{\alpha}, & \frac{\partial}{\partial \alpha}\psi_{\alpha}, & \frac{\partial}{\partial \beta}\psi_{\alpha} \\ \frac{\partial}{\partial \nu_1}\psi_{\beta}, & \frac{\partial}{\partial \nu_0}\psi_{\beta}, & \frac{\partial}{\partial \alpha}\psi_{\beta}, & \frac{\partial}{\partial \beta}\psi_{\beta} \end{bmatrix} \begin{matrix} \text{-- } (1) \\ \text{-- } (2) \\ \text{-- } (3) \\ \text{-- } (4) \end{matrix}$$

절편 trt

1) $$\psi_{\nu_1} = \nu_1 - \frac{(1+e^{\beta^T X_T})\{E_T Y_T - (E_T - \frac{e^{\beta^T X_T}}{1+e^{\beta^T X_T}}) \cdot (1, 1, X_T^T)\alpha\}}{e^{\beta^T X_T}}$$

$$= \nu_1 - \frac{(1+e^{\beta^T X_T})(E_T Y_T) \cancel{-} E_T(1+e^{\beta^T X_T}) \cdot (1,1,X_T^T)\alpha \cancel{+} e^{\beta^T X_T} \cdot (1,1,X_T^T)\alpha\}}{e^{\beta^T X_T}}$$

$$= \nu_1 - \frac{(1+e^{\beta^T X_T}) E_T Y_T}{e^{\beta^T X_T}} + \frac{E_T(1+e^{\beta^T X_T})}{e^{\beta^T X_T}} \cdot (1,1,X_T^T)\alpha - (1,1,X_T^T)\alpha$$

$$; \ \frac{\partial}{\partial \nu_1}\psi_{\nu_1} = 1 \quad / \quad \frac{\partial}{\partial \nu_0}\psi_{\nu_1} = 0$$

; $\frac{\partial}{\partial \alpha}\psi_{v_1} = \frac{E_T(1+e^{\beta^T X_T})}{e^{\beta^T X_T}} \cdot (1,1,X_T^T) - (1,1,X_T^T) = \left(\underbrace{\frac{E_T(1+e^{\beta^T X_T})}{e^{\beta^T X_T}} - 1}_{\text{Scalar}} \right)(1,1,X_T^T)$

Our situation ; (B_T, C_T)

; ψ_{v_1} 에서 β와 관련된 부분 $\Rightarrow -\frac{E_T Y_T}{e^{X_T^T\beta}} + \frac{E_T(1,1,X_T^T)\alpha}{e^{X_T^T\beta}} - \frac{E_T(Y_T-(1,1,X_T^T)\alpha)}{e^{X_T^T\beta}} =$

$$\frac{\partial}{\partial \beta}\psi_{v_1} = \frac{E_T Y_T \cdot \cancel{e^{\beta^T X_T}} \cdot X_T}{(e^{\beta^T X_T})^{\cancel{2}}} - \frac{E_T(1,1,X_T^T)\alpha \cdot \cancel{e^{\beta^T X_T}} \cdot X_T}{(e^{\beta^T X_T})^{\cancel{2}}} + \frac{E_T(Y_T-(1,1,X_T^T)\alpha)\,\cancel{e^{\beta^T X_T}} \cdot X_T}{(e^{\beta^T X_T})^{\cancel{2}}}$$

$$= \frac{E_T(Y_T-(1,1,X_T^T)\alpha + Y_T-(1,1,X_T^T)\alpha) \cdot X_T}{e^{\beta^T X_T}} = \frac{2 \cdot E_T(Y_T-(1,1,X_T^T)\alpha)X_T}{e^{\beta^T X_T}}$$

PS model의 model matrix

2) $\psi_{v_0} = v_0 - (1+e^{\beta^T X_T})(1-E_T)Y_T + (1+e^{\beta^T X_T}) \cdot E_T \cdot (1,0,X_T^T)\alpha - e^{\beta^T X_T} \cdot (1,0,X_T^T)\alpha$

; $\frac{\partial}{\partial v_1}\psi_{v_0} = 0$, $\frac{\partial}{\partial v_0}\psi_{v_0} = 1$, $\frac{\partial}{\partial \alpha}\psi_{v_0} = (1+e^{\beta^T X_T}) \cdot E_T \cdot (1,0,X_T^T) - e^{\beta^T X_T}(1,0,X_T^T)$

$= \{(1+e^{\beta^T X_T}) \cdot E_T - e^{\beta^T X_T}\}(1,0,X_T^T)$

; ψ_{v_0} 중 β와 관련된 부분 $\Rightarrow -e^{\beta^T X_T}(1-E_T)Y_T + e^{\beta^T X_T} \cdot E_T(1,0,X_T^T)\alpha - e^{\beta^T X_T}(1,0,X_T^T)\alpha$

$$\frac{\partial}{\partial \beta}\psi_{v_0} = -e^{\beta^T X_T}(1-E_T)Y_T \cdot X_T + e^{\beta^T X_T} \cdot E_T \cdot (1,0,X_T^T)\alpha \cdot X_T - e^{\beta^T X_T}(1,0,X_T^T)\alpha \cdot X_T$$

$$= e^{\beta^T X_T} \cdot X_T(-(1-E_T)Y_T + E_T(1,0,X_T^T)\alpha - (1,0,X_T^T)\alpha)$$

PS model의 model matrix

$$= e^{\beta^T X_T}(-(1-E_T)Y_T - (1,0,X_T^T)\alpha(1-E_T))X_T = -e^{X_T^T\beta}((1-E_T)(Y_T + (1,0,X_T^T)\alpha) \cdot X_T$$

3) $\psi_\alpha = X_T(Y_T - X_T^T\alpha)$ (X_T는 Outcome model의 model matrix $\Rightarrow (1, E, \underset{\sim}{X})$)

; $\frac{\partial}{\partial v_1}\psi_\alpha = 0$, $\frac{\partial}{\partial v_0}\psi_\alpha = 0$, $\frac{\partial}{\partial \alpha}\psi_\alpha = -X_T X_T^T$, $\frac{\partial}{\partial \beta}\psi_\alpha = \underset{\sim}{O}$ (← 원소 개수는 $\underset{\sim}{\beta}$ 원소의 개수)

4) $\psi_\beta = E_T \cdot X_T - \frac{e^{X_T^T\beta}}{1+e^{X_T^T\beta}} \cdot X_T$ (X_T는 PS model의 model matrix $\Rightarrow (1, \underset{\sim}{X})$)

; $\frac{\partial}{\partial v_1}\psi_\beta = \frac{\partial}{\partial v_0}\psi_\beta = 0$, $\frac{\partial}{\partial \alpha}\psi_\beta = \underset{\sim}{O}$ (← 원소 개수는 $\underset{\sim}{\alpha}$ 원소의 개수)

$\frac{\partial}{\partial \beta}\psi_\beta = \frac{-e^{X_T^T\beta}}{1+e^{X_T^T\beta}} \cdot X_T \cdot X_T^T$