Sandwich Variance of DR ATT Estimator

- 9월 28일 Version

Estimating equation)

$$
\Psi_T(\theta) = \begin{array}{c} \left[\begin{array}{c}
v_1 - Y_T \cdot E_T \\[2ex]
\dfrac{v_0 - \{ Y_T \cdot (1-E_T) \cdot e_T + E[Y_T \mid E=0, X_{c,T}] \cdot (E_T - e_T) \}}{1-e_T} \\[2ex]
X_{o,T}(y_T - X_{o,T}^T \alpha) \\[2ex]
\left(E_T - \dfrac{e^{X_{p,T}^T \beta}}{1+e^{X_{p,T}^T \beta}} \right) \cdot X_{p,T}
\end{array} \right] \end{array}
\begin{array}{l}
\text{---} \psi_{v_1} \\[2ex]
\text{---} \psi_{v_0} \\[2ex]
\text{---} \psi_{\alpha} \\[2ex]
\text{---} \psi_{\beta}
\end{array}
$$

$\theta = (v_1, v_0, \alpha^T, \beta^T)^T$ / X_c = Covariates set, X_o = Outcome model 의 Design matrix

X_p = PS model 의 Design matrix 라고 표기 $\Rightarrow$ $(1, 0, X_c^T) = X\_0$ 라 표기

$$
\frac{\partial}{\partial \theta'} \psi_T(\hat{\theta}) = \left[\begin{array}{cccc}
\frac{\partial}{\partial v_1} \psi_{v_1}, & \frac{\partial}{\partial v_0} \psi_{v_1}, & \frac{\partial}{\partial \alpha} \psi_{v_1}, & \frac{\partial}{\partial \beta} \psi_{v_1} \\
\frac{\partial}{\partial v_1} \psi_{v_0}, & \frac{\partial}{\partial v_0} \psi_{v_0}, & \frac{\partial}{\partial \alpha} \psi_{v_0}, & \frac{\partial}{\partial \beta} \psi_{v_0} \\
\frac{\partial}{\partial v_1} \psi_{\alpha}, & \frac{\partial}{\partial v_0} \psi_{\alpha}, & \frac{\partial}{\partial \alpha} \psi_{\alpha}, & \frac{\partial}{\partial \beta} \psi_{\alpha} \\
\frac{\partial}{\partial v_1} \psi_{\beta}, & \frac{\partial}{\partial v_0} \psi_{\beta}, & \frac{\partial}{\partial \alpha} \psi_{\beta}, & \frac{\partial}{\partial \beta} \psi_{\beta}
\end{array} \right]
\begin{array}{l}
\text{--- (1)} \\
\text{--- (2)} \\
\text{--- (3)} \\
\text{--- (4)}
\end{array}
$$

1) $\psi_{v_1} = v_1 - Y_T \cdot E_T$

Outcome model 의 Parameter 개수

$\therefore \frac{\partial}{\partial v_1} \psi_{v_1} = 1$ / $\frac{\partial}{\partial v_0} \psi_{v_1} = 0$ / $\frac{\partial}{\partial \alpha^T} \psi_{v_1} = \overbrace{\underset{\sim}{0}^T}^{q}$ / $\frac{\partial}{\partial \beta^T} \psi_{v_1} = \underbrace{\underset{\sim}{0}^T}_{e}$

Propensity score model 의 parameter 개수

(1) = $(0, 0, \underset{\sim}{0}^T, \underset{\sim}{0}^T)$

2) $\psi_{v_0} = v_0 - \dfrac{\left\{ Y_T \cdot (1-E_T)\cdot \left(\dfrac{e^{X_{p,T}^T\beta}}{1+e^{X_{p,T}^T\beta}} \right) + X\_0^T \cdot \alpha \cdot \left(E_T - \dfrac{e^{X_{p,T}^T\beta}}{1+e^{X_{p,T}^T\beta}} \right) \right\}}{\dfrac{1}{1+e^{X_{p,T}^T\beta}}}$

$$= v_0 - (1+e^{X_{p,T}^T\beta}) \left\{ Y_T \cdot (1-E_T) \cdot \left(\frac{e^{X_{p,T}^T\beta}}{1+e^{X_{p,T}^T\beta}} \right) + X\_0^T \cdot \alpha \left(E_T - \frac{e^{X_{p,T}^T\beta}}{1+e^{X_{p,T}^T\beta}} \right) \right\}$$

$$= v_0 - Y_T \cdot (1-E_T) \cdot e^{X_{p,T}^T\beta} - (1+e^{X_{p,T}^T\beta}) \cdot (\underbrace{X\_0^T \cdot \alpha}_{\alpha_0 + 0 + \alpha_2 \cdot X_1 + \cdots + \alpha_c \cdot X_c}) E_T + e^{X_{p,T}^T\beta} \cdot (\underbrace{X\_0^T \cdot \alpha}_{\alpha_0 + 0 + \alpha_2 \cdot X_1 + \cdots + \alpha_c \cdot X_c})$$

$\therefore \dfrac{\partial}{\partial v_1}\psi_{v_0} = 0, \quad \dfrac{\partial}{\partial v_0}\psi_{v_0} = 1, \quad \dfrac{\partial}{\partial \underset{\sim}{\alpha}^T}\psi_{v_0} = -(1+e^{X_{p,T}^T\beta}) \cdot E_T \cdot (X\_0)^T + e^{X_{p,T}^T\beta} \cdot (X\_0)^T$

$$= \left\{ -(1+e^{X_{p,T}^T\beta}) \cdot E_T + e^{X_{p,T}^T\beta} \right\} \cdot (X\_0)^T$$

$$\frac{\partial}{\partial \underset{\sim}{\beta}^T}\psi_{v_0} = -Y_T \cdot (1-E_T) \cdot e^{X_{p,T}^T\beta} \cdot (X_{p,T})^T - e^{X_{p,T}^T\beta} \cdot (X\_0^T \cdot \alpha) \cdot E_T \cdot X_{p,T}^T + e^{X_{p,T}^T\beta} \cdot (X\_0^T \cdot \alpha) \cdot X_{p,T}^T$$

$$= e^{X_{p,T}^T \cdot \beta} \left\{ -Y_T (1-E_T) - (X\_0^T \cdot \alpha) \cdot E_T + (X\_0^T \cdot \alpha) \right\} \cdot X_{p,T}^T$$

$$= e^{X_{p,T}^T\beta} \left\{ (1-E_T)(-Y_T + (X\_0^T \cdot \alpha)) \right\} X_{p,T}^T$$

(2) = (0, 0, ▨ $(1, 0, X_c^T)$, △ $(1, X_c^T)$)

3) $\psi_\alpha = X_{0,T}(Y_T - X_{0,T}^T \cdot \alpha)$

$\therefore \dfrac{\partial}{\partial v_1}\psi_\alpha = \underbrace{\underset{\sim}{0}}_{\alpha\text{의 개수}}, \quad \dfrac{\partial}{\partial v_0}\psi_\alpha = \overbrace{\underset{\sim}{0}}^{\alpha\text{의 개수}}, \quad \dfrac{\partial}{\partial \underset{\sim}{\alpha}^T}\psi_\alpha = -X\_0 \cdot X\_0^T = -\begin{bmatrix} 1 \\ 0 \\ X_c \end{bmatrix} \begin{bmatrix} 1 & 0 & X_c^T \end{bmatrix}$

$\dfrac{\partial}{\partial \underset{\sim}{\beta}^T}\psi_\alpha = \square \xrightarrow{} \text{Zero matrix}$

(α 개수 X $\underset{\sim}{\beta}$ 개수)

/ (3) = $\left[\underset{\sim}{0}, \underset{\sim}{0}, -\begin{bmatrix} 1 \\ 0 \\ X_c \end{bmatrix} \begin{bmatrix} 1 & 0 & X_c^T \end{bmatrix}, \boxed{0} \right]$

4) $\psi_\beta = \left(E_T - \dfrac{e^{X_{p,T}^T \beta}}{1 + e^{X_{p,T}^T \beta}} \right) \cdot X_{p,T}$

- $\dfrac{\partial}{\partial \nu_1} \psi_\beta = \underbrace{\underset{\sim}{0}}$, $\dfrac{\partial}{\partial \nu_0} \psi_\beta = \underbrace{\underset{\sim}{0}}$, $\dfrac{\partial}{\partial \underset{\sim}{\alpha}^T} \psi_\beta = \boxed{0}$ ($\underset{\sim}{\beta}$ 개수 x $\underset{\sim}{\alpha}$ 개수)

PS model parameter 의 개수

$$\frac{\partial}{\partial \underset{\sim}{\beta}^T} \psi_\beta = - \frac{e^{X_{p,T}^T \beta} \cdot (1 + e^{X_{p,T}^T \beta}) \cdot X_{p,T} - e^{X_{p,T}^T \beta} \cdot e^{X_{p,T}^T \beta} \cdot X_{p,T}}{(1 + e^{X_{p,T}^T \beta})^2} \cdot X_{p,T}^T = \frac{-e^{X_{p,T}^T \beta}}{(1 + e^{X_{p,T}^T \beta})^2} \cdot X_{p,T} \cdot X_{p,T}^T$$

$$= - \frac{e^{X_{p,T}^T \beta}}{(1 + e^{X_{p,T}^T \cdot \beta})^2} \cdot \begin{bmatrix} 1 \\ X_c \end{bmatrix} \begin{bmatrix} 1 & X_c^T \end{bmatrix}$$

$$(4) = \left[\underset{\sim}{0}, \underset{\sim}{0}, \boxed{0}, - \frac{e^{X_{p,T}^T \beta}}{(1 + e^{X_{p,T}^T \cdot \beta})^2} \cdot \begin{bmatrix} 1 \\ X_c \end{bmatrix} \begin{bmatrix} 1 & X_c^T \end{bmatrix} \right]$$