Variance Estimator of DR Estimator : Sandwich Variance

: 공통사항 정리

: 교수님 comment도 정리(—— 로 표기)

1) Treatment model : $\log\left(\frac{P(E=1 \mid X)}{1-P(E=1 \mid X)}\right) = \beta^T X \leftrightarrow e(X_i) = \frac{e^{\beta^T x_i}}{1+e^{\beta^T x_i}}$

2) Outcome model : $y_i = \alpha^T x_i + \varepsilon_i \longleftrightarrow E[y_i \mid x_i] = \alpha^T x_i$

[Score function of treatment model]

$E_i \sim Ber(p_i)$ / $logit(p_i) = \beta^T X_i$ model 가정 : $L(\beta; \underset{\sim}{X}) \propto \prod_{i=1}^{n} p_i^{E_i} (1-p_i)^{1-E_i}$

$\log L(\beta; \underset{\sim}{X}) \propto \sum_{i=1}^{n} E_i \log p_i + (1-E_i)\log(1-p_i)$

$\left(= \sum_{i=1}^{n} E_i \log\left(\frac{e^{\beta^T x_i}}{1+e^{\beta^T x_i}}\right) + (1-E_i)\log\left(\frac{1}{1+e^{\beta^T x_i}}\right)\right)$

$\left(= \sum_{i=1}^{n} \beta^T x_i - \log(1+e^{\beta^T x_i})\right)$

$S(\beta; \underset{\sim}{X}) = \frac{\partial}{\partial \beta}\log(L(\beta; \underset{\sim}{X})) = \frac{X}{1+e^{\beta^T x}}$

[Score function of Outcome regression model]

$Y \sim N(X\alpha, \sigma^2)$ / $f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-x\alpha)^2}{2\sigma^2}}$ → unknown Variance는 Constant 취급

$L(\alpha; \underset{\sim}{X}) \propto \prod_{i=1}^{n} e^{-\frac{(y_i - x_i\alpha)^2}{2\sigma^2}} \Rightarrow \log L(\alpha; \underset{\sim}{X}) \propto \prod_{i=1}^{n} -\frac{(y_i-\alpha^T x_i)^2}{2\sigma^2}$

$\frac{\partial}{\partial \alpha}\log L(\alpha; \underset{\sim}{X}) \propto \sum_{i=1}^{n} -\frac{2(y_i-\alpha^T x_i)\cdot(-x_i)}{2\sigma^2} \quad \left(= \sum_{i=1}^{n} \frac{x_i(y_i-\alpha^T x_i)}{\sigma^2}\right)$

$S(\alpha; \underset{\sim}{X}) \propto X(Y-\alpha^T X)$

Sandwich Variance Estimator of DR Estimator of ATE

: 9월 1일 Version / comment는 ——— 로 표기

$$\psi_T(\theta) = \begin{bmatrix} v_1 - \dfrac{\{E_T Y_T - (E_T - e_T)\, E[Y_T \mid E=1, X_T]}{e_T} \\[2ex] v_0 - \dfrac{\{(1-E_T) Y_T - (E_T - e_T)\, E[Y_T \mid E=0, X_T]}{1-e_T} \\[2ex] X_T (Y_T - \alpha^T X_T) \\[2ex] \dfrac{X_T}{1+e^{\beta^T X_T}} \end{bmatrix} \begin{array}{l} \text{---}(\psi_{v_1}) \\[4ex] \text{---}(\psi_{v_0}) \\[4ex] \text{---}(\psi_\alpha) \\[4ex] \text{---}(\psi_\beta) \end{array} , \quad \underset{\sim}{\theta} = (v_1, v_0, \underset{\sim}{\alpha}, \underset{\sim}{\beta})$$

절편

처치

$$1)\ \psi_{v_1} = v_1 - \frac{(1+e^{\beta^T X_T})\left\{E_T Y_T - \left(E_T - \dfrac{e^{\beta^T X_T}}{1+e^{\beta^T X_T}}\right)\cdot (1, 1, X_T^T)\alpha\right\}}{e^{\beta^T X_T}}$$

$$= v_1 - \frac{(1+e^{\beta^T X_T})(E_T Y_T) - E_T(1+e^{\beta^T X_T})\cdot(1,1,X_T^T)\alpha + e^{\beta^T X_T}\cdot(1,1,X_T^T)\alpha\}}{e^{\beta^T X_T}}$$

절편

처치

$$2)\ \psi_{v_0} = v_0 - (1+e^{\beta^T X_T})\left\{(1-E_T) Y_T - \left(E_T - \frac{e^{\beta^T X_T}}{1+e^{\beta^T X_T}}\right)\cdot(1, 0, X_T^T)\alpha\right.$$

$$= v_0 - (1+e^{\beta^T X_T})(1-E_T)Y_T + (1+e^{\beta^T X_T})\cdot E_T \cdot (1,0,X_T^T)\alpha - e^{\beta^T X_T}\cdot(1,0,X_T^T)\alpha$$

$$\frac{\partial}{\partial \theta'}\psi_T(\hat{\theta}) = \begin{bmatrix} \frac{\partial}{\partial v_1}\psi_{v_1}, & \frac{\partial}{\partial v_0}\psi_{v_1}, & \frac{\partial}{\partial \alpha}\psi_{v_1}, & \frac{\partial}{\partial \beta}\psi_{v_1} \\ \frac{\partial}{\partial v_1}\psi_{v_0}, & \frac{\partial}{\partial v_0}\psi_{v_0}, & \frac{\partial}{\partial \alpha}\psi_{v_0}, & \frac{\partial}{\partial \beta}\psi_{v_0} \\ \frac{\partial}{\partial v_1}\psi_{\alpha}, & \frac{\partial}{\partial v_0}\psi_{\alpha}, & \frac{\partial}{\partial \alpha}\psi_{\alpha}, & \frac{\partial}{\partial \beta}\psi_{\alpha} \\ \frac{\partial}{\partial v_1}\psi_{\beta}, & \frac{\partial}{\partial v_0}\psi_{\beta}, & \frac{\partial}{\partial \alpha}\psi_{\beta}, & \frac{\partial}{\partial \beta}\psi_{\beta} \end{bmatrix} \begin{array}{l} \text{--}(1) \\ \text{--}(2) \\ \text{--}(3) \\ \text{--}(4) \end{array}$$

$$(1)\ \frac{\partial}{\partial v_1}\psi_{v_1} = 1, \quad \frac{\partial}{\partial v_0}\psi_{v_1} = 0, \quad \frac{\partial}{\partial \alpha}\psi_{v_1} = \frac{E_T(1+e^{\beta^T X_T})(1,1,X_T^T)}{e^{\beta^T X_T}} - (1,1,X_T^T)$$

$$= (1,1,X_T^T)\left(\frac{E_T(1+e^{\beta^T X_T})}{e^{\beta^T X_T}} - 1\right)$$

$$\frac{\partial}{\partial \beta}\psi_{\nu_1} = \frac{+E_T(Y_T-(1,1,X_T^T)\alpha)\cdot \cancel{e^{\beta^T X_T}}\cdot X_T}{(e^{\beta^T X_T})^{\cancel{2}}} = \frac{E_T(Y_T-(1,1,X_T^T)\alpha)}{e^{\beta^T X_T}}\cdot X_T$$

(2) $\frac{\partial}{\partial \nu_1}\psi_{\nu_0} = 0$, $\frac{\partial}{\partial \nu_0}\psi_{\nu_0} = 1$, $\frac{\partial}{\partial \alpha}\psi_{\nu_0} = E_T\cdot(1,0,X_T^T)$

$$\frac{\partial}{\partial \beta}\psi_{\nu_0} = -e^{\beta^T X_i}(1-E_T)Y_T\cdot X_T + e^{\beta^T X_T}X_T\cdot E_T\cdot(1,0,X_T^T)\alpha - e^{\beta^T X_T}\cdot(1,0,X_T^T)\alpha\cdot X_T$$

$$= e^{\beta^T X_T}\cdot X_T\left\{(E_T-1)Y_T + E_T\cdot(1,0,X_T^T)\alpha - (1,0,X_T^T)\alpha\right\}$$

$$= -e^{\beta^T X_T}\cdot X_T\left\{(1-E_T)(Y_T+\alpha^T(1,0,X_T))\right\}$$

(3) $\frac{\partial}{\partial \nu_1}\psi_\alpha = 0$, $\frac{\partial}{\partial \nu_0}\psi_\alpha = 0$, $\frac{\partial}{\partial \alpha}\psi_\alpha = -X_T X_T^T$, $\frac{\partial}{\partial \beta}\psi_\alpha = 0$

(4) $\frac{\partial}{\partial \nu_1}\psi_\beta = 0$, $\frac{\partial}{\partial \nu_0}\psi_\beta = 0$, $\frac{\partial}{\partial \alpha}\psi_\beta = 0$, $\frac{\partial}{\partial \beta}\psi_\beta = \frac{-e^{\beta^T X_T}\cdot X_T\cdot X_T^T}{(1+e^{\beta^T X_T})^2}$

Sandwich Variance Estimator of DR ATT Estimator

: 9월 1일 meeting 후 Version. / comment는 ——— 로 표기

$$\psi_T(\theta) = \begin{bmatrix} v_1 - Y_T E_T \\ v_0 - \dfrac{Y_T(1-E_T)e(X_T) + E[Y_T \mid E=0, X_T](E_T - e(X_T))}{1-e(X_T)} \\ X_T(Y_T - \alpha^T X_T) \\ \dfrac{X_T}{1+e^{\beta^T X_T}} \end{bmatrix} \begin{array}{l} \text{---}(\psi_{v_1}) \\ \text{---}(\psi_{v_0}) \\ \text{---}(\psi_\alpha) \\ \text{---}(\psi_\beta) \end{array}, \quad \underset{\sim}{\theta} = (v_1, v_0, \underset{\sim}{\alpha}, \underset{\sim}{\beta})$$

$$* \ \psi_{v_0} = v_0 - \frac{\overbrace{Y_T(1-E_T)\cdot \dfrac{e^{\beta^T X_T}}{1+e^{\beta^T X_T}} + (1,0,X_T^T)\alpha \cdot \left(E_T - \dfrac{e^{\beta^T X_T}}{1+e^{\beta^T X_T}}\right)}^{= \frac{(1+e^{\beta^T X_T})E_T - e^{\beta^T X_T}}{1+e^{\beta^T X_i}}}}{\underbrace{1 - \dfrac{e^{\beta^T X_T}}{1+e^{\beta^T X_T}}}_{= \frac{1}{1+e^{\beta^T X_i}}}}$$

$$= v_0 - Y_T(1-E_T)e^{\beta^T X_T} - (1,0,X_T^T)\alpha(1+e^{\beta^T X_T})\cdot E_T + (1,0,X_T^T)\alpha e^{\beta^T X_T}$$

$$\frac{\partial}{\partial \theta'}\psi_T(\hat{\theta}) = \begin{bmatrix} \frac{\partial}{\partial v_1}\psi_{v_1}, & \frac{\partial}{\partial v_0}\psi_{v_1}, & \frac{\partial}{\partial \alpha}\psi_{v_1}, & \frac{\partial}{\partial \beta}\psi_{v_1} \\ \frac{\partial}{\partial v_1}\psi_{v_0}, & \frac{\partial}{\partial v_0}\psi_{v_0}, & \frac{\partial}{\partial \alpha}\psi_{v_0}, & \frac{\partial}{\partial \beta}\psi_{v_0} \\ \frac{\partial}{\partial v_1}\psi_{\alpha}, & \frac{\partial}{\partial v_0}\psi_{\alpha}, & \frac{\partial}{\partial \alpha}\psi_{\alpha}, & \frac{\partial}{\partial \beta}\psi_{\alpha} \\ \frac{\partial}{\partial v_1}\psi_{\beta}, & \frac{\partial}{\partial v_0}\psi_{\beta}, & \frac{\partial}{\partial \alpha}\psi_{\beta}, & \frac{\partial}{\partial \beta}\psi_{\beta} \end{bmatrix} \begin{array}{l} \text{--(1)} \\ \text{--(2)} \\ \text{--(3)} \\ \text{--(4)} \end{array}$$

(1) $\frac{\partial}{\partial v_1}\psi_{v_1} = 1$, $\frac{\partial}{\partial v_0}\psi_{v_1} = 0$, $\frac{\partial}{\partial \alpha}\psi_{v_1} = 0$, $\frac{\partial}{\partial \beta}\psi_{v_1} = 0$

(2) $\frac{\partial}{\partial v_1}\psi_{v_0} = 0$, $\frac{\partial}{\partial v_0}\psi_{v_0} = 1$

$$\frac{\partial}{\partial \alpha}\psi_{v_0} = -(1,0,X_T^T)(1+e^{\beta^T X_T})E_T + (1,0,X_T^T)e^{\beta^T X_T} = -(1,0,X_T^T)E_T - (1,0,X_T^T)e^{\beta^T X_T}\cdot E_T$$

$$+(1,0,X_T^T)e^{\beta^T X_T}$$

$$= -(1,0,X_T^T)E_T + (1,0,X_T^T)e^{\beta^T X_T}(1-E_T)$$

$$\frac{\partial}{\partial \beta}\psi_{v_0} = -Y_T(1-E_T)e^{\beta^T X_T}\cdot X_T - (1,0,X_T^T)\alpha\cdot e^{\beta^T X_T}\cdot X_T\cdot E_T + (1,0,X_T^T)\alpha\, e^{\beta^T X_T}\cdot X_T$$

$$= e^{\beta^T X_T}\cdot X_T \left\{ -Y_T(1-E_T) - (1,0,X_T^T)\alpha E_T + (1,0,X_T^T)\alpha \right\}$$

$$= e^{\beta^T X_T}\cdot X_T \left\{ -Y_T(1-E_T) + (1,0,X_T^T)\alpha(1-E_T) \right\} = -e^{\beta^T X_T}\cdot X_T\cdot(1-E_T)(Y_T - (1,0,X_T^T)\alpha)$$

(3) $\frac{\partial}{\partial v_1}\psi_\alpha = 0$, $\frac{\partial}{\partial v_0}\psi_\alpha = 0$, $\frac{\partial}{\partial \alpha}\psi_\alpha = -X_T X_T^T$, $\frac{\partial}{\partial \beta}\psi_\alpha = 0$

(4) $\frac{\partial}{\partial v_1}\psi_\beta = 0$, $\frac{\partial}{\partial v_0}\psi_\beta = 0$, $\frac{\partial}{\partial \alpha}\psi_\beta = 0$, $\frac{\partial}{\partial \beta}\psi_\beta = \frac{-e^{\beta^T X_T}\cdot X_T X_T^T}{(1+e^{\beta^T X_T})^2}$