Variance Estimator of DR Estimator : Sandwich Variance

: 8월 24일 Version (DR of ATE)

1) Treatment model : $\log\left(\frac{P(E=1|X)}{1-P(E=1|X)}\right) = \beta^T X \leftrightarrow e(X_T) = \frac{e^{\beta^T X_i}}{1+e^{\beta^T X_i}}$

2) Outcome model : $y_T = \alpha^T x_T + \varepsilon_T \longleftrightarrow E[y_T | x_T] = \alpha^T x_T$

[Score function of treatment model]

$E_T \sim Ber(p_T)$ / $logit(p_T) = \beta^T X_i$ model 가정 : $L(\beta; \underset{\sim}{X}) \propto \prod_{i=1}^{n} p_i^{E_i} (1-p_i)^{1-E_i}$

$$\log L(\beta; \underset{\sim}{X}) \propto \sum_{T=1}^{n} E_T \log p_T + (1-E_T)\log(1-p_T)$$

$$\left(= \sum_{T=1}^{n} E_T \log\left(\frac{e^{\beta^T X_i}}{1+e^{\beta^T X_i}}\right) + (1-E_T)\log\left(\frac{1}{1+e^{\beta^T X_i}}\right)\right)$$

$$\left(= \sum_{T=1}^{n} \beta^T X_T - \log(1+e^{\beta^T X_i})\right)$$

$$S(\beta; \underset{\sim}{X}) = \frac{\partial}{\partial\beta}\log(L(\beta; \underset{\sim}{X})) = \frac{X}{1+e^{\beta^T X}}$$

[Score function of Outcome regression model]

$$Y \sim N(X\alpha, \sigma^2) \;/\; f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-X\alpha)^2}{2\sigma^2}}$$

$$L(\alpha; \underset{\sim}{X}) \propto \prod_{T=1}^{n} e^{-\frac{(y_T - X_T\alpha)^2}{2\sigma^2}} \Rightarrow \log L(\alpha; \underset{\sim}{X}) \propto \prod_{T=1}^{n} -\frac{(y_T-\alpha^T X_T)^2}{2\sigma^2}$$

$$\frac{\partial}{\partial\alpha}\log L(\alpha; \underset{\sim}{X}) \propto \sum_{T=1}^{n} -\frac{2(y_T-\alpha^T X_T)\cdot(-X_T)}{2\sigma^2} \quad \left(= \sum_{T=1}^{n} \frac{X_T(y_T-\alpha^T X_T)}{\sigma^2}\right)$$

$$S(\alpha; \underset{\sim}{X}) \propto X(Y-\alpha^T X)$$

$$\psi_T(\hat{\theta}) = \begin{bmatrix} \nu_1 - \{E_T Y_T - (E_T - \hat{e}_T)\,\widehat{E[Y_T | E_i=1, X_T]}\}/\hat{e}_i \\ \nu_0 - \{(1-E_T)Y_T + (E_T-\hat{e}_T)\,\widehat{E[Y_T | E_T=0, X_T]}\}/(1-\hat{e}_T) \\ X_T(y_T - \alpha^T X_T) \\ X_T\left(\frac{1}{1+e^{\beta^T X_T}}\right) \end{bmatrix}$$

Sandwich Variance Estimator of DR Estimator of ATE

: 9월 1일 Version

$$\psi_T(\hat{\theta}) = \begin{array}{cl} \left[\begin{matrix} v_1 - \dfrac{\{E_T Y_T - (E_T - \hat{e}_T)\, \widehat{E[Y_T \mid E=1, X_T]}}{\hat{e}_T} \\[2ex] v_0 - \dfrac{\{(1-E_T) Y_T - (E_T - \hat{e}_T)\, \widehat{E[Y_T \mid E=0, X_T]}}{1-\hat{e}_T} \\[2ex] X_T (Y_T - \alpha^T X_T) \\[2ex] \dfrac{X_T}{1+e^{\beta^T X_T}} \end{matrix}\right] & \begin{matrix} (\psi_{v_1}) \\[5ex] (\psi_{v_0}) \\[4ex] (\psi_{\alpha}) \\[4ex] (\psi_{\beta}) \end{matrix} \end{array} \quad , \quad \underset{\sim}{\theta} = (v_1, v_0, \underset{\sim}{\alpha}, \underset{\sim}{\beta})$$

1) $$\psi_{v_1} = v_1 - \frac{(1+e^{\beta^T X_T}) \{E_T Y_T - (E_T - \frac{e^{\beta^T X_T}}{1+e^{\beta^T X_T}}) \cdot \alpha^T(\overset{\text{절편}}{1}, \overset{\text{trt}}{1}, X_T)\}}{e^{\beta^T X_T}}$$

$$= v_1 - \frac{(1+e^{\beta^T X_T})(E_T Y_T) - E_T(1+e^{\beta^T X_T}) \cdot \alpha^T(1,1,X_T) + e^{\beta^T X_T} \cdot \alpha^T(1,1,X_T)\}}{e^{\beta^T X_T}}$$

2) $$\psi_{v_0} = v_0 - (1+e^{\beta^T X_T}) \{(1-E_T) Y_T - (E_T - \frac{e^{\beta^T X_T}}{1+e^{\beta^T X_T}}) \cdot \alpha^T(\overset{\text{절편}}{1}, \overset{\text{trt}}{0}, X_T)$$

$$= v_0 - (1+e^{\beta^T X_T})(1-E_T) Y_T + (1+e^{\beta^T X_T}) \cdot E_T \cdot \alpha^T(1,0,X_T) - e^{\beta^T X_T} \cdot \alpha^T(1,0,X_T)$$

$$\frac{\partial}{\partial \theta'} \psi_T(\hat{\theta}) = \begin{array}{cl} \left[\begin{matrix} \frac{\partial}{\partial v_1}\psi_{v_1}, & \frac{\partial}{\partial v_0}\psi_{v_1}, & \frac{\partial}{\partial \alpha}\psi_{v_1}, & \frac{\partial}{\partial \beta}\psi_{v_1} \\[1ex] \frac{\partial}{\partial v_1}\psi_{v_0}, & \frac{\partial}{\partial v_0}\psi_{v_0}, & \frac{\partial}{\partial \alpha}\psi_{v_0}, & \frac{\partial}{\partial \beta}\psi_{v_0} \\[1ex] \frac{\partial}{\partial v_1}\psi_{\alpha}, & \frac{\partial}{\partial v_0}\psi_{\alpha}, & \frac{\partial}{\partial \alpha}\psi_{\alpha}, & \frac{\partial}{\partial \beta}\psi_{\alpha} \\[1ex] \frac{\partial}{\partial v_1}\psi_{\beta}, & \frac{\partial}{\partial v_0}\psi_{\beta}, & \frac{\partial}{\partial \alpha}\psi_{\beta}, & \frac{\partial}{\partial \beta}\psi_{\beta} \end{matrix}\right] & \begin{matrix} (1) \\[1ex] (2) \\[1ex] (3) \\[1ex] (4) \end{matrix} \end{array}$$

(1) $\frac{\partial}{\partial v_1}\psi_{v_1} = 1$, $\frac{\partial}{\partial v_0}\psi_{v_0} = 0$, $\frac{\partial}{\partial \alpha}\psi_{v_1} = \frac{E_T(1+e^{\beta^T X_T})(1,1,X_T)}{e^{\beta^T X_T}} - (1,1,X_T)$

$$= (1,1,X_T)\left(\frac{E_T(1+e^{\beta^T X_T})}{e^{\beta^T X_T}} - 1\right)$$

$$\frac{\partial}{\partial \beta}\psi_{v_1} = \frac{-E_T(Y_T-\alpha^T(1,1,X_T))\cdot e^{\beta^T X_T}\cdot X_T}{(e^{\beta^T X_T})^2} = \frac{-E_T\cdot X_T(Y_T-\alpha^T(1,1,X_T))}{e^{\beta^T X_T}}$$

(2) $\frac{\partial}{\partial v_1}\psi_{v_0}=0$, $\frac{\partial}{\partial v_0}\psi_{v_0}=1$, $\frac{\partial}{\partial \alpha}\psi_{v_0}=E_T\cdot(1,0,X_T)$

$$\frac{\partial}{\partial \beta}\psi_{v_0} = -e^{\beta^T X_i}(1-E_T)Y_T\cdot X_T + e^{\beta^T X_T}X_T\cdot E_T\cdot\alpha^T(1,0,X_T) - e^{\beta^T X_T}\cdot\alpha^T(1,0,X_T)\cdot X_T$$

$$= e^{\beta^T X_T}\cdot X_T\{(E_T-1)Y_T + E_T\cdot\alpha^T(1,0,X_T) - \alpha^T(1,0,X_T)\}$$

$$= e^{\beta^T X_T}\cdot X_T\{(E_T-1)(Y_T+\alpha^T(1,0,X_T))\}$$

(3) $\frac{\partial}{\partial v_1}\psi_\alpha=0$, $\frac{\partial}{\partial v_0}\psi_\alpha=0$, $\frac{\partial}{\partial \alpha}\psi_\alpha=-X_T^T\cdot X_T$, $\frac{\partial}{\partial \beta}\psi_\alpha=0$

(4) $\frac{\partial}{\partial v_1}\psi_\beta=0$, $\frac{\partial}{\partial v_0}\psi_\beta=0$, $\frac{\partial}{\partial \alpha}\psi_\beta=0$, $\frac{\partial}{\partial \beta}\psi_\beta=\frac{-X_T\cdot e^{\beta^T X_T}\cdot X_T}{(1+e^{\beta^T X_T})^2}$

Sandwich Variance Estimator of DR ATT Estimator

: 9월 1일 Version

$$\psi_T(\hat{\theta}) = \begin{bmatrix} \nu_1 - Y_T E_T \\ \nu_0 - \dfrac{Y_T(1-E_T)\hat{e}(X_T) + \hat{E}[Y_T \mid E=0, X_T](E_T - \hat{e}(X_T))}{1-\hat{e}(X_T)} \\ X_T(Y_T - \hat{\alpha}^T X_T) \\ \dfrac{X_T}{1+e^{\hat{\beta}^T X_T}} \end{bmatrix} \begin{array}{l} \text{--- } (\psi_{\nu_1}) \\ \text{--- } (\psi_{\nu_0}) \\ \text{--- } (\psi_\alpha) \\ \text{--- } (\psi_\beta) \end{array} , \quad \underset{\sim}{\theta} = (\nu_1, \nu_0, \underset{\sim}{\alpha}, \underset{\sim}{\beta})$$

$$* \ \psi_{\nu_0} = \nu_0 - \frac{Y_T(1-E_T)\cdot \frac{e^{\beta^T X_T}}{1+e^{\beta^T X_T}} + \alpha^T(1,0,X_T)\cdot \overbrace{\left(E_T - \frac{e^{\beta^T X_T}}{1+e^{\beta^T X_T}}\right)}^{= \frac{(1+e^{\beta^T X_T})E_T - e^{\beta^T X_T}}{1+e^{\beta^T X_i}}}}{\underbrace{1 - \frac{e^{\beta^T X_T}}{1+e^{\beta^T X_T}}}_{= \frac{1}{1+e^{\beta^T X_i}}}}$$

$$= \nu_0 - Y_T(1-E_T)e^{\beta^T X_T} - \alpha^T(1,0,X_T)(1+e^{\beta^T X_T})\cdot E_T + \alpha^T(1,0,X_T)e^{\beta^T X_T}$$

$$\frac{\partial}{\partial \theta'}\psi_T(\hat{\theta}) = \begin{bmatrix} \frac{\partial}{\partial \nu_1}\psi_{\nu_1}, & \frac{\partial}{\partial \nu_0}\psi_{\nu_1}, & \frac{\partial}{\partial \alpha}\psi_{\nu_1}, & \frac{\partial}{\partial \beta}\psi_{\nu_1} \\ \frac{\partial}{\partial \nu_1}\psi_{\nu_0}, & \frac{\partial}{\partial \nu_0}\psi_{\nu_0}, & \frac{\partial}{\partial \alpha}\psi_{\nu_0}, & \frac{\partial}{\partial \beta}\psi_{\nu_0} \\ \frac{\partial}{\partial \nu_1}\psi_{\alpha}, & \frac{\partial}{\partial \nu_0}\psi_{\alpha}, & \frac{\partial}{\partial \alpha}\psi_{\alpha}, & \frac{\partial}{\partial \beta}\psi_{\alpha} \\ \frac{\partial}{\partial \nu_1}\psi_{\beta}, & \frac{\partial}{\partial \nu_0}\psi_{\beta}, & \frac{\partial}{\partial \alpha}\psi_{\beta}, & \frac{\partial}{\partial \beta}\psi_{\beta} \end{bmatrix} \begin{array}{l} \text{--- } (1) \\ \text{--- } (2) \\ \text{--- } (3) \\ \text{--- } (4) \end{array}$$

(1) $\frac{\partial}{\partial \nu_1}\psi_{\nu_1} = 1$, $\frac{\partial}{\partial \nu_0}\psi_{\nu_1} = 0$, $\frac{\partial}{\partial \alpha}\psi_{\nu_1} = 0$, $\frac{\partial}{\partial \beta}\psi_{\nu_1} = 0$

(2) $\frac{\partial}{\partial \nu_1}\psi_{\nu_0} = 0$, $\frac{\partial}{\partial \nu_0}\psi_{\nu_0} = 1$

$$\frac{\partial}{\partial \alpha}\psi_{v_0} = -(1,0,X_T)(1+e^{\beta^T X_T})E_T + (1,0,X_T)e^{\beta^T X_T} = -(1,0,X_T)E_T - (1,0,X_T)e^{\beta^T X_T}\cdot E_T$$

$$+(1,0,X_T)e^{\beta^T X_T}$$

$$= -(1,0,X_T)E_T + (1,0,X_T)e^{\beta^T X_T}(1-E_T)$$

$$\frac{\partial}{\partial \beta}\psi_{v_0} = -Y_T(1-E_T)e^{\beta^T X_T}\cdot X_T - \alpha^T(1,0,X_T)\cdot e^{\beta^T X_T}\cdot X_T\cdot E_T + \alpha^T(1,0,X_T)e^{\beta^T X_T}\cdot X_T$$

$$= e^{\beta^T X_T}\cdot X_T\left\{-Y_T(1-E_T) - \alpha^T(1,0,X_T)E_T + \alpha^T(1,0,X_T)\right\}$$

$$= e^{\beta^T X_T}\cdot X_T\left\{-Y_T(1-E_T) + \alpha^T(1,0,X_T)(1-E_T)\right\} = -e^{\beta^T X_T}\cdot X_T\cdot(1-E_T)(Y_T - \alpha^T(1,0,X_T))$$

(3) $\frac{\partial}{\partial v_1}\psi_\alpha = 0$, $\frac{\partial}{\partial v_0}\psi_\alpha = 0$, $\frac{\partial}{\partial \alpha}\psi_\alpha = -X_T X_T^T$, $\frac{\partial}{\partial \beta}\psi_\alpha = 0$

(4) $\frac{\partial}{\partial v_1}\psi_\beta = 0$, $\frac{\partial}{\partial v_0}\psi_\beta = 0$, $\frac{\partial}{\partial \alpha}\psi_\beta = 0$, $\frac{\partial}{\partial \beta}\psi_\beta = \frac{-X_T\cdot e^{\beta^T X_T}\cdot X_T}{(1+e^{\beta^T X_T})^2}$