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Mathematics for Economists May 3, 2012

Concavity and Sufficiency of the Kuhn-Tucker Conditions

So far, all of the theorems we have established for each of the programming problems we have considered have been **necessary** FOC. That is, any solution to the programming problem must also be a solution to the FOC. Without further restrictions, however, there is no guarantee that a solution to the FOC is also a solution to the actual programming problem.

Example: Consider the problem:

$$\begin{gathered}\text{Choose } x \in \mathbb{R}^2_+ \\ \text{to maximize } u(x) \equiv x_1 + x_2 \\ \text{subject to } g(x) \equiv 9 - (2 + x_1)(1 + x_2) \geq 0\end{gathered}$$

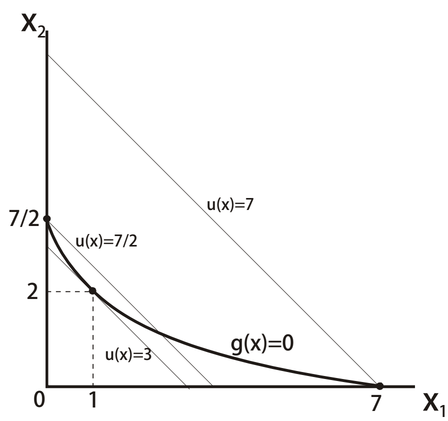
The Lagrangian for this problem is

$$\mathcal{L}(x, \lambda) = x_1 + x_2 + \lambda \ (9 - (2 + x_1)(1 + x_2)).$$

Applying the FOC conditions, we have

$$\begin{aligned}\mathcal{L}_{x_1}(x,\lambda) &= 1 - \lambda(1 + x_2) \leq 0, \quad x_1 \geq 0 \quad \text{with at least one equality} \\ \mathcal{L}_{x_2}(x,\lambda) &= 1 - \lambda(2 + x_1) \leq 0, \quad x_2 \geq 0 \quad \text{with at least one equality} \\ \mathcal{L}_{\lambda}(x,\lambda) &= 9 - (2 + x_1)(1 + x_2) = 0.\end{aligned}$$

These relations have three solutions: (a) $x = (7, 0)$ and $\lambda = 1$, (b) $x = \left(0, \frac{7}{2}\right)$ and $\lambda = \frac{1}{2}$ and (c) $x = (1, 2)$ and $\lambda = \frac{1}{3}$, which are are illustrated in the figure below.



The boundary solution $x = (7, 0)$ with $u(x) = 7$ attains a global maximum. The other boundary solution $x = \left(0, \frac{7}{2}\right)$ with $u(x) = \frac{7}{2}$ attains a local maximum but not a global maximum. The interior solution $x = (1, 2)$ with $u(x) = 3$ attains a global minimum.

It is apparent from this example, that to ensure a solution to the Kuhn-Tucker conditions identifies a global maximum, we need to impose additional restrictions on the objective function u and/or the constraint functions in g. It is possible to derive SOC that incorporate the second derivatives of u and g to provide sufficient conditions for a local maximum, but by themselves these conditions do not guarantee a global maximum. A simpler and more powerful approach is to use the concepts of concave and convex functions.

Some Preliminary Results

Our main theorem uses the following properties of concave functions that we established in earlier lectures.

Lemma 01: Suppose $f_1, ..., f_m$ are concave functions. Then for any $\alpha_1, ..., \alpha_m$, for which each $\alpha_i \geq 0$, $f \equiv \sum_{i=1}^{m} \alpha_i f_i$ is also a concave function.

Lemma 02: Suppose $f : \mathbb{R}_+^n \to \mathbb{R}$ is a concave function. Then $f(z) - f(x) \leq f_x(x)\ (z - x)$ for all $x, z \in \mathbb{R}_+^n$.

We also use the fact that a function f is convex if and only if $-f$ is concave.

Sufficient Conditions

Let $u : \mathbb{R}_+^n \to \mathbb{R}$ and $g : \mathbb{R}_+^n \to \mathbb{R}^m$ be continuously differentiable and suppose $c \in \mathbb{R}^m$. We consider the problem

$$\begin{array}{c} \text{Choose } x \in \mathbb{R}_+^n \\ \text{to maximize } u(x) \\ \text{subject to } g(x) \geq 0 \end{array} \tag{Problem K-T}$$

Let $\mathcal{L}(x, \lambda) = u(x) + \lambda \cdot g(x)$ denote the Lagrangian for this problem.

Theorem 1: Suppose u and each g_i are concave functions. If (x, λ) satisfies

$$\begin{aligned} \mathcal{L}_x(x,\lambda) &= u_x(x) + \lambda \cdot g_x \leq 0,\ x \geq 0,\ \text{and } \mathcal{L}_x(x,\lambda)\ x = 0 \\ \mathcal{L}_\lambda(x,\lambda) &= g(x) \geq 0,\ \lambda \geq 0,\ \text{and } \mathcal{L}_\lambda(x,\lambda)\ \lambda = 0 \end{aligned} \tag{K-T}$$

then x is a solution to the problem K-T.

Proof. Suppose the (K-T) conditions are satisfied at $(x, \lambda) \in \mathbb{R}^n_+$. Now consider any $z \in \mathbb{R}^n_+$ such that $g(z) \geq 0$. We will show that $u(z) \leq u(x)$.

Our first step is to observe that if u and each g_i are concave, then Lemma 01 implies that $\mathcal{L}(x, \lambda) = u(x) + \lambda \cdot g(x) = u(x) + \sum_{i=1}^m \lambda_i \cdot g_i(x)$ is a concave function.

It then follows from Lemma 02 that

$$\mathcal{L}(z, \lambda) - \mathcal{L}(x, \lambda) \leq \mathcal{L}_x(x, \lambda)\ (z - x) = \mathcal{L}_x(x, \lambda)\ z - \mathcal{L}_x(x, \lambda)\ x)$$

Now consider right hand side. Observe first that the (K-T) conditions imply that $\mathcal{L}_x(x, \lambda)\ x = 0$. Next observe that $z \in \mathbb{R}^n_+ \geq 0$ and $\mathcal{L}_x(x, \lambda) \leq 0$ (from (K-T)) imply

$$\mathcal{L}_x(x, \lambda)\ z = \sum_{i=1}^n \mathcal{L}_{x_i}(x, \lambda)\ z_i \leq 0.$$

We conclude that

$$\mathcal{L}(z, \lambda) - \mathcal{L}(x, \lambda) = u(z) + \lambda \cdot g(z) - (u(x) - \lambda \cdot g(x) \leq 0$$

or equivalently,

$$u(z) - u(x) \leq \lambda \cdot g(x) - \lambda \cdot g(z)$$

All that remains is to show that the right hand side is nonpositive. From the (K-T), we have $\lambda \cdot g(x) = 0$. Also $\lambda \geq 0$ (from K-T) and $g(z) \geq 0$ implies

$$\lambda \cdot g(z) = \sum_{i=1}^m \lambda_i g_i(z) \geq 0$$

and therefore,

$$u(z) - u(x) \leq 0.$$

■

Combined with the Necessity of the Kuhn-Tucker conditions established in earlier notes, Theorem 1 is generally called the **Kuhn-Tucker theorem.**

Uniqueness

Let $X = \{x \in \mathbb{R}^n_+ : g(x) \geq 0\}$ be the set of all feasible points for problem K-T.

Lemma 1: If each g_i is convex, then X is a convex set.

Proof. For each g_i, let $X_i = \{x \in \mathbb{R}^n_+ : g_i(x) \geq 0\}$. Then since g_i concave implies that g_i is quasi-concave, X_i is just the better set of a quasi-concave function and is therefore a convex set. But $X = \cap_{i=1}^m X_i$. Therefore, since the intersection of convex sets is a convex set, it follows that X is a convex set. ■

Theorem 2: Suppose each g_i is concave. (a) If u is concave, then the set of solutions to the K-T conditions are convex. (b) If u is strictly concave, then there is at most one solution to the K-T conditions.

Proof. Observe first that $z \in X$ if and only if $z \in \mathbb{R}^n$ and each $g_i(z) \geq 0$. But $g_i(x) \geq 0$ for all i is equivalent to $g(x) \geq 0_m$. Therefore the KT problem is equivalent to the problem

$$\text{Choose } x \in X \text{ to maximize } u(x).$$

To prove (a), observe that from earlier notes we know that the set of maximizers of a quasi-concave function on a convex set is convex. Therefore, the set of solutions to the K-T conditions must be convex

To prove (b), observe that from earlier notes we know that the maximizer of strictly quasi-concave function on a convex set is unique. Therefore, the solution to the K-T conditions must be unique. ■

If u is concave, it is also possible to establish conditions for uniqueness if each of the g_i are strictly convex.

Theorem 3: Suppose u is concave and each g_i is strictly convex. If there is an $x^* \notin X$, such that $u(x^*) \geq \sup_{x \in X} u(x)$, then there is at most one solution to the K-T conditions.

Proof. Left as an optional exercise. ■

The importance of Theorems 2 and 3 is that in searching for a solution to a K-T problem, it is sufficient to find just one solution to the K-T problems. Once that solution has been found, we know that we do not have to look for any more solutions.