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MAT 280
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Lecture 11 Notes

These notes correspond to Section 12.2 in Stewart and Sections 5.3 and 5.4 in Marsden and Tromba.

Double Integrals over More General Regions

We have learned how to integrate a function $f(x, y)$ of two variables over a rectangle R. However, it is important to be able to integrate such functions over more general regions, in order to be able to compute the volume of a wider variety of solids.

To that end, given a region $D \subset \mathbb{R}^2$, contained within a rectangle R, we define the double integral of $f(x, y)$ over D by

$$\int\int_D f(x, y)\, dA = \int\int_R F(x, y)\, dA$$

where

$$F(x, y) = \begin{cases} f(x, y) & (x, y) \in D \\ 0 & (x, y) \in R, \notin D \end{cases}.$$

It is possible to use Fubini's Theorem to compute integrals over certain types of general regions. We say that a region D is of *type I* if it lies between the graphs of two continuous functions of x, and is also bounded by two vertical lines. Specifically,

$$D = \{(x, y) \,|\, a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$

To integrate $f(x, y)$ over such a region, we can apply Fubini's Theorem. We let $R = [a, b] \times [c, d]$ be a rectangle that contains D. Then we have

$$\begin{aligned} \int\int_D f(x, y)\, dA &= \int\int_R F(x, y)\, dA \\ &= \int_a^b \int_c^d F(x, y)\, dy\, dx \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy\, dx \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx. \end{aligned}$$

This is valid because $F(x, y) = 0$ when $y < g_1(x)$ or $y > g_2(x)$, because in these cases, (x, y) lies outside of D. The resulting iterated integral can be evaluated in the same way as iterated integrals over rectangles; the only difference is that when the limits of the inner integral are substituted for y in the antiderivative of $f(x, y)$ with respect to y, the limits are functions of x, rather than constants.

A similar approach can be applied to a *region of type II*, which is bounded on the left and right by continuous functions of y, and bounded above and below by vertical lines. Specifically, D is a region of type II if

$$D = \{(x, y)|h_1(y) \leq x \leq h_2(y), \quad c \leq y \leq d\}.$$

Using Fubini's Theorem, we obtain

$$\int\int_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Example We wish to compute the volume of the solid under the plane $x+y+z = 8$, and bounded by the surfaces $y = x$ and $y = x^2$. These surfaces intersect along the lines $x = 0$, $y = 0$ and $x = 1$, $y = 1$. It follows that the volume V of the solid is given by the double integral

$$\int_0^1 \int_{x^2}^x 8 - x - y\, dy\, dx.$$

Note that $g_2(x) = x$ is the upper limit of integration, because $x^2 \leq x$ when $0 \leq x \leq 1$. We have

$$\begin{aligned}
V &= \int_0^1 \int_{x^2}^x 8 - x - y\, dy\, dx \\
&= \int_0^1 \left(8y - xy - \frac{y^2}{2}\right)\Big|_{x^2}^x dx \\
&= \int_0^1 \left(8x - x^2 - \frac{x^2}{2}\right) - \left(8x^2 - x^3 - \frac{x^4}{2}\right) dx \\
&= \int_0^1 \frac{x^4}{2} + x^3 - \frac{19x^2}{2} + 8x\, dx \\
&= \left(\frac{x^5}{10} + \frac{x^4}{4} - \frac{19x^3}{6} + 4x^2\right)\Big|_0^1 \\
&= \frac{1}{10} + \frac{1}{4} - \frac{19}{6} + 4 \\
&= \frac{71}{60}.
\end{aligned}$$

□

Note that it is sometimes necessary to determine the intersections of surfaces that define a solid, in order to obtain the limits of integration.

To compute the volume of a solid that is bounded above and below (along the z-direction) by two different surfaces, we can add the volume of the solid bounded by the top surface and the plane $z = 0$ to the volume of the solid bounded above by $z = 0$ and below by the lower surface, which is equivalent to subtracting the volume of the solid bounded above by the lower surface and below by $z = 0$.

Example We will compute the volume V of the solid in the first octant bounded by the planes $z = 10 + x + y$, $z = 2 - x - y$, and $x = 0$, as well as the surfaces $y = \sin x$ and $y = \cos x$. As these surfaces intersect along the line $y = \sqrt{2}/2$, $x = \pi/4$, this volume is given by the double integral

$$\begin{aligned} V &= \int_0^{\pi/4} \int_{\sin x}^{\cos x} (10 + x + y) - (2 - x - y)\, dy\, dx \\ &= \int_0^{\pi/4} \int_{\sin x}^{\cos x} 8 + 2x + 2y\, dy\, dx \\ &= \int_0^{\pi/4} \left(8y + 2xy + y^2\right)\Big|_{\sin x}^{\cos x}\, dx \\ &= \int_0^{\pi/4} (2x + 8)(\cos x - \sin x) + \cos^2 x - \sin^2 x\, dx \\ &= \int_0^{\pi/4} (2x + 8)(\cos x - \sin x) + \cos 2x\, dx \\ &= \left(2x \sin x + 2x \cos x + 6 \sin x + 10 \cos x + \frac{1}{2}\sin 2x\right)\Bigg|_0^{\pi/4} \\ &= \frac{\pi\sqrt{2}}{2} + 8\sqrt{2} - \frac{19}{2}. \end{aligned}$$

The final anti-differentiation requires integration by parts,

$$\int u\, dv = uv - \int v\, du,$$

with $u = x$ and $dv = (\cos x - \sin x)\, dx$. The function $z = 10 + x + y$ is the "top" plane because for $0 \leq x \leq \pi/4$, $\sin x \leq y \leq cosx$, $10 + x + y \geq 2 - x - y$. $\square$

By setting the integrand $f(x, y) \equiv 1$ on a region D, and integrating over D, we can obtain $A(D)$, the area of D.

Example We will compute the area of a half-circle by integrating $f(x, y) \equiv 1$ over a region D that is bounded by the planes $z = 0$, $z = 1$, and $y = 0$, and the surface $y = \sqrt{1 - x^2}$. This surface intersects the plane $y = 0$ along the lines $y = 0$, $x = 1$ and $y = 0$, $x = -1$. Therefore the area is given by

$$A(D) = \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} 1\, dy\, dx = \int_{-1}^{1} y|_0^{\sqrt{1-x^2}}\, dx = \int_{-1}^{1} \sqrt{1 - x^2}\, dx.$$

To evaluate this integral, we use the trigonometric substitution $x = \sin\theta$, for which $dx = \cos\theta\, d\theta$, which yields

$$A(D) = \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta = \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Bigg|_{-\pi/2}^{\pi/2} = \frac{\pi}{2}.$$

□

Changing the Order of Integration

In some cases, a region can be classified as being of *either* type I or type II, and therefore a function can be integrated over the region in two different ways. However, one approach or the other may be impractical, due to the complexity, or even impossibility, of carrying out the anti-differentiation. Therefore, it is important to be able to change the order of integration if necessary.

Example Consider the double integral

$$\int\int_D e^{y^3}\, dA$$

where $D = \{(x,y) \,|\, 0 \leq x \leq 1, \sqrt{x} \leq y \leq 1\}$. This region is defined as a region of type I, so it is natural to attempt to evaluate the iterated integral

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dy\, dx.$$

Unfortunately, it is impossible to anti-differentiate e^{y^3} with respect to y. However, the region D is also a region of type II, as it can be redefined as

$$D = \{(x,y) \,|\, 0 \leq y \leq 1, 0 \leq x \leq y^2\}.$$

We then have

$$\begin{aligned}
\int\int_D e^{y^3}\, dA &= \int_0^1 \int_0^{y^2} e^{y^3}\, dx\, dy \\
&= \int_0^1 xe^{y^3}\Big|_0^{y^2}\, dy \\
&= \int_0^1 y^2 e^{y^3}\, dy \\
&= \frac{1}{3}\int_0^1 e^u\, du, \quad u = y^3, \\
&= \frac{1}{3}e^u\Big|_0^1 \\
&= \frac{1}{3}(e-1).
\end{aligned}$$

It should be noted that usually, when changing the order of integration, it is necessary to use the *inverse functions* of the functions that define the curved portions of the boundary, in order to obtain the limits of the integration of the new inner integral.

The Mean Value Theorem for Integrals

It is important to note that all of the properties of double integrals that have been previously discussed, including linearity, homogeneity, monotonicity, and additivity, apply to double integrals over non-rectangular regions as well. One additional property, that is a consequence of monotonicity, is that if $f(x,y) \geq m$ on a region D, and $f(x,y) \leq M$ on D, then

$$mA(D) \leq \int\int_D f(x,y)\,dA \leq MA(D),$$

where, as before, $A(D)$ is the area of D. Furthermore, if f is continuous on D, then, by the *Mean Value Theorem for Double Integrals*, we have

$$\int\int_D f(x,y)\,dA = f(x_0,y_0)A(D),$$

where (x_0,y_0) is some point in D. This is a generalization of the Mean Value Theorem for Integrals, which is closely related to the Mean Value Theorem for derivatives.

Example Consider the double integral

$$\int\int_D e^y\,dA$$

where D is the triangle defined by $D = \{(x,y)\,|\,0 \leq x \leq 1, 0 \leq y \leq 4x\}$. The area of this triangle is given by $A(D) = \frac{1}{2}bh$, where b, the base, is 1 and h, the height, is 4, which yields $A(D) = 2$. Because $1 \leq e^y \leq e^4$ when $0 \leq y \leq 4$, it follows that

$$2 \leq \int\int_D e^y\,dA \leq 2e^4 \approx 109.2.$$

The exact value is $\frac{1}{4}(e^4 - 5) \approx 12.4$, which is between the above lower and upper bounds. □

Practice Problems

Practice problems from the recommended textbooks are:

- Stewart: Section 12.2, Exercises 1-27 odd, 31-41 odd
- Marsden/Tromba: Section 5.3, Exercises 1, 3, 5, 13; Section 5.4, Exercise 1