Jim Lambers
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Lecture 12 Notes

These notes correspond to Section 2.4 in the text.

Error Analysis for Iterative Methods

In general, nonlinear equations cannot be solved in a finite sequence of steps. As linear equations can be solved using *direct methods* such as Gaussian elimination, nonlinear equations usually require *iterative methods*. In iterative methods, an approximate solution is refined with each iteration until it is determined to be sufficiently accurate, at which time the iteration terminates. Since it is desirable for iterative methods to converge to the solution as rapidly as possible, it is necessary to be able to measure the speed with which an iterative method converges.

To that end, we assume that an iterative method generates a sequence of iterates $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ that converges to the exact solution $\mathbf{x}^*$. Ideally, we would like the error in a given iterate $\mathbf{x}_{k+1}$ to be much smaller than the error in the previous iterate $\mathbf{x}_k$. For example, if the error is raised to a power greater than 1 from iteration to iteration, then, because the error is typically less than 1, it will approach zero very rapidly. This leads to the following definition.

Definition *(Rate of Convergence) Let $\{\mathbf{x}_k\}_{k=0}^{\infty}$ be a sequence in $\mathbb{R}^n$ that converges to $\mathbf{x}^* \in \mathbb{R}^n$ and assume that $\mathbf{x}_k \neq \mathbf{x}^*$ for each k. We say that the* **rate of convergence** *of $\{\mathbf{x}_k\}$ to $\mathbf{x}^*$ is of* **order** *r, with* **asymptotic error constant** *C, if*

$$\lim_{k\to\infty} \frac{\|\mathbf{x}_{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}_k - \mathbf{x}^*\|^r} = C,$$

where $r \geq 1$ and $C > 0$.

If $r = 1$, we say that convergence is *linear*. If $r = 2$, then the method converges *quadratically*, and if $r = 3$, we say it converges *cubically*, and so on.

Of course, we cannot use the error $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}^*$ to determine the rate of convergence experimentally because we do not know the solution $\mathbf{x}^*$. Instead, we can use the relative change in successive iterations, but it is advisable to also compute $\|f(\mathbf{x}_k)\|$ after each iteration to ensure that the iterates are actually converging to a solution of $f(\mathbf{x}) = \mathbf{0}$.

In the remainder of this lecture, we will focus on the case where f is a scalar-valued function of a single variable. We will examine the various methods we have discussed for solving $f(x) = 0$ and attempt to determine their rate of convergence analytically. In this case, we can work with the error $e_k = x_k - x^*$, where x_k is the kth iterate that is computed using the given method, and x^* is the exact solution to which the sequence of iterates $\{x_k\}$ converges.

Bisection

For this method, it is easier to determine the rate of convergence if we use a different measure of the error in each iterate x_k. Since each iterate is contained within an interval $[a_k, b_k]$ where $b_k - a_k = 2^{-k}(b-a)$, with $[a,b]$ being the original interval, it follows that we can bound the error $x_k - x^*$ by $e_k = b_k - a_k$. Using this measure, we can easily conclude that bisection converges linearly, with asymptotic error constant $1/2$.

Fixed-point Iteration

Suppose that we are using Fixed-point Iteration to solve the equation $g(x) = x$, where g is continuously differentiable on an interval $[a,b]$ Starting with the formula for computing iterates in Fixed-point Iteration,

$$x_{k+1} = g(x_k),$$

we can use the Mean Value Theorem to obtain

$$e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*) = g'(\xi_k)(x_k - x^*) = g'(\xi_k)e_k,$$

where ξ_k lies between x_k and x^*.

It follows that if g maps $[a,b]$ into itself, and $|g'(x)| \leq k < 1$ on (a,b) for some constant k, then for any initial iterate $x_0 \in [a,b]$, Fixed-point Iteration converges linearly with asymptotic error constant $|g'(x^*)|$, since, by the definition of ξ_k and the continuity of g',

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|} = \lim_{k\to\infty} |g'(\xi_k)| = |g'(x^*)|.$$

Recall that the conditions we have stated for linear convergence are nearly identical to the conditions for g to have a unique fixed point in $[a,b]$. The only difference is that now, we also require g' to be continuous on $[a,b]$.

Now, suppose that in addition to the previous conditions on g, we assume that $g'(x^*) = 0$, and that g is twice continuously differentiable on $[a,b]$. Then, using Taylor's Theorem, we obtain

$$e_{k+1} = g(x_k) - g(x^*) = g'(x^*)(x_k - x^*) + \frac{1}{2}g''(\xi_k)(x_k - x^*)^2 = \frac{1}{2}g''(\xi_k)e_k^2,$$

where ξ_k lies between x_k and x^*. It follows that for any initial iterate $x_0 \in [a,b]$, Fixed-point Iteration converges at least quadratically, with asymptotic error constant $|g''(x^*)/2|$. This discussion implies the following general result.

Theorem *Let $g(x)$ be a function that is n times continuously differentiable on an interval $[a,b]$. Furthermore, assume that $g(x) \in [a,b]$ for $x \in [a,b]$, and that $|g'(x)| \leq k$ on (a,b) for some constant $k < 1$. If the unique fixed point x^* in $[a,b]$ satisfies*

$$g'(x^*) = g''(x^*) = \cdots = g^{(n-1)}(x^*) = 0,$$

then for any $x_0 \in [a,b]$, Fixed-point Iteration converges to x^* of order n, with asymptotic error constant $|g^{(n)}(x^*)/n!|$.

Newton's Method

Using the same approach as with Fixed-point Iteration, we can determine the convergence rate of Newton's Method applied to the equation $f(x) = 0$, where we assume that f is continuously differentiable near the exact solution x^*, and that f'' exists near x^*. Using Taylor's Theorem, we obtain

$$\begin{aligned}
e_{k+1} &= x_{k+1} - x^* \\
&= x_k - \frac{f(x_k)}{f'(x_k)} - x^* \\
&= e_k - \frac{f(x_k)}{f'(x_k)} \\
&= e_k - \frac{1}{f'(x_k)}\left[f(x^*) - f'(x_k)(x^* - x_k) - \frac{1}{2}f''(\xi_k)(x_k - x^*)^2\right] \\
&= e_k + \frac{1}{f'(x_k)}\left[f'(x_k)(x^* - x_k) + \frac{1}{2}f''(\xi_k)(x_k - x^*)^2\right] \\
&= e_k + \frac{1}{f'(x_k)}\left[-f'(x_k)e_k + \frac{1}{2}f''(\xi_k)e_k^2\right] \\
&= e_k - e_k + \frac{f''(\xi_k)}{2f'(x_k)}e_k^2 \\
&= \frac{f''(\xi_k)}{2f'(x_k)}e_k^2
\end{aligned}$$

where ξ_k is between x_k and x^*. We conclude that if $f'(x^*) \neq 0$, then Newton's Method converges quadratically, with asymptotic error constant $|f''(x^*)/2f'(x^*)|$. It is easy to see from this constant, however, that if $f'(x^*)$ is very small, or zero, then convergence can be very slow or may not even occur.

Example Suppose that Newton's Method is used to find the solution of $f(x) = 0$, where $f(x) = x^2 - 2$. We examine the error $e_k = x_k - x^*$, where $x^* = \sqrt{2}$ is the exact solution. We have

k	x_k	$\|e_k\|$
0	1	0.41421356237310
1	1.5	0.08578643762690
2	1.41666666666667	0.00245310429357
3	1.41421568627457	0.00000212390141
4	1.41421356237469	0.00000000000159

We can determine analytically that Newton's Method converges quadratically, and in this example, the asymptotic error constant is $|f''(\sqrt{2})/2f'(\sqrt{2})| \approx 0.35355$. Examining the numbers in the table above, we can see that the number of correct decimal places approximately doubles with each iteration, which is typical of quadratic convergence. Furthermore, we have

$$\frac{|e_4|}{|e_3|^2} \approx 0.35352,$$

so the actual behavior of the error is consistent with the behavior that is predicted by theory. □

The Secant Method

The convergence rate of the Secant Method can be determined using a result, which we will not prove here, stating that if $\{x_k\}_{k=0}^{\infty}$ is the sequence of iterates produced by the Secant Method for solving $f(x) = 0$, and if this sequence converges to a solution x^*, then for k sufficiently large,

$$|x_{k+1} - x^*| \approx S|x_k - x^*||x_{k-1} - x^*|$$

for some constant S.

We assume that $\{x_k\}$ converges to x^* of order α. Then, dividing both sides of the above relation by $|x_k - x^*|^\alpha$, we obtain

$$\frac{|x_{k+1} - x^*|}{|x_k - x^*|^\alpha} \approx S|x_k - x^*|^{1-\alpha}|x_{k-1} - x^*|.$$

Because α is the rate of convergence, the left side must converge to a positive constant C as $k \to \infty$. It follows that the right side must converge to a positive constant as well, as must its reciprocal. In other words, there must exist positive constants C_1 and C_2

$$\frac{|x_k - x^*|}{|x_{k-1} - x^*|^\alpha} \to C_1, \quad \frac{|x_k - x^*|^{\alpha-1}}{|x_{k-1} - x^*|} \to C_2.$$

This can only be the case if there exists a nonzero constant β such that

$$\frac{|x_k - x^*|}{|x_{k-1} - x^*|^\alpha} = \left(\frac{|x_k - x^*|^{\alpha-1}}{|x_{k-1} - x^*|}\right)^\beta,$$

which implies that

$$1 = (\alpha - 1)\beta \quad \text{and} \quad \alpha = \beta.$$

Eliminating β, we obtain the equation

$$\alpha^2 - \alpha - 1 = 0,$$

which has the solutions

$$\alpha_1 = \frac{1 + \sqrt{5}}{2} \approx 1.618, \quad \alpha_2 = \frac{1 - \sqrt{5}}{2} \approx -0.618.$$

Since we must have $\alpha > 1$, the rate of convergence is 1.618.