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MAT 460/560
Fall Semester 2009-10
Lecture 10 Notes

These notes correspond to Section 2.3 in the text.

Newton's Method

In the previous lecture, we developed a simple method, bisection, for approximately solving the equation $f(x) = 0$. Unfortunately, this method, while guaranteed to find a solution on an interval that is known to contain one, is not practical because of the large number of iterations that are necessary to obtain an accurate approximate solution.

To develop a more effective method for solving this problem of computing a solution to $f(x) = 0$, we can address the following questions:

- Are there cases in which the problem is easy to solve, and if so, how do we solve it in such cases?
- Is it possible to apply our method of solving the problem in these “easy” cases to more general cases?

In this course, we will see that these questions are useful for solving a variety of problems. For the problem at hand, we ask whether the equation $f(x) = 0$ is easy to solve for any particular choice of the function f . Certainly, if f is a linear function, then it has a formula of the form $f(x) = m(x - a) + b$, where m and b are constants and $m \neq 0$. Setting $f(x) = 0$ yields the equation

$$m(x - a) + b = 0,$$

which can easily be solved for x to obtain the unique solution

$$x = a - \frac{b}{m}.$$

We now consider the case where f is not a linear function. Using Taylor's theorem, it is simple to construct a linear function that approximates $f(x)$ near a given point x_0 . This function is simply the first Taylor polynomial of $f(x)$ with center x_0 ,

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0).$$

This function has a useful geometric interpretation, as its graph is the tangent line of $f(x)$ at the point $(x_0, f(x_0))$.

We can obtain an approximate solution to the equation $f(x) = 0$ by determining where the linear function $P_1(x)$ is equal to zero. If the resulting value, x_1 , is not a solution, then we can repeat

this process, approximating f by a linear function near x_1 and once again determining where this approximation is equal to zero. The resulting algorithm is *Newton's method*, which we now describe in detail.

Algorithm (Newton's Method) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. The following algorithm computes an approximate solution x^* to the equation $f(x) = 0$.

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Choose an initial guess  $x_0$ 
for  $k = 0, 1, 2, \dots$  do
    if  $f(x_k)$  is sufficiently small then
         $x^* = x_k$ 
        return  $x^*$ 
    end
     $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$ 
    if  $|x_{k+1} - x_k|$  is sufficiently small then
         $x^* = x_{k+1}$ 
        return  $x^*$ 
    end
end

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When Newton's method converges, it does so very rapidly. However, it can be difficult to ensure convergence, particularly if $f(x)$ has horizontal tangents near the solution x^* . Typically, it is necessary to choose a starting iterate x_0 that is close to x^* . As the following result indicates, such a choice, if close enough, is indeed sufficient for convergence.

Theorem (*Convergence of Newton's Method*) Let f be twice continuously differentiable on the interval $[a, b]$, and suppose that $f(c) = 0$ and $f'(c) \neq 0$ for some $c \in [a, b]$. Then there exists a $\delta > 0$ such that Newton's Method applied to $f(x)$ converges to c for any initial guess x_0 in the interval $[c - \delta, c + \delta]$.

Example We will use of Newton's Method in computing $\sqrt{2}$. This number satisfies the equation $f(x) = 0$ where

$$f(x) = x^2 - 2.$$

Since $f'(x) = 2x$, it follows that in Newton's Method, we can obtain the next iterate x_{n+1} from the previous iterate x_n by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = x_n - \frac{x_n^2}{2x_n} + \frac{2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}.$$

We choose our starting iterate $x_0 = 1$, and compute the next several iterates as follows:

$$x_1 = \frac{1}{2} + \frac{1}{1} = 1.5$$

$$\begin{aligned}
x_2 &= \frac{1.5}{2} + \frac{1}{1.5} = 1.41666667 \\
x_3 &= 1.41421569 \\
x_4 &= 1.41421356 \\
x_5 &= 1.41421356.
\end{aligned}$$

Since the fourth and fifth iterates agree in to eight decimal places, we assume that 1.41421356 is a correct solution to $f(x) = 0$, to at least eight decimal places. The first two iterations are illustrated in Figure 1. \square

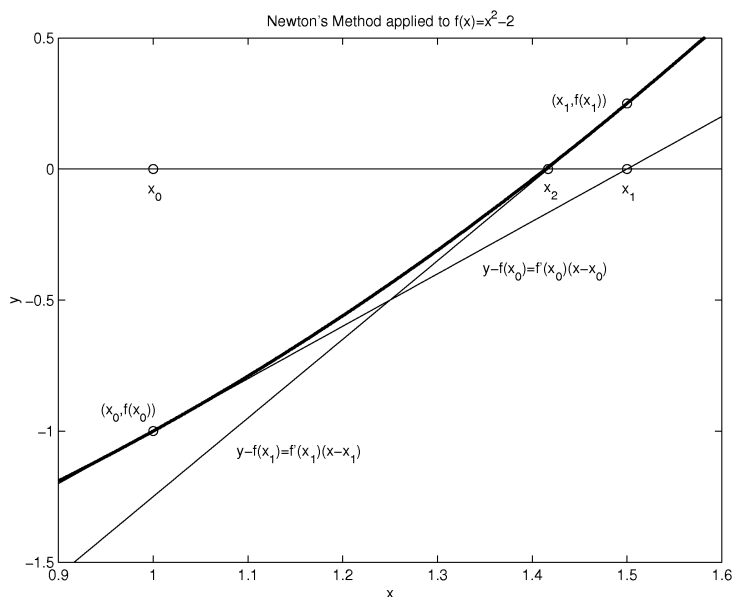


Figure 1: Newton's Method applied to $f(x) = x^2 - 2$. The bold curve is the graph of f . The initial iterate x_0 is chosen to be 1. The tangent line of $f(x)$ at the point $(x_0, f(x_0))$ is used to approximate $f(x)$, and it crosses the x -axis at $x_1 = 1.5$, which is much closer to the exact solution than x_0 . Then, the tangent line at $(x_1, f(x_1))$ is used to approximate $f(x)$, and it crosses the x -axis at $x_2 = 1.41\bar{6}$, which is already very close to the exact solution.

Example Newton's Method can be used to compute the reciprocal of a number a *without performing any divisions*. The solution, $1/a$, satisfies the equation $f(x) = 0$, where

$$f(x) = a - \frac{1}{x}.$$

Since

$$f'(x) = \frac{1}{x^2},$$

it follows that in Newton's Method, we can obtain the next iterate x_{n+1} from the previous iterate x_n by

$$x_{n+1} = x_n - \frac{a - 1/x_n}{1/x_n^2} = x_n - \frac{a}{1/x_n} + \frac{1/x_n}{1/x_n^2} = 2x_n - ax_n^2.$$

Note that no divisions are necessary to obtain x_{n+1} from x_n . This iteration was actually used on older IBM computers to implement division in hardware.

We use this iteration to compute the reciprocal of $a = 12$. Choosing our starting iterate to be 0.1, we compute the next several iterates as follows:

$$\begin{aligned} x_1 &= 2(0.1) - 8(0.1)^2 = 0.08 \\ x_2 &= 2(0.12) - 8(0.12)^2 = 0.0832 \\ x_3 &= 0.0833312 \\ x_4 &= 0.083333333279 \\ x_5 &= 0.083333333333. \end{aligned}$$

We conclude that 0.083333333333 is an accurate approximation to the correct solution.

Now, suppose we repeat this process, but with an initial iterate of $x_0 = 1$. Then, we have

$$\begin{aligned} x_1 &= 2(1) - 12(1)^2 = -10 \\ x_2 &= 2(-6) - 12(-6)^2 = -1220 \\ x_3 &= 2(-300) - 12(-300)^2 = -17863240 \end{aligned}$$

It is clear that this sequence of iterates is not going to converge to the correct solution. In general, for this iteration to converge to the reciprocal of a , the initial iterate x_0 must be chosen so that $0 < x_0 < 2/a$. This condition guarantees that the next iterate x_1 will at least be positive. The contrast between the two choices of x_0 are illustrated in Figure 2. \square

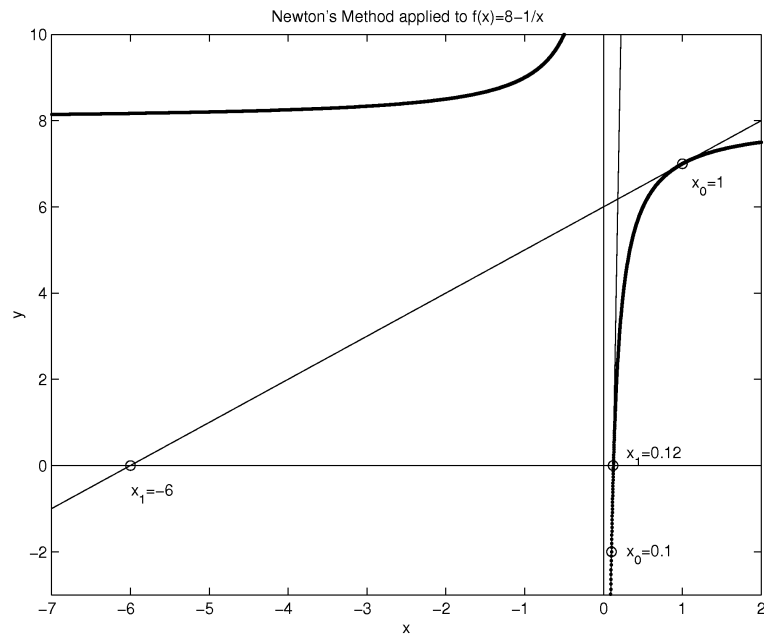


Figure 2: Newton's Method used to compute the reciprocal of 8 by solving the equation $f(x) = 8 - 1/x = 0$. When $x_0 = 0.1$, the tangent line of $f(x)$ at $(x_0, f(x_0))$ crosses the x -axis at $x_1 = 0.12$, which is close to the exact solution. When $x_0 = 1$, the tangent line crosses the x -axis at $x_1 = -6$, which causes searching to continue on the wrong portion of the graph, so the sequence of iterates does not converge to the correct solution.