Fall Semester '07-'08
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LECTURE 9: CONSTRAINED OPTIMIZATION

I. INTRODUCTION

- The next section of optimization is to move from the world of unconstrained optimization to the world of constrained optimization. The primary techniques, dealing with Lagrange multipliers should be familiar from Econ 201. We will spend some of this lecture reviewing basic concepts from that class, but also dig a little bit deeper into the underlying theory. As always, we will then do some applications drawn from various fields of economics.
- Constrained optimization is the mathematics concept most central to economics. Economics is inherently a framework for studying a world in which individuals, firms and governments make decisions that seem best to them given the inherent limitations on their behavior. The most common constraints are limitations on time, money and other resources.
- Most of these constraints tend to be binding, or in other words we tend to use up all the available resources in making the best decision. So the main techniques we will cover deal with optimization with equality constraints: i.e. maximizing some measure of well being subject to a constraint that holds exactly. We may also talk about techniques for optimization with inequality constraints in a subsequent lecture.

II. SOLUTION USING THE SUBSTITUTION METHOD

- A typical constrained optimization problem is of the form: $\max\limits_{x_1,\cdots,x_n} f\left(x_1,\cdots,x_n;\beta\right)$ subject to the existence of m constraints $g^j\left(x_1,\cdots,x_n;\beta\right) = c_j$ for $i = 1,2,\cdots,m$ where the c_j are arbitrary constants and β represents exogenous variables, of which there may be several. In this setting, f is known as the objective function.
- How does the presence of a constraint change the optimization problem? Essentially, what the constraint does is impose restrictions on the domain of the function. So the solution to a constrained optimization problem is the optimum value that the function takes on, over the restricted part of the domain that is consistent with the constraint.
- The unconstrained minimum (or maximum) is typically smaller (or larger) than the constrained minimum (or maximum). It's rarely that the constrained and unconstrained solutions match up.
- We can solve constrained optimization problems in two ways: through the substitution method or by use of the Lagrange multiplier method. We will first illustrate the substitution method using a simple mathematical example.
- Consider the following problem: $\min\limits_{x,y} f(x,y) \equiv x^2 + 2y^2 + 2xy - 18$ subject to the constraint $g(x,y) \equiv x - y = 1$.
- We are looking for the minimum value for $f(x,y)$ over the domain of x,y that satisfy $x-y=1$

- The substitution method is very simple: essentially, we substitute the constraints into the objective function to reduce the problem to an unconstrained problem.

- Substituting in the constraints gives us the following optimization

$$\min_y g(y) = (1+y)^2 + 2y^2 + 2y(1+y) - 18 \Rightarrow \min_y g(y) = 5y^2 + 4y - 17$$

- Note that we now have a unconstrained maximization over a single variable, y, instead of a constrained maximization over two variables x, y.

- Also note that this is a *different* unconstrained problem. The unconstrained minimum of $g(y) = 5y^2 + 4y - 17$ will be different from the unconstrained minimization of $f(x,y) = x^2 + 2y^2 + 2xy - 18$

- Since this is an unconstrained problem the FOC and SOC from the last class can be used here to derive the candidate solution and to verify it is indeed a minimum.

- The FOC is $g'(y^*) = 0 \Rightarrow 10y^* + 4 = 0 \Rightarrow y^* = -0.4$. The SOC is $g''(y^*) = 10 > 0$, which shows that the function is strictly convex, and thus the solution is a global minimum.

- We can now calculate the corresponding value of $x^* = 1 + y^*$ to be $x^* = 0.6$. The value of the constrained minimum is $f(x^*, y^*) = 0.6^2 + 2(0.4)^2 + 2(0.4)(0.6) - 18 = -17.8$

- Note that even though the substitution converted the problem to an unconstrained optimization, we still are looking over a restricted domain. The unconstrained optimization of $\min_{x,y} f(x,y) \equiv x^2+2y^2+2xy-18$ yields the FOCs $2x+2y=0$ and $4y+2x=0$ which implies that the unconstrained optimum is at $x^*=0$ and $y^*=0$. At these values, the unconstrained minimum of the function is -18. Since $(x^*=0, y^*=0)$ is not in the portion of the domain that satisfies the constraint, it was not considered in the constrained optimization.

III. SOLUTION USING THE LAGRANGE MULTIPLIER METHOD

- Substituting the constraint is not always easy to do, so instead we typically solve constrained optimization problems using the Lagrange multiplier method.

- Given a constrained optimization problem of the form: $\max\limits_{x_1,\cdots,x_n} f(x_1, \cdots, x_n; \beta)$ subject to the existence of m constraints $g^j(x_1, \cdots, x_n; beta) = c_j$ we write down a function known as the Lagrangian function.

$$\mathcal{L} = f(x_1, \cdots, x_n; \beta) + \sum_{j=1}^{m} \lambda_j \left[c_j - g^j(x_1, \cdots, x_n; \beta)\right]$$

- Note that the constraints are written in the form $c_j - g^j(x_1, \cdots, x_n; \beta)$ rather than in the form $g^j(x_1, \cdots, x_n; \beta) - c_j$. We will discuss the reason for writing the constraint in this fashion in a subsequent lecture.

First Order Conditions

- We can then treat the constrained optimization decision as an unconstrained optimization of the Lagrangian function with respect to x_i for $i = 1, 2, \cdots, n$ and λ_j for $j = 1, 2, \cdots, m$, i.e. by solving

$$\max_{x_1,\cdots,x_n,\lambda_1,\cdots,\lambda_j} \mathcal{L} \equiv \max_{x_1,\cdots,x_n,\lambda_1,\cdots,\lambda_j} f(x_1, \cdots, x_n; \beta) + \sum_{j=1}^{m} \lambda_j \left(c_j - g^j(x_1, \cdots, x_n; \beta)\right)$$

- The FOCs are of the form:

$$\begin{aligned} f_i(x_1, \cdots, x_n; \beta) - \sum_{j=1}^{m} \lambda_j \left[\frac{\partial g^j(x_1, \cdots, x_n; \beta)}{\partial x_i}\right] &= 0 \\ g^j(x_1, \cdots, x_n; \beta) &= c_j \end{aligned}$$

- Note that there are $(n+m)$ FOC, for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$

Example

In terms of our example $\min_{x,y} f(x, y) \equiv x^2 + 2y^2 + 2xy - 18$ subject to the constraint $g(x, y) \equiv x - y = 1$ we would write the Lagrangian as

$$\min_{x,y,\lambda} \mathcal{L} \equiv \min_{x,y,\lambda} x^2 + 2y^2 + 2xy - 18 + \lambda(1 - x + y)$$

- Since there are two variables $(i = 2)$ and one constraint $(j = 1)$, we should have 3 FOC.

$$\begin{aligned} 2x + 2y - \lambda &= 0 \\ 4y + 2x + \lambda &= 0 \\ 1 - x + y &= 0 \end{aligned}$$

- In matrix notation we have

$$\begin{bmatrix} 2 & 2 & -1 \\ 2 & 4 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Using Cramer's Rule we can solve for:

$$x = \frac{\begin{vmatrix} 0 & 2 & -1 \\ 0 & 4 & 1 \\ 1 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 2 & -1 \\ 2 & 4 & 1 \\ 1 & -1 & 0 \end{vmatrix}} = \frac{6}{10}, y = \frac{\begin{vmatrix} 2 & 0 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 2 & -1 \\ 2 & 4 & 1 \\ 1 & -1 & 0 \end{vmatrix}} = \frac{-4}{10}, \lambda = \frac{\begin{vmatrix} 2 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 2 & -1 \\ 2 & 4 & 1 \\ 1 & -1 & 0 \end{vmatrix}} = \frac{4}{10}$$

Second Order Conditions

- We can also think about the second order conditions that ensure whether the solution is a minimum or a maximum.

- Under the Lagrange multiplier method, we reduced the constrained optimization problem $\max_{x_1,\cdots,x_n} f(x_1,\cdots,x_n)$ subject to $g^j(x_1,\cdots,x_n) = c_j$ for $j = 1\cdots m$ into the unconstrained maximization problem

$$\mathcal{L} = \max_{x_1,\cdots,x_n,\lambda_1,\cdots,\lambda_j} f(x_1,\cdots,x_n) + \sum_{j=1}^{m} \lambda_j \left(c_j - g^j(x_1,\cdots,x_n)\right)$$

- Technically, it may seem like we should be able to use the results from the SOC for an unconstrained optimization problem to solve for the SOC for a constrained optimization problem. In other words, we can derive the Hessian, then check whether either the sufficient condition for a local maximum ($|H_1| < 0, |H_2| > 0, |H_3| < 0, \cdots, (-1)^n |H_n| > 0$) or a local minimum ($|H_1| > 0, |H_2| > 0, |H_3| > 0, \cdots, |H_n| > 0$) was satisfied.

- Unfortunately, we need to change this approach when dealing with this type of a problem. This change occurs mainly because the Lagrange multipliers only appear once, in front of each constraint. As a result, the $L_{\lambda_j\lambda_j}$ terms are all zero. Since the Hessian made no claims about the proper ordering of variables, we could end up with the first few principal minors being zero!

- Instead, in the constrained case, we construct a matrix known as the **Bordered Hessian**. The Bordered Hessian will be an $(n+m)\times(n+m)$ Hessian matrix, where we take derivatives with respect to the Lagrange multipliers first.

$$H^B = \begin{bmatrix} L_{\lambda_1\lambda_1} & L_{\lambda_1\lambda_2} & \cdots & L_{\lambda_1\lambda_m} & L_{\lambda_1 x_1} & L_{\lambda_1 x_2} & \cdots & L_{\lambda_1 x_n} \\ L_{\lambda_2\lambda_1} & L_{\lambda_2\lambda_2} & \cdots & L_{\lambda_2\lambda_m} & L_{\lambda_2 x_1} & L_{\lambda_2 x_2} & \cdots & L_{\lambda_2 x_n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ L_{\lambda_m\lambda_1} & L_{\lambda_m\lambda_2} & \cdots & L_{\lambda_m\lambda_m} & L_{\lambda_m x_1} & L_{\lambda_m x_2} & \cdots & L_{\lambda_m x_n} \\ L_{x_1\lambda_1} & L_{x_1\lambda_2} & \cdots & L_{x_1\lambda_m} & L_{x_1 x_1} & L_{x_1 x_2} & \cdots & L_{x_1 x_n} \\ L_{x_2\lambda_1} & L_{x_2\lambda_2} & \cdots & L_{x_2\lambda_m} & L_{x_2 x_1} & L_{x_2 x_2} & \cdots & L_{x_2 x_n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ L_{x_n\lambda_1} & L_{x_n\lambda_2} & \cdots & L_{x_n\lambda_m} & L_{x_n x_1} & L_{x_n x_2} & \cdots & L_{x_n x_n} \end{bmatrix}$$

- The sufficient conditions involve only the last (largest) n principal minors needs to be applied. In other words, we omit the first m principal minors (which happen to be all zeros anyway!!!!!!!!)

- If the last n minors, evaluated at the extreme point, alternate in sign, provided that that the determinant of the Hessian itself has the sign of $(-1)^n$, then the point is a local maximum.

- On the other hand if the largest n minors, evaluated at the extreme point, all have the same sign, and that sign is $(-1)^m$, then the point is a local minimum.

- With a single constraint this means that a local minimum would require that all the principal minors be negative (unlike in the unconstrained case where they would be positive), while a local maximum would require that the principal minors alternate in sign with the last one being positive if there are an even number of FOC or negative if there are a odd number of FOC (this would be identical to the unconstrained case).

Example 1: Profit Maximization

- Consider a monopolist firm that can sell a given quantity $\bar{Q}$ of its products in 2 markets: one with the inverse demand curve $P_A = \alpha - \beta Q_A$ and the other with the inverse demand curve $P_B = \gamma - \delta Q_B$
- Assume that the total cost function for this company is given by $C(\bar{Q}) = a\bar{Q}^2$ where $\bar{Q} = Q_A + Q_B$. Solve for the profit maximizing quantity of goods produced, and also calculate how much will be sold in each of the markets.
- The profit function $\Pi(Q_A, Q_B) = [\alpha - \beta Q_A]\, Q_A + [\gamma - \delta Q_B]\, Q_B - a\bar{Q}^2$
- The maximization decision can be written as

$$\max_{Q_A, Q_B, \lambda} [\alpha - \beta Q_A]\, Q_A + [\gamma - \delta Q_B]\, Q_B - a\bar{Q}^2 \text{ subject to } \bar{Q} = Q_A + Q_B$$

- The Lagrangian function for this problem is

$$\max_{Q_A, Q_B, \lambda} \mathcal{L} = \max_{Q_A, Q_B, \lambda} [\alpha - \beta Q_A]\, Q_A + [\gamma - \delta Q_B]\, Q_B - a\bar{Q}^2 + \lambda[\bar{Q} - Q_A - Q_B]$$

- The FOCs are

$$\begin{aligned} \alpha - 2\beta Q_A - \lambda &= 0 \\ \gamma - 2\delta Q_B - \lambda &= 0 \\ \bar{Q} - Q_A - Q_B &= 0 \end{aligned}$$

- In matrix notation we have

$$\begin{bmatrix} 2\beta & 0 & 1 \\ 0 & 2\delta & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q_A \\ Q_B \\ \lambda \end{bmatrix} = \begin{bmatrix} \alpha \\ \gamma \\ \bar{Q} \end{bmatrix}$$

- Using Cramer's Rule we can solve for:

$$Q_A = \frac{\begin{vmatrix} \alpha & 0 & 1 \\ \gamma & 2\delta & 1 \\ \bar{Q} & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 2\beta & 0 & 1 \\ 0 & 2\delta & 1 \\ 1 & 1 & 0 \end{vmatrix}} = \frac{(\alpha - \gamma) + 2\delta\bar{Q}}{2(\beta + \delta)}, Q_B = \frac{\begin{vmatrix} 2\beta & \alpha & 1 \\ 0 & \gamma & 1 \\ 1 & \bar{Q} & 0 \end{vmatrix}}{\begin{vmatrix} 2\beta & 0 & 1 \\ 0 & 2\delta & 1 \\ 1 & 1 & 0 \end{vmatrix}} = \frac{(\gamma - \alpha) + 2\beta\bar{Q}}{2(\beta + \delta)}$$

$$\lambda = \frac{\begin{vmatrix} 2\beta & 0 & \alpha \\ 0 & 2\delta & \gamma \\ 1 & 1 & \bar{Q} \end{vmatrix}}{\begin{vmatrix} 2\beta & 0 & 1 \\ 0 & 2\delta & 1 \\ 1 & 1 & 0 \end{vmatrix}} = \frac{(\alpha\delta + \beta\gamma) - 2\delta\beta\bar{Q}}{2(\beta+\delta)}$$

- We can verify the second-order conditions here by constructing the Bordered Hessian

$$H^B = \begin{bmatrix} \Pi_{\lambda\lambda} & \Pi_{\lambda Q_A} & \Pi_{\lambda Q_B} \\ \Pi_{Q_A\lambda} & \Pi_{Q_A Q_A} & \Pi_{Q_A Q_B} \\ \Pi_{Q_B\lambda} & \Pi_{Q_B Q_A} & \Pi_{Q_B Q_B} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -2\beta & 0 \\ -1 & 0 & -2\delta \end{bmatrix}$$

- The two (n-m=3-1=2) largest principal minors are: $\left|H_2^B\right| = \begin{vmatrix} 0 & -1 \\ -1 & -2\beta \end{vmatrix} = -1$ and

$$\left|H_3^B\right| = \begin{vmatrix} 0 & -1 & -1 \\ -1 & -2\beta & 0 \\ -1 & 0 & -2\delta \end{vmatrix} = 2\beta + 2\delta > 0$$

- Since the principal minors alternate in sign with the determinant of the Bordered Hessian itself being of sign $(-1)^2$, i.e. positive, therefore, the solution is a local maximum.

Example 2: Utility Maximization

- Consider a consumer whose utility function is $U(C_1, C_2) = 2\sqrt{C_1 C_2}$ with an income of \$Y where the price of good 1 is P_1 and the price of good 2 is P_2.
- The maximization decision can be written as

$$\max_{C_1,C_2} U(C_1, C_2) \equiv 2\sqrt{C_1 C_2} \text{ subject to } P_1C_1 + P_2C_2 = Y$$

- The Lagrangian function for this problem is

$$\max_{C_1,C_2,\lambda} \mathcal{L} \equiv \max_{C_1,C_2,\lambda} 2\sqrt{C_1C_2} + \lambda(Y - P_1C_1 - P_2C_2)$$

- The FOCs are

$$\begin{aligned} \sqrt{\frac{C_2}{C_1}} - \lambda P_1 &= 0 \\ \sqrt{\frac{C_1}{C_2}} - \lambda P_2 &= 0 \\ Y - P_1C_1 - P_2C_2 &= 0 \end{aligned}$$

- Since this is a non-linear system of equations, we will not be able to set it up in matrix notation and use Cramer's Rule to solve it. However, the problem is simple enough to solve by hand: use equation 1 and 2 to express $\frac{C_2}{C_1} = \frac{P_1}{P_2}$ or $C_2 = \left(\frac{P_1}{P_2}\right) C_1$. Plug this into equation 3 to get $C_1^* = \frac{Y}{2P_1}$ and $C_2^* = \frac{Y}{2P_2}$. We can also solve easily for $\lambda^* = \frac{1}{\sqrt{P_1P_2}}$.

- The optimal level of utility that this consumer can achieve, given the constraints is a utility level of $U^* = \frac{Y}{\sqrt{P_1 P_2}}$
- We can verify the second-order conditions here by constructing the Bordered Hessian

$$H^B = \begin{bmatrix} U_{\lambda\lambda} & U_{\lambda C_1} & U_{\lambda C_2} \\ U_{C_1\lambda} & U_{C_1C_1} & U_{C_1C_2} \\ U_{C_2\lambda} & U_{C_2C_1} & U_{C_2C_2} \end{bmatrix} = \begin{bmatrix} 0 & -P_1 & -P_2 \\ -P_1 & -\frac{1}{2C_1}\sqrt{\frac{C_2}{C_1}} & \frac{1}{2\sqrt{C_2C_1}} \\ -P_2 & \frac{1}{2\sqrt{C_2C_1}} & -\frac{1}{2C_2}\sqrt{\frac{C_1}{C_2}} \end{bmatrix}$$

- The two largest (n-m=3-1) principal minors are: $\left|H_2^B\right| = \begin{vmatrix} 0 & -P_1 \\ -P_1 & -\frac{1}{2C_1}\sqrt{\frac{C_2}{C_1}} \end{vmatrix} = -P_1^2$ and

$$\left|H_3^B\right| = \begin{vmatrix} 0 & -P_1 & -P_2 \\ -P_1 & -\frac{1}{2C_1}\sqrt{\frac{C_2}{C_1}} & \frac{1}{2\sqrt{C_2C_1}} \\ -P_2 & \frac{1}{2\sqrt{C_2C_1}} & -\frac{1}{2C_2}\sqrt{\frac{C_1}{C_2}} \end{vmatrix} = \frac{P_1P_2}{\sqrt{C_2C_1}} + \frac{P_2^2}{2C_1}\sqrt{\frac{C_2}{C_1}} + \frac{P_1^2}{2C_2}\sqrt{\frac{C_1}{C_2}} > 0$$

- Since the principal minors alternate in sign with the determinant of the Bordered Hessian itself being of sign $(-1)^2$, i.e. positive, therefore, the solution is a local maximum.

IV. QUASI-CONCAVITY/QUASI-CONVEXITY

- The last task is to derive a set of conditions for global minimums and global maximums for constrained optimization problems. These will be analogous to the unconstrained case, where testing for global minima and maxima required that we test whether the function f was convex or concave, respectively.
- In the constrained optimization case, we need to do something similar, except that we need to test whether f satisfies properties known as "quasi-concavity" and "quasi-convexity".

Quasiconcavity

- Quasi-concavity is a weaker form of concavity. A function $f(x)$ is said to be **quasiconcave** if for any points x_1 and x_2 in the domain of the function $f(\lambda x_1 + (1-\lambda)x_2) \geq \min(f(x_1), f(x_2))$. In other words if you take any two points of the function, all intermediate points are at least as high as the lower point.
- A function $f(x)$ is said to be **strictly quasiconcave** if $f(\lambda x_1 + (1-\lambda)x_2) > \min(f(x_1), f(x_2))$. In other all intermediate points are strictly higher than the lower point.
- Note that this is a much weaker condition than concavity. If the function is concave, then for any two points all intermediate points are higher than the line connecting the two points. A quasiconcave function allows the intermediate points to be less than the line as long as they stay above the lower point.
- We can illustrate this with a simple example: Consider the function $y = x^3$ this is clearly not a concave function except in the sub region $x < 0$. However, it satisfies the conditions for quasi-concavity since the value of the function at any intermediate point between two points is higher than the lower point.

- Intuitively, quasiconcave functions are functions that either have only one peak or are monotonic. The function itself can have concave and convex parts.
- Note that concavity is a sufficient but not necessary condition for quasiconcavity, i.e. concavity implies quasiconcavity but not vice versa; therefore we can't have functions that are concave but not quasiconcave.

Quasiconvexity

- A function $f(x)$ is said to be **quasiconvex** if $f(\lambda x_1 + (1-\lambda)x_2) \leq \max(f(x_1), f(x_2))$. In other words if you take any two points of the function, all intermediate points are at least as low as the higher point.
- A function $f(x)$ is said to be **strictly quasiconvex** if $f(\lambda x_1+(1-\lambda)x_2) < \max(f(x_1), f(x_2))$. In other words if you take any two points of the function, all intermediate points are strictly lower than the higher point.
- Again, this is a much weaker condition than convexity. With a convex function then for any two points of the function all intermediate points are lower than the line connecting the two points. A quasiconvex function allows the intermediate points to be higher than the line as long as they stay below the higher end point.
- We can illustrate quasiconvexity with a simple example: Consider the function $y = x^3$ again. This is clearly not a convex function except in the sub region $x > 0$. However, it satisfies the conditions for quasi-concavity since the value of the function at any intermediate point between two points is lower than the higher point.
- Intuitively, quasiconvex functions are functions that either have only one valley or are monotonic. The function itself can have concave and convex parts.

A Derivative Test for Quasiconvexity/Quasiconvexity

- If the function is differentiable, we can verify quasiconcavity and quasiconvexity by constructing an object known as the Bordered Determinant (which is distinct from the Bordered Hessian!!!!).
- Given a function of the form $f(x_1, x_2, \cdots, x_n)$, the Bordered Determinant is of the form

$$B = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

- Essentially this is the determinant of a matrix with a border made up of first partials, with a zero as the first element.
- The function f is quasiconcave if the principal minors $|B_2| \leq 0, |B_3| \geq 0, |B_4| \leq 0, \cdots, (-1)^n |B_n| \leq 0$

- The definition of strict quasiconcavity requires that the $\leq$ and $\geq$ signs be changed to $<$ and $>$ respectively.

- The function f is quasiconvex if the principal minors $|B_2| \leq 0, |B_3| \leq 0, |B_4| \leq 0, \cdots, |B_n| \leq 0$

- The definition of strict quasiconvexity requires that the $\leq$ signs be changed to $<$ signs.

Second Order Condition for Global Constrained Maxima/Minima

- Given a maximization problem of the form $\max_{x_1, x_2, \cdots x_n} f(x_1, x_2, \cdots x_n)$ subject to $g(x_1, x_2, \cdots x_n)$, the sufficient SOC for the solution to be a constrained global maximum is that f be quasiconcave and that the constraint be linear. If f is strictly quasiconcave, then the function has a unique constrained global maximum.

- Given a minimization problem of the form $Maxf(x_1, x_2, \cdots x_n)$ subject to $g(x_1, x_2, \cdots x_n)$, the sufficient SOC for the solution to be a constrained minimum is that f be quasiconvex and that the constraint be linear. If f is strictly quasiconvex, then the function has a unique constrained global minimum.

Example 1: Utility Maximization

- The most common application of constrained optimization is in analyzing the behavior of individuals who maximize utility subject to their budget constraint.

- Consider the following simple example of a consumer whose utility function is $U(C_1, C_2) = 2\sqrt{C_1 C_2}$ with an income of \$Y where the price of good 1 is \$3 and the price of good 2 is \$1.

- We can construct the Bordered Determinant using the fact that $U_{C_1} = \sqrt{\frac{C_2}{C_1}}$,$U_{C_2} = \sqrt{\frac{C_1}{C_2}}$,$U_{C_1,C_1} = -\frac{1}{2C_1}\sqrt{\frac{C_2}{C_1}}$,$U_{C_2,C_2} = -\frac{1}{2C_2}\sqrt{\frac{C_1}{C_2}}$, $U_{C_1,C_2} = U_{C_2,C_1} = \frac{1}{2\sqrt{C_2 C_1}}$:

$$B = \begin{vmatrix} 0 & \sqrt{\frac{C_2}{C_1}} & \sqrt{\frac{C_1}{C_2}} \\ \sqrt{\frac{C_2}{C_1}} & -\frac{1}{2C_1}\sqrt{\frac{C_2}{C_1}} & \frac{1}{2\sqrt{C_2 C_1}} \\ \sqrt{\frac{C_1}{C_2}} & \frac{1}{2\sqrt{C_2 C_1}} & -\frac{1}{2C_2}\sqrt{\frac{C_1}{C_2}} \end{vmatrix}$$

- The principal minors are: $|B_2| = \begin{vmatrix} 0 & \sqrt{\frac{C_2}{C_1}} \\ \sqrt{\frac{C_2}{C_1}} & -\frac{1}{2C_1}\sqrt{\frac{C_2}{C_1}} \end{vmatrix} = -\frac{C_2}{C_1} < 0$ and
$|B_3| = \begin{vmatrix} 0 & \sqrt{\frac{C_2}{C_1}} & \sqrt{\frac{C_1}{C_2}} \\ \sqrt{\frac{C_2}{C_1}} & -\frac{1}{2C_1}\sqrt{\frac{C_2}{C_1}} & \frac{1}{2\sqrt{C_2 C_1}} \\ \sqrt{\frac{C_1}{C_2}} & \frac{1}{2\sqrt{C_2 C_1}} & -\frac{1}{2C_2}\sqrt{\frac{C_1}{C_2}} \end{vmatrix} = \frac{2}{\sqrt{C_2 C_1}} > 0$

- The principal minors alternate in sign, therefore, the utility function is quasi-concave. Combined with the fact that the constraint is linear, we know that the solution is a global maximum.

Examples 2: Profit Maximization

- Consider a monopolist firm with the profit function $\Pi(Q_A, Q_B) = [\alpha - \beta Q_A]\, Q_A + [\gamma - \delta Q_B]\, Q_B - a\bar{Q}^2$
- We can construct the Bordered Determinant using the fact that $\Pi_{Q_A} = \alpha - 2\beta Q_A$,$\Pi_{Q_B} = \gamma - 2\delta Q_B$,$\Pi_{Q_A,Q_A} = -2\beta$,$\Pi_{Q_B,Q_B} = -2\delta$, $\Pi_{Q_A,Q_B} = \Pi_{Q_B,Q_A} = 0$:

$$B = \begin{vmatrix} 0 & \alpha - 2\beta Q_A & \gamma - 2\delta Q_B \\ \alpha - 2\beta Q_A & -2\beta & 0 \\ \gamma - 2\delta Q_B & 0 & -2\delta \end{vmatrix}$$

- The principal minors are: $|B_2| = \begin{vmatrix} 0 & \alpha - 2\beta Q_A \\ \alpha - 2\beta Q_A & -2\beta \end{vmatrix} = -(\alpha - 2\beta Q_A)^2 < 0$ and $|B_3| = \begin{vmatrix} 0 & \alpha - 2\beta Q_A & \gamma - 2\delta Q_B \\ \alpha - 2\beta Q_A & -2\beta & 0 \\ \gamma - 2\delta Q_B & 0 & -2\delta \end{vmatrix} = (2\beta)(\gamma - 2\delta Q_A)^2 + (2\delta)(\alpha - 2\beta Q_A)^2 > 0$
- The principal minors alternate in sign, therefore, the profit function is quasi-concave. Combined with the fact that the constraint is linear, we know that the solution is a global maximum.