8.2 CONJUGATES AND DIVISION OF COMPLEX NUMBERS

In Section 8.1, we mentioned that the complex zeros of a polynomial with real coefficients occur in conjugate pairs. For instance, we saw that the zeros of $p(x) = x^2 - 6x + 13$ are $3 + 2i$ and $3 - 2i$.

In this section, we examine some additional properties of complex conjugates. We begin with the definition of the conjugate of a complex number.

Definition of the Conjugate of a Complex Number

The **conjugate** of the complex number $z = a + bi$, is denoted by $\bar{z}$ and is given by

$$\bar{z} = a - bi.$$

From this definition, we can see that the conjugate of a complex number is found by changing the sign of the imaginary part of the number, as demonstrated in the following example.

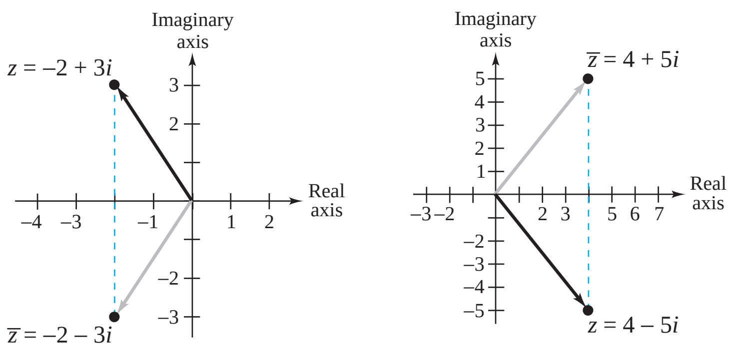
EXAMPLE 1 ***Finding the Conjugate of a Complex Number***

Complex Number	*Conjugate*
(a) $z = -2 + 3i$	$\bar{z} = -2 - 3i$
(b) $z = 4 - 5i$	$\bar{z} = 4 + 5i$
(c) $z = -2i$	$\bar{z} = 2i$
(d) $z = 5$	$\bar{z} = 5$

REMARK: In part (d) of Example 1, note that 5 is its own complex conjugate. In general, it can be shown that a number is its own complex conjugate if and only if the number is real. (See Exercise 29.)

Geometrically, two points in the complex plane are conjugates if and only if they are reflections about the real (horizontal) axis, as shown in Figure 8.5.

Figure 8.5



Conjugate of a Complex Number

Complex conjugates have many useful properties. Some of these are given in Theorem 8.1.

Theorem 8.1
Properties of Complex Conjugates

For a complex number $z = a + bi$, the following properties are true.

1. $z\bar{z} = a^2 + b^2$
2. $z\bar{z} \geq 0$
3. $z\bar{z} = 0$ if and only if $z = 0$.
4. $\overline{(\bar{z})} = z$

Proof To prove the first property, we let $z = a + bi$. Then $\bar{z} = a - bi$ and

$$z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2.$$

The second and third properties follow directly from the first. Finally, the fourth property follows the definition of complex conjugate. That is,

$$\overline{(\bar{z})} = \overline{(\overline{a + bi})} = \overline{a - bi} = a + bi = z.$$

EXAMPLE 2 ***Finding the Product of Complex Conjugates***

Find the product of $z = 1 - 2i$ and its complex conjugate.

Solution Since $\bar{z} = 1 + 2i$ we have

$$z\bar{z} = (1 - 2i)(1 + 2i) = 1^2 + 2^2 = 1 + 4 = 5.$$

The Modulus of a Complex Number

Since a complex number can be represented by a vector in the complex plane, it makes sense to talk about the *length* of a complex number. We call this length the modulus of the complex number.

Definition of the Modulus of a Complex Number

The **modulus** of the complex number $z = a + bi$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{a^2 + b^2}.$$

REMARK: The modulus of a complex number is also called the **absolute value** of the number. In fact, when $z = a + 0i$ is a real number, we have

$$|z| = \sqrt{a^2 + 0^2} = |a|.$$

EXAMPLE 3 ***Finding the Modulus of a Complex Number***

For $z = 2 + 3i$ and $w = 6 - i$, determine the following.

(a) $|z|$ (b) $|w|$ (c) $|zw|$

Solution (a) $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$

(b) $|w| = \sqrt{6^2 + (-1)^2} = \sqrt{37}$

(c) Since $zw = (2 + 3i)(6 - i) = 15 + 16i$, we have

$$|zw| = \sqrt{15^2 + 16^2} = \sqrt{481}.$$

Note that in Example 3, $|zw| = |z|\,|w|$. In Exercise 30, you are asked to show that this multiplicative property of the modulus always holds. The modulus of a complex number is related to its conjugate in the following way.

Theorem 8.2

The Modulus of a Complex Number

For a complex number z,

$$|z|^2 = z\bar{z}.$$

Proof Let $z = a + bi$, then $\bar{z} = a - bi$ and we have

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2.$$

Division of Complex Numbers

One of the most important uses of the conjugate of a complex number is in performing division in the complex number system. To define division of complex numbers, let us consider $z = a + bi$ and $w = c + di$ and assume that c and d are not both 0. If the quotient

$$\frac{z}{w} = x + yi$$

is to make sense, it would have to be true that

$$z = w(x + yi) = (c + di)(x + yi) = (cx - dy) + (dx + cy)i.$$

But, since $z = a + bi$, we can form the following linear system.

$$\begin{aligned} cx - dy &= a \\ dx + cy &= b \end{aligned}$$

Solving this system of linear equations for x and y we get

$$x = \frac{ac + bd}{w\overline{w}} \quad \text{and} \quad y = \frac{bc - ad}{w\overline{w}}.$$

Now, since $z\overline{w} = (a + bi)(c - di) = (ac + bd) + (bc - ad)i$, we obtain the following definition.

Definition of Division of Complex Numbers

The **quotient** of the complex numbers $z = a + bi$ and $w = c + di$ is defined to be

$$\frac{z}{w} = \frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i = \frac{1}{|w|^2}(z\overline{w})$$

provided $c^2 + d^2 \neq 0$.

REMARK: If $c^2 + d^2 = 0$, then $c = d = 0$, and therefore $w = 0$. In other words, just as is the case with real numbers, division of complex numbers by zero is not defined.

In practice, the quotient of two complex numbers can be found by multiplying the numerator and the denominator by the conjugate of the denominator, as follows.

$$\begin{aligned}\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right) &= \frac{(a + bi)(c - di)}{(c + di)(c - di)}\\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}\\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i\end{aligned}$$

EXAMPLE 4 ***Division of Complex Numbers***

(a) $\dfrac{1}{1 + i} = \dfrac{1}{1 + i}\left(\dfrac{1 - i}{1 - i}\right) = \dfrac{1 - i}{1^2 - i^2} = \dfrac{1 - i}{2} = \dfrac{1}{2} - \dfrac{1}{2}i$

(b) $\dfrac{2 - i}{3 + 4i} = \dfrac{2 - i}{3 + 4i}\left(\dfrac{3 - 4i}{3 - 4i}\right) = \dfrac{2 - 11i}{9 + 16} = \dfrac{2}{25} - \dfrac{11}{25}i$

Now that we are able to divide complex numbers, we can find the (multiplicative) inverse of a complex matrix, as demonstrated in Example 5.

EXAMPLE 5 ***Finding the Inverse of a Complex Matrix***

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 - i & -5 + 2i \\ 3 - i & -6 + 2i \end{bmatrix}$$

and verify your solution by showing that $AA^{-1} = I_2$.

Solution Using the formula for the inverse of a 2×2 matrix given in Section 2.3, we have

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} -6 + 2i & 5 - 2i \\ -3 + i & 2 - i \end{bmatrix}.$$

Furthermore, because

$$\begin{aligned} |A| &= (2 - i)(-6 + 2i) - (-5 + 2i)(3 - i) \\ &= (-12 + 6i + 4i + 2) - (-15 + 6i + 5i + 2) \\ &= 3 - i \end{aligned}$$

we can write

$$\begin{aligned} A^{-1} &= \frac{1}{3 - i}\begin{bmatrix} -6 + 2i & 5 - 2i \\ -3 + i & 2 - i \end{bmatrix} \\ &= \frac{1}{3 - i}\left(\frac{1}{3 + i}\right)\begin{bmatrix} (-6 + 2i)(3 + i) & (5 - 2i)(3 + i) \\ (-3 + i)(3 + i) & (2 - i)(3 + i) \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} -20 & 17 - i \\ -10 & 7 - i \end{bmatrix}. \end{aligned}$$

To verify our solution, we multiply A and A^{-1} as follows.

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 2 - i & -5 + 2i \\ 3 - i & -6 + 2i \end{bmatrix} \frac{1}{10}\begin{bmatrix} -20 & 17 - i \\ -10 & 7 - i \end{bmatrix} \\ &= \frac{1}{10}\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

TECHNOLOGY NOTE

If your computer or graphing utility can perform operations with complex matrices, then you can verify the result of Example 5. For instance, on the HP 48G, you would enter the matrix A on the stack and then press the $1/x$ key. On the TI-85, if you have stored the matrix A, then you should evaluate A^{-1}.

The last theorem in this section summarizes some useful properties of complex conjugates.

Theorem 8.3

Properties of Complex Conjugates

For the complex numbers z and w, the following properties are true.

1. $\overline{z + w} = \bar{z} + \bar{w}$
2. $\overline{z - w} = \bar{z} - \bar{w}$
3. $\overline{zw} = \bar{z}\,\bar{w}$
4. $\overline{z/w} = \bar{z}/\bar{w}$

Proof To prove the first property, let $z = a + bi$ and $w = c + di$. Then

$$\begin{aligned}\overline{z + w} &= \overline{(a + c) + (b + d)i}\\ &= (a + c) - (b + d)i\\ &= (a - bi) + (c - di)\\ &= \bar{z} + \bar{w}.\end{aligned}$$

The proof of the second property is similar. The proofs of the other two properties are left to you.

SECTION 8.2 ❑ EXERCISES

In Exercises 1–4, find the complex conjugate $\bar{z}$ and graphically represent both z and $\bar{z}$.

1. $z = 6 - 3i$ **2.** $z = 2 + 5i$

3. $z = -8i$ **4.** $z = 4$

In Exercises 5–10, find the indicated modulus, where $z = 2 + i$, $w = -3 + 2i$, and $v = -5i$.

5. $|z|$ **6.** $|z^2|$ **7.** $|zw|$

8. $|wz|$ **9.** $|v|$ **10.** $|zv^2|$

11. Verify that $|wz| = |w|\,|z| = |zw|$, where $z = 1 + i$ and $w = -1 + 2i$.

12. Verify that $|zv^2| = |z|\,|v^2| = |z|\,|v|^2$, where $z = 1 + 2i$ and $v = -2 - 3i$.

In Exercises 13–18, perform the indicated operations.

13. $\dfrac{2 + i}{i}$ **14.** $\dfrac{1}{6 + 3i}$

15. $\dfrac{3 - \sqrt{2}i}{3 + \sqrt{2}i}$ **16.** $\dfrac{5 + i}{4 + i}$

17. $\dfrac{(2 + i)(3 - i)}{4 - 2i}$ **18.** $\dfrac{3 - i}{(2 - i)(5 + 2i)}$

In Exercises 19 and 20, find the following powers of the complex number z.

(a) z^2 (b) z^3 (c) z^{-1} (d) z^{-2}

19. $z = 2 - i$ **20.** $z = 1 + i$

In Exercises 21–26, determine whether the complex matrix A has an inverse. If A is invertible, find its inverse and verify that $AA^{-1} = I$.

21. $A = \begin{bmatrix} 6 & 3i \\ 2 - i & i \end{bmatrix}$ **22.** $A = \begin{bmatrix} 2i & -2 - i \\ 3 & 3i \end{bmatrix}$

23. $A = \begin{bmatrix} 1 - i & 2 \\ 1 & 1 + i \end{bmatrix}$ **24.** $A = \begin{bmatrix} 1 - i & 2 \\ 0 & 1 + i \end{bmatrix}$

25. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - i & 0 \\ 0 & 0 & 1 + i \end{bmatrix}$ **26.** $A = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$

In Exercises 27 and 28, determine all values of the complex number z for which A is singular. (Hint: Set $\det(A) = 0$ and solve for z.)

27. $A = \begin{bmatrix} 5 & z \\ 3i & 2 - i \end{bmatrix}$ **28.** $A = \begin{bmatrix} 2 & 2i & 1 + i \\ 1 - i & -1 + i & z \\ 1 & 0 & 0 \end{bmatrix}$

29. Prove that $z = \bar{z}$ if and only if z is real.

30. Prove that for any two complex numbers z and w, the following are true.

(a) $|zw| = |z|\,|w|$

(b) If $w \neq 0$, then $|z/w| = |z|/|w|$.

31. Describe the set of points in the complex plane that satisfy the following.

(a) $|z| = 3$ (b) $|z - 1 + i| = 5$

(c) $|z - i| \leq 2$ (d) $2 \leq |z| \leq 5$

32. Describe the set of points in the complex plane that satisfy the following.

(a) $|z| = 4$ (b) $|z - i| = 2$

(c) $|z + 1| \leq 1$ (d) $|z| > 3$

33. (a) Evaluate $(1/i)^n$ for $n = 1, 2, 3, 4,$ and 5.

(b) Calculate $(1/i)^{57}$ and $(1/i)^{1995}$.

(c) Find a general formula for $(1/i)^n$ for any positive integer n.

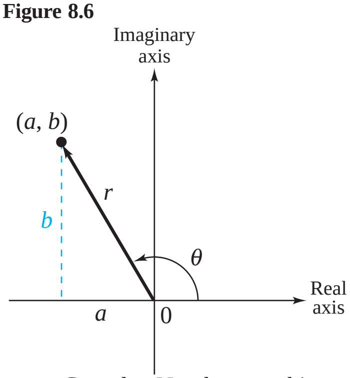
34. (a) Verify that $\left(\frac{1+i}{\sqrt{2}}\right)^2 = i$.

(b) Find the two square roots of i.

(c) Find all zeros of the polynomial $x^4 + 1$.

8.3 POLAR FORM AND DEMOIVRE'S THEOREM

Figure 8.6



Complex Number: $a + bi$
Rectangular Form: (a, b)
Polar Form: (r, θ)

At this point we can add, subtract, multiply, and divide complex numbers. However, there is still one basic procedure that is missing from our algebra of complex numbers. To see this, consider the problem of finding the square root of a complex number such as i. When we use the four basic operations (addition, subtraction, multiplication, and division), there seems to be no reason to guess that

$$\sqrt{i} = \frac{1+i}{\sqrt{2}}. \quad \text{That is,} \quad \left(\frac{1+i}{\sqrt{2}}\right)^2 = i.$$

To work effectively with *powers* and *roots* of complex numbers, it is helpful to use a polar representation for complex numbers, as shown in Figure 8.6. Specifically, if $a + bi$ is a nonzero complex number, then we let θ be the angle from the positive x-axis to the radial line passing through the point (a, b) and we let r be the modulus of $a + bi$. Thus,

$$a = r\cos\theta, \qquad b = r\sin\theta, \qquad \text{and} \qquad r = \sqrt{a^2 + b^2}$$

and we have $a + bi = (r\cos\theta) + (r\sin\theta)i$ from which we obtain the following **polar form** of a complex number.

Definition of Polar Form of a Complex Number

The **polar form** of the nonzero complex number $z = a + bi$ is given by

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = b/a$. The number r is the **modulus** of z and θ is called the **argument** of z.