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Lecture 7 Notes

These notes correspond to Section 1.3 in the text.

Approximations in Numerical Analysis, cont'd

Convergence

Many algorithms in numerical analysis are *iterative methods* that produce a sequence $\{\alpha_n\}$ of approximate solutions which, ideally, converges to a limit α that is the exact solution as n approaches ∞. Because we can only perform a finite number of iterations, we cannot obtain the exact solution, and we have introduced computational error.

If our iterative method is properly designed, then this computational error will approach zero as n approaches ∞. However, it is important that we obtain a sufficiently accurate approximate solution using as few computations as possible. Therefore, it is not practical to simply perform enough iterations so that the computational error is determined to be sufficiently small, because it is possible that another method may yield comparable accuracy with less computational effort.

The total computational effort of an iterative method depends on both the effort per iteration and the number of iterations performed. Therefore, in order to determine the amount of computation that is needed to attain a given accuracy, we must be able to measure the error in α_n as a function of n. The more rapidly this function approaches zero as n approaches ∞, the more rapidly the sequence of approximations $\{\alpha_n\}$ converges to the exact solution α, and as a result, fewer iterations are needed to achieve a desired accuracy. We now introduce some terminology that will aid in the discussion of the convergence behavior of iterative methods.

Definition *(Big-O Notation) Let f and g be two functions defined on a domain $D \subseteq \mathbb{R}$ that is not bounded above. We write that $f(n) = O(g(n))$ if there exists a positive constant c such that*

$$|f(n)| \leq c|g(n)|, \quad n \geq n_0,$$

for some $n_0 \in D$.

As sequences are functions defined on $\mathbb{N}$, the domain of the natural numbers, we can apply big-O notation to sequences. Therefore, this notation is useful to describe the rate at which a sequence of computations converges to a limit.

Definition *(Rate of Convergence) Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences that satisfy*

$$\lim_{n\to\infty} \alpha_n = \alpha, \quad \lim_{n\to\infty} \beta_n = 0,$$

where α is a real number. We say that $\{\alpha_n\}$ converges to α with **rate of convergence** $O(\beta_n)$ *if* $\alpha_n - \alpha = O(\beta_n)$.

We say that an iterative method converges rapidly, in some sense, if it produces a sequence of approximate solutions whose rate of convergence is $O(\beta_n)$, where the terms of the sequence β_n approach zero rapidly as n approaches ∞. Intuitively, if two iterative methods for solving the same problem perform a comparable amount of computation during each iteration, but one method exhibits a faster rate of convergence, then that method should be used because it will require less overall computational effort to obtain an approximate solution that is sufficiently accurate.

Example Consider the sequence $\{\alpha_n\}_{n=1}^{\infty}$ defined by

$$\alpha_n = \frac{n+1}{n+2}, \quad n = 1, 2, \ldots$$

Then, we have

$$\begin{aligned}
\lim_{n\to\infty} \alpha_n &= \lim_{n\to\infty} \frac{n+1}{n+2}\frac{1/n}{1/n} \\
&= \lim_{n\to\infty} \frac{1+1/n}{1+2/n} \\
&= \frac{\lim_{n\to\infty}(1+1/n)}{\lim_{n\to\infty}(1+2/n)} \\
&= \frac{1+\lim_{n\to\infty} 1/n}{1+\lim_{n\to\infty} 2/n} \\
&= \frac{1+0}{1+0} \\
&= 1.
\end{aligned}$$

That is, the sequence $\{\alpha_n\}$ converges to $\alpha = 1$. To determine the rate of convergence, we note that

$$\alpha_n - \alpha = \frac{n+1}{n+2} - 1 = \frac{n+1}{n+2} - \frac{n+2}{n+2} = \frac{-1}{n+2},$$

and since

$$\left|\frac{-1}{n+2}\right| \leq \left|\frac{1}{n}\right|$$

for any positive integer n, it follows that

$$\alpha_n = \alpha + O\left(\frac{1}{n}\right).$$

On the other hand, consider the sequence $\{\alpha_n\}_{n=1}^{\infty}$ defined by

$$\alpha_n = \frac{2n^2+4n}{n^2+2n+1}, \quad n = 1, 2, \ldots$$

Then, we have

$$\begin{aligned}
\lim_{n\to\infty} \alpha_n &= \lim_{n\to\infty} \frac{2n^2+4n}{n^2+2n+1}\frac{1/n^2}{1/n^2} \\
&= \lim_{n\to\infty} \frac{2+4/n}{1+2/n+1/n^2} \\
&= \frac{\lim_{n\to\infty}(2+4/n)}{\lim_{n\to\infty}(1+2/n+1/n^2)} \\
&= \frac{2+\lim_{n\to\infty} 4/n}{1+\lim_{n\to\infty}(2/n+1/n^2)} \\
&= 2.
\end{aligned}$$

That is, the sequence $\{\alpha_n\}$ converges to $\alpha = 2$. To determine the rate of convergence, we note that

$$\alpha_n - \alpha = \frac{2n^2+4n}{n^2+2n+1} - 2 = \frac{2n^2+4n}{n^2+2n+1} - \frac{2n^2+4n+2}{n^2+2n+1} = \frac{-2}{n^2+2n+1},$$

and since

$$\left|\frac{-2}{n^2+2n+1}\right| = \left|\frac{2}{(n+1)^2}\right| \leq \left|\frac{2}{n^2}\right|$$

for any positive integer n, it follows that

$$\alpha_n = \alpha + O\left(\frac{1}{n^2}\right).$$

□

We can also use big-O notation to describe the rate of convergence of a *function* to a limit.

Example Consider the function $f(h) = 1 + 2h$. Since this function is continuous for all h, we have

$$\lim_{h\to 0} f(h) = f(0) = 1.$$

It follows that

$$f(h) - f_0 = (1 + 2h) - 1 = 2h = O(h),$$

so we can conclude that as $h \to 0$, $1 + 2h$ converges to 1 of order $O(h)$. □

Example Consider the function $f(h) = 1 + 4h + 2h^2$. Since this function is continuous for all h, we have

$$\lim_{h\to 0} f(h) = f(0) = 1.$$

It follows that

$$f(h) - f_0 = (1 + 4h + 2h^2) - 1 = 4h + 2h^2.$$

To determine the rate of convergence as $h \to 0$, we consider h in the interval $[-1, 1]$. In this interval, $|h^2| \leq |h|$. It follows that

$$\begin{aligned} |4h + 2h^2| &\leq |4h| + |2h^2| \\ &\leq |4h| + |2h| \\ &\leq 6|h|. \end{aligned}$$

Since there exists a constant C (namely, 6) such that $|4h+2h^2| \leq C|h|$ for h satisfying $|h| \leq h_0$ for some h_0 (namely, 1), we can conclude that as $h \to 0$, $1 + 4h + 2h^2$ converges to 1 of order $O(h)$.

In general, when $f(h)$ denotes an approximation that depends on h, and

$$f_0 = \lim_{h \to 0} f(h)$$

denotes the exact value, $f(h) - f_0$ represents the absolute error in the approximation $f(h)$. When this error is a polynomial in h, as in this example and the previous example, the rate of convergence is $O(h^k)$ where k is the smallest exponent of h in the error. This is because as $h \to 0$, the smallest power of h approaches zero more slowly than higher powers, thereby making the dominant contribution to the error.

By contrast, when determining the rate of convergence of a sequence $\{\alpha_n\}$ as $n \to \infty$, the *highest* power of n determines the rate of convergence. As powers of n are negative if convergence occurs at all as $n \to \infty$, and powers of h are positive if convergence occurs at all as $h \to 0$, it can be said that for either types of convergence, it is the exponent that is closest to zero that determines the rate of convergence. □

Example Consider the function $f(h) = \cos h$. Since this function is continuous for all h, we have

$$\lim_{h \to 0} f(h) = f(0) = 1.$$

Using Taylor's Theorem, with center $h_0 = 0$, we obtain

$$f(h) = f(0) + f'(0)h + \frac{f''(\xi(h))}{2}h^2,$$

where $\xi(h)$ is between 0 and h. Substituting $f(h) = \cos h$ into the above, we obtain

$$\cos h = 1 - (\sin 0)h + \frac{-\cos \xi(h)}{2}h^2,$$

or

$$\cos h = 1 - \frac{\cos \xi(h)}{2}h^2.$$

Because $|\cos x| \leq 1$ for all x, we have

$$|\cos h - 1| = \left| -\frac{\cos \xi(h)}{2}h^2 \right| \leq \frac{1}{2}h^2,$$

so we can conclude that as $h \to 0$, $\cos h$ converges to 1 of order $O(h^2)$. □