These notes correspond to Section 1.1 in the text.

Review of Calculus

Among the mathematical problems that can be solved using techniques from numerical analysis are the basic problems of differential and integral calculus:

- computing the instantaneous rate of change of one quantity with respect to another, which is a *derivative*, and
- computing the total change in a function over some portion of its domain, which is a *definite integral*.

Calculus also plays an essential role in the development and analysis of techniques used in numerical analysis, including those techniques that are applied to problems not arising directly from calculus. Therefore, it is appropriate to review some basic concepts from calculus before we begin our study of numerical analysis.

Limits

The basic problems of differential and integral calculus described in the previous paragraph can be solved by computing a sequence of approximations to the desired quantity and then determining what value, if any, the sequence of approximations approaches. This value is called a *limit* of the sequence. As a sequence is a function, we begin by defining, precisely, the concept of the limit of a function.

Definition *We write*

$$\lim_{x \to a} f(x) = L$$

*if for any open interval I_1 containing L, there is some open interval I_2 containing a such that $f(x) \in I_1$ whenever $x \in I_2$, and $x \neq a$. We say that L is the **limit of** $f(x)$ **as** x **approaches** a.*
    We write

$$\lim_{x \to a^-} f(x) = L$$

*if, for any open interval I_1 containing L, there is an open interval I_2 of the form (c, a), where $c < a$, such that $f(x) \in I_1$ whenever $x \in I_2$. We say that L is the **limit of** $f(x)$ **as** x **approaches** a **from the left**, or the **left-hand limit of** $f(x)$ **as** x **approaches** a.*

Similarly, we write

$$\lim_{x \to a^+} f(x) = L$$

*if, for any open interval I_1 containing L, there is an open interval I_2 of the form (a, c), where $c > a$, such that $f(x) \in I_1$ whenever $x \in I_2$. We say that L is the **limit of** $f(x)$ **as** x **approaches** a **from the right**, or the **right-hand limit of** $f(x)$ **as** x **approaches** a.*

We can make the definition of a limit a little more concrete by imposing sizes on the intervals I_1 and I_2, as long as the interval I_1 can still be of arbitrary size. It can be shown that the following definition is equivalent to the previous one.

Definition *We write*

$$\lim_{x \to a} f(x) = L$$

if, for any $\epsilon > 0$, there exists a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Similar definitions can be given for the left-hand and right-hand limits.

Note that in either definition, the point $x = a$ is specifically excluded from consideration when requiring that $f(x)$ be close to L whenever x is close to a. This is because the concept of a limit is only intended to describe the behavior of $f(x)$ *near* $x = a$, as opposed to its behavior *at* $x = a$. Later in this lecture we discuss the case where the two distinct behaviors coincide.

Limits at Infinity

The concept of a limit defined above is useful for describing the behavior of a function $f(x)$ as x approaches a *finite* value a. However, suppose that the function f is a *sequence*, which is a function that maps $\mathbb{N}$, the set of natural numbers, to $\mathbb{R}$, the set of real numbers. We will denote such a sequence by $\{f_n\}_{n=0}^\infty$, or simply $\{f_n\}$. In numerical analysis, it is sometimes necessary to determine the value that the terms of a sequence $\{f_n\}$ approach as $n \to \infty$. Such a value, if it exists, is not a limit, as defined previously. However, it is natural to use the notation of limits to describe this behavior of a function. We therefore define what it means for a sequence $\{f_n\}$ to have a limit as n becomes infinite.

Definition *(Limit at Infinity) Let $\{f_n\}$ be a sequence defined for all integers not less than some integer n_0. We say that the **limit of** $\{f_n\}$ **as** n **approaches** ∞ is equal to L, and write*

$$\lim_{n \to \infty} f_n = L,$$

if for any open interval I containing L, there exists a number M such that $f_n \in I$ whenever $x > M$.

Example Let the sequence $\{f_n\}_{n=1}^\infty$ be defined by $f_n = 1/n$ for every positive integer n. Then

$$\lim_{n \to \infty} f_n = 0,$$

since for any $\epsilon > 0$, no matter how small, we can find a positive integer n_0 such that $|f_n| < \epsilon$ for all $n \geq n_0$. In fact, for any given ϵ, we can choose $n_0 = \lceil 1/\epsilon \rceil$, where $\lceil x \rceil$, known as the *ceiling function*, denotes the smallest integer that is greater than or equal to x. □

Continuity

In many cases, the limit of a function $f(x)$ as x approached a could be obtained by simply computing $f(a)$. Intuitively, this indicates that f has to have a graph that is one continuous curve, because any "break" or "jump" in the graph at $x = a$ is caused by f approaching one value as x approaches a, only to actually assume a different value at a. This leads to the following precise definition of what it means for a function to be continuous at a given point.

Definition *(Continuity) We say that a function f is* **continuous at** *a if*

$$\lim_{x \to a} f(x) = f(a).$$

We also say that $f(x)$ has the **Direct Subsitution Property** *at $x = a$.*
    We say that a function f is **continuous from the right** *at a if*

$$\lim_{x \to a^+} f(x) = f(a).$$

Similarly, we say that f is **continuous from the left** *at a if*

$$\lim_{x \to a^-} f(x) = f(a).$$

The preceding definition describes continuity at a single point. In describing where a function is continuous, the concept of continuity over an interval is useful, so we define this concept as well.

Definition *(Continuity on an Interval) We say that a function f is* **continuous on the interval** *(a, b) if f is continuous at every point in (a, b). Similarly, we say that f is continuous on*

1. *$[a, b)$ if f is continuous on (a, b), and continuous from the right at a.*

2. *$(a, b]$ if f is continuous on (a, b), and continuous from the left at b.*

3. *$[a, b]$ if f is continuous on (a, b), continuous from the right at a, and continuous from the left at b.*

In numerical analysis, it is often necessary to construct a continuous function, such as a polynomial, based on data obtained by measurements and problem-dependent constraints. In this course, we will learn some of the most basic techniques for constructing such continuous functions by a process called *interpolation*.

The Intermediate Value Theorem

Suppose that a function f is continuous on some closed interval $[a, b]$. The graph of such a function is a continuous curve connecting the points $(a, f(a))$ with $(b, f(b))$. If one were to draw such a graph, their pen would not leave the paper in the process, and therefore it would be impossible to "avoid" any y-value between $f(a)$ and $f(b)$. This leads to the following statement about such continuous functions.

Theorem *(Intermediate Value Theorem) Let f be continuous on $[a, b]$. Then, on (a, b), f assumes every value between $f(a)$ and $f(b)$; that is, for any value y between $f(a)$ and $f(b)$, $f(c) = y$ for some c in (a, b).*

The Intermediate Value Theorem has a very important application in the problem of finding solutions of a general equation of the form $f(x) = 0$, where x is the solution we wish to compute and f is a given continuous function. Often, methods for solving such an equation try to identify an interval $[a, b]$ where $f(a) > 0$ and $f(b) < 0$, or vice versa. In either case, the Intermediate Value Theorem states that f must assume every value between $f(a)$ and $f(b)$, and since 0 is one such value, it follows that the equation $f(x) = 0$ must have a solution somewhere in the interval (a, b).

We can find an approximation to this solution using a procedure called *bisection*, which repeatedly applies the Intermediate Value Theorem to smaller and smaller intervals that contain the solution. We will study bisection, and other methods for solving the equation $f(x) = 0$, in this course.

Derivatives

The basic problem of differential calculus is computing the instantaneous rate of change of one quantity y with respect to another quantity x. For example, y may represent the position of an object and x may represent time, in which case the instantaneous rate of change of y with respect to x is interpreted as the velocity of the object.

When the two quantities x and y are related by an equation of the form $y = f(x)$, it is certainly convenient to describe the rate of change of y with respect to x in terms of the function f. Because the instantaneous rate of change is so commonplace, it is practical to assign a concise name and notation to it, which we do now.

Definition *(Derivative) The* **derivative** *of a function $f(x)$ at $x = a$, denoted by $f'(a)$, is*

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

provided that the above limit exists. When this limit exists, we say that f is **differentiable** *at a.*

Remark Given a function $f(x)$ that is differentiable at $x = a$, the following numbers are all equal:

- the derivative of f at $x = a$, $f'(a)$,

- the slope of the tangent line of f at the point $(a, f(a))$, and
- the instantaneous rate of change of $y = f(x)$ with respect to x at $x = a$.

This can be seen from the fact that all three numbers are defined in the same way. □

Many functions can be differentiated using differentiation rules such as those learned in a calculus course. However, many functions cannot be differentiated using these rules. For example, we may need to compute the instantaneous rate of change of a quantity $y = f(x)$ with respect to another quantity x, where our only knowledge of the function f that relates x and y is a set of pairs of x-values and y-values that may be obtained using measurements. In this course we will learn how to approximate the derivative of such a function using this limited information. The most common methods involve constructing a continuous function, such as a polynomial, based on the given data, using interpolation. The polynomial can then be differentiated using differentiation rules. Since the polynomial is an approximation to the function $f(x)$, its derivative is an approximation to $f'(x)$.

Differentiability and Continuity

Consider a tangent line of a function f at a point $(a, f(a))$. When we consider that this tangent line is the limit of secant lines that can cross the graph of f at points on *either side* of a, it seems impossible that f can fail to be continuous at a. The following result confirms this: a function that is differentiable at a given point (and therefore has a tangent line at that point) *must* also be continuous at that point.

Theorem *If f is differentiable at a, then f is continuous at a.*

It is important to keep in mind, however, that the *converse* of the above statement, "if f is continuous, then f is differentiable", is not true. It is actually very easy to find examples of functions that are continuous at a point, but fail to be differentiable at that point. As an extreme example, it is known that there is a function that is continuous everywhere, but is differentiable *nowhere*.

Example The functions $f(x) = |x|$ and $g(x) = x^{1/3}$ are examples of functions that are continuous for all x, but are not differentiable at $x = 0$. The graph of the *absolute value function* $|x|$ has a sharp corner at $x = 0$, since the one-sided limits

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = -1, \quad \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = 1$$

do not agree, but in general these limits must agree in order for $f(x)$ to have a derivative at $x = 0$.

The cube root function $g(x) = x^{1/3}$ is not differentiable at $x = 0$ because the tangent line to the graph at the point $(0, 0)$ is vertical, so it has no finite slope. We can also see that the derivative does not exist at this point by noting that the function $g'(x) = (1/3)x^{-2/3}$ has a vertical asymptote at $x = 0$. □