Jim Lambers
MAT 460/560
Fall Semester 2009-10
Lecture 13 Notes

These notes correspond to Section 2.5 in the text.

Accelerating Convergence

Suppose that a sequence $\{x_k\}_{k=0}^\infty$ converges linearly to a limit x^*, in such a way that if k is sufficiently large, then $x_k - x^*$ has the same sign; that is, $\{x_k\}$ converges *monotonically* to x^*. It follows from the linear convergence of $\{x_k\}$ that for sufficiently large k,

$$\frac{x_{k+2} - x^*}{x_{k+1} - x^*} \approx \frac{x_{k+1} - x^*}{x_k - x^*}.$$

Solving for x^* yields

$$x^* \approx x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}.$$

Therefore, we can construct an alternative sequence $\{\hat{x}_k\}_{k=0}^\infty$, where

$$\hat{x}_k = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k},$$

that also converges to x^*. This sequence has the following desirable property.

Theorem *Suppose that the sequence $\{x_k\}_{k=0}^\infty$ converges linearly to a limit x^* and that for k sufficiently large, $(x_{k+1} - x^*)(x_k - x^*) > 0$. Then, if the sequence $\{\hat{x}_k\}_{k=0}^\infty$ is defined by*

$$\hat{x}_k = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}, \quad k = 0, 1, 2, \ldots,$$

then

$$\lim_{k\to\infty} \frac{\hat{x}_k - x^*}{x^k - x^*} = 0.$$

In other words, the sequence $\{\hat{x}_k\}$ converges to x^* more rapidly than $\{x_k\}$ does.

If we define the *forward difference operator* Δ by

$$\Delta x_k = x_{k+1} - x_k,$$

then

$$\Delta^2 x_k = \Delta(x_{k+1} - x_k) = (x_{k+2} - x_{k+1}) - (x_{k+1} - x_k) = x_{k+2} - 2x_{k+1} + x_k,$$

and therefore $\hat{x}_k$ can be rewritten as

$$\hat{x}_k = x_k - \frac{(\Delta x_k)^2}{\Delta^2 x_k}, \quad k = 0, 1, 2, \ldots$$

For this reason, the method of accelerating the convergence of $\{x_k\}$ by constructing $\{\hat{x}_k\}$ is called *Aitken's* Δ^2 *Method*.

A slight variation of this method, called *Steffensen's Method*, can be used to accelerate the convergence of Fixed-point Iteration, which, as previously discussed, is linearly convergent. The basic idea is as follows:

1. Choose an initial iterate x_0
2. Compute x_1 and x_2 using Fixed-point Iteration
3. Use Aitken's Δ^2 Method to compute $\hat{x}_0$ from x_0, x_1, and x_2
4. Repeat steps 2 and 3 with $x_0 = \hat{x}_0$.

The principle behind Steffensen's Method is that $\hat{x}_0$ is thought to be a better approximation to the fixed point x^* than x_2, so it should be used as the next iterate for Fixed-point Iteration.

Example We wish to find the unique fixed point of the function $f(x) = \cos x$ on the interval $[0, 1]$. If we use Fixed-point Iteration with $x_0 = 0.5$, then we obtain the following iterates from the formula $x_{k+1} = g(x_k) = \cos(x_k)$. All iterates are rounded to five decimal places.

$$\begin{aligned} x_1 &= 0.87758 \\ x_2 &= 0.63901 \\ x_3 &= 0.80269 \\ x_4 &= 0.69478 \\ x_5 &= 0.76820 \\ x_6 &= 0.71917. \end{aligned}$$

These iterates show little sign of converging, as they are oscillating around the fixed point.

If, instead, we use Fixed-point Iteration with acceleration by Aitken's Δ^2 method, we obtain a new sequence of iterates $\{\hat{x}_k\}$, where

$$\begin{aligned} \hat{x}_k &= x_k - \frac{(\Delta x_k)^2}{\Delta^2 x_k} \\ &= x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}, \end{aligned}$$

for $k = 0, 1, 2, \ldots$. The first few iterates of this sequence are

$$\begin{aligned}
\hat{x}_0 &= 0.73139 \\
\hat{x}_1 &= 0.73609 \\
\hat{x}_2 &= 0.73765 \\
\hat{x}_3 &= 0.73847 \\
\hat{x}_4 &= 0.73880.
\end{aligned}$$

Clearly, these iterates are converging much more rapidly than Fixed-point Iteration, as they are not oscillating around the fixed point, but convergence is still linear.

Finally, we try Steffensen's Method. We begin with the first three iterates of Fixed-point Iteration,

$$x_0^{(0)} = x_0 = 0.5, \quad x_1^{(0)} = x_1 = 0.87758, \quad x_2^{(0)} = x_2 = 0.63901.$$

Then, we use the formula from Aitken's Δ^2 Method to compute

$$x_0^{(1)} = x_0^{(0)} - \frac{(x_1^{(0)} - x_0^{(0)})^2}{x_2^{(0)} - 2x_1^{(0)} + x_0^{(0)}} = 0.73139.$$

We use this value to restart Fixed-point Iteration and compute two iterates, which are

$$x_1^{(1)} = \cos(x_0^{(1)}) = 0.74425, \quad x_2^{(1)} = \cos(x_1^{(1)}) = 0.73560.$$

Repeating this process, we apply the formula from Aitken's Δ^2 Method to the iterates $x_0^{(1)}$, $x_1^{(1)}$ and $x_2^{(1)}$ to obtain

$$x_0^{(2)} = x_0^{(1)} - \frac{(x_1^{(1)} - x_0^{(1)})^2}{x_2^{(1)} - 2x_1^{(1)} + x_0^{(1)}} = 0.739076.$$

Restarting Fixed-point Iteration with $x_0^{(2)}$ as the initial iterate, we obtain

$$x_1^{(2)} = \cos(x_0^{(2)}) = 0.739091, \quad x_2^{(2)} = \cos(x_1^{(2)}) = 0.739081.$$

The most recent iterate $x_2^{(2)}$ is correct to five decimal places.

Using all three methods to compute the fixed point to ten decimal digits of accuracy, we find that Fixed-point Iteration requires 57 iterations, so x_58 must be computed. Aitken's Δ^2 Method requires us to compute 25 iterates of the modified sequence $\{\hat{x}_k\}$, which in turn requires 27 iterates of the sequence $\{x_k\}$, where the first iterate x_0 is given. Steffensen's Method requires us to compute $x_2^{(3)}$, which means that only 11 iterates need to be computed. $\Box$