Chapter 4

Constrained Optimization

4.1   Equality Constraints (Lagrangians)

Suppose we have a problem:

Maximize $5 - (x_1 - 2)^2 - 2(x_2 - 1)^2$
subject to
$\quad x_1 + 4x_2 = 3$

If we ignore the constraint, we get the solution $x_1 = 2, x_2 = 1$, which is too large for the constraint. Let us penalize ourselves λ for making the constraint too big. We end up with a function

$$L(x_1, x_2, \lambda) = 5 - (x_1 - 2)^2 - 2(x_2 - 1)^2 + \lambda(3 - x_1 - 4x_2)$$

This function is called the *Lagrangian* of the problem. The main idea is to adjust λ so that we use exactly the right amount of the resource.

$\lambda = 0$ leads to $(2, 1)$.

$\lambda = 1$ leads to $(3/2, 0)$ which uses too little of the resource.

$\lambda = 2/3$ gives $(5/3, 1/3)$ and the constraint is satisfied exactly.

We now explore this idea more formally. Given a nonlinear program (P) with equality constraints:

Minimize (or maximize) $f(x)$
subject to
$\quad g_1(x) = b_1$
$\quad g_2(x) = b_2$
$\quad \vdots$
$\quad g_m(x) = b_m$

a solution can be found using the *Lagrangian*:

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i(b_i - g_i(x))$$

(Note: this can also be written $f(x) - \sum_{i=1}^{m} \lambda_i(g_i(x) - b_i)$).
Each λ_i gives the price associated with constraint i.

The reason L is of interest is the following:

Assume $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ maximizes or minimizes $f(x)$ subject to the constraints $g_i(x) = b_i$, for $i = 1, 2, \ldots, m$. Then either

(i) the vectors $\nabla g_1(x^*), \nabla g_2(x^*), \ldots, \nabla g_m(x^*)$ are linearly dependent, or

(ii) there exists a vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)$ such that $\nabla L(x^*, \lambda^*) = 0$. I.e.

$$\frac{\partial L}{\partial x_1}(x^*, \lambda^*) = \frac{\partial L}{\partial x_2}(x^*, \lambda^*) = \cdots = \frac{\partial L}{\partial x_n}(x^*, \lambda^*) = 0$$

$$\text{and} \quad \frac{\partial L}{\partial \lambda_1}(x^*, \lambda^*) = \cdots = \frac{\partial L}{\partial \lambda_m}(x^*, \lambda^*) = 0$$

Of course, Case (i) above cannot occur when there is only one constraint. The following example shows how it might occur.

Example 4.1.1
Minimize $x_1 + x_2 + x_3^2$
  subject to
   $x_1 = 1$
   $x_1^2 + x_2^2 = 1$.

It is easy to check directly that the minimum is acheived at $(x_1, x_2, x_3) = (1, 0, 0)$. The associated Lagrangian is

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = x_1 + x_2 + x_3^2 + \lambda_1(1 - x_1) + \lambda_2(1 - x_1^2 - x_2^2).$$

Observe that
$$\frac{\partial L}{\partial x_2}(1, 0, 0, \lambda_1, \lambda_2) = 1 \quad \text{for all } \lambda_1, \lambda_2,$$

and consequently $\frac{\partial L}{\partial x_2}$ *does not* vanish at the optimal solution. The reason for this is the following. Let $g_1(x_1, x_2, x_3) = x_1$ and $g_2(x_1, x_2, x_3) = x_1^2 + x_2^2$ denote the left hand sides of the constraints. Then $\nabla g_1(1, 0, 0) = (1, 0, 0)$ and $\nabla g_2(1, 0, 0) = (2, 0, 0)$ are linearly dependent vectors. So Case (i) occurs here!

Nevertheless, Case (i) will not concern us in this course. When solving optimization problems with equality constraints, we will only look for solutions x^* that satisfy Case (ii).

Note that the equation
$$\frac{\partial L}{\partial \lambda_i}(x^*, \lambda^*) = 0$$

is nothing more than

$$b_i - g_i(x^*) = 0 \quad \text{or} \quad g_i(x^*) = b_i.$$

In other words, taking the partials with respect to λ does nothing more than return the original constraints.

Once we have found candidate solutions x^*, it is not always easy to figure out whether they correspond to a minimum, a maximum or neither. The following situation is one when we can conclude. If $f(x)$ is concave and all of the $g_i(x)$ are linear, then any feasible x^* with a corresponding λ^* making $\nabla L(x^*, \lambda^*) = 0$ maximizes $f(x)$ subject to the constraints. Similarly, if $f(x)$ is convex and each $g_i(x)$ is linear, then any x^* with a λ^* making $\nabla L(x^*, \lambda^*) = 0$ minimizes $f(x)$ subject to the constraints.

Example 4.1.2
Minimize $2x_1^2 + x_2^2$
  subject to
    $x_1 + x_2 = 1$

$$L(x_1, x_2, \lambda) = 2x_1^2 + x_2^2 + \lambda_1(1 - x_1 - x_2)$$

$$\frac{\partial L}{\partial x_1}(x_1^*, x_2^*, \lambda^*) = 4x_1^* - \lambda_1^* = 0$$

$$\frac{\partial L}{\partial x_2}(x_1^*, x_2^*, \lambda^*) = 2x_2^* - \lambda_1^* = 0$$

$$\frac{\partial L}{\partial \lambda}(x_1^*, x_2^*, \lambda^*) = 1 - x_1^* - x_2^* = 0$$

Now, the first two equations imply $2x_1^* = x_2^*$. Substituting into the final equation gives the solution $x_1^* = 1/3$, $x_2^* = 2/3$ and $\lambda^* = 4/3$, with function value $2/3$.

Since $f(x_1, x_2)$ is convex (its Hessian matrix $H(x_1, x_2) = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ is positive definite) and $g(x_1, x_2) = x_1 + x_2$ is a linear function, the above solution minimizes $f(x_1, x_2)$ subject to the constraint.

4.1.1 Geometric Interpretation

There is a geometric interpretation of the conditions an optimal solution must satisfy. If we graph Example 4.1.2, we get a picture like that in Figure 4.1.

Now, examine the gradients of f and g at the optimum point. They must point in the same direction, though they may have different lengths. This implies:

$$\nabla f(x^*) = \lambda^* \nabla g(x^*)$$

which, along with the feasibility of x^*, is exactly the condition $\nabla L(x^*, \lambda^*) = 0$ of Case (ii).

4.1.2 Economic Interpretation

The values λ_i^* have an important economic interpretation: If the right hand side b_i of Constraint i is increased by Δ, then the optimum objective value increases by approximately $\lambda_i^* \Delta$.

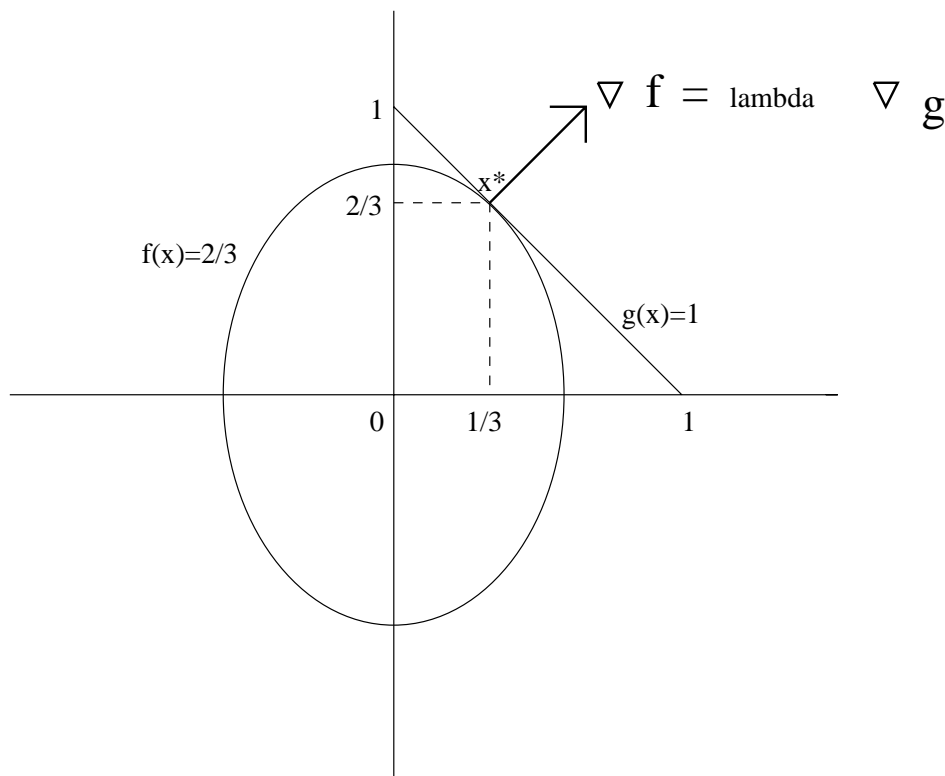In particular, consider the problem

Maximize $p(x)$
subject to
  $g(x) = b$,

Figure 4.1: Geometric interpretation

where $p(x)$ is a profit to maximize and b is a limited amount of resource. Then, the optimum Lagrange multiplier λ^* is the marginal value of the resource. Equivalently, if b were increased by Δ, profit would increase by $\lambda^*\Delta$. This is an important result to remember. It will be used repeatedly in your Managerial Economics course.

Similarly, if

Minimize $c(x)$
subject to
$\quad d(x) = b$,

represents the minimum cost $c(x)$ of meeting some demand b, the optimum Lagrange multiplier λ^* is the marginal cost of meeting the demand.

In Example 4.1.2

Minimize $2x_1^2 + x_2^2$
subject to
$\quad x_1 + x_2 = 1$,

if we change the right hand side from 1 to 1.05 (i.e. $\Delta = 0.05$), then the optimum objective function value goes from $\frac{2}{3}$ to roughly

$$\frac{2}{3} + \frac{4}{3}(0.05) = \frac{2.2}{3}.$$

If instead the right hand side became 0.98, our estimate of the optimum objective function value would be

$$\frac{2}{3} + \frac{4}{3}(-0.02) = \frac{1.92}{3}$$

Example 4.1.3 *Suppose we have a refinery that must ship finished goods to some storage tanks. Suppose further that there are two pipelines, A and B, to do the shipping. The cost of shipping x units on A is ax^2; the cost of shipping y units on B is by^2, where $a > 0$ and $b > 0$ are given. How can we ship Q units while minimizing cost? What happens to the cost if Q increases by r %?*

Minimize $ax^2 + by^2$
Subject to
$x + y = Q$

$$L(x, y, \lambda) = ax^2 + by^2 + \lambda(Q - x - y)$$

$$\frac{\partial L}{\partial x}(x^*, y^*, \lambda^*) = 2ax^* - \lambda^* = 0$$

$$\frac{\partial L}{\partial y}(x^*, y^*, \lambda^*) = 2by^* - \lambda^* = 0$$

$$\frac{\partial L}{\partial \lambda}(x^*, y^*, \lambda^*) = Q - x^* - y^* = 0$$

The first two constraints give $x^* = \frac{b}{a}y^*$, which leads to

$$x^* = \frac{bQ}{a+b}, \quad y^* = \frac{aQ}{a+b}, \quad \lambda^* = \frac{2abQ}{a+b}$$

and cost of $\frac{abQ^2}{a+b}$. The Hessian matrix $H(x_1, x_2) = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$ is positive definite since $a > 0$ and $b > 0$. So this solution minimizes cost, given a, b, Q.

If Q increases by $r\%$, then the RHS of the constraint increases by $\Delta = rQ$ and the minimum cost increases by $\lambda^*\Delta = \frac{2abrQ^2}{a+b}$. That is, the minimum cost increases by $2r\%$.

Example 4.1.4 How should one divide his/her savings between three mutual funds with expected returns 10%, 10% and 15% repectively, so as to minimize risk while achieving an expected return of 12%. We measure risk as the variance of the return on the investment (you will learn more about measuring risk in 45-733): when a fraction x of the savings is invested in Fund 1, y in Fund 2 and z in Fund 3, where $x + y + z = 1$, the variance of the return has been calculated to be

$$400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz.$$

So your problem is

$$\begin{array}{rcccccccc} \min & 400x^2 & + & 800y^2 & + & 200xy & + & 1600z^2 & + & 400yz \\ s.t. & x & + & y & + & 1.5z & = & 1.2 \\ & x & + & y & + & z & = & 1 \end{array}$$

Using the Lagrangian method, the following optimal solution was obtained

$$x = 0.5 \quad y = 0.1 \quad z = 0.4 \quad \lambda_1 = 1800 \quad \lambda_2 = -1380$$

where λ_1 is the Lagrange multiplier associated with the first constraint and λ_2 with the second constraint. The corresponding objective function value (i.e. the variance on the return) is 390. If an expected return of 12.5% was desired (instead of 12%), what would be (approximately) the correcponding variance of the return?

Since $\Delta = 0.05$, the variance would increase by

$$\Delta \lambda_1 = 0.05 \times 1800 = 90.$$

So the answer is 390+90=480.

Exercise 34 Record'm Records needs to produce 100 gold records at one or more of its three studios. The cost of producing x records at studio 1 is $10x$; the cost of producing y records at studio 2 is $2y^2$; the cost of producing z records at studio 3 is $z^2 + 8z$.
  (a) Formulate the nonlinear program of producing the 100 records at minimum cost.
  (b) What is the Lagrangian associated with your formulation in (a)?
  (c) Solve this Lagrangian. What is the optimal production plan?
  (d) What is the marginal cost of producing one extra gold record?
  (e) Union regulations require that exactly 60 hours of work be done at studios 2 and 3 combined. Each gold record requires 4 hours at studio 2 and 2 hours at studio 3. Formulate the problem of finding the optimal production plan, give the Lagrangian, and give the set of equations that must be solved to find the optimal production plan. It is not necessary to actually solve the equations.

Exercise 35   (a) Solve the problem

$$\begin{array}{ll} \max & 2x + y \\ \text{subject to} & 4x^2 + y^2 = 8 \end{array}$$

  (b) Estimate the change in the optimal objective function value when the right hand side increases by 5%, i.e. the right hand side increases from 8 to 8.4.

Exercise 36

(a) Solve the following constrained optimization problem using the method of Lagrange multipliers.

$$\begin{array}{ll} \max & \ln x + 2 \ln y + 3 \ln z \\ \text{subject to} & x + y + z = 60 \end{array}$$

(b) Estimate the change in the optimal objective function value if the right hand side of the constraint increases from 60 to 65.

4.2 Equality and Inequality Constraints

How do we handle both equality and inequality constraints in (P)? Let (P) be:

Maximize $f(x)$
Subject to
$g_1(x) = b_1$
$\vdots$
$g_m(x) = b_m$
$h_1(x) \leq d_1$
$\vdots$
$h_p(x) \leq d_p$

If you have a program with $\geq$ constraints, convert it into $\leq$ by multiplying by -1. Also convert a minimization to a maximization.

The Lagrangian is

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x)) + \sum_{j=1}^{p} \mu_j (d_j - h_j(x))$$

The fundamental result is the following:

Assume $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ maximizes $f(x)$ subject to the constraints $g_i(x) = b_i$, for $i = 1, 2, \ldots, m$ and $h_j(x) \leq d_j$, for $j = 1, 2, \ldots, p$. Then either

(i) the vectors $\nabla g_1(x^*), \ldots, \nabla g_m(x^*), \nabla h_1(x^*), \ldots, \nabla h_p(x^*)$ are linearly dependent, or

(ii) there exists vectors $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)$ and $\mu^* = (\mu_1^*, \ldots, \mu_p^*)$ such that

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) - \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*) = 0$$

$$\mu_j^* (h_j(x^*) - d_j) = 0 \quad \text{(Complementarity)}$$

$$\mu_j^* \geq 0$$

In this course, we will not concern ourselves with Case (i). We will only look for candidate solutions x^* for which we can find λ^* and μ^* satisfying the equations in Case (ii) above.

In general, to solve these equations, you begin with complementarity and note that either μ_j^* must be zero or $h_j(x^*) - d_j = 0$. Based on the various possibilities, you come up with one or more candidate solutions. If there is an optimal solution, then one of your candidates will be it.

The above conditions are called the *Kuhn–Tucker (or Karush–Kuhn–Tucker)* conditions. Why do they make sense?

For x^* optimal, some of the inequalities will be tight and some not. Those not tight can be ignored (and will have corresponding price $\mu_j^* = 0$). Those that are tight can be treated as equalities which leads to the previous Lagrangian stuff. So

$$\mu_j^*(h_j(x^*) - d_j) = 0 \quad \text{(Complementarity)}$$

forces either the price μ_j^* to be 0 or the constraint to be tight.

Example 4.2.1

Maximize $x^3 - 3x$
    Subject to
      $x \leq 2$

The Lagrangian is

$$L = x^3 - 3x + \mu(2 - x)$$

So we need

$$3x^2 - 3 - \mu = 0$$
$$x \leq 2$$
$$\mu(2 - x) = 0$$
$$\mu \geq 0$$

Typically, at this point we must break the analysis into cases depending on the complementarity conditions.

If $\mu = 0$ then $3x^2 - 3 = 0$ so $x = 1$ or $x = -1$. Both are feasible. $f(1) = -2$, $f(-1) = 2$.

If $x = 2$ then $\mu = 9$ which again is feasible. Since $f(2) = 2$, we have two solutions: $x = -1$ and $x = 2$.

Example 4.2.2 *Minimize $(x - 2)^2 + 2(y - 1)^2$*
    Subject to
      $x + 4y \leq 3$
      $x \geq y$

First we convert to standard form, to get

Maximize $-(x - 2)^2 - 2(y - 1)^2$
Subject to
    $x + 4y \leq 3$
    $-x + y \leq 0$

$$L(x, y, \mu_1, \mu_2) = -(x - 2)^2 - 2(y - 1)^2 + \mu_1(3 - x - 4y) + \mu_2(0 + x - y)$$

which gives optimality conditions

$$-2(x - 2) - \mu_1 + \mu_2 = 0$$
$$-4(y - 1) - 4\mu_1 - \mu_2 = 0$$
$$\mu_1(3 - x - 4y) = 0$$

$$\mu_2(x - y) = 0$$
$$x + 4y \leq 3$$
$$-x + y \leq 0$$
$$\mu_1, \mu_2 \geq 0$$

Since there are two complementarity conditions, there are four cases to check:

$\mu_1 = 0, \mu_2 = 0$: gives $x = 2$, $y = 1$ which is not feasible.

$\mu_1 = 0, x - y = 0$: gives $x = 4/3, y = 4/3, \mu_2 = -4/3$ which is not feasible.

$\mu_2 = 0, 3 - x - 4y = 0$ gives $x = 5/3, y = 1/3, \mu_1 = 2/3$ which is O.K.

$3 - x - 4y = 0, x - y = 0$ gives $x = 3/5, y = 3/5, \mu_1 = 22/25, \mu_2 = -48/25$ which is not feasible.

Since it is clear that there is an optimal solution, $x = 5/3, y = 1/3$ is it!

Economic Interpretation

The economic interpretation is essentially the same as the equality case. If the right hand side of a constraint is changed by a small amount Δ, then the optimal objective changes by $\mu^* \Delta$, where μ^* is the optimal Lagrange multiplier corresponding to that constraint. Note that if the constraint is not tight then the objective does not change (since then $\mu^* = 0$).

Handling Nonnegativity

A special type of constraint is nonnegativity. If you have a constraint $x_k \geq 0$, you can write this as $-x_k \leq 0$ and use the above result. This constraint would get a Lagrange multiplier of its own, and would be treated just like every other constraint.

An alternative is to treat nonnegativity implicitly. If x_k must be nonnegative:

1. Change the equality associated with its partial to a less than or equal to zero:

$$\frac{\partial f(x)}{\partial x_k} - \sum_i \lambda_i \frac{\partial g_i(x)}{\partial x_k} - \sum_j \mu_j \frac{\partial h_j(x)}{\partial x_k} \leq 0$$

2. Add a new complementarity constraint:

$$\left(\frac{\partial f(x)}{\partial x_k} - \sum_i \lambda_i \frac{\partial g_i(x)}{\partial x_k} - \sum_j \mu_j \frac{\partial h_j(x)}{\partial x_k} \right) x_k = 0$$

3. Don't forget that $x_k \geq 0$ for x to be feasible.

Sufficiency of conditions

The Karush–Kuhn–Tucker conditions give us candidate optimal solutions x^*. When are these conditions sufficient for optimality? That is, given x^* with λ^* and μ^* satisfying the KKT conditions, when can we be certain that it is an optimal solution?

The most general condition available is:

1. $f(x)$ is concave, and

2. the feasible region forms a convex region.

While it is straightforward to determine if the objective is concave by computing its Hessian matrix, it is not so easy to tell if the feasible region is convex. A useful condition is as follows:

The feasible region is convex if all of the $g_i(x)$ are linear and all of the $h_j(x)$ are convex. If this condition is satisfied, then any point that satisfies the KKT conditions gives a point that maximizes $f(x)$ subject to the constraints.

Example 4.2.3 *Suppose we can buy a chemical for \$10 per ounce. There are only 17.25 oz available. We can transform this chemical into two products: A and B. Transforming to A costs \$3 per oz, while transforming to B costs \$5 per oz. If x_1 oz of A are produced, the price we command for A is \$30 $- x_1$; if x_2 oz of B are produced, the price we get for B is \$50 $- x_2$. How much chemical should we buy, and what should we transform it to?*

There are many ways to model this. Let's let x_3 be the amount of chemical we purchase. Here is one model:

Maximize $x_1(30 - x_1) + x_2(50 - 2x_2) - 3x_1 - 5x_2 - 10x_3$
Subject to
$\quad x_1 + x_2 - x_3 \leq 0$
$\quad x_3 \leq 17.25$

The KKT conditions are the above feasibility constraints along with:
$30 - 2x_1 - 3 - \mu_1 = 0$
$50 - 4x_2 - 5 - \mu_1 = 0$
$-10 + \mu_1 - \mu_2 = 0$
$\mu_1(-x_1 - x_2 + x_3) = 0$
$\mu_2(17.25 - x_3) = 0$
$\mu_1, \mu_2 \geq 0$

There are four cases to check:
$\mu_1 = 0, \mu_2 = 0$. This gives us $-10 = 0$ in the third constraint, so is infeasible.
$\mu_1 = 0, x_3 = 17.25$. This gives $\mu_2 = -10$ so is infeasible.
$-x_1 - x_2 + x_3 = 0, \mu_2 = 0$. This gives $\mu_1 = 10, x_1 = 8.5, x_2 = 8.75, x_3 = 17.25$, which is feasible. Since the objective is concave and the constraints are linear, this must be an optimal solution. So there is no point in going through the last case $(-x_1 - x_2 + x_3 = 0, x_3 = 17.25)$. We are done with $x_1 = 8.5, x_2 = 8.75$, and $x_3 = 17.25$.

What is the value of being able to purchase 1 more unit of chemical?

This question is equivalent to increasing the right hand side of the constraint $x_3 \leq 17.25$ by 1 unit. Since the corresponding lagrangian value is 0, there is no value in increasing the right hand side.

Review of Optimality Conditions.

The following reviews what we have learned so far:

Single Variable (Unconstrained)
Solve $f'(x) = 0$ to get candidate x^*.
If $f''(x^*) > 0$ then x^* is a local min.
  $f''(x^*) < 0$ then x^* is a local max.
If $f(x)$ is convex then a local min is a global min.
  $f(x)$ is concave then a local max is a global max.

Multiple Variable (Unconstrained)
Solve $\nabla f(x) = 0$ to get candidate x^*.
If $H(x^*)$ is positive definite then x^* is a local min.
  $H(x^*)$ is negative definite x^* is a local max.
If $f(x)$ is convex then a local min is a global min.
  $f(x)$ is concave then a local max is a global max.

Multiple Variable (Equality constrained)   Form Lagrangian $L(x, \lambda) = f(x) + \sum_i \lambda_i (b_i - g_i(x))$
Solve $\nabla L(x, \lambda) = 0$ to get candidate x^* (and λ^*).
Best x^* is optimum if optimum exists.

Multiple Variable (Equality and Inequality constrained)
Put into standard form (maximize and $\leq$ constraints)
Form Lagrangian $L(x, \lambda) = f(x) + \sum_i \lambda_i (b_i - g_i(x)) + \sum_j \mu_j (d_j - h_j(x))$
Solve
$$\nabla f(x) - \sum_i \lambda_i \nabla g_i(x) - \sum_j \mu_j \nabla h_j(x) = 0$$
$$g_i(x) = b_i$$
$$h_j(x) \leq d_j$$
$$\mu_j (d_j - h_j(x)) = 0$$
$$\mu_j \geq 0$$
to get candidates x^* (and λ^*, μ^*).
Best x^* is optimum if optimum exists.
Any x^* is optimum if $f(x)$ concave, $g_i(x)$ convex, $h_j(x)$ linear.

4.3 Exercises

Exercise 37 Solve the following constrained optimization problem using the method of Lagrange multipliers.

$$\begin{aligned} \max \quad & 2 \ln x_1 + 3 \ln x_2 + 3 \ln x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 2x_3 = 10 \end{aligned}$$

Exercise 38 Find the two points on the ellipse given by $x_1^2 + 4x_2^2 = 4$ that are at minimum distance of the point $(1, 0)$. Formulate the problem as a minimization problem and solve it by solving the Lagrangian equations. [*Hint:* To minimize the distance d between two points, one can also minimize d^2. The formula for the distance between points (x_1, x_2) and (y_1, y_2) is $d^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2$.]

Exercise 39 Solve using Lagrange multipliers.

a) $\min x_1^2 + x_2^2 + x_3^2$ subject to $x_1 + x_2 + x_3 = b$, where b is given.

b) $\max \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}$ subject to $x_1 + x_2 + x_3 = b$, where b is given.

c) $\min c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2$ subject to $x_1 + x_2 + x_3 = b$, where $c_1 > 0$, $c_2 > 0$, $c_3 > 0$ and b are given.

d) $\min x_1^2 + x_2^2 + x_3^2$ subject to $x_1 + x_2 = b_1$ and $x_2 + x_3 = b_2$, where b_1 and b_2 are given.

Exercise 40 Let a, b and c be given positive scalars. What is the change in the optimum value of the following constrained optimization problem when the right hand side of the constraint is increased by 5%, i.e. a is changed to $a + \frac{5}{100}a$.

$$\begin{aligned} \max \quad & by - x^4 \\ \text{s.t.} \quad & x^2 + cy = a \end{aligned}$$

Give your answer in terms of a, b and c.

Exercise 41 You want to invest in two mutual funds so as to maximize your expected earnings while limiting the variance of your earnings to a given figure s^2. The expected yield rates of Mutual Funds 1 and 2 are r_1 and r_2 respectively, and the variance of earnings for the portfolio (x_1, x_2) is $\sigma^2 x_1^2 + \rho x_1 x_2 + \sigma^2 x_2^2$. Thus the problem is

$$\begin{aligned} \max \quad & r_1 x_1 + r_2 x_2 \\ \text{s.t.} \quad & \sigma^2 x_1^2 + \rho x_1 x_2 + \sigma^2 x_2^2 = s^2 \end{aligned}$$

(a) Use the method of Lagrange multipliers to compute the optimal investments x_1 and x_2 in Mutual Funds 1 and 2 respectively. Your expressions for x_1 and x_2 should not contain the Lagrange multiplier λ.

(b) Suppose both mutual funds have the same yield r. How much should you invest in each?

Exercise 42 You want to minimize the surface area of a cone-shaped drinking cup having fixed volume V_0. Solve the problem as a constrained optimization problem. To simplify the algebra, minimize the square of the area. The area is $\pi r \sqrt{r^2 + h^2}$. The problem is,

$$\begin{aligned} \min \quad & \pi^2 r^4 + \pi^2 r^2 h^2 \\ \text{s.t.} \quad & \frac{1}{3} \pi r^2 h = V_0. \end{aligned}$$

Solve the problem using Lagrange multipliers.
        Hint. You can assume that $r \neq 0$ in the optimal solution.

Exercise 43 A company manufactures two types of products: a standard product, say A, and a more sophisticated product B. If management charges a price of p_A for one unit of product A and a price of p_B for one unit of product B, the company can sell q_A units of A and q_B units of B, where

$$q_A = 400 - 2p_A + p_B, \qquad q_B = 200 + p_A - p_B.$$

Manufacturing one unit of product A requires 2 hours of labor and one unit of raw material. For one unit of B, 3 hours of labor and 2 units of raw material are needed. At present, 1000 hours of labor and 200 units of raw material are available. Substituting the expressions for q_A and q_B, the problem of maximizing the company's total revenue can be formulated as:

$$
\begin{aligned}
\max \quad & 400p_A + 200p_B - 2p_A^2 - p_B^2 + 2p_A p_B \\
\text{s.t.} \quad & -p_A - p_B \leq -400 \\
& -p_B \leq -600
\end{aligned}
$$

(a) Use the Khun-Tucker conditions to find the company's optimal pricing policy.

(b) What is the maximum the company would be willing to pay for

- another hour of labor,
- another unit of raw material?