A. proof of theorem 1

$$u(n) = K_p \cdot e(n) + K_i \cdot T \sum_{i=0}^{n} e(i) + \frac{K_d}{T}\big(e(n) - e(n-1)\big)$$

The equation has been transformed to compute K_p, K_i and K_d as integers.

$$= \left((K_p \cdot 10^m \cdot 10^{-m}) e(n) + (K_i \cdot 10^m \cdot 10^{-m} \cdot T) \sum_{i=0}^{n} e(i) + \frac{K_d \cdot 10^m \cdot 10^{-m}}{T}\big(e(n) - e(n-1)\big) \right)$$

$K_p' = K_p \cdot 10^m$, $K_i' = K_i \cdot 10^m$ and $K_d' = K_d \cdot 10^m$

$$= \left(K_p' \cdot e(n) + K_i' \cdot T \sum_{i=1}^{n} e(i) + \frac{K_d'}{T}\big(e(n) - e(n-1)\big) \right) \cdot 10^{-m}$$

Given that $T = \frac{1}{2^k}$, the equation is represented in the form of $\frac{1}{T}$ as follows:

$$= \left(\frac{K_p'}{T} \cdot e(k) + K_i' \sum_{i=1}^{n} e(i) + \frac{K_d'}{T^2}\big(e(k) - e(k-1)\big) \right) 10^{-m} \cdot T$$

The equation is substituted as $e(n) = \frac{P_r}{2^N} \cdot \hat{e}(n)$

$$= \left(\frac{K_p'}{T}\left(\hat{e}(n) \cdot \frac{P_r}{2^N}\right) + K_i' \sum_{i=1}^{n} \left(\hat{e}(i) \cdot \frac{P_r}{2^N}\right) + \frac{K_d'}{T^2}\left(\big(\hat{e}(n) - \hat{e}(n-1)\big) \cdot \frac{P_r}{2^N}\right) \right) 10^{-m} \cdot T$$

Given the binomial expansion of $\frac{P_r}{2^N}$, the following equation is given:

$$= \left(\frac{K_p'}{T} \cdot \hat{e}(n) + K_i' \sum_{i=1}^{n} \hat{e}(i) + \frac{K_d'}{T^2}\big(\hat{e}(n) - \hat{e}(n-1)\big) \right) \left(\frac{P_r}{2^N} \cdot T \cdot 10^{-m} \right)$$

The constant term C_t is defined as $C_t = \left(\frac{P_r}{2^N} \cdot T \cdot 10^{-m} \right)$, the following equation is given:

$$= C_t \left(\frac{K_p'}{T} \hat{e}(n) + K_i' \sum_{i=1}^{n} \hat{e}(i) + \frac{K_d'}{T^2}\big(\hat{e}(n) - \hat{e}(n-1)\big) \right)$$

By representing the constant term C_t as a floating-point type and, I_t as an integer type, the flowing equation:

$$= C_t \cdot I_t$$

A. QED

B. proof of theorem 2

$u(n) = u(n-1) + b_0 \cdot e(n) + b_1 \cdot e(n-1) + b_2 \cdot e(n-2)$

$b_0 = K_p + \frac{T \cdot K_i}{2} + \frac{K_d}{T}$

$b_1 = \frac{K_i \cdot T}{2} - K_p - \frac{2K_d}{T}$

$b_2 = \frac{K_d}{T}$

By substituting $e(n) = \frac{P_r}{2^N} \cdot \hat{e}(n)$, the following equation is given:

$u(n) = u(n-1) + \frac{P_r}{2^N}\left(b_0 \cdot \hat{e}(n) + b_1 \cdot \hat{e}(n-1) + b_2 \cdot \hat{e}(n-2)\right)$

b_0, b_1, and b_2 are defined for integer calculations and follow as:

$b_0 = \left(\frac{T}{2} \cdot 10^{-m}\right)\left(\frac{2K_p'}{T} + K_i' + \frac{2K_d'}{T^2}\right)$

$b_1 = \left(\frac{T}{2} \cdot 10^{-m}\right)\left(K_i' - \frac{2K_p'}{T} - \frac{4K_d'}{T^2}\right)$

$b_2 = \left(\frac{T}{2} \cdot 10^{-m}\right)\left(\frac{2K_d'}{T^2}\right)$

The constant term C_t is defined as $C_t = \left(\frac{P_r}{2^N}\right)\left(\frac{T}{2} \cdot 10^{-m}\right)$, the following equation is given:

$u(n) = u(n-1) + C_t\left(b_o'\hat{e}(n) + b_1'\hat{e}(n-1) + b_2' \cdot \hat{e}(n-2)\right)$

$b_o' = \frac{2K_p'}{T} + K_i' + \frac{2K_d'}{T^2}$

$b_1' = K_i' - \frac{2K_p'}{T} - \frac{4K_d'}{T^2},$

$b_2' = \frac{2K_d'}{T^2}$

By representing the constant term C_t as a floating-point type and, I_t as an integer type, the following equation is given:

$u(n) = u(n-1) + C_t I_t$

B. QED