20170269 김은진 homework 5

Part A. Algorithms.

1. Procedure duplicates ($a_1, a_2, a_3 \cdots a_n$: integer n nondecreasing order)

$k := 0$ {this counts the duplicates}.
$j := 2$
while $j \le n$.
 if $a_j = a_{j-1}$ then
  $k := k+1$
  $c_k = a_j$
  while $j \le n$ and $a_j = c_k$
   $j := j+1$
 $j := j+1$
{$c_1, c_2, \cdots c_k$ is the desired list}.

2.

input ; $a_1, a_2, a_3, \ldots, a_n$: integers.
output ; location of maximum among inputs.

∴ This function is function that return location of maximum from Inputs (integers)

3.

a) 3 quarters, 1 dimes, 2 pennies.
b) 1 quarter, 2 dimes, 4 pennies.
c) 3 quarters, 2 dimes, 4 pennies.
d) 1 quarter, 1 nickels, 3 pennies.

4)

a) 2 quarters, 1 pennies.
b) 2 quarters, 1 dime, 1 nickel, and 4 pennies.
c) 3 quarters, 1 pennies.
d) 2 quarters, 1 dime.

Part B. The Growth of Functions.

1.

(a) $f(x) = 10$.
$1 < x$ for all $x > 1$.
$10 \le C \cdot x$.

Witness. $C = 10$, $k = 1$.
∴ $f(x)$ is $O(x)$.

(b) $f(x) = 3x + 17$.
$3x+17 \le 3x+17x$
$= 10x$.
Witness $C = 10$, $k = 1$.
∴ $f(x)$ is $O(x)$.

(c) $f(x) = x^2+x+1$.
$x^2+x+1 \le x^2+x^2+x^2$
$= 3x^2$.
Witness $C = 3$, $k = 1$.
∴ $f(x)$ is not $O(x)$.
($f(x)$ is $O(x^2)$).

(d) $f(x) = 5\log x$.
$5\log x < 5x$ for all $x > 1$.
$5\log x \le 5x$
Winess $C = 5$, $k = 1$.
∴ $f(x)$ is $O(x)$.

(e) $f(x) = \lfloor x \rfloor$

$\lfloor x \rfloor \le x$ for all $x \ge 1$.

$\lfloor x \rfloor \le x$

Witness $C=1$, $k=1$.

$\therefore f(x)$ is $O(x)$.

(f) $f(x) = \lceil x/2 \rceil$.

$\lceil x/2 \rceil \le x$ for all $x \ge 1$.

$\lceil x/2 \rceil \le x$.

Witness $C=1$, $k=1$.

$f(x)$ is $O(x)$.

2.

(a) $f(x) = 17x+11$.

$$17x+11 \le 17x^2+11x^2 = 28x^2$$

Witness $C=28$, $k=1$.

$\therefore f(x)$ is $O(x^2)$.

(b) $f(x) = x^2+1000$.

$$x^2+1000 \le x^2+1000x^2 = 1001x^2$$

Witness $C=1001$, $k=1$.

(c) $f(x) = x \log x$.

$x \log x \le x^2$ for all $x \ge 1$.

$x \log x \le x^2$.

Witness $C=1$, $k=1$.

$\therefore f(x)$ is $O(x^2)$.

(d) $f(x) = \frac{x^4}{2}$.

$x^4/2 \ge x^2$.

$\therefore f(x)$ is not $O(x^2)$.

(e) $f(x) = 2^x$.

$2^x \ge x^2$.

$\therefore f(x)$ is not $O(x^2)$

(f) $f(x) = \lfloor x \rfloor \cdot \lceil x \rceil$

$\lfloor x \rfloor \cdot \lceil x \rceil \le x^2$.

Witness $C=1$, $k=1$

$\therefore f(x)$ is $O(x^2)$.

3.

Let $f(x) = x^4+9x^3+4x+7$.

$|f(x)| \le C \cdot |g(x)|$.

$1 < x^4$ for all $x > 1$.

$$x^4+9x^3+4x+7 \le x^4+9x^4+4x^4+7x^4 = 21\underbrace{x^4}_{g(x)}$$

Witness $C=21$, $k=1$.

$C=21$, $g(x)=x^4$.

$\therefore f(x)$ is $O(x^4)$.

4.

Let $f(x) = 2^x+17$.

$|f(x)| \le C \cdot |g(x)|$.

$2^x < 3^x$ for all $x > 1$

$$2^x+17 \le 3^x+17 \cdot 3^x = 18 \cdot 3^x$$

Witness. $C=18$, $k=0$.

$C=18$, $g(x)=3^x$.

$\therefore f(x)$ is $O(3^x)$.

5.

$\frac{1}{x+1} < \frac{1}{x}$ for all $x>1$

$1<x<x^2$ for all $x>1$.

$$\frac{x^2+1}{x+1} \le \frac{x^2+x^2}{x} = \frac{2x^2}{x} = 2x$$

Witness, $C=2$, $k=1$

$\therefore f(x)$ is $O(x)$.

6. $\frac{1}{2x+1} < \frac{1}{2x}$ for all $x>1$.

$2x<2x^3$ for all $x>1$.

$$\frac{x^3+2x}{2x+1} \le \frac{x^3+2x^3}{2x} = \frac{3x^3}{2x} = \frac{3}{2}x^2$$

Witness $C=\frac{3}{2}$, $k=1$

$\therefore f(x)$ is $O(x^2)$.

7.

(a)

$\log x < x$ for all $x>0$

$$2x^3+x^2\log x \le 2x^3+x^2\cdot x = 2x^3+x^3 = 3x^3$$

Witness $C=3$, $k=1$.

$\therefore f(x)$ is $O(x^3)$, $n=3$.

(b) $(\log x)^4 < x^3$ for all $x>1$.

$$3x^3+(\log x)^4 \le 3x^3+x^3 = 4x^3$$

Witness $C=4$, $k=1$.

$\therefore f(x)$ is $O(x^3)$, $n=3$.

(c)

$\frac{1}{x^3+1} < \frac{1}{x^3}$ for all $x>1$.

$1<x^2<x^4$ for all $x>1$.

$$\frac{x^4+x^2+1}{x^3+1} \le \frac{x^4+x^4+x^4}{x^3} = \frac{3x^4}{x^3} = 3x$$

Witness $C=3$, $k=1$.

$\therefore f(x)$ is $O(x)$, $n=1$.

(d).

$\frac{1}{x^4+1} < \frac{1}{x^4}$ for all $x>1$.

$5\log x < 5x^4$ for all $x>0$.

$$\frac{x^4+5\log x}{x^4+1} \le \frac{x^4+5x^4}{x^4} = 6$$

$C=6$, $k=0$.

$\therefore f(x)$ is $O(x^0)$
$= O(1)$. $n=0$.

8.

(a) $2x^2 < 2x^4$ for all $x>1$.

$\log x < x$ for all $x>1$.

$$2x^2 + x^3\log x \le 2x^4 + x^3 \cdot x$$
$$= 2x^4 + x^4$$
$$= 3x^4.$$

Witness. $c=3$, $k=1$.

$\therefore f(x)$ is $O(x^4)$, $n=4$.

(b)

$(\log x)^4 < x^5$ for all $x>1$.

$$3x^5 + (\log x)^4 \le 3x^5 + x^5$$
$$= 4x^5$$

Witness. $c=4$, $k=1$.

$\therefore f(x)$ is $O(x^5)$, $n=5$

(c)

$\frac{1}{x^4+1} < \frac{1}{x^4}$ for all $x>1$.

$1 < x^2 < x^4$ for all $x>1$.

$$\frac{x^4+x^2+1}{x^4+1} \le \frac{x^4+x^4+x^4}{x^4}$$
$$= \frac{3x^4}{x^4} = 3.$$

Witness $c=3$, $k=0$.

$\therefore f(x)$ is $O(x^0)$; $n=0$.

(d)

$\frac{1}{x^4+1} < \frac{1}{x^4}$ for all $x>1$.

$5\log x < 5x < 5x^2 < 5x^3$.
for all $x>1$.

$$\frac{x^3+5\log x}{x^4+1} \le \frac{x^3+5x^3}{x^4}$$
$$= \frac{6x^3}{x^4}$$
$$= \frac{6}{x}.$$

Witness $c=6$, $k=1$.

$\therefore f(x)$ is $O(x^{-1})$, $n=-1$.

9.

Let $f(x) = x^2+4x+17$.

$1 < x < x^2 < x^3$ for all $x>1$.

$$x^2+4x+17 \le x^3+4x^3+17x^3.$$
$$= 22x^3.$$

Witness $c=22$, $k=1$.

$\therefore f(x)$ is $O(x^3)$.

If x^3 is $O(x^2+4x+17)$.

$$x^3 \le c \cdot (x^2+4x+17).$$

$x < x^2$ for all $x>1$.

$$c \cdot (x^2+4x+17) \le c(x^2+4x^2+17x^2).$$
$$= c \cdot 22x^2.$$

$$x^3 \le c(x^2+4x+17) \le 22 \cdot c \cdot x^2$$
$$x^3 \le 22 \cdot c \cdot x^2$$

$x \le 22c$. (c is constant value.)

$\hookrightarrow$ Impossible

∴ x^3 is not $O(x^2+4x+17)$.

10.

Let $f(x)$ is x^3.

$x^3 < x^4$ for all $x>1$.

$x^3 \le x^4$

Witness $C=1$, $k=1$.

∴ $f(x)$ is $O(x^4)$.

if x^4 is $O(x^3)$.

$\Rightarrow x^4 \le C\cdot x^3$.

$x \le C$. but C is constant value.

∴ x^4 is not $O(x^3)$

11.

(a) $(n^2+8)(n+1)$

$O(n^2)\cdot O(n) = O(n^3)$.

(b) $(n\log n+n^2)(n^3+2)$

$O(n^2)\cdot O(n^3) = O(n^5)$

(c) $(n!+2^n)(n^3+\log(n^2+1))$

$O(n!)\cdot O(n^3) = O(n^3 n!)$

12.

(a) $(n^3+n^2\log n)(\log n+1)+$

$O(n^3)$ $O(\log n)$ $\rightarrow O(n^3\log n)$.

$(17\log n+19)(n^3+2)$.

$O(\log n)$ $O(n^3)$

$\hookrightarrow O(n^3\log n)$

same $\Rightarrow O(n^3\log n)$.

(b) $(2^n+n^2)(n^3+3^n)$.

$O(2^n)$ $O(3^n)$.

$\Rightarrow O(6^n)$.

(c) $(n^n+n2^n+5^n)(n!+5^n)$.

$O(n^n)$ $O(n!)$

$= O(n!\cdot n^n)$.

13.

(a) $n\log(n^2+1)+n^2\log n$.

$n\log(n^2+1) \le n\log(n^2+n^2)$

$= n\log 2n^2$

$= n(\log 2+\log n^2)$

$= n\log 2 + 2n\log n$.

$n\log(n^2+1) \Rightarrow O(n\log n)$

$n^2\log n \Rightarrow O(n^2\log n)$

∴ $O(n^2\log n)$.

(b) $(n\log n+1)^2+(\log n+1)(n^2+1)$.

$\Downarrow$

$(n\log n+1)(n\log n+1)+(\log n+1)(n^2+1)$.

$O(n\log n)$ $O(n\log n)$ $O(\log n)$ $O(n^2)$.

$O((n\log n)^2)$. $O(n^2\log n)$.

∴ $O(n^2(\log n)^2)$.

(c). $n^{2^n}+n^{n^2}$

$O(n^{2^n})$ $O(n^{n^2})$.

$2^n > n^2$ for all $x\ge 1$.

∴ $O(n^{2^n})$.