31. *General Decomposition Theory of Ordered Exponentials*

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Abstract: A general decomposition theory of ordered exponentials is presented by reducing the problem to the decomposition of ordinary exponential operators in terms of the super-operator $\mathcal{T}$ defined by $F(t)\exp(\Delta t\mathcal{T})G(t) = F(t+\Delta t)G(t)$. It is proved that $\mathrm{T}\left(\exp\int_t^{t+\Delta t}\mathscr{H}(s)ds\right) = \exp[\Delta t(\mathscr{H}(t) + \mathcal{T})]$. Here T denotes the time ordering.

Key words: Exponential operator; ordered exponential; decomposition; time-dependent Hamiltonian; Trotter formula.

1. Introduction. An exponential operator has been used conveniently to describe a formal solution of a time-evolution equation of the form

$$\frac{\partial}{\partial t}P(t) = \mathscr{H}P(t). \tag{1}$$

The formal solution of (1) is given by

$$P(t) = \mathrm{e}^{t\mathscr{H}}P(0). \tag{2}$$

If $\mathscr{H}$ is composed of the two non-commutable oprators A and B, namely $\mathscr{H} = A + B$, then we have the following decomposition formula

$$e^{t(A+B)} = e^{t_1A}e^{t_2B}e^{t_3A}e^{t_4B}\cdots e^{t_MA} + O(t^{m+1}) \tag{3}$$

for any positive integer m with some appropriate parameters[1)–8)] $\{t_j\}$.

When $\mathscr{H}$ depends on time t explicitly, namely $\mathscr{H} = \mathscr{H}(t)$, the situation becomes more complicated. A formal solution of the equation

$$\frac{\partial}{\partial t}\Psi(t) = \mathscr{H}(t)\Psi(t) \tag{4}$$

is given by $\Psi(t) = U(t, 0)\Psi(0)$ using the following ordered exponential[9)]

$$\begin{aligned} U(t, 0) &= \mathrm{T}\left(\exp\int_0^t\mathscr{H}(s)ds\right) \\ &= 1 + \int_0^t\mathscr{H}(s_1)ds_1 + \int_0^t ds_1\int_0^{s_1}ds_2\, ds_2\mathscr{H}(s_1)\mathscr{H}(s_2) + \cdots. \end{aligned} \tag{5}$$

Here T denotes the time ordering.

At a glance, it seems very complicated[10)–15)] to decompose such ordered exponential operators. One of the purposes of the present paper is to give a general theory of decomposing such ordered exponentials using the ordinary decomposition formulas of exponential operators already given by the present author.[1)–8)] It is shown from the present general theory that an m-th order decomposition formula of the ordered exponential (5) is immediately obtained by replacing A and B of the ordinary m-th order decomposition formula for $\mathscr{H} = A + B$ by $A(t_{j,\mathrm{mid}})$ and $B(t_{j,\mathrm{mid}})$, in each first- or second-order approximate basis, where $t_{j,\mathrm{mid}}$ denotes the middle point of each time separation.

For example, the first-order decomposition formula $U_1(t+\Delta t, t)$ of the ordered exponential

$$U(t+\Delta t, t) = \mathrm{T}\left(\exp\int_t^{t+\Delta t}\mathscr{H}(s)ds\right) = \mathrm{T}\left(\exp\int_t^{t+\Delta t}(A(s) + B(s))ds\right) \tag{6}$$

is given by

$$U_1(t+\Delta t,\, t) = e^{\Delta tA\left(t+\frac{1}{2}\Delta t\right)} e^{\Delta tB\left(t+\frac{1}{2}\Delta t\right)} \tag{7}$$

corresponding to the ordinary decomposition formula

$$e^{\Delta t(A+B)} = e^{\Delta tA} e^{\Delta tB} + O((\Delta t)^2). \tag{8}$$

Similarly the second-order decomposition $U_2(t+\Delta t,\, t)$ is also given by

$$U_2(t+\Delta t,\, t) = e^{\frac{1}{2}\Delta tA\left(t+\frac{1}{2}\Delta t\right)} e^{\Delta tB\left(t+\frac{1}{2}\Delta t\right)} e^{\frac{1}{2}\Delta tA\left(t+\frac{1}{2}\Delta t\right)}, \tag{9}$$

corresponding to the ordinary symmetrized decomposition

$$e^{\Delta t(A+B)} = e^{\frac{1}{2}\Delta tA} e^{\Delta tB} e^{\frac{1}{2}\Delta tA} + O((\Delta t)^3). \tag{10}$$

Of course, these low-order decomposition formulas can be derived in an elementary way, but the derivation is much more complicated than the present general procedure.

2. Super-operator formula of ordered exponentials. In order to reduce the decomposition problem of ordered exponentials to that of ordinary exponential oprators, we intoduce the following super-operator $\mathcal{T}$

$$F(t)e^{\Delta t\mathcal{T}}G(t) = F(t+\Delta t)G(t) \tag{11}$$

for any (even non-differentiable) operators $F(t)$ and $G(t)$. Of course, if $F(t)=1$, we have

$$1\cdot e^{\Delta t\mathcal{T}}G(t) = e^{\Delta t\mathcal{T}}G(t) = G(t). \tag{12}$$

When $F(t)$ is differentiable with respect to time t, the super-operator $\mathcal{T}$ is expressed by the following differential opertor

$$\mathcal{T} = \frac{\overleftarrow{\partial}}{\partial t}, \tag{13}$$

where the arrow in (13) denotes the differentiation of the operators preceding this symbol.

Then, we find the following formula.

Formula 1. Any ordered exponential can be expressed by an ordinary exponential operator in terms of the super-operator $\mathcal{T}$ as

$$\mathrm{T}\left(\exp\int_t^{t+\Delta t}\mathcal{H}(s)ds\right) = \exp\left[\Delta t(\mathcal{H}(t)+\mathcal{T})\right]. \tag{14}$$

The proof of this formula is given as follows. Using the Trotter formula

$$e^{x(A+B)} = \lim_{n\to\infty}\left(e^{\frac{x}{n}A}e^{\frac{x}{n}B}\right)^n, \tag{15}$$

the right-hand side of (14) is expressed as

$$\begin{aligned} e^{\Delta t(\mathcal{H}(t)+\mathcal{T})} &= \lim_{n\to\infty}\left(e^{\frac{\Delta t}{n}\mathcal{H}(t)}e^{\frac{\Delta t}{n}\mathcal{T}}\right)^n, \\ &= \lim_{n\to\infty} e^{\frac{\Delta t}{n}\mathcal{H}(t+\Delta t)}\cdots e^{\frac{\Delta t}{n}\mathcal{H}\left(t+\frac{2\Delta t}{n}\right)}e^{\frac{\Delta t}{n}\mathcal{H}\left(t+\frac{\Delta t}{n}\right)} \\ &= \mathrm{T}\left(\exp\int_t^{t+\Delta t}\mathcal{H}(s)ds\right). \end{aligned} \tag{16}$$

Here we have used the following relations recursively

$$e^{\frac{\Delta t}{n}\mathcal{H}(t)}e^{\frac{\Delta t}{n}\mathcal{T}} = e^{\frac{\Delta t}{n}\mathcal{H}\left(t+\frac{\Delta t}{n}\right)},\quad e^{\frac{\Delta t}{n}\mathcal{H}\left(t+\frac{\Delta t}{n}\right)}e^{\frac{\Delta t}{n}\mathcal{H}(t)}e^{\frac{\Delta t}{n}\mathcal{T}} = e^{\frac{\Delta t}{n}\mathcal{H}\left(t+\frac{2\Delta t}{n}\right)}e^{\frac{\Delta t}{n}\mathcal{H}\left(t+\frac{\Delta t}{n}\right)},\ \text{etc.} \tag{17}$$

There is an alternative derivation of (14) as follows. Note that the left-hand side of (14) is expanded as (5). Then. we expand the right-hand side of (14) in a power series of $\mathcal{H}$:

$$\begin{aligned} &\exp[\Delta t(\mathcal{H}(t)+\mathcal{T})] = e^{\Delta t\mathcal{T}}\,\mathrm{T}\left(\exp\int_0^{\Delta t}\mathcal{H}_t[u]du\right) \\ &= e^{\Delta t\mathcal{T}}\left(1+\int_0^{\Delta t}\mathcal{H}_t[u]du + \int_0^{\Delta t}du_1\int_0^{u_1}du_2\,\mathcal{H}_t[u_1]\mathcal{H}_t[u_2] + \cdots\right) \\ &= e^{\Delta t\mathcal{T}}\left(1+\int_0^{\Delta t}e^{-u\mathcal{T}}\mathcal{H}(t)e^{u\mathcal{T}}du + \int_0^{\Delta t}du_1\int_0^{u_1}du_2 e^{-u_1\mathcal{T}}\mathcal{H}(t)e^{(u_1-u_2)\mathcal{T}}\mathcal{H}(t)e^{u_2\mathcal{T}} + \cdots\right) \\ &= 1+\int_0^{\Delta t}\mathcal{H}(t+u)du + \int_0^{\Delta t}du_1\int_0^{u_1}du_2\,\mathcal{H}(t+u_1)\mathcal{H}(t+u_2) + \cdots \\ &= 1+\int_t^{t+\Delta t}\mathcal{H}(s)ds + \int_t^{t+\Delta t}ds_1\int_t^{s_1}ds_2\,\mathcal{H}(s_1)\mathcal{H}(s_2) + \cdots \end{aligned}$$

$$= \mathrm{T}\left(\exp \int_t^{t+\Delta t} \mathscr{H}(s)ds\right). \tag{18}$$

Here we have used the notation

$$\mathscr{H}_t[u] \equiv e^{-u\mathscr{T}} \mathscr{H}(t) e^{u\mathscr{T}}. \tag{19}$$

The transformation of the n-th order term in (18) is confirmed more explicitly as

$$\begin{aligned}
&e^{\Delta t\mathscr{T}} \int_0^{\Delta t} du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-1}} du_n \mathscr{H}_t[u_1]\mathscr{H}_t[u_2] \cdots \mathscr{H}_t[u_n] \\
&= \int_0^{\Delta t} du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-1}} du_n \mathscr{H}(t+u_1)\mathscr{H}(t+u_2) \cdots \mathscr{H}(t+u_n) \\
&= \int_0^{t+\Delta t} ds_1 \int_t^{s_1} ds_2 \cdots \int_t^{s_{n-1}} ds_n \mathscr{H}(s_1)\mathscr{H}(s_2) \cdots \mathscr{H}(s_n).
\end{aligned} \tag{20}$$

The formula (14) will be used effectively in the succeeding section.

3. General decomposition scheme of ordered exponentials. In the present section, we explain how to decompose ordered exponentials into the product of ordinary exponential operators. In general, we consider the Hamiltonian $\mathscr{H}(t) = A_1(t) + \cdots + A_q(t)$. Then, the corresponding ordered exponential is expressed as

$$\mathrm{T}\left(\exp \int_t^{t+\Delta t} \mathscr{H}(s)ds\right) = \exp[\Delta t(A_1(t) + \cdots + A_q(t) + \mathscr{T})], \tag{21}$$

using the super-operator $\mathscr{T}$ defined by (11). According to the general decomposition theory[1)–5)] of the ordinary exponential operator, the first-order decomposition of (21) is given by

$$\begin{aligned}
U_1(t+\Delta t, t) &= e^{\frac{\Delta t}{2}\mathscr{T}} e^{\Delta t A_1(t)} \cdots e^{\Delta t A_q(t)} e^{\frac{\Delta t}{2}\mathscr{T}} \\
&= e^{\Delta t A_1\left(t+\frac{\Delta t}{2}\right)} \cdots e^{\Delta t A_q\left(t+\frac{\Delta t}{2}\right)}.
\end{aligned} \tag{22}$$

The case $q = 2$ is reduced to (7). There is also another first-order decomposition of the form

$$U'_1(t+\Delta t, t) = e^{\Delta t A_1(t)} \cdots e^{\Delta t A_q(t)} e^{\Delta t\mathscr{T}} = e^{\Delta t A_1(t+\Delta t)} \cdots e^{\Delta t A_q(t+\Delta t)}. \tag{23}$$

There are many other first-order decomposition formulas, as is easily seen from the above two examples (22) and (23). Thus, the way of decomposition is not unique. However, the decomposition (22) with the middlet-point values $\left\{A_j\left(t+\dfrac{\Delta t}{2}\right)\right\}$ may be the most accurate first-order formula.

Similarly, the second-order decomposition $U_2(t+\Delta t, t)$ is derived as

$$\begin{aligned}
U_2(t+\Delta t, t) &= e^{\frac{\Delta t}{2}\mathscr{T}} e^{\frac{\Delta t}{2}A_1(t)} \cdots e^{\frac{\Delta t}{2}A_{q-1}(t)} e^{\Delta t A_q(t)} e^{\frac{\Delta t}{2}A_{q-1}(t)} \cdots e^{\frac{\Delta t}{2}A_1(t)} e^{\frac{\Delta t}{2}\mathscr{T}} \\
&= e^{\frac{\Delta t}{2}A_1\left(t+\frac{\Delta t}{2}\right)} \cdots e^{\frac{\Delta t}{2}A_{q-1}\left(t+\frac{\Delta t}{2}\right)} e^{\Delta t A_q\left(t+\frac{\Delta t}{2}\right)} e^{\frac{\Delta t}{2}A_{q-1}\left(t+\frac{\Delta t}{2}\right)} \cdots e^{\frac{\Delta t}{2}A_1\left(t+\frac{\Delta t}{2}\right)}.
\end{aligned} \tag{24}$$

The case $q = 2$ is nothing but (9). Higher-order decomposition formulas can be constructed as the product of first-order approximants $\{Q(p_j\Delta t)\}$, namely

$$U_m(t+\Delta t, t) = Q(p_r\Delta t)Q(p_{r-1}\Delta t) \cdots Q(p_2\Delta t)Q(p_1\Delta t) \tag{25}$$

for some appropriate positive integer r, where $p_1 + p_2 + \cdots + p_r = 1$. Here, $Q(x)$ is given by the following super-operator

$$Q(x) = e^{\frac{x}{2}\mathscr{T}} e^{xA_1(t)} \cdots e^{xA_q(t)} e^{\frac{x}{2}\mathscr{T}}. \tag{26}$$

The parameters $\{p_j\}$ are determined[1)–5)] so that $U_m(t+\Delta t, t)$ may become of the m-th order of Δt, as will be discussed later generally. Thus, we obtain

$$U_m(t+\Delta t, t) = Q(p_r\Delta t; t_r) \cdots Q(p_2\Delta t; t_2)Q(p_1\Delta t; t_1), \tag{27}$$

where

$$t_j = t + \left(p_1 + p_2 + \cdots + p_{j-1} + \frac{1}{2}p_j\right)\Delta t \text{ and } Q(\Delta t; t) = e^{\Delta t A_1(t)} \cdots e^{\Delta t A_q(t)}. \tag{28}$$

There is also another decomposition of the form

$$U'_m(t+\Delta t, t) = Q(p_r\Delta t; t'_r) \cdots Q(p_2\Delta t; t'_2)Q(p_1\Delta t; t'_1), \tag{29}$$

Where $t'_j = t + (p_1 + p_2 + \cdots + p_j)\Delta t$.

Higher-order decomposition formulas can also be constructed as the product of second-order symmetric approximants $\{S_2(p_j\Delta t)\}$, namely

$$U_m^{(s)}(t+\Delta t,\, t) = S_2(p_r\Delta t)\cdots S_2(p_2\Delta t)S_2(p_1\Delta t), \tag{30}$$

where $S_2(x)$ denotes the following super-operator

$$S_2(x) = e^{\frac{x}{2}\mathcal{T}}e^{\frac{x}{2}A_1(t)}\cdots e^{\frac{x}{2}A_{q-1}(t)}e^{xA_q(t)}e^{\frac{x}{2}A_{q-1}(t)}\cdots e^{\frac{x}{2}A_1(t)}e^{\frac{x}{2}\mathcal{T}}. \tag{31}$$

The parameters $\{p_j\}$ here are different from those in (25) and they can be also determined[1)-5)] so that $U_m^{(s)}(t+\Delta t,\, t)$ may become of the m-th order of Δt. Thus, we obtain

$$U_m^{(s)}(t+\Delta t,\, t) = S(p_r\Delta t\,;t_r)\cdots S(p_2\Delta t\,;t_2)S(p_1\Delta t\,;t_1), \tag{32}$$

where

$$S(x\,;t) = e^{\frac{x}{2}A_1(t)}\cdots e^{\frac{x}{2}A_{q-1}(t)}e^{xA_q(t)}e^{\frac{x}{2}A_{q-1}(t)}\cdots e^{\frac{x}{2}A_1(t)}. \tag{33}$$

The above scheme can be extended to expressing any kind of higher-order decomposition as the product of higher-order approximants $\{F_s(p_j\Delta t)\}$ for $s\geq 3$. Thus, we arrive finally at the following conclusion.

Middle-point decomposition scheme. A typical m-th order decomposition formula of the ordered exponential (21) is obtained immediately from the corresponding m-th order decomposition formula for the ordinary exponential operator, by replacing $\{A_j\}$ in $Q(x)$ and $S(x)$ by the middle-point operators $\{A_j(t_{j,\mathrm{mid}})\}$ at each time separation for $Q(x)$ (or $\tilde{Q}(x)=Q^{-1}(x)$ and $S(x)$.

Of course, there are many other kinds of decomposition, as was mentioned already, but the abve decomposition scheme is systematic and it will be very convenient for practical applications.

4. General decomposition theory of ordinary exponential operators. As the decomposition problem of ordered exponentials has been reduced to that of ordinary exponential operators in the preceding section, we present here a general decomposition theory[1)-5)] of ordinary exponential operators.

Let $G_j(x)\,(j=1,2,\cdots,r)$ be first-order approximants of the original exponential operator

$$e^{x\mathcal{H}} \equiv e^{x(A_1+A_2+\cdots+A_q)}. \tag{34}$$

Furthermore we assume that $G_j(x)$ is expressed as

$$G_j(x) = e^{x\mathcal{H}+\varepsilon_{j2}x^2R_2+\varepsilon_{j3}x^3R_3+\cdots}. \tag{35}$$

That is, each $G_j(x)$ differs from each other only by the modular factors $\{\varepsilon_{jk}\}$. For example, the tilde operator $\tilde{G}(x)\equiv G^{-1}(-x)$ is expressed[4)] as

$$\begin{aligned}\tilde{G}(x) &= [e^{(-x)\mathcal{H}+(-x)^2R_2+(-x)^3R_3+\cdots}]^{-1}\\ &= e^{x\mathcal{H}-x^2R_2+x^3R_3-x^4R_4+\cdots-(-1)^nx^nR_n+\cdots}.\end{aligned} \tag{36}$$

That is, we may put $\varepsilon_{jk}=1$ for $G(x)$ and $\varepsilon_{jk}=(-1)^{k-1}$ for $\tilde{G}(x)$.

Now we make the following general decomposition

$$F_m(x) = G_1(p_1x)G_2(p_2x)\cdots G_r(p_rx) \tag{37}$$

with (35), and with $p_1+p_2+\cdots+p_r=1$. The parameters $\{p_r\}$ should be determined[1)-5)] such that $F_m(x)$ is of the m-th order as follows. Using the "time-ordering" operation P with respect to the subscript j, we rewrite $F_m(x)$ as

$$\begin{aligned}F_m(x) &= \Pi_{j=1}^r G_j(p_jx) = \Pi_{j=1}^r \exp[(xp_j)\mathcal{H}+(xp_j)^2\varepsilon_{j2}R_2+\cdots]\\ &= \mathrm{P}\exp(\Sigma_{j=1}^r[(xp_j)\mathcal{H}+(xp_j)^2\varepsilon_{j2}R_2+\cdots])\\ &= \Sigma\frac{x^{n_1+2n_2+3n_3+\cdots}}{n_1!n_2!n_3!\cdots}\mathrm{PS}(Y_1^{n_1}Y_2^{n_2}Y_3^{n_3}\cdots).\end{aligned} \tag{38}$$

Here, the symbol S denotes Kubo's symmetrization operation[9)] with respect to the operators $\{Y_n\}$, and

$$Y_1 = \sum_{j=1}^r(p_j\mathcal{H}) \text{ and } Y_n = \sum_{j=1}^r\varepsilon_{jn}(p_j^nR_n). \tag{39}$$

Here, p_j and p_j^n in (39) should not be separated from $\mathscr{H}$ and R_n, respectively, until the "time-ordering" operation P is performed.

As was shown in the previous papers[3)–5)], Eq. (38) can be easily rewritten as

$$F_m(x) = e^{x\mathscr{H}} + \sum_{n_1,n_2,\cdots}' \frac{x^{n_1+2n_2+\cdots}}{n_1!n_2!} \mathrm{PS}(Y_1^{n_1}Y_2^{n_2}\cdots), \tag{40}$$

where the symbol $\sum'$ in (40) denotes the summation over n_1, n_2, $\cdots$ excluding the case $n_2 = n_3 = \cdots = 0$. Thus, the condition for $F_m(x)$ to be correct up to the m-th order is given by the following requirement

$$C_m(n_1, n_2, n_3, \cdots) \equiv \mathrm{PS}(Y_1^{n_1}Y_2^{n_2}Y_3^{n_3}\cdots) = 0 \tag{41}$$

for all non-negative integers n_1, n_2, $\cdots$ under the condition that $n_1 + 2n_2 + \cdots \leq m$, excluding the case $n_2 = n_3 = \cdots = 0$.

It should be remarked here that $\{C_m(n_1, n_2, n_3, \cdots)\}$ make[3)] the free Lie algebra under the condition that $C_k(n_1, n_2, n_3, \cdots) \equiv 0$ for all k's satisfying the inequality $k \leq m-1$. This situation greatly simplifies explicit derivations of equations for determining the parameters $\{p_j\}$, as was shown in Ref.3. The minimal number of the product factors, $r_{\min}$, is thus expressed[3)] in terms of the Möbius function for any m-th order decomposition.

The above decomposition scheme is quite general. In fact, it is reduced to decompositions on the symmetric basis $S(x)$ defined by[2)–5)]

$$S(x) = G\left(\frac{x}{2}\right)\tilde{G}\left(\frac{x}{2}\right), \tag{42}$$

if we put $G_2(x) = \tilde{G}_1(x)$, $G_4(x) = \tilde{G}_3(x), \cdots$, and $p_2 = p_1$, $p_4 = p_3, \cdots$ in (37). Many other non-symmetric decompositions can be obtained from (37). It should be noted here that many symmetric decompositions can also be derived using the first-order non-symmetric bases $\{G(x)$ and $\tilde{G}(x)\}$. This corresponds to the products obtained by Yevick *et al.*[12)–14)] in the case of $\mathscr{H} = A + B$. Thus, all the known formulas can be derived from the above general decomposition scheme.

5. Some applications of the general scheme. In the preceding two sections, we have given how to construct a general decomposition of an ordinary exponential operator and how to reinterpret it in the case of time-dependent Hamiltonians. In the present section, we explain some examples.

If the Hamiltonian $\mathscr{H}(t)$ is composed of the two operators A and $B(t)$ (where A does not depend on time t), then the situation becomes simpler. For example, when an m-th order decomposition of an ordinary exponential operator is given in the form

$$Q^{(m)}(x) = \prod_{j=1}^{r} e^{b_jxB}e^{a_jxA} \tag{43}$$

for some appropriate decomposition parameters $\{a_j\}$ and $\{b_j\}$, the corresponding m-th order decomposition of the ordered exponential (21) is given by

$$U_m(t+\Delta t, t) = \prod_{j=1}^{r} \exp[b_j\Delta tB(t + c_j\Delta t)]\exp(a_j\Delta tA). \tag{44}$$

Here the parameters $\{c_j\}$ are given by $c_j = a_1 + a_2 + \cdots + a_{j-1} + \frac{1}{2}a_j$ for the middle-point decomposition scheme, and $c_j = a_1 + a_2 + \cdots + a_j$ for the decomposition scheme (29). The second special case was derived by Glasner *et al.*[14)] analytically for $m = 4$ and numerically for $m = 6$. Thus, their conjecture on the general case m has now been confirmed by the present general theory.

For the ordinary exponential operators, the following third-order decomposition formula has been proposed by the present author[4)]:

$$F_3(x) = e^{xp_5A}e^{x(p_4+p_5)B}e^{x(p_3+p_4)A}e^{x(p_2+p_3)B}e^{x(p_1+p_2)A}e^{xp_1B}, \tag{45}$$

where $p_1 = p_5 = 0.2683300957817599\cdots$, $p_2 = p_4 = 0.651331427235699\cdots$, and $p_3 = -0.8393230460347997$. The corresponding third-order decomposition $U_3(t + \Delta t, t)$ is given as

$$U_3(t+\Delta t, t) = Q\Big(p_1\Delta t\,; t+\Big(1-\frac{1}{2}p_1\Big)\Delta t\Big)\tilde{Q}\Big(p_2\Delta t\,; t+\Big(1-p_1-\frac{1}{2}p_2\Big)\Delta t\Big)$$
$$\times\, Q\Big(p_3\Delta t\,; t+\frac{1}{2}\Delta t\Big)\tilde{Q}\Big(p_2\Delta t\,; t+\Big(p_1+\frac{1}{2}p_2\Big)\Delta t\Big)Q\Big(p_1\Delta t\,; t+\frac{1}{2}p_1\Delta t\Big), \tag{46}$$

where

$$Q(x\,; t) = e^{xA(t)}e^{xB(t)} \text{ and } \tilde{Q}(x, t) = e^{xB(t)}e^{xA(t)}. \tag{47}$$

The above formula (46) has been derived by Terai and Ono[15] by extending the present author's formula (45) to the case of time-dependent Hamiltonians. They have applied the formula (46) to the dynamics of solitons. Other kinds of time-dependent decomposition formulas have been used by Bandrauk *et al.*[11] in quantum optics.

6. Discussion. The propagator $U(t', t)$ for a finite time difference $(t'-t)$ can be calculated as $U(t', t) = U(t', t+(n-1)\Delta t)U(t+(n-1)\Delta t, t+(n-2)\Delta t)\cdots U(t+2\Delta t, t+\Delta t)U(t+\Delta t, t)$ with $t'-t\equiv n\Delta t$. In the case of classical Hamiltonian dynamics described by a Hamiltonian $\mathscr{H}(\mathbf{p}, \mathbf{q}, t)$, it is well known that the system can be transformed into another "time-independent" system by introducing[16] the canonically conjugate pair (φ, p_φ). The new Hamiltonian $\tilde{\mathscr{H}}$ is given by

$$\tilde{\mathscr{H}} = \mathscr{H} + \frac{\partial}{\partial t}(\mathscr{F}) = \mathscr{H} + p_\varphi \tag{48}$$

for the generating function $\mathscr{F} = p_\varphi t$. The equation of motion of the new variable φ is given by $\varphi = \partial\mathscr{F}/\partial p_\varphi = t$. Thus, we can apply the previous time-independent decomposition formulas even to time-dependent problems. This corresponds to the present case of time-dependent quantum Hamiltonians. However, the present quantum formulation will be more transparent. The present scheme can also be extended to a more general case in which the time-dependent operators $\{A_j(t)\}$ are determined self-consistently, by using lower-order formulas to calculate $\{A_j(t+\frac{1}{2}\Delta t)\}$ in higher-order formulas, as in Ref. 15.

The convergence of the preset general decomposition of ordered exponentials is also proven in the same way as in the case[7] of ordinary exponential operators in a Banach space in which $\mathscr{H}(t)$ is bounded in the whole region of time.

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