Homework 6

CFRM 505

Due Wednesday, March 15 at 11:59pm

1 Problem 1

Consider the problem of estimating

$$\theta = \mathbb{P}[X > a]$$

where X is some random variable and a is a constant. We will assume that a is very large, so that θ is very small. Define

$$f_X(x;\beta) = \frac{e^{\beta x} f_X(x)}{Z(\beta)}$$

where

$$Z(\beta) = \int_{-\infty}^{\infty} e^{\beta x} f_X(x)\,\mathrm{d}x$$

and let $\mathbb{E}_\beta$ denote expectation with respect to the density $f_X(x;\beta)$. (This is the same as our notation from class.) You can assume that the integral in $Z(\beta)$ converges for any value of β.

In class, we showed that

$$\theta = \mathbb{E}_\beta\left[1_{(a,\infty)}(X)e^{-\beta X + \ln Z(\beta)}\right]$$

and we derived a formula for the variance of our importance sampling estimator in the form

$$\mathrm{Var}_\beta\left[1_{(a,\infty)}(X)e^{-\beta X + \ln Z(\beta)}\right] = F(\beta) - \theta^2$$

where

$$F(\beta) = \int_{-\infty}^{\infty} 1_{(a,\infty)}(x)e^{-\beta x + \ln Z(\beta)} f_X(x)\,\mathrm{d}x$$

(You should confirm to yourself that these match the general formulas we had in class, but there is no need to re-derive them on the homework.)

1.1 Part 1

In order to minimize the variance of our importance sampling estimator, we need to minimize $F(\beta)$. Under some mild assumptions about X and a, one can show that there is a unique, optimal value of β and that it satisfies

$$F'(\beta) = 0$$

Show that the optimal β satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \ln Z(\beta) = \mathbb{E}_{-\beta}\left[X \mid X > a\right]$$

(You can assume that all integrals involved converge uniformly, and so you are free to interchange derivatives and integrals.)

1.2 Part 2

Under some mild assumptions about X, one can show that

$$\lim_{a\to\infty} \mathbb{E}_{-\beta}\left[X \mid X > a\right] = a$$

for $\beta \geq 0$. Combined with part 1, this means that we can approximate the optimal β by solving

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \ln Z(\beta) = a$$

Use this formula to approximate the optimal value of β when $X \sim \mathrm{Exp}(1)$ and $a = 10$. (We solved this problem in class and found that the true optimal value was $\beta \approx 0.90499$.)

(In this particular problem, you should find that solving this formula for β is quite simple. In general, you would typically have to resort to root-finding algorithms like `scipy.optimize.root` to solve this equation.)

2 Problem 2

Suppose $X \sim \mathrm{Exp}(2)$ and $Y \sim \mathrm{Exp}(1)$ are independent random variables. and define $Z = \max\{X, Y\}$.

Use exponential tilting on *both* X and Y to estimate

$$\mathbb{P}\left[Z > 20\right]$$

You do not have to calculate the opitmal value(s) of β, but you should give some justification for why your choice is reasonable.

Report your estimate along with a 95% confidence interval.

3 Problem 3

Suppose $X \sim N(0,1)$ and $Y \sim N(0,4)$ are independent random variables. Consider the probability

$$\theta = \mathbb{P}[V > 10]$$

where $V = X + Y$.

3.1 Part 1

Estimate θ using direct Monte Carlo simulation. Make sure you use a large enough sample size to get a valid estimate. Report your estimate along with a 95% confidence interval.

3.2 Part 2

Estimate θ using exponential tilting in V. You can use the fact that the moment generating function of a normal random variable with mean μ and variance σ^2 is given by

$$Z(\beta) = e^{\beta\mu + \sigma^2\beta^2/2}$$

You should approximate the optimal value of β using the technique outlined in problem 1. Report your estimate along with a 95% confidence interval. How much did this reduce the error from part 1?

3.3 Part 3

Repeat part 2, but this time approximate the optimal value of β using the method described in lecture 15. (I.e., approximate $F(\beta)$ and then minimize that function numerically.) Report your estimate along with a 95% confidence interval. How much did this reduce the error from part 1?

4 Problem 4

Consider the random variable

$$L = \sum_{i=1}^{m} X_i$$

where $X_i = c_i Y_i$ and the c_i are positive constants and $Y_i \sim \text{Bern}(p_i)$ are independent (but not necessarily identically distributed). We want to estimate

$$\theta = \mathbb{P}[L > \ell]$$

We already did this in class when the c_i and p_i were all the same, but now we will look at a case where they vary with i. In particular, suppose that $m = 10$ and

$$c_i = i + 1 \quad \text{and} \quad p_i = \frac{1}{i+1}$$

for $i = 1, \ldots, 10$ and $\ell = 40$.

In each part, you should implement your method with at least $100,000$ samples.

4.1 Part 1

Estimate θ using direct Monte Carlo simulation. Report your estimate along with a 95% confidence interval. (You can use the code from class as a template, so this should be easy.)

4.2 Part 2

Estimate θ using exponential tilting in each of the X_i's. Use the same value of β for each X_i. You should approximate the optimal value of β using either the approach from problem 1 or from lecture 15. (You can use the code from class as a template. The hard part of this problem is finding the optimal β.)

Hint: The moment generating function of a sum of random variables is the product of their moment generating functions.